A linear programming bound for sum-rank-metric codes

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Sum-rank metric and the size of a sum-rank-metric code

The **sum-rank-metric space** is

$$\mathbb{F}_q^{n_1 \times m_1} \times \cdots \times \mathbb{F}_q^{n_t \times m_t}$$

with **sum-rank distance** between $\mathrm{X} := (X_1, \ldots, X_t)$ and $\mathrm{Y} := (Y_1, \ldots, Y_t)$:

$$\mathrm{srkd}(\mathrm{X},\mathrm{Y}) = \sum_{i=1}^{t} \mathrm{rk}(X_i - Y_i).$$

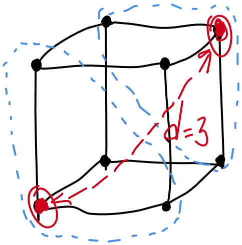
- t = the number of matrices in the tuple;
- $\mathbf{n} = [n_1, \ldots, n_t]$ and $\mathbf{m} = [m_1, \ldots, m_t]$ are sizes of the matrices;
- q = the size of the field.

A **sum-rank-metric code** $\mathcal{C}$ **with minimum distance** d is a subset of a sum-rank-metric space such that:

$$\min_{X,Y\in\mathcal{C}} \mathrm{srkd}(X,Y) = d.$$

NB! The code is non-linear in general.

Question: What is the maximal size of a sum-rank-metric code with minimum distance d?



Many upper bounds on the code size are known:

- **Byrne, Gluesing-Luerssen, Ravagnani, 2021**: classical coding arguments (Singleton, Gilbert-Varshamov, sphere-packing bounds, etc.)
- **Abiad, K, Ravagnani, 2024**: a spectral-graph-theoretical bound on the independence number.
- **Ott, Puchinger, Bossert, 2021**; **Abiad, Reijnders, Tait, 2025+**: simplifications/improvements on GV and SP bounds.
- $\cdots$
- **Abiad, Gavrilyuk, K, Ponomarenko, 2025**: Delsarte's LP approach in sum-rank metric.

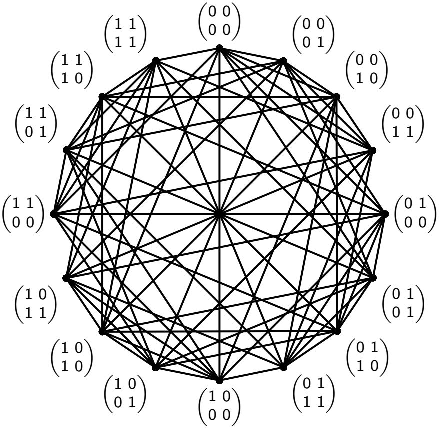
Outline

Sum-rank-metric graph

Sum-rank-metric graph $\Gamma := \Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$, $\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$, with $m_i \geq n_i$ and $m_1 \geq \cdots \geq m_t$:

- *vertices* of Γ = elements of the metric space (t-tuples of matrices from $\mathbb{F}_q^{\mathbf{n} \times \mathbf{m}}$);
- *edge* if the distance is exactly 1: for $\mathrm{X} := (X_1, \ldots, X_t)$ and $\mathrm{Y} := (Y_1, \ldots, Y_t)$,

$$\mathrm{srkd}(\mathrm{X},\mathrm{Y}) = \sum_{i=1}^{t} \mathrm{rk}(X_i - Y_i) = 1.$$



Sum-rank-metric graph
$\Gamma := \Gamma(2, 2, \mathbb{F}_2)$:

$V(\Gamma) =$ matrices 2×2 over $\mathbb{F}_2$.

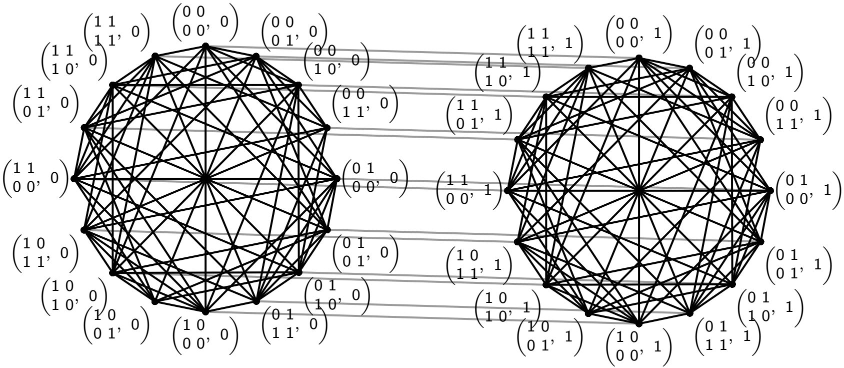
$A \sim B$ if $\mathrm{rk}(A - B) = 1$.

If $t = 1$ it is also a **bilinear forms graph**.

(Byrne, Gluesing-Luerssen, Ravagnani, 2022)
Geodesic distance between A and B in $\Gamma =$ sum-rank distance $\mathrm{srkd}(A, B)$.

Sum-rank-metric graph $\Gamma := \Gamma([2,1],[2,1],\mathbb{F}_2)$:

- vertices: (A_1, A_2), A_1 is size 2×2 over $\mathbb{F}_2$, $A_2 \in \{0,1\}$;
- edges: $(A_1, A_2) \sim (B_1, B_2)$ if $rk(A_1 - B_1) + rk(A_2 - B_2) = 1$.



$\Rightarrow$ a *Cartesian product* of the first graph $\Gamma(2,2,\mathbb{F}_2)$ and $\Gamma(1,1,\mathbb{F}_2) = K_2$.

Let $\mathbf{n} = [n_1, \ldots, n_t]$, $\mathbf{m} = [m_1, \ldots, m_t]$.

(Abiad, K, Ravagnani, 2024) The sum-rank-metric graph $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ is the Cartesian product of bilinear forms graphs $\Gamma(n_i, m_i, \mathbb{F}_q)$ for $i = 1, \ldots, t$.

The graph $\Gamma(n_i, m_i, \mathbb{F}_q)$ is a *bilinear forms graph* for $i = 1, \ldots, t$, which are very well understood. Many properties of $\Gamma(\mathbf{n}, \mathbf{m}, \mathbb{F}_q)$ can be derived through the Cartesian product connection.

Delsarte's LP approach

The Delsarte's LP bound is an efficient tool that has been used to estimate the maximal size of the code in multiple metrics:

- Hamming codes **(Delsarte, 1973)**;
- rank-metric codes **(Delsarte, 1978)**;
- bilinear alternating forms **(Delsarte, Goethals, 1975)**;
- Lee codes **(Astola, 1982)**;
- permutation codes **(Dukes, Ihringer, Lindzey, 2020)**;
- ...
- sum-rank-metric codes?

$\mathcal{A} = (X, \mathcal{R})$ is a **symmetric association scheme** on set X with relations $\mathcal{R} = \{R_0, \ldots, R_n\}$ that form a partition of $X \times X$ such that:

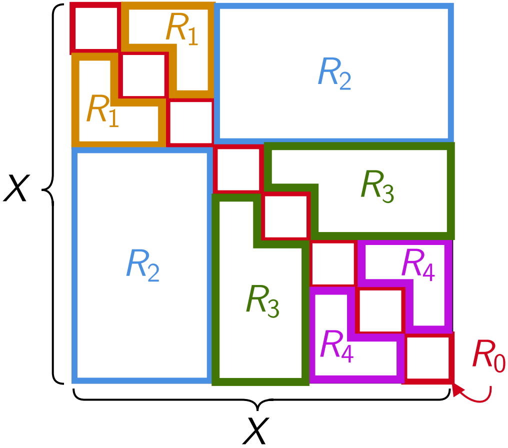
(1) R_0 consists of all (x, x) for $x \in X$.

(2) $(x, y) \in R_i$ means $(y, x) \in R_i$ for any R_i, x, y.

(3) If $(x, y) \in R_k$, then the number of z such that $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant $p^k_{i,j}$ that does not depend on the choice of x, y.

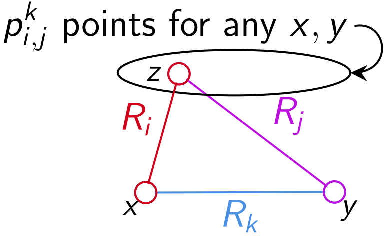
(1) R_0 consists of all (x, x) for $x \in X$.
$\Rightarrow$ R_0 is the 'diagonal' reflexive relation.

(2) $(x, y) \in R_i \Leftrightarrow (y, x) \in R_i$ for any R_i, x, y.
$\Rightarrow$ R_i's are symmetric.



For points $x, y, z \in X$:

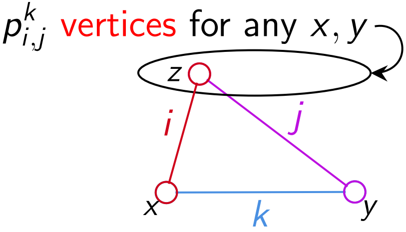
(3) If $(x, y) \in R_k$, then the number of z such that $(x, z) \in R_i$ and $(y, z) \in R_j$ does not depend on the choice of x, y.



The $p^k_{i,j}$ are **intersection numbers** of the scheme.

For vertices $x, y, z \in V(G)$:

(3) If $d_G(x,y) = k$, then the number of z such that $d_G(x,z) = i$ and $d_G(y,z) = j$ does not depend on the choice of x, y.



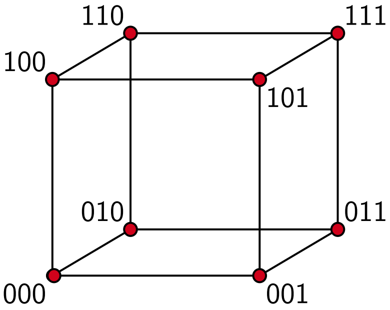
If (3) holds, the graph G is called **distance-regular**.
The $p^k_{i,j}$ are **intersection numbers** of the graph.

If G is a distance-regular graph, then $(V(G), \mathcal{R})$ is a symmetric association scheme if we define relations by:

$$(x, y) \in R_i \Leftrightarrow d_G(x, y) = i.$$

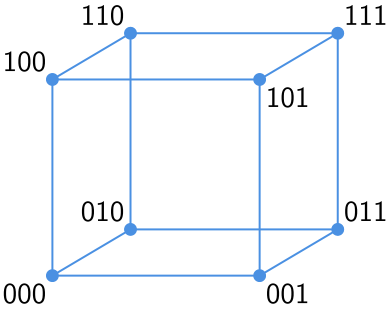
A well-known example of distance-regular graphs are *Hamming graphs*.

Vertices at distance 0:



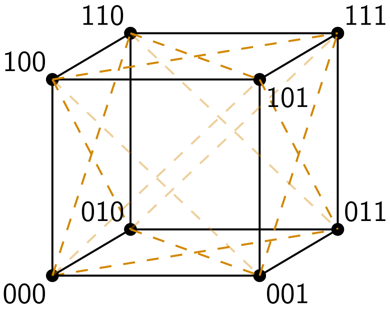
	000	001	010	011	100	101	110	111
000	R_0							
001		R_0						
010			R_0					
011				R_0				
100					R_0			
101						R_0		
110							R_0	
111								R_0

Vertices at distance 1:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1		R_1			
001	R_1	R_0		R_1		R_1		
010	R_1		R_0	R_1			R_1	
011		R_1	R_1	R_0				R_1
100	R_1				R_0	R_1	R_1	
101		R_1			R_1	R_0		R_1
110			R_1		R_1		R_0	R_1
111				R_1		R_1	R_1	R_0

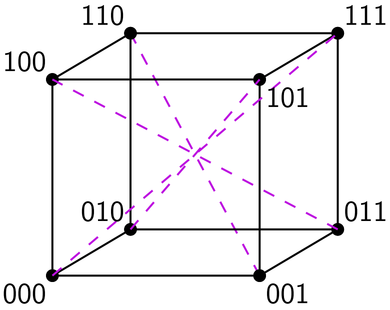
Vertices at distance 2:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1	R_2	R_1	R_2	R_2	
001	R_1	R_0	R_2	R_1	R_2	R_1		R_2
010	R_1	R_2	R_0	R_1	R_2		R_1	R_2
011	R_2	R_1	R_1	R_0		R_2	R_2	R_1
100	R_1	R_2	R_2		R_0	R_1	R_1	R_2
101	R_2	R_1		R_2	R_1	R_0	R_2	R_1
110	R_2		R_1	R_2	R_1	R_2	R_0	R_1
111		R_2	R_2	R_1	R_2	R_1	R_1	R_0

Example: Hamming scheme

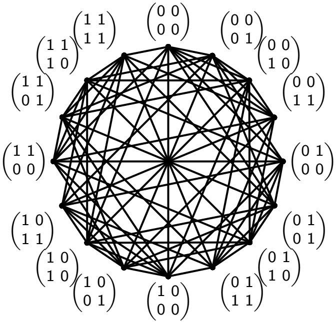
Vertices at distance 3:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1	R_2	R_1	R_2	R_2	R_3
001	R_1	R_0	R_2	R_1	R_2	R_1	R_3	R_2
010	R_1	R_2	R_0	R_1	R_2	R_3	R_1	R_2
011	R_2	R_1	R_1	R_0	R_3	R_2	R_2	R_1
100	R_1	R_2	R_2	R_3	R_0	R_1	R_1	R_2
101	R_2	R_1	R_3	R_2	R_1	R_0	R_2	R_1
110	R_2	R_3	R_1	R_2	R_1	R_2	R_0	R_1
111	R_3	R_2	R_2	R_1	R_2	R_1	R_1	R_0

Bilinear forms graphs are distance-regular.

A symmetric association scheme defined on a bilinear forms graph is called a **bilinear forms scheme**.



Delsarte's LP bound

Let $\mathcal{A} = (X; R_0, \ldots, R_n)$ be an association scheme, and let $\Delta \subseteq X$. We define the **distribution vector a** of Δ with entries $a_i = \dfrac{|(\Delta \times \Delta) \cap R_i|}{|\Delta|}$, $i = 0, \ldots, n$.

(Delsarte, 1973) For the second eigenmatrix Q of $\mathcal{A}$, we have $\mathbf{a}Q \geq \mathbf{0}$ for any Δ.

In bilinear forms schemes, the j-th column of Q consists of the eigenvalues of the adjacency matrix of R_j (known closed expressions in q, n, m).

$\Rightarrow$ **Delsarte's LP bound** on the size of a code with minimum distance d:

$$\begin{array}{lll} \text{maximize} & \sum_{i=0}^{n} a_i (= |\Delta|) & \\ \text{subject to} & \mathbf{a}Q \geq \mathbf{0}, & \\ & \mathbf{a} \geq \mathbf{0}, & \\ & a_0 = 1, & \\ & a_i = 0, & 0 < i < d. \end{array}$$

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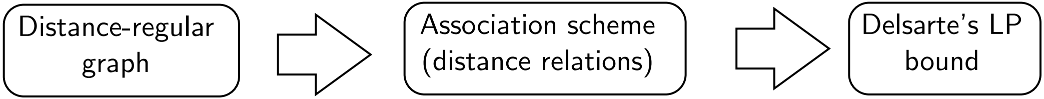
Delsarte's LP bound, 1973:

$$
\begin{array}{lll}
\text{maximize} & \sum_{i=0}^{n} a_i (= |\Delta|) & \\
\text{subject to} & \mathbf{a}Q \geq \mathbf{0}, & \\
 & \mathbf{a} \geq \mathbf{0}, & \\
 & a_0 = 1, & \\
 & a_i = 0, & 0 < i < d.
\end{array}
$$

When an association scheme is defined, one can use *Delsarte's LP* to upper bound the size of the code with given minimum distance.



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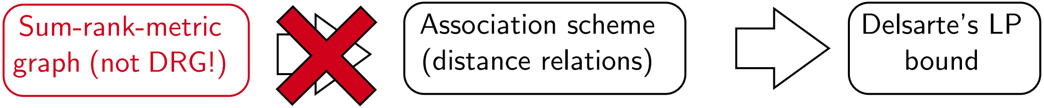
↑ The main challenge in applying Delsarte's approach in sum-rank!

But can we still apply Delsarte's LP bound?



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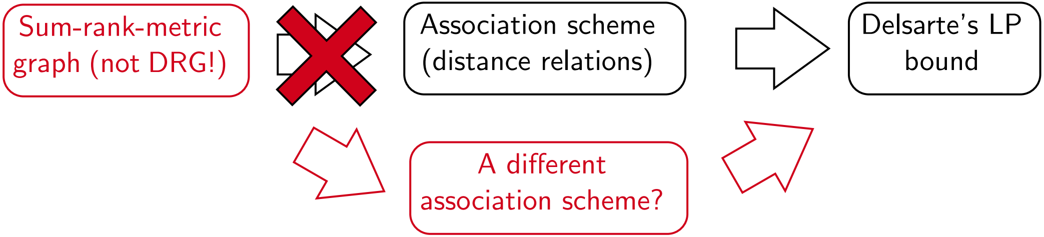

Constructing a sum-rank-metric scheme

Given two association schemes

$$\mathcal{A}_1 = (X, \{S_0, \ldots, S_{D_1}\}) \text{ and } \mathcal{A}_2 = (Y, \{T_0, \ldots, T_{D_2}\}),$$

the **direct product** $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the association scheme $(X \times Y, \mathcal{R})$ such that:

- $\mathcal{R} = \{R_{0,0}, R_{0,1}, \ldots, R_{0,D_2},\ R_{1,0}, \ldots, R_{1,D_2},\ \ldots,\ R_{D_1,0}, \ldots, R_{D_1,D_2}\}$;
- If $(x_1, x_2) \in S_i$ and $(y_1, y_2) \in T_j$, then $((x_1, y_1), (x_2, y_2)) \in R_{i,j}$.

The Q-eigenmatrix of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the Kronecker product of the Q-eigenmatrices of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Example: The Hamming cube

$$\begin{pmatrix} & 0 & 1 \\ 0 & R_0 & R_1 \\ 1 & R_1 & R_0 \end{pmatrix}$$

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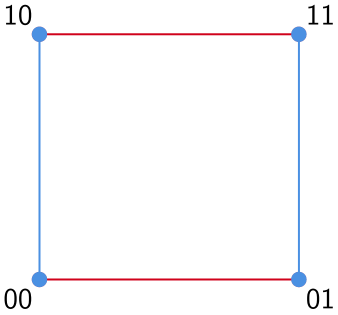
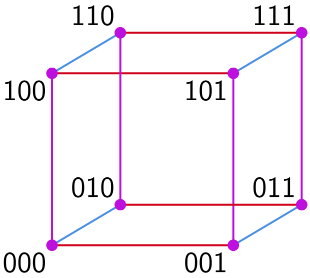
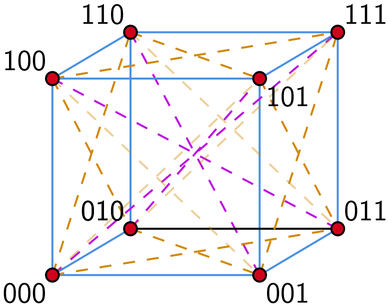


$$
\begin{pmatrix}
 & 00 & 01 & 10 & 11 \\
00 & R_{0,0} & R_{0,1} & R_{1,0} & R_{1,1} \\
01 & R_{0,1} & R_{0,0} & R_{1,1} & R_{1,0} \\
10 & R_{1,0} & R_{1,1} & R_{0,0} & R_{0,1} \\
11 & R_{1,1} & R_{1,0} & R_{0,1} & R_{0,0}
\end{pmatrix}
$$



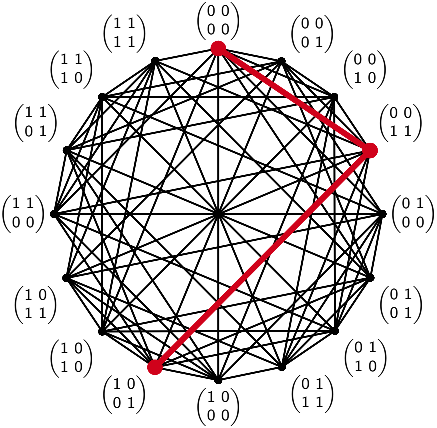
	000	001	010	011	100	101	110	111
000	$R_{0,0,0}$	$R_{0,0,1}$	$R_{0,1,0}$	$R_{0,1,1}$	$R_{1,0,0}$	$R_{1,0,1}$	$R_{1,1,0}$	$R_{1,1,1}$
001	$R_{0,0,1}$	$R_{0,0,0}$	$R_{0,1,1}$	$R_{0,1,0}$	$R_{1,0,1}$	$R_{1,0,0}$	$R_{1,1,1}$	$R_{1,1,0}$
010	$R_{0,1,0}$	$R_{0,1,1}$	$R_{0,0,0}$	$R_{0,0,1}$	$R_{1,1,0}$	$R_{1,1,1}$	$R_{1,0,0}$	$R_{1,0,1}$
011	$R_{0,1,1}$	$R_{0,1,0}$	$R_{0,0,1}$	$R_{0,0,0}$	$R_{1,1,1}$	$R_{1,1,0}$	$R_{1,0,1}$	$R_{1,0,0}$
100	$R_{1,0,0}$	$R_{1,0,1}$	$R_{1,1,0}$	$R_{1,1,1}$	$R_{0,0,0}$	$R_{0,0,1}$	$R_{0,1,0}$	$R_{0,1,1}$
101	$R_{1,0,1}$	$R_{1,0,0}$	$R_{1,1,1}$	$R_{1,1,0}$	$R_{0,0,1}$	$R_{0,0,0}$	$R_{0,1,1}$	$R_{0,1,0}$
110	$R_{1,1,0}$	$R_{1,1,1}$	$R_{1,0,0}$	$R_{1,0,1}$	$R_{0,1,0}$	$R_{0,1,1}$	$R_{0,0,0}$	$R_{0,0,1}$
111	$R_{1,1,1}$	$R_{1,1,0}$	$R_{1,0,1}$	$R_{1,0,0}$	$R_{0,1,1}$	$R_{0,1,0}$	$R_{0,0,1}$	$R_{0,0,0}$

The Hamming scheme based on distances in the graph:



	000	001	010	011	100	101	110	111
000	R_0	R_1	R_1	R_2	R_1	R_2	R_2	R_3
001	R_1	R_0	R_2	R_1	R_2	R_1	R_3	R_2
010	R_1	R_2	R_0	R_1	R_2	R_3	R_1	R_2
011	R_2	R_1	R_1	R_0	R_3	R_2	R_2	R_1
100	R_1	R_2	R_2	R_3	R_0	R_1	R_1	R_2
101	R_2	R_1	R_3	R_2	R_1	R_0	R_2	R_1
110	R_2	R_3	R_1	R_2	R_1	R_2	R_0	R_1
111	R_3	R_2	R_2	R_1	R_2	R_1	R_1	R_0

For example, in $\Gamma(2,2,\mathbb{F}_2) \times \Gamma(1,1,\mathbb{F}_2)$, the tuples $\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0\right)$ and $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)$ are in the relation $R_{2,1}$.



$\Gamma(2,2,\mathbb{F}_2)$

$K_2 = \Gamma(1,1,\mathbb{F}_2)$

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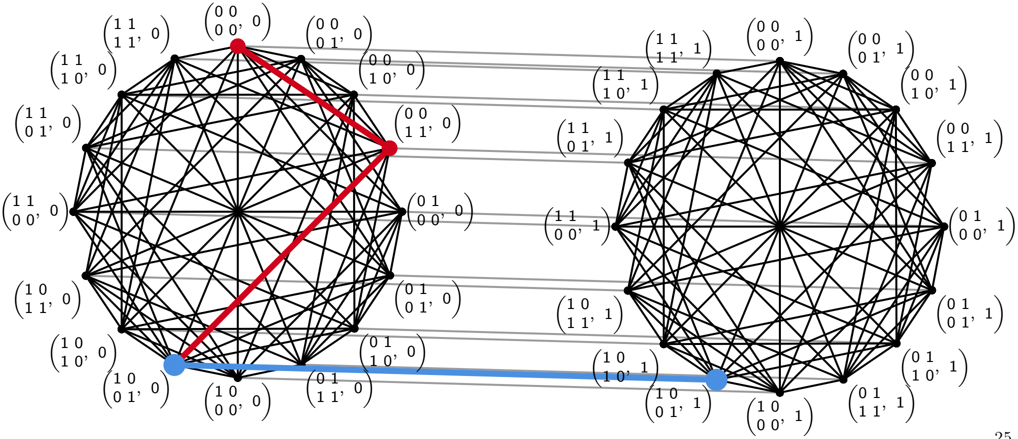

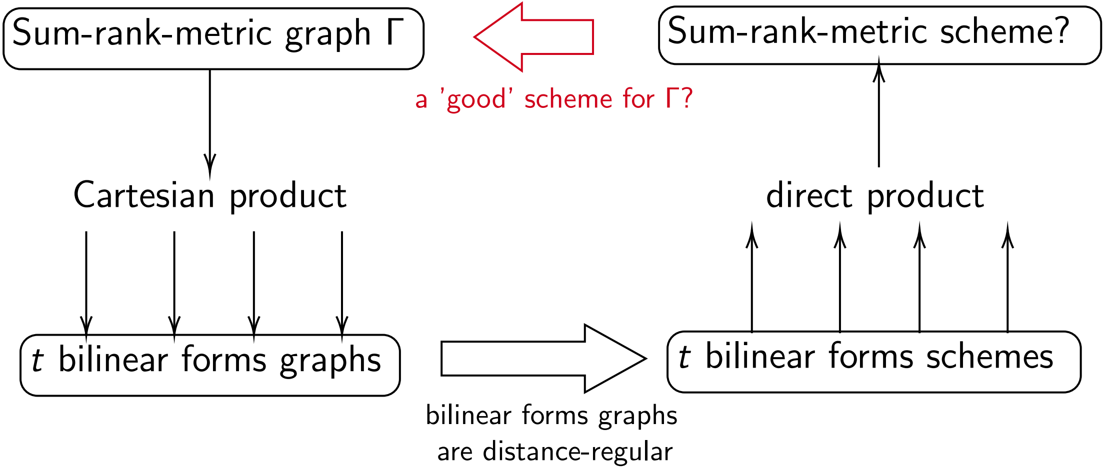
Sum-rank-metric graph Γ
Sum-rank-metric scheme?
a 'good' scheme for Γ?
Cartesian product
direct product
t bilinear forms graphs
t bilinear forms schemes
bilinear forms graphs
are distance-regular

(Abiad, Gavrilyuk, K, Ponomarenko, 2025) If the graph G is a sum-rank-metric graph which is a Cartesian product of bilinear forms graphs $G_1, \ldots, G_t$, then the *Weisfeiler-Leman closure (coherent closure)* $\mathrm{WL}(G)$ is a *fusion* of the direct product of bilinear forms schemes corresponding to $G_1, \ldots, G_t$.

(Abiad, Gavrilyuk, K, Ponomarenko, 2025, simplified) The product of bilinear forms schemes is 'good' for sum-rank-metric graphs.

$\Rightarrow$ We can define an association scheme for a sum-rank-metric graph G and apply Delsarte's LP bound to it.

Bound comparison: computational results

For $|V| \leq 10^7$ and $t \leq 7$ Delsarte's LP is never strictly outperformed.

t	q	**n**	**m**	d	$\|V\|$	Delsarte LP	RT_d	iS_d	iH_d	iE_d	S_d	SP_d	PSP_d
2	2	[2, 2]	[2, 2]	3	256	**10**	**11**	16	19	34	16	13	13
3	2	[2, 2, 1]	[2, 2, 1]	3	512	**20**	**25**	64	64	151	32	**25**	**25**
3	2	[2, 2, 1]	[2, 2, 1]	4	512	**6**	10	16	64	27	**8**	25	18
3	2	[2, 2, 1]	[2, 2, 2]	3	1024	**34**	**38**	64	64	151	64	46	46
3	2	[2, 2, 1]	[2, 2, 2]	4	1024	**8**	**15**	16	64	27	16	46	36
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	3	512	**24**	**28**	64	64	151	32	30	30
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	4	512	**6**	11	16	64	27	**8**	30	32
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	3	1024	**42**	**44**	64	64	151	64	53	53
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	4	1024	**10**	18	**16**	64	27	**16**	53	64
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	3	1024	**40**	**46**	256	215	529	64	48	48
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	4	1024	**12**	19	64	215	119	**16**	48	36
5	2	[2, 1, 1, 1, 1]	[2, 1, 1, 1, 1]	5	256	**2**	5	16	26	19	4	4	**3**
5	2	[2, 1, 1, 1, 1]	[3, 1, 1, 1, 1]	5	1024	**2**	8	64	336	240	4	6	**3**
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	3	1024	**49**	**56**	256	215	529	64	**56**	**56**
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	4	1024	**13**	22	64	215	119	**16**	56	64
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	4	512	**12**	**16**	256	512	407	**16**	34	32
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	5	512	**4**	8	64	77	99	8	6	**5**
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	5	1024	**6**	11	64	77	99	**8**	9	**8**
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	6	1024	**2**	7	16	77	14	4	9	**3**

(Schrijver, 1979) The Delsarte's LP does not perform worse than Lovász θ_k bound.

The talk is based on:

Abiad, A., Gavrilyuk, A.L., Khramova, A.P., Ponomarenko I. The linear programming bound for sum-rank-metric codes. *IEEE Transactions on Information Theory* (2025)

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Thank you for your attention!

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Delsarte's LP takes into account relations between pairs of codewords (association scheme, Bose-Mesner algebra, Q-matrix).

A more refined approach: consider triples of codewords (Terwilliger algebra, semidefinite matrix)
$\Longrightarrow$ **Schrijver's SDP bound (2005)**.

We can apply existing symmetry reduction methods to computationally obtain new bounds on smaller examples.

Appendix: SDP bound (a step beyond LP)

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We can apply existing symmetry reduction methods to computationally obtain new bounds on smaller examples.

Appendix: Computational comparison with SDP

bold entries in the SDP column = SDP strictly outperforms any other known bound (otherwise coincides with Delsarte's LP).

t	q	**n**	**m**	d	$\|V\|$	SDP	DLP$_d$	RT$_d$	iS$_d$	iH$_d$	iE$_d$	S$_d$	SP$_d$	PSP$_d$
2	2	[2, 2]	[2, 2]	3	256	**9**	**10**	**11**	16	19	34	16	13	13
3	2	[2, 2, 1]	[2, 2, 1]	3	512	**19**	**20**	**25**	64	64	151	32	**25**	**25**
3	2	[2, 2, 1]	[2, 2, 1]	4	512	**5**	**6**	10	16	64	27	**8**	25	18
3	2	[2, 2, 1]	[2, 2, 2]	3	1024	34	**34**	**38**	64	64	151	64	46	46
3	2	[2, 2, 1]	[2, 2, 2]	4	1024	8	**8**	**15**	16	64	27	16	46	36
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	3	512	**23**	**24**	**28**	64	64	151	32	30	30
4	2	[2, 1, 1, 1]	[2, 2, 2, 1]	4	512	6	**6**	11	16	64	27	**8**	30	32
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	3	1024	42	**42**	**44**	64	64	151	64	53	53
4	2	[2, 1, 1, 1]	[2, 2, 2, 2]	4	1024	**9**	**10**	18	**16**	64	27	**16**	53	64
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	3	1024	**39**	**40**	**46**	256	215	529	64	48	48
4	2	[2, 2, 1, 1]	[2, 2, 1, 1]	4	1024	**11**	**12**	19	64	215	119	**16**	48	36
5	2	[2, 1, 1, 1, 1]	[2, 1, 1, 1, 1]	5	256	2	**2**	5	16	26	19	4	4	**3**
5	2	[2, 1, 1, 1, 1]	[3, 1, 1, 1, 1]	5	1024	2	**2**	8	64	336	240	4	6	**3**
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	3	1024	**48**	**49**	**56**	256	215	529	64	**56**	**56**
5	2	[2, 1, 1, 1, 1]	[2, 2, 2, 1, 1]	4	1024	**12**	**13**	22	64	215	119	**16**	56	64
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	4	512	**11**	**12**	**16**	256	512	407	**16**	34	32
6	2	[2, 1, 1, 1, 1, 1]	[2, 1, 1, 1, 1, 1]	5	512	4	**4**	8	64	77	99	8	6	**5**
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	5	1024	**5**	**6**	11	64	77	99	**8**	9	**8**
6	2	[2, 1, 1, 1, 1, 1]	[2, 2, 1, 1, 1, 1]	6	1024	2	**2**	7	16	77	14	4	9	**3**

Appendix 2: The Q-eigenmatrix in a bilinear forms graph

By considering the Bose-Mesner algebra arising from association schemes, one can derive the Q-**eigenmatrix** of the bilinear forms scheme.
For a graph with eigenvalues $\theta_0, \ldots, \theta_n$ and intersection numbers a_i, b_i, c_i:

$$P_{ij} = p_j(\theta_i) = \frac{1}{c_j}\left((\theta_i - a_{j-1})p_{j-1}(\theta_i) - b_{j-2}p_{j-2}(\theta_i)\right), \quad p_0(\theta_i) = 1,\ p_1(\theta_i) = \theta_i.$$

In bilinear forms, $P_{ij} = Q_{ij}$, and the values θ_i, a_i, b_i, c_i are all expressed in the parameters of the graph:

$$\theta_i = \frac{(q^{n-i}-1)(q^m - q^i) - q^i + 1}{q-1}, \qquad c_i = p^i_{1,i-1} = \frac{q^{i-1}(q^i - 1)}{q-1},$$

$$b_i = p^i_{1,i+1} = \frac{q^{2i}(q^{m-i}-1)(q^{n-i}-1)}{q-1}, \qquad a_i = p^i_{1,i} = b_0 - b_i - c_i.$$