# Maximally $h$-Scattered Subspaces

Open Problems on Rank-Metric Codes, OpeRa 2026 – Bordeaux

Joint work with Bartoli, Csajbók, Marino, Neri

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# The Ambient Space

Fix a prime power $q$ and integers $k, m$.

**Ambient space**

We consider

$$V(k, q^m) := \mathbb{F}_{q^m}^k$$

as a vector space over $\mathbb{F}_q$.

- We study $\mathbb{F}_q$-subspaces

$$U \leq \mathbb{F}_{q^m}^k$$

  interacting with $\mathbb{F}_{q^m}$-subspaces.

# (h,r)-Evasive and $h$-Scattered Subspaces

**Definition**

Let $k, n \in \mathbb{N}$ and let $h, r$ be integers with

$$h < k, \qquad h \leq r.$$

An $\mathbb{F}_q$-subspace

$$U \subseteq V(k, q^m)$$

that generates the whole space $V(k, q^m)$, is called $(h, r)$-*evasive* if, for every $h$-dimensional $\mathbb{F}_{q^m}$-subspace $H \subseteq V(k, q^m)$,

$$\dim_{\mathbb{F}_q}(U \cap H) \leq r.$$

- If $h = r$, we say that $U$ is $h$-**scattered**.
- If $h = r = 1$, we simply say **scattered**.

# Maximum vs. Maximally $h$-Scattered

**Upper bound (maximum) [Csajbók, Marino, Polverino, Zullo 2021]**

If $U \subseteq V(k, q^m)$ is $h$-scattered and it is not a subgeometry, then

$$\dim_{\mathbb{F}_q} U \leq \frac{km}{h+1}.$$

If equality holds, $U$ is called **maximum $h$-scattered**.

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## Inclusion maximality (maximally)

An $h$-scattered subspace $U$ is **maximally $h$-scattered** if it is not properly contained in any larger $h$-scattered subspace of the same ambient space.

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## Key point

*Maximally* (by inclusion) does **not** mean *maximum* (by dimension).

# The Case $h = 1$: A First Phenomenon

**First non-maximum example [Lavrauw 2016]**

The first example of a maximally scattered subspace which is *not* maximum scattered was constructed in

$$V(2, 2^6).$$

It is a maximally scattered subspace of dimension 5, obtained computationally using the GAP package `FinInG`.

# The Case $h = 1$: A First Phenomenon

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It is a maximally scattered subspace of dimension 5, obtained computationally using the GAP package *`FinInG`*.

## Lower bound (scattered case) [Blokhuis, Lavrauw 2000]

If $U \leq V(k, q^m)$ is maximally scattered, then

$$\dim_{\mathbb{F}_q}(U) \geq \left\lceil \frac{km - m}{2} \right\rceil + 1.$$

# A General Lower Bound for any $h$

**Theorem [Bartoli, Csajbók, G., Marino, Neri]**

If $U \leq V(k, q^m)$ is maximally $h$-scattered, then

$$\dim_{\mathbb{F}_q}(U) \geq \left\lceil \frac{km - hm}{h+1} \right\rceil + h.$$

# Why am I talking about this in OpeRa©?

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Maximally $(k-1)$-scattered subspaces in $V(k,q^m)$ corresponds to non-square MRD codes which are not extendable to square MRD codes.

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**Example: Maximally scattered in $V(2, q^m)$**

$U$ is a maximally scattered subspace in $V(2, q^m)$ of dimension $m-1$

$$U = \{(x, f(x)) \ : \ x \in S\} \longrightarrow \mathcal{C}_U = \{M_{a,b} \ : \ a, b \in \mathbb{F}_{q^m}\} \subseteq \mathbb{F}_q^{m\times(m-1)}$$

where $M_{a,b}$ is the matrix associated to the $q$-linearized function

$$\phi_{a,b} : S \longrightarrow \mathbb{F}_{q^m},$$

where $\phi_{a,b}(x) = ax + bf(x)$.

# A Lunardon–Polverino Type Polynomial

Let $m = 5$.

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**Classical situation**

For $\mathrm{N}_{\mathbb{F}_{q^5}/\mathbb{F}_q}(\delta) \neq 1$ the subspace

$$\left\{(x, x^q + \delta x^{q^4}) \,:\, x \in \mathbb{F}_{q^5}\right\}$$

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## Our viewpoint

We choose $\delta$ with $\mathrm{N}_{\mathbb{F}_{q^5}/\mathbb{F}_q}(\delta) = 1$.

$$U_\delta := \left\{(x, x^q + \delta x^{q^4}) : x \in \mathbb{F}_{q^5},\ \mathrm{Tr}_{q^5/q}(x) = 0\right\}$$

# Characterisation via a Polynomial Condition

Let $\delta \in \mathbb{F}_{q^5}$ with $\mathrm{N}_{\mathbb{F}_{q^5}/\mathbb{F}_q}(\delta) = 1$ and define

$$f_\delta(y) = y^2 + \left(\delta^{q^3+q^2-q+1} - \delta^{q^2+q+1} + \delta^{q+1} - \delta - 1\right)y - \delta^{q^3+q^2+q+2} - \delta^{q+1} + \delta \ \in \ \mathbb{F}_{q^5}[y].$$

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**Proposition [Bartoli, Csajbók, G., Marino, Neri]**

The subspace $U_\delta$ is scattered of dimension 4 if and only if $f_\delta(y)$ has no roots in $\mathbb{F}_{q^5}$.

# Two Different Problems

**Step 1: Scatteredness**

We can find many $\delta \in \mathbb{F}_{q^5}$ such that $f_\delta(y)$ has no roots in $\mathbb{F}_{q^5}$. Hence there are many **rank** 4 **scattered** subspaces.

# Two Different Problems

**Step 1: Scatteredness**

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**Step 2: Maximality (the hard part)**

We must prove that $U_\delta$ is *not contained* in any scattered subspace of dimension 5.

# Classification of Maximum Scattered in $V(2,q^5)$ [Lia, Longobardi, Zanella]

**(C1) Pseudoregulus type**

$$PR_s := \{(x, x^{q^s}) : x \in \mathbb{F}_{q^5}\},\ s \in \{1, \dots, 4\};$$

**(C2) Lunardon–Polverino type**

$$LP_{s,\eta} := \{(x, x^{q^s} + \eta x^{q^{5-s}}) : x \in \mathbb{F}_{q^5}\},\ s \in \{1,2\},\ \mathrm{N}_{q^5/q}(\eta) \neq 0, 1;$$

**(C3) Trace-type constructions**

$$W_{\eta,\rho} := \{(\eta(x^q - x) + \mathrm{Tr}_{q^5/q}(\rho x), x^q - x^{q^4}) : x \in \mathbb{F}_{q^5}\},\ \eta \neq 0,\ \mathrm{Tr}_{q^5/q}(\eta) = 0 \neq \mathrm{Tr}_{q^5/q}(\rho);$$

**(C4) Norm-type constructions**

$$Z_k := \{(x, k(x^q + x^{q^3}) + x^{q^2} + x^{q^4}) : x \in \mathbb{F}_{q^5}\},\ \mathrm{N}_{q^5/q}(k) = 1.$$

It is conjectured that the last two families are empty (proven for $q \leq 25$).

# Not contained in the Pseudoregulus type

**Proposition [Bartoli, Csajbók, G., Marino, Neri]**

The subspace

$$U_\delta := \{(x, x^q + \delta x^{q^4}) : \mathrm{Tr}_{q^5/q}(x) = 0\} \leq V(2, q^5),$$

$N_{q^5/q}(\delta) = 1$, is not contained up to $\Gamma L(2, q^5)$-equivalence in any scattered subspace of the type

$$\mathrm{PR}_s := \{(x, x^{q^s}) : x \in \mathbb{F}_{q^5}\}, \quad s = 1, \dots 4.$$

**Proof**: Direct computation

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$$\mathrm{LP}_{s,\eta} := \{(x, x^{q^s} + \eta x^{q^{5-s}}) : x \in \mathbb{F}_{q^5}\}, \quad s = 1, 2, \quad N_{q^5/q}(\eta) \neq 1.$$

**Proof**: Direct computation

## (C1) Pseudoregulus type

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## (C3) Trace-type constructions

$$W_{\eta,\rho} := \{(\eta(x^q - x) + \mathrm{Tr}_{q^5/q}(\rho x), x^q - x^{q^4}) : x \in \mathbb{F}_{q^5}\},\ \eta \neq 0,\ \mathrm{Tr}_{q^5/q}(\eta) = 0 \neq \mathrm{Tr}_{q^5/q}(\rho);$$

## (C4) Norm-type constructions

$$Z_k := \{(x, k(x^q + x^{q^3}) + x^{q^2} + x^{q^4}) : x \in \mathbb{F}_{q^5}\},\ \mathrm{N}_{q^5/q}(k) = 1.$$

# Depending on the classification

**Theorem**

If $(C3)$ and $(C4)$ are empty and $\delta \in \mathbb{F}_{q^5}$ is such that $U_\delta$ is scattered, then $U_\delta$ is maximally scattered.

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If $(C3)$ and $(C4)$ are empty and $\delta \in \mathbb{F}_{q^5}$ is such that $U_\delta$ is scattered, then $U_\delta$ is maximally scattered.

Since we don't have these assumptions, we need to make the problem "easier":

let $q = 3^e$, $\delta = 1$, then $f_\delta(y) = y^2 - y - 1 = 0$.

$y^2 - y - 1$ has no roots in $\mathbb{F}_{q^5}$ if and only if $q = 3^{2h+1}$.

# Not contained in the (C3) type

**Proposition [Bartoli, Csajbók, G., Marino, Neri]**

Let $q = 3^{2h+1}$. The subspace

$$U_1 := \{(x, x^q + x^{q^4}) : \mathrm{Tr}_{q^5/q}(x) = 0\} \leq V(2, q^5),$$

is not contained in any of the subspaces in the $\Gamma L(2, q^5)$-orbit of

$$W_{\eta,\rho} := \{(\eta(x^q - x) + \mathrm{Tr}_{q^5/q}(\rho x), x^q - x^{q^4}) : x \in \mathbb{F}_{q^5}\},$$

for every $\eta, \rho \in \mathbb{F}_{q^5}$ with $\mathrm{Tr}_{q^5/q}(\eta) = 0 \neq \mathrm{Tr}_{q^5/q}(\rho)$.

**Proof**: Direct computation (a lot of computation).

### (C1) Pseudoregulus type

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### (C4) Norm-type constructions

$$Z_k := \{(x, k(x^q + x^{q^3}) + x^{q^2} + x^{q^4}) : x \in \mathbb{F}_{q^5}\},\ \mathrm{N}_{q^5/q}(k) = 1.$$

# Not contained in the (C4) type

## Proposition [Bartoli, Csajbók, G., Marino, Neri]

Let $q = 3^{2h+1}$. If $U_1$ is contained, up to $\Gamma L(2, q^5)$ equivalence in $Z_k$, $N_{q^5/q}(k) = 1$, then $u_1(k)u_2(k) = 0$ and $t(k) = 0$, where

$$
\begin{aligned}
u_1(k) &:= k^{2+2q+2q^2} + 2k^{1+2q+2q^2} + 2k^{1+2q+q^2} + k^{1+q+2q^2} + 2k^{1+q^2} + k^{q+q^2} + 2k^{q^2} + 1;\\
u_2(k) &:= k^{2+2q+2q^2} + 2k^{2+2q+q^2} + k^{2+q+q^2} + 2k^{1+2q+q^2} + k^{1+q} + 2k^{1+q^2} + 2k + 1;\\
t(k) &:= k^{1+q+q^2}(k^{3+3q+3q^2+q^3} + 2k^{3+3q+2q^2+q^3} + 2k^{2+3q+3q^2+q^3} + k^{2+2q+q^2+q^3} + k^{2+2q+q^2}\\
&\quad +2k^{2+q+2q^2+q^3} + k^{2+q+q^2+q^3} + 2k^{2+q+q^2} + k^{1+3q+2q^2+q^3} + k^{1+2q+3q^2+q^3} + 2k^{1+2q+2q^2}\\
&\quad +2k^{1+2q+q^2+q^3} + 2k^{1+q+3q^2+q^3} + k^{1+q+2q^2} + 2k^{1+q} + k^{1+2q^2+q^3} + 2k^{1+q^3} + k\\
&\quad +2k^{2q+2q^2+q^3} + k^{q+2q^2+q^3} + k^{q+q^2} + 2k^{q^2+q^3} + 2k^{q^2} + k^{q^3}).
\end{aligned}
$$

**Proof**: Direct computation (again, a lot of computation).

## Not contained in the (C4) type

**Proposition [Bartoli, Csajbók, G., Marino, Neri]**

Let $q = 3^{2h+1}$, $h > 1$. If $U_1$ is scattered and it is contained in

$$Z_k := \{(x, k(x^q + x^{q^3}) + x^{q^2} + x^{q^4}) : x \in \mathbb{F}_{q^5}\},$$

with $N_{q^5/q}(k) = 1$, then $Z_k$ is not scattered.

**Sketch of the Proof**: We need to show that there exists an element $\ell \in \mathbb{F}_{q^5}$ such that

$$\dim_{\mathbb{F}_q}(\mathrm{Ker}(\ell x + k(x^q + x^{q^3}) + x^{q^2} + x^{q^4})) = 2.$$

# Sketch of proof

$$M = \begin{pmatrix} \ell & k & 1 & k & 1 \\ 1 & \ell^q & k^q & 1 & k^q \\ k^{q^2} & 1 & \ell^{q^2} & k^{q^2} & 1 \\ 1 & k^{q^3} & 1 & \ell^{q^3} & k^{q^3} \\ k^{q^4} & 1 & k^{q^4} & 1 & \ell^{q^4} \end{pmatrix}.$$

We have that $\mathrm{rank}(M) = 3$ holds if and only if $\det(M) = 0$ and $\det(M_{1,2,3,4}^{2,3,4,5}) = 0$.

# Sketch of proof

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These conditions, together with their Frobenius conjugates, yield six equations in the variables $\ell_i \in \mathbb{F}_q$, where $\ell = \sum_{i=0}^{4} \ell_i \xi^{q^i}$ and $\{\xi, \xi^q, \dots, \xi^{q^4}\}$ is a normal basis of $\mathbb{F}_{q^5}$ over $\mathbb{F}_q$.
$\longrightarrow$ Variety $\mathcal{V}$ (actually a curve).

# Sketch of proof

**Hasse-Weil bound for curves**

$\mathcal{V}$ absolutely irreducible curve of degree $d$ defined over $\mathbb{F}_q$.
The number $\#\mathcal{C}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of $\mathcal{V}$ is

$$|\#\mathcal{V}(\mathbb{F}_q) - (q+1)| \leq (d-1)(d-2)\sqrt{q}.$$

We prove that $\mathcal{V}$ is irreducible, using Hasse-Weil bound we obtain that when $q = 3^{2h+1}$ with $h > 1$, then $\mathcal{V}$ has at least a point $(\ell_0, \ell_1, \ell_2, \ell_3, \ell_4)$, and so that there exists an $\ell$ such that $\mathrm{rank}(M) = 3 \longrightarrow Z_k$ is not scattered. □

# Main result

**Main Theorem**

Let $q = 3^{2h+1}$, with $h > 1$. The subspace

$$U_1 := \{(x, x^q + x^{q^4}) : \mathrm{Tr}_{q^5/q}(x) = 0\} \leq V(2, q^5)$$

is maximally scattered.

# Open problems

- Are the families (C3) and (C4) empty for any $q$?

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  is maximally scattered ($m$ odd)?
  (No partial classification of scattered subspaces in $V(2, q^m)$).

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- $$U_\delta = \left\{(x, x^q + \delta x^{q^{m-1}}) : x \in \mathbb{F}_{q^m}, \mathrm{Tr}_{q^m/q}(x) = 0\right\}$$
  is maximally scattered ($m$ odd)?
  (No partial classification of scattered subspaces in $V(2, q^m)$).
- Maximally $h$-scattered subspaces in $V(k, q^n)$ whith $h > 1$ or $k \geq 3$? We found a sporadic example of a maximally $2$-scattered in $V(3, 2^7)$ of dimension $6$ using MAGMA.

# Thanks!

Question?