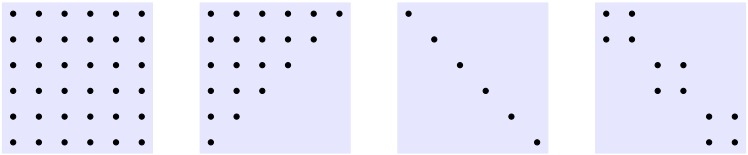
Zero-Pattern Rank-Metric Codes

Giuseppe Marino & Alessandro Neri

OpeRa 2026 - Open Problems on Rank-Metric Codes
Bordeaux – February 24th, 2026



Università degli Studi di Napoli
Federico II

Part 0 – Warm Up
Linear Spaces of Matrices: A Few Questions

Questions for the Audience

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- **Question 2.1: What is the largest dimension?** **1**
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$$\frac{(n-r+1)(n-r+2)}{2}.$$

- (Etzion, Gorla, Ravagnani, Wachter-Zeh 2016) if $|\mathbb{F}| \geq n-1$;
- (N., Stanojkovski 2024) for every $\mathbb{F}$.

Part I – Introduction
Ferrers Diagrams Rank-Metric Codes

Ferrers Diagrams

A **Ferrers (Young) diagram** $\mathcal{D}$ of order n is a subset of $[n]^2$ s.t.:

- If $(i,j) \in \mathcal{D}$, then $(i',j') \in \mathcal{D}$ for every $i' \in [i], j' \in [j]$;

Example

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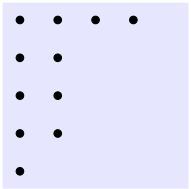

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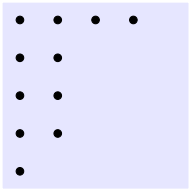

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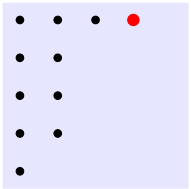

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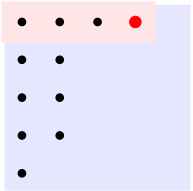

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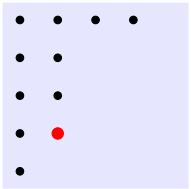

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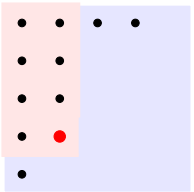

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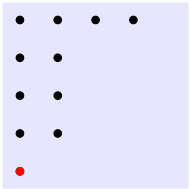

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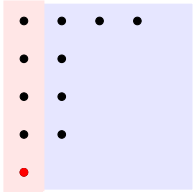

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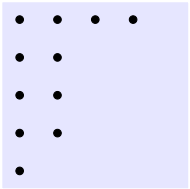

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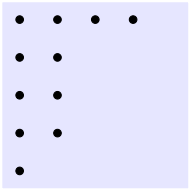

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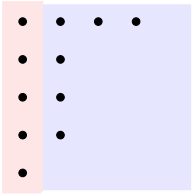

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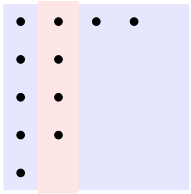

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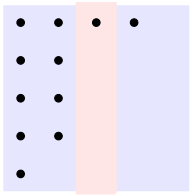

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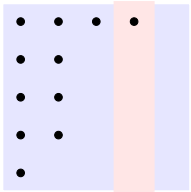

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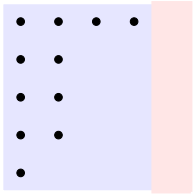

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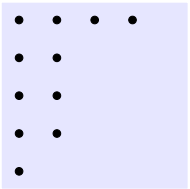

$$\mathbb{F}^{\mathcal{D}} = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 & 0 & 0 \\ a_{4,1} & a_{4,2} & 0 & 0 & 0 \\ a_{5,1} & 0 & 0 & 0 & 0 \end{pmatrix} \ : \ a_{i,j} \in \mathbb{F} \right\}.$$

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Etzion and Silberstein (2009): **Multilevel construction** of subspace codes in network coding.

Part II
The Etzion-Silberstein Conjecture

A Combinatorial Bound on the Parameters

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Singleton-like Bound (Etzion, Silberstein (2009))

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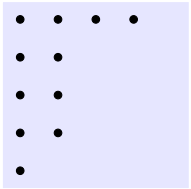

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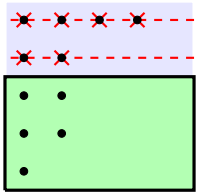

$$\nu_0(\mathcal{D}, 3) = 5$$

A Combinatorial Bound on the Parameters

Singleton-like Bound (Etzion, Silberstein (2009)

Let $\mathcal{C}$ be an $[\mathcal{D}, k, r]_{\mathbb{F}}$ code. Then

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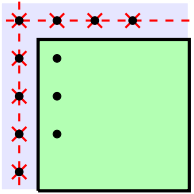

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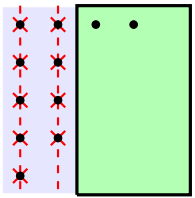

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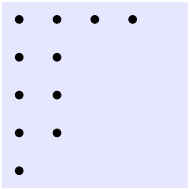

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Etzion–Silberstein Conjecture

Etzion-Silberstein (ES) Conjecture (2009)

For every **finite field** $\mathbb{F}$ the Singleton-like bound is tight:

$$\forall\, \mathcal{D}, r, \quad \exists \text{ a } [\mathcal{D}, \nu_{\min}(\mathcal{D}, r), r]_{\mathbb{F}} \text{ code}$$

Maximum Ferrers Diagram (MFD) code

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Maximum Ferrers Diagram (MFD) code

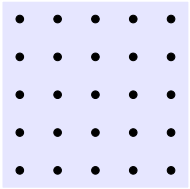
Constructive proofs for some special cases (Etzion, Silberstein, 2009), (Etzion, Gorla, Ravagnani, Wachter-Zeh, 2016), (Gorla, Ravagnani, 2017), (Antrobus, Gluesing-Luerssen, 2019), (Liu, Chang, Feng, 2019), (N., Stanojkovski, 2024), (Calderini, Messia, N., 2026+)

Still widely open!

Two Special Cases

Two Special Cases

(1) **Full size diagrams**: $\mathcal{D} = (n, n, \ldots, n) = [n]^2$



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ES conjecture holds! ✓

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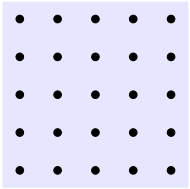

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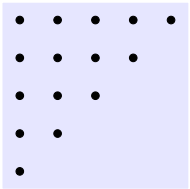

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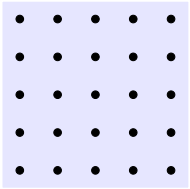

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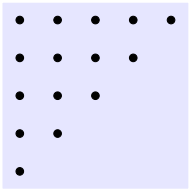

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Question

Do you wanna know more?

Listen to **Marta** and **Hugo** on Thursday!

Part III
Zero-Pattern Matrix Spaces

Other Diagrams

Question

Why **only** Ferrers diagrams?

- Application: Multilevel Construction of Subspace codes.
  Ferrers diagram matrix spaces correspond to **Schubert cells** in $\mathrm{Gr}_q(n, 2n)$
- Theoretical: Optimal codes exist (?) over **every finite field**. (ES conjecture)

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- Application: Multilevel Construction of Subspace codes.
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- Theoretical: Optimal codes exist (?) over **every finite field**. (ES conjecture)

So, why not other subsets $\mathcal{D} \subseteq [n]^2$?!

Zero-Pattern Rank-Metric Codes: Subspaces of $\mathbb{F}^{\mathcal{D}}$, where $\mathcal{D} \subset [n]^2$ is **any** subset.

Hamming and Sum-Rank Metric

We retrieve **Hamming** and **sum-rank** metric!

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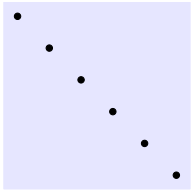

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Hamming metric

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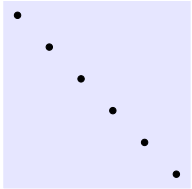
We retrieve **Hamming** and **sum-rank** metric!



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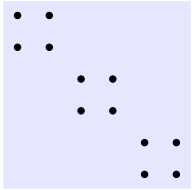
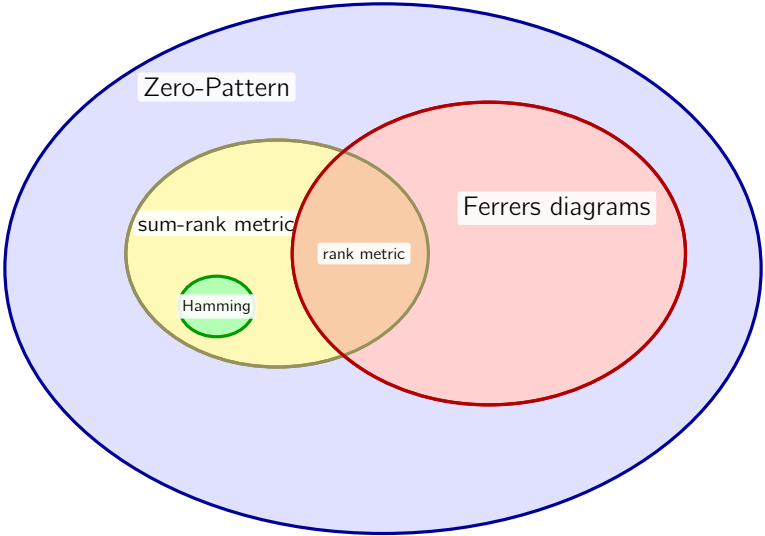


$$(\mathbb{F}^{\mathcal{D}}, \mathsf{d}_{\mathrm{rk}}) \cong ((\mathbb{F}^{m\times m})^{\ell}, \mathsf{d}_{\mathrm{srk}})$$

Sum-rank metric

Relations Between Families of Codes



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Bound on the parameters?

Same idea: canceling **any** $r-1$ between rows and cols

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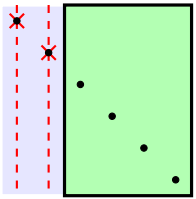

$$k \leq n - r + 1$$

MDS Codes

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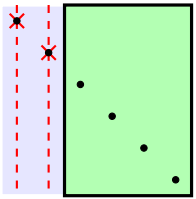
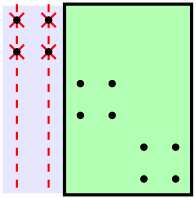


$k \leq n - r + 1$ **MDS Codes**



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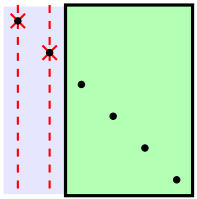

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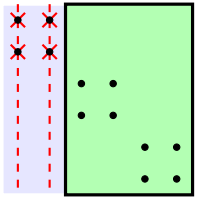

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Achtung!

Existence for $|\mathbb{F}| > \#\text{blocks}$

- **MDS conjecture**
- **MSRD conjecture?**

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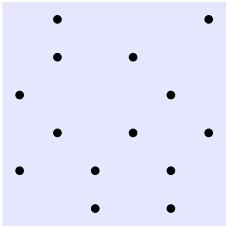

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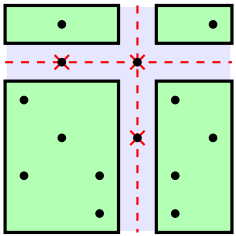

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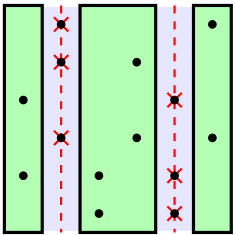

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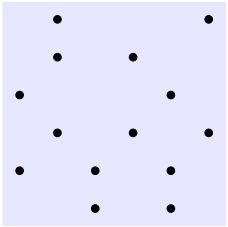

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Is it tight? Under which conditions?

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Let $\mathcal{D} \subseteq [n]^2$. Define the **associated bipartite graph** as

$$G = ([n] \sqcup [n], \mathcal{D})$$

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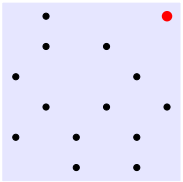

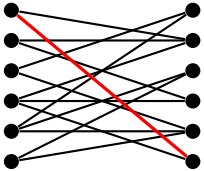

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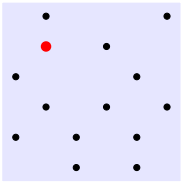

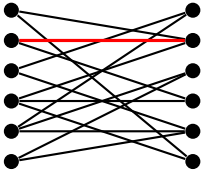

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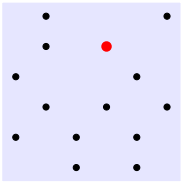

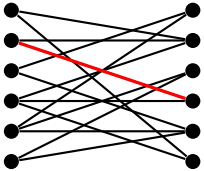

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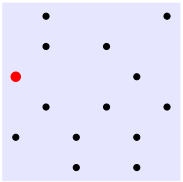

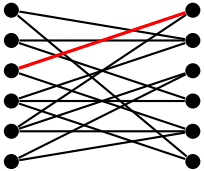

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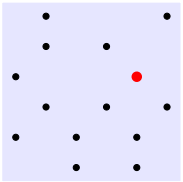

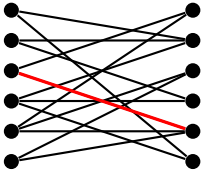

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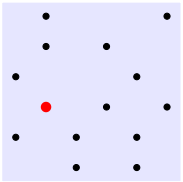

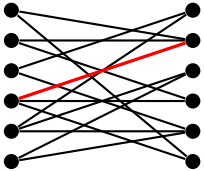

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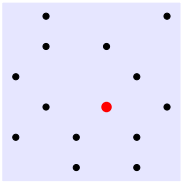

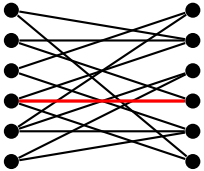

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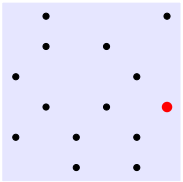

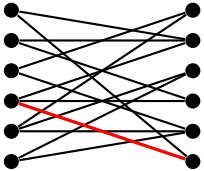

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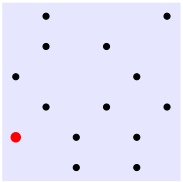

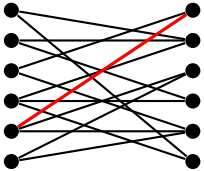

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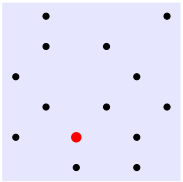
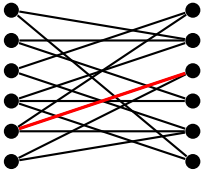
$$G = ([n] \sqcup [n], \mathcal{D})$$





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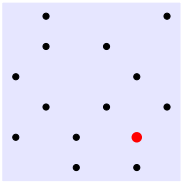

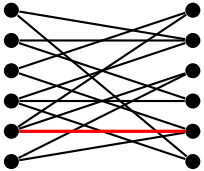

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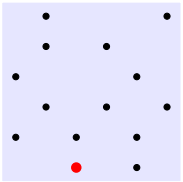

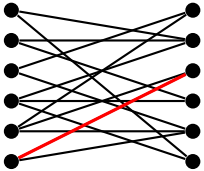

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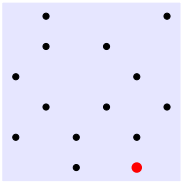

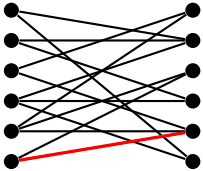

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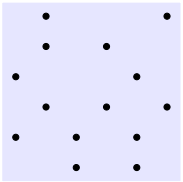

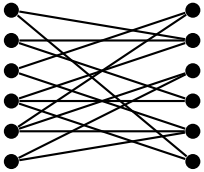

Properties of $\mathbb{F}^{\mathcal{D}}$ can be read from G.

Y. Li, Y. Qiao, A. Wigderson, Y. Wigderson, C. Zhang. "Connections between graphs and matrix spaces", Israel Journal of Mathematics, 256(2), 513-580. 2023.

Permutation Equivalence $\leftrightarrow$ Graph Isomoprhism

$G = ([n] \sqcup [n], \mathcal{D}_1)$ and $H = ([n] \sqcup [n], \mathcal{D}_{"})$ are isomorphic if and only if

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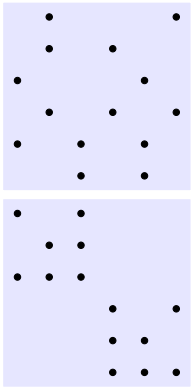
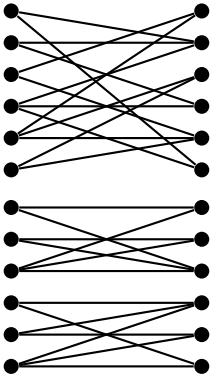
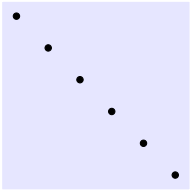




$\longleftarrow$ **Canonical form?**
Can we study each connected component independently?

Number of Connected Components?



$k \leq n - r + 1$ **MDS Codes**

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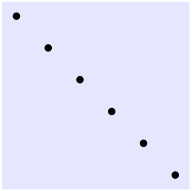

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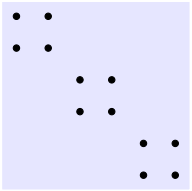

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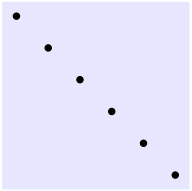

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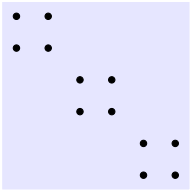

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Achtung!

Existence for

$|\mathbb{F}| > \#$**conn. comp**

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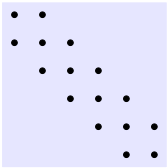

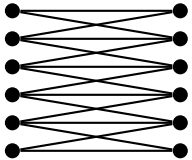

The End?

Thank you! Grazie!
Merci!

COMBINATORICS 2026

NAPLES, ITALY - MAY 25-29 2026

SPEAKERS

Anurag Bishnoi
Alain Couvreur
Tao Feng
Sam Mattheus
Gretchen L. Matthews
Maria Montanucci
Valentina Pepe
Martin Škoviera
Tommaso Traetta
Yue Zhou