Local codes in rank metric

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supervised by Olivier Ruatta and Ilaria Zappatore

OpeRa 2026 - Bordeaux
25 February 2026

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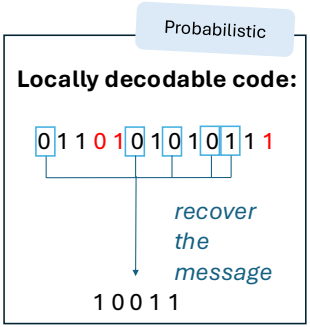

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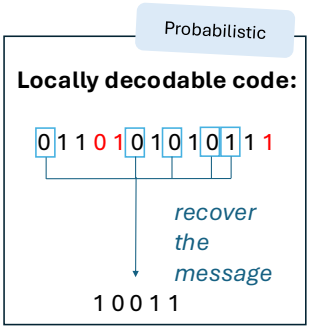

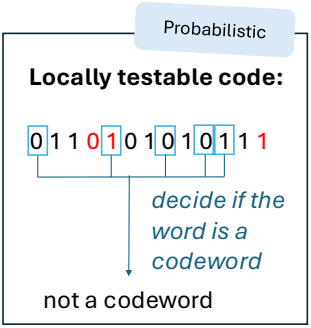

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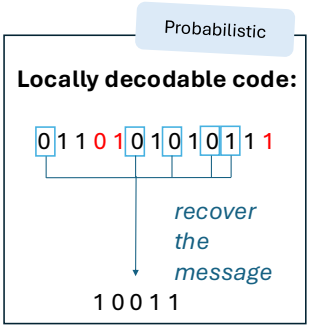

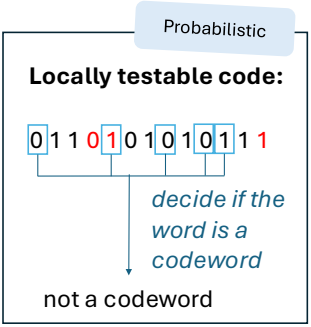

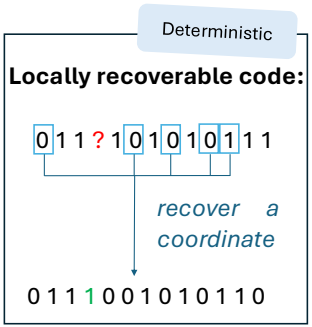

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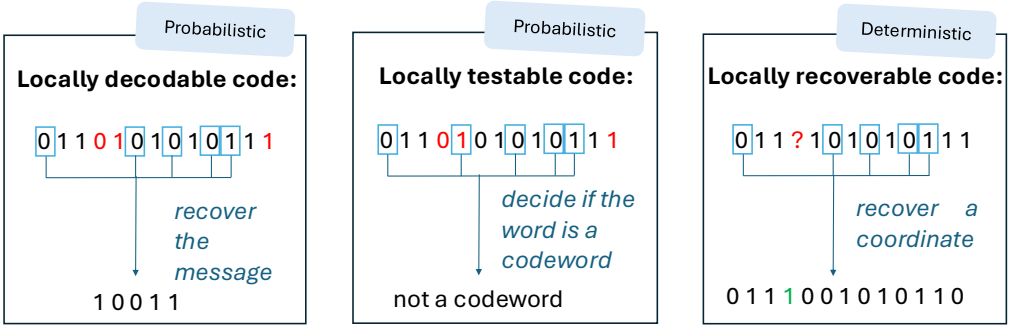

And in Rank metric ? in [Kad+16] and [BDM24]: they use coordinates as in Hamming metric.

Hamming metric vs rank metric

$\boldsymbol{y} \in \mathbb{F}_{q^m}^n$

	Hamming metric	**Rank metric**
support of $\boldsymbol{y}$	$\{i \in [n],\ y_i \neq 0\}$	$< y_1, \ \ldots\ , y_n >_{\mathbb{F}_q}$

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Queries in Hamming metric

A query on $\boldsymbol{v} \in \mathbb{F}_q^n$: one coordinate v_i of $\boldsymbol{v}$.

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$$v_i = \left\langle \boldsymbol{e}^{(i)}, \mathbf{v} \right\rangle = \underbrace{0, \dots, 0, 1, 0, \dots, 0}_{\text{Hamming weight 1}} \times \begin{pmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix}$$

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A query on $\boldsymbol{v} \in \mathbb{F}_q^n$: computation of a scalar product between $\boldsymbol{v}$ and a vector of Hamming weight one.

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A **rank** query on $\boldsymbol{v} \in \mathbb{F}_{q^m}^n$: computation of a scalar product between $\boldsymbol{v}$ and a vector of **rank** weight one.

A rank query on $\boldsymbol{v}$ is an element of the support of $\boldsymbol{v}$:

$\boldsymbol{v} \in \mathbb{F}_{q^m}^n$, $\boldsymbol{w} \in \mathbb{F}_{q^m}^n$ with $w_R(\boldsymbol{w}) = 1$.
Since $w_R(\boldsymbol{w}) = 1$, $\boldsymbol{w} = \lambda(w_1, \ldots, w_n)$, with $\lambda \in \mathbb{F}_{q^m}^*$ and $(w_1, \ldots, w_n) \in \mathbb{F}_q^n \setminus \{0\}$.

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \lambda(v_1 w_1 + \cdots + v_n w_n) \in \lambda \cdot \mathrm{supp}(\boldsymbol{v}).$$

Link with puncturing

$\boldsymbol{v} \in \mathbb{F}_{q^m}^n$.

Hamming puncturing

$I \subset [n]$. The Hamming punctured vector of $\boldsymbol{v}$ on I is the vector $\boldsymbol{v}_{[n]\setminus I} = (v_j)_{j \in [n]\setminus I}$.

A set of s Hamming queries $v_{i_1}, \ldots, v_{i_s}$ on $\boldsymbol{v} \Leftrightarrow$ the Hamming puncturing of $\boldsymbol{v}$ on $[n]\setminus I$, with $I = \{i_1, \ldots, i_s\}$.

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Rank puncturing [Ner19]

$A \in \mathrm{GL}_n(\mathbb{F}_q)$, $I \subset [n]$. The **rank punctured vector** of $\boldsymbol{v}$ with respect to A and I is the vector $(\boldsymbol{v}A)_{[n]\setminus I}$.

A set of s rank queries $q_1, \dots, q_s$ obtained from s linearly independent vectors of rank weigth one $w_1, \dots, w_s$ on $\boldsymbol{v} \Leftrightarrow$ the rank puncturing of $\boldsymbol{v}$ with respect to a set I of size $n-s$ and a matrix $A \in \mathrm{GL}_n(\mathbb{F}_q)$:

$$(\langle \boldsymbol{v}, w_1\rangle, \dots, \langle \boldsymbol{v}, w_s\rangle) = \boldsymbol{v} \cdot \Big(w_1 | \dots | w_s | (*)\Big)_{\{1,\dots,s\}}$$

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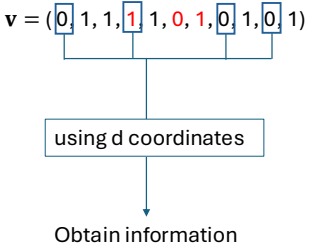
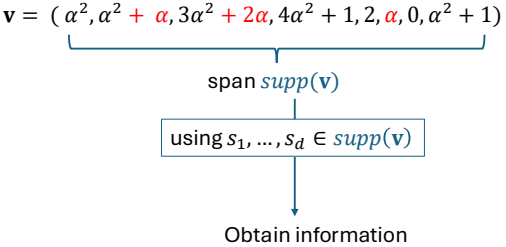
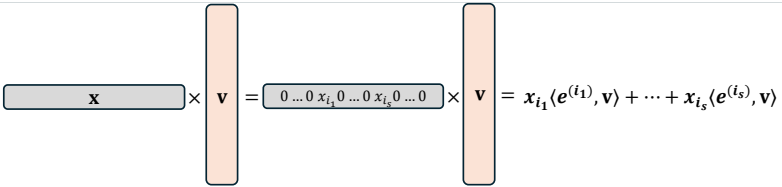




Scalar product and queries, Hamming version

$\boldsymbol{x} \in \mathbb{F}_q^n$ with $w_H(\boldsymbol{x}) = s$, $\boldsymbol{v} \in \mathbb{F}_q^n$.

We can compute $\langle \boldsymbol{x}, \boldsymbol{v} \rangle$ with s Hamming queries on $\boldsymbol{v}$:



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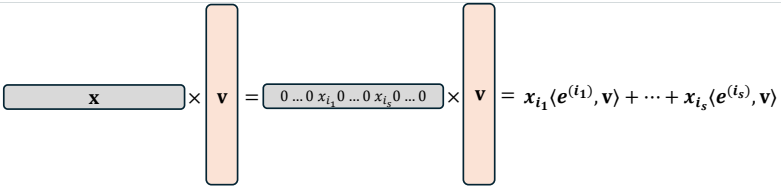

C a code such that $x \in C^{\perp}$.

- Local testability: to decide if $\boldsymbol{v} \in \mathbb{F}_q^n$ belongs to C with s Hamming queries on $\boldsymbol{v}$:
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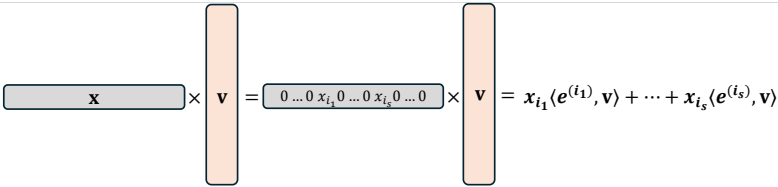

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- Local recoverability: $\forall i \in \mathit{supp}(\boldsymbol{x}) = \{i_1, \ldots, i_s\}$, $\forall c \in C$:

$$c_i = x_i^{-1} \sum_{j \in \mathrm{supp}(\boldsymbol{x}) \setminus \{i\}} x_j c_j.$$

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This means that $\mathrm{Mat}(\boldsymbol{x})_{\{\alpha_1,\ldots,\alpha_s\}} = \begin{pmatrix} x_{1,1} & \cdots & x_{n,1} \\ \vdots & \ddots & \vdots \\ x_{1,s} & \cdots & x_{n,s} \end{pmatrix}$.

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We have

$$\langle \boldsymbol{x}, \boldsymbol{v} \rangle = \alpha_1 \underbrace{\langle (x_{1,1}, \ldots, x_{n,1}), \boldsymbol{v} \rangle}_{\text{one rank query on } \boldsymbol{v}} + \cdots + \alpha_s \underbrace{\langle (x_{1,s}, \ldots, x_{n,s}), \boldsymbol{v} \rangle}_{\text{one rank query on } \boldsymbol{v}}$$

Application to local testability and local recoverability

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C code such that $x \in C^\perp$.

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- Local recoverability:
  For every $\boldsymbol{c} \in C$, every element of the support of $\boldsymbol{c}$ can be computed with $s-1$ other elements (not necessarily distinct) of the support of $\boldsymbol{c}$.
  For every $\boldsymbol{w} \in \mathbb{F}_q^n$, there exists $\alpha_1, \ldots, \alpha_s \in \mathbb{F}_{q^m}$ linearly independent over $\mathbb{F}_q$ and $(x_{i,j}) \in \mathbb{F}_q^{n \times (s-1)}$ such that for every $\boldsymbol{c} \in C$,

$$\langle \boldsymbol{c}, \boldsymbol{w} \rangle = -\alpha_i^{-1} \Big(\sum_{j=1, j \neq i}^{s-1} \alpha_j \langle (x_{1,j}, \ldots, x_{n,j}), \boldsymbol{c} \rangle \Big).$$

  (current work with Julien Lavauzelle and Ilaria Zappatore.)

Conclusion

- Extension of the notion of query to the rank metric using the notion of scalar product.
- This definition fits with the notion of rank puncturing.
- "*Part of information*" becomes "'*elements of the support*".
- In progress, with this notion of query:
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Thank you for your attention !

References I

[BDM24] Luca Bastioni, Mohamed O. Darwish, and Giacomo Micheli. *Optimal Rank-Metric Codes with Rank-Locality from Drinfeld Modules*. 2024. arXiv: 2407.06081 [cs.IT]. URL: https://arxiv.org/abs/2407.06081.

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