On the Etzion and Silberstein conjecture for block Ferrers diagrams

Opera 2026

(joint work with Marco Calderini and Alessandro Neri)

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Notation

Let $n \in \mathbb{N}$, denote by $[n] = \{1, \dots, n\}$.

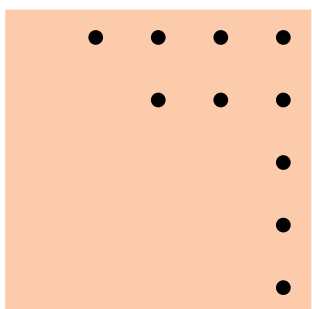
Ferrers diagram

A (top-right justified) *Ferrers diagram* $\mathcal{F}$ of order n is a subset of $[n]^2$ with the following properties

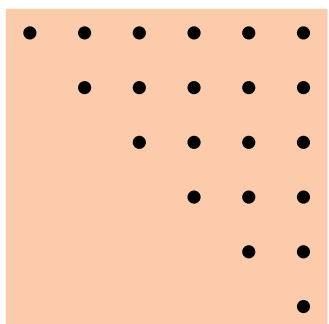
1. if $(i,j) \in \mathcal{F}$ and $j < n$, then $(i, j+1) \in \mathcal{F}$,
2. if $(i,j) \in \mathcal{F}$ and $i > 1$, then $(i-1, j) \in \mathcal{F}$.

Example

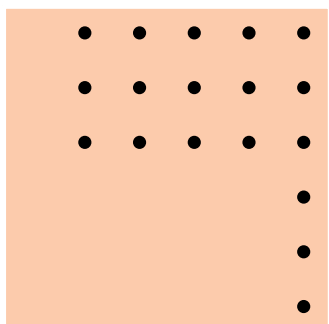
$\mathcal{F}_1 = [0,1,2,2,5]$



$\mathcal{F}_2 = [1,2,3,4,5,6]$



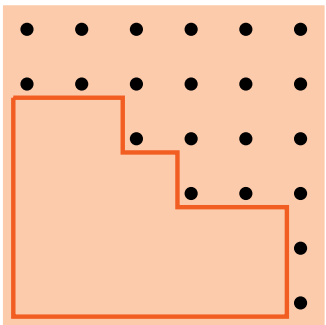
$\mathcal{F}_3 = [0,3,3,3,3,6]$



Notation

What is the relation between a diagram and a code?

$$\mathbb{F}_q^{\mathcal{F}} := \{ M \in \mathbb{F}_q^{n \times n} \ : \ \mathrm{supp}(M) \subseteq \mathcal{F} \}.$$



$\Rightarrow$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Ferrers diagrams rank-metric code

Let $\mathcal{F}$ be a Ferrers diagram of order n. An $[\mathcal{F}, k, d]_q$ *Ferrers diagram rank-metric code* is a k-dimensional linear subspace $\mathcal{C}$ of $\mathbb{F}_q^{\mathcal{F}}$. The parameter d is the *minimum rank distance* of $\mathcal{C}$,

$$d := \min\{ \mathrm{rk}(A) \ : \ A \in \mathcal{C}, A \neq 0 \}.$$

The Etzion-Silberstein conjecture

$\nu_{\min}(\mathcal{F}, d)$

Let $\mathcal{F} = [c_1, \ldots, c_n]$ be a Ferrers diagram of order n and let $d \in [n]$. For $j \in \{0, \ldots, d-1\}$ we define

$$\nu_j(\mathcal{F}, d) = \sum_{i=1}^{n-j} \max\{0,\ c_i - d + 1 + j\},$$

$$\nu_{\min}(\mathcal{F}, d) = \min\{\nu_0(\mathcal{F}, d), \ldots, \nu_{d-1}(\mathcal{F}, d)\}.$$

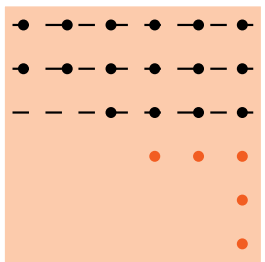
j-Singleton

The pair $(\mathcal{F}, d)$ is called *j-Singleton* if

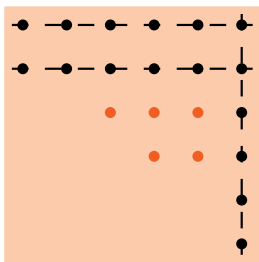
$$\nu_{\min}(\mathcal{F}, d) = \nu_j(\mathcal{F}, d).$$

Example $\mathcal{F} = [2, 2, 3, 4, 4, 6]$

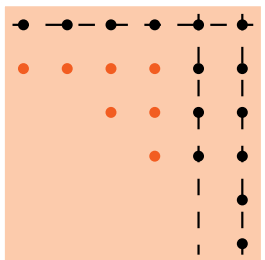
$\nu_0(\mathcal{F}, 4) = 5$



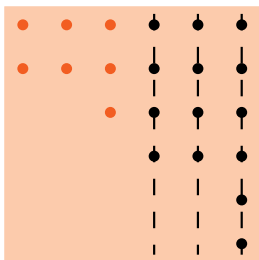
$\nu_1(\mathcal{F}, 4) = 5$



$\nu_2(\mathcal{F}, 4) = 7$



$\nu_3(\mathcal{F}, 4) = 7$



$\downarrow$

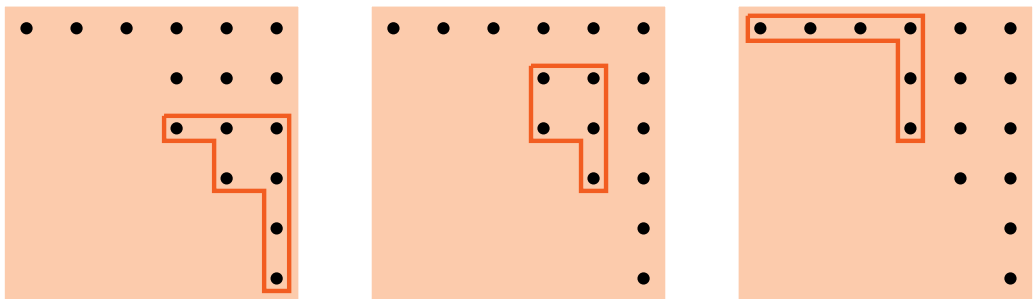
$\nu_{\min}(\mathcal{F}, 4) = 5$, the pair $(\mathcal{F}, 4)$ is 0-Singleton and 1-Sinlgeton.

The Etzion-Silberstein conjecture

Notation Let $n \in \mathbb{N}$, $d \in [n]$ and $j \in \{0, \ldots, d-1\}$.

$$\mathcal{S}_{n,d,j} = \{ (i,l) \in [n]^2 \ : \ i \in \{d-j, \ldots, n\}, l \in [n-j] \}.$$

$\mathcal{F} \cap \mathcal{S}_{6,3,0}$, $\mathcal{F} \cap \mathcal{S}_{6,3,1}$, $\mathcal{F} \cap \mathcal{S}_{6,3,2}$



Remark

$$\nu_j(\mathcal{F}, d) = |\mathcal{F} \cap \mathcal{S}_{n,d,j}| \Rightarrow \nu_{\min}(\mathcal{F}, d) = \min_{j \in \{0,\ldots,d-1\}} |\mathcal{F} \cap \mathcal{S}_{n,d,j}|$$

The Etzion-Silberstein conjecture

Theorem (Etzion-Silberstein, 2009)

Let $\mathcal{F}$ be a Ferrers diagram of order n and let $\mathcal{C}$ be an $[\mathcal{F}, d]_q$ code. Then $\dim(\mathcal{C}) \leq \nu_{\min}(\mathcal{F}, d)$.

Optimality

Let $\mathcal{F}$ be a Ferrers diagram of order n, an $[\mathcal{F}, d]_q$ code $\mathcal{C}$ is called *maximum Ferrers diagram* (MFD) if $\dim(\mathcal{C}) = \nu_{\min}(\mathcal{F}, d)$.

Etzion-Silberstein conjecture

For every Ferrers diagram $\mathcal{F}$ of order n, every integer $1 \leq d \leq n$ and every finite field $\mathbb{F}_q$ there exists an MFD $[\mathcal{F}, d]_q$ code.

Sum-rank metric codes (square case)

Notation Let $m, t \in \mathbb{N}$, $(\mathbb{F}_q^{m\times m})^t := \bigoplus_{i=1}^t \mathbb{F}_q^{m\times m}$.

Sum-rank weight

For any $X = (X_1, \ldots, X_t) \in (\mathbb{F}_q^{m\times m})^t$, the *sum-rank weight* of X is

$$\mathrm{w}_{\mathrm{srk}}(X) := \sum_{i=1}^{t} \mathrm{rk}(X_i).$$

Sum-rank metric code

An $[(m \times m)^t, k, d]_q$ *sum-rank metric code* $\mathcal{C}$ is a k-dimensional linear subspace of $(\mathbb{F}_q^{m\times m})^t$. The parameter d is the *minimum sum-rank distance* of $\mathcal{C}$,

$$d := \min\{\, \mathrm{w}_{\mathrm{srk}}(X) \,:\, X \in \mathcal{C}, X \neq 0 \,\}.$$

Sum-rank metric codes (square case)

Theorem

Let $\mathcal{C}$ be an $[(m \times m)^t, k, d]_q$ code. Then $k \leq m(tm - d + 1)$.

Maximum sum-rank distance code

Let $\mathcal{C}$ be an $[(m \times m)^t, k, d]_q$ code. The code $\mathcal{C}$ is a *Maximum Sum-Rank Distance (MSRD) code* if

$$k = m(tm - d + 1).$$

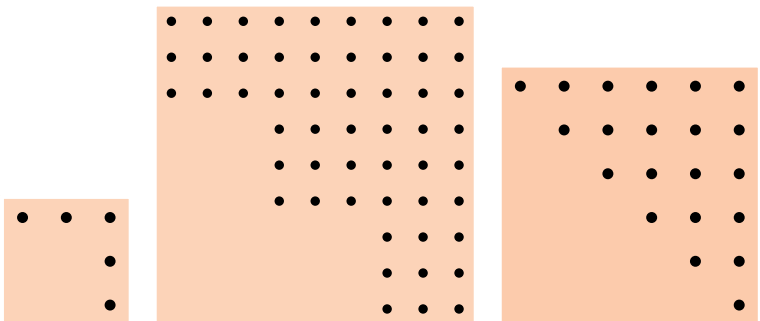
Theorem (Martinez-Peñas)

If $t \leq q - 1$, then there exists an $[(m \times m)^t, k, d]_q$ MSRD code.

Research context

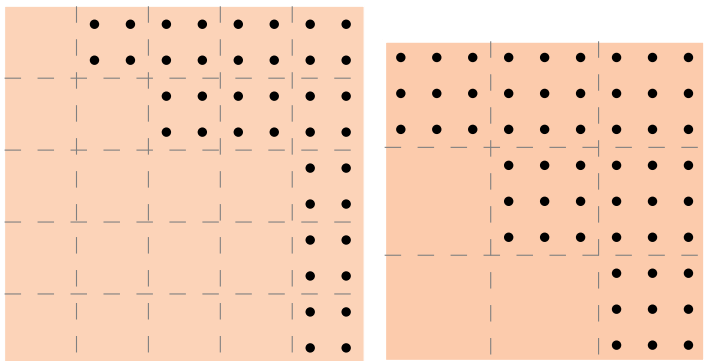
Previous works*

- Key tool:
  Hamming distance.
- Key property:
  MDS-constructibility.
  $\nu_{\min}(\mathcal{F}, \delta) = \sum_{i=1}^{n} \max\{ |D_i \cap \mathcal{F}| - \delta + 1, 0 \}$
- Key result:
  Optimality.



Our work

- Key tool:
  Sum-rank distance.
- Key property:
  MSRD-constructibility.
- Key result:
  Optimality.



*(Etzion, Silberstein, 2009), (Etzion, Gorla, Ravagnani, Wachter-Zeh, 2016), (Antrobus, Gluesing-Luerssen, 2019), (Neri, Stanojkovski, 2024)

Block Ferrers diagrams

Notation Let $m, s \in \mathbb{N}$.

Block

The (i,j)-th *block* of side m is the set
$Q_{i,j}^{(m)} := ([im] \setminus [(i-1)m]) \times ([jm] \setminus [(j-1)m])$.

Block Ferrers diagram

A Ferrers diagram $\mathcal{F}$ of order sm is an *m-block Ferrers diagram* if for every $i,j \in [s]$, $Q_{i,j}^{(m)} \cap \mathcal{F} \in \{\emptyset, Q_{i,j}^{(m)}\}$.

Block set

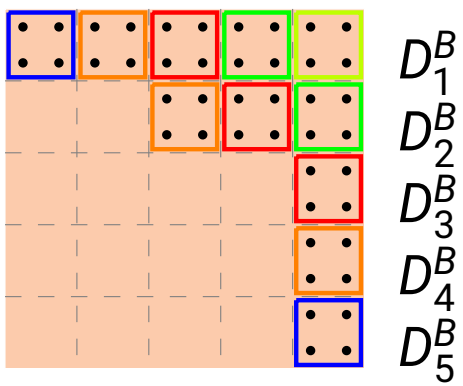
The *block set* of $\mathcal{F}$ is
$\mathbf{B}(\mathcal{F}) = \{Q_{i,j}^{(m)} : i,j \in [s], Q_{i,j}^{(m)} \cap \mathcal{F} = Q_{i,j}^{(m)}\}$.

Notation
Block diagonals of $[sm]^2$, for $i \in [s]$,

$$D_i^B := \{Q_{l,s-i+l}^{(m)} : 1 \le l \le i\}.$$

The diagonals of an arbitrary Ferrers diagram $\mathcal{F}$ are

$$D_i^B \cap \mathbf{B}(\mathcal{F}), \quad i \in [s].$$



Construction

The construction

Let $\mathcal{F}$ be an m-block Ferrers diagram of order sm, let $d \in [sm]$. We define $D^B_{\max} := \max_{i\in[s]}|D^B_i \cap \mathbf{B}(\mathcal{F})|$ and set $q \geq D^B_{\max} + 1$. Let $A_j \subseteq (\mathbb{F}_q^{m\times m})^j$ be an $[(m\times m)^j, m(mj-d+1), d]_q$ MSRD code for $j \in \{\lceil \frac{d-1}{m} \rceil, \ldots, D^B_{\max}\}$ and $A_j := \{0\}$ for all $j \in [\lceil \frac{d-1}{m} \rceil - 1]$. We define

$$\mathcal{C} := \left\{C \in \mathbb{F}_q^{\mathcal{F}} \ : \text{ the } i\text{-th block diagonal of } C \text{ is in } A_{|D^B_i \cap \mathbf{B}(\mathcal{F})|}, \forall i \in [s]\right\}.$$

MSRD-constructibility

A pair $(\mathcal{F}, d)$ is *MSRD-constructible* if

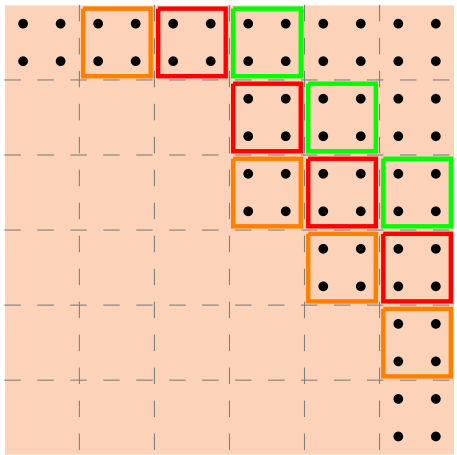
$$\nu_{\min}(\mathcal{F}, d) = m \sum_{i=1}^{s} \max\{m|D^B_i \cap \mathbf{B}(\mathcal{F})| - d + 1, 0\}.$$

Construction

Example $\mathcal{F} = [2, 2, 2, 2, 2, 2, 6, 6, 8, 8, 12, 12]$

Hypotheses:

- Distance $d = 6$.
- $\mathbb{F}_q$ with $q \geq 5$.



- $D_4^B, D_5^B \to [(2 \times 2)^4, 6, 6]_q$ MSRD code.
- $D_3^B \to [(2 \times 2)^3, 2, 6]_q$ MSRD code.

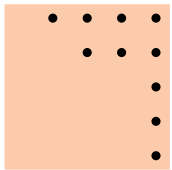
$$2 \sum_{i=1}^{6} \max\{2|D_i^B \cap \mathbf{B}(\mathcal{F})| - 6 + 1, 0\} = 14 = \nu_{\min}(\mathcal{F}, 6).$$

Theorem

Let $\mathcal{F}$ be an m-block Ferrers diagram of order sm and $d \in [sm]$ such that $(\mathcal{F}, d)$ is MSRD-constructible. Then the construction gives an MFD $[\mathcal{F}, d]_q$ code for every $q \geq D_{\max}^B + 1$.

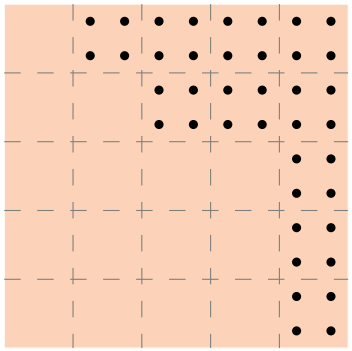
Connection-MDS/MSRD

$\mathcal{F} = [0, 1, 2, 2, 5]$



$\Updownarrow$

$\mathcal{F}^2 = [0, 0, 2, 2, 4, 4, 4, 4, 10, 10]$



Idea

Expand each dot into a block of the desired size

$\downarrow$

Result

MDS-constructibility

$? \Updownarrow ?$

MSRD-constructibility

The map ϕ_m

$$\phi_m : \quad \mathcal{F} \to \mathbf{B}([sm]^2)$$

$$(i,j) \mapsto Q_{i,j}^{(m)}$$

Connection-MDS/MSRD

Block triangular Ferrers diagram

The m-block Ferrers diagram $\mathcal{T}^B_{s,m}$ of order sm is *block triangular* when

$$Q^{(m)}_{i,j} \cap \mathcal{T}^B_{s,m} = \begin{cases} Q^{(m)}_{i,j} & \text{if } i,j \in [s] \text{ and } i \leq j \\ \emptyset & \text{otherwise.} \end{cases}$$

Theorem

Let $\mathcal{T}^B_{s,m}$ be the m-block triangular Ferrers diagram of order sm and $d \in [sm]$, then $(\mathcal{T}^B_{s,m}, d)$ is MSRD-constructible.

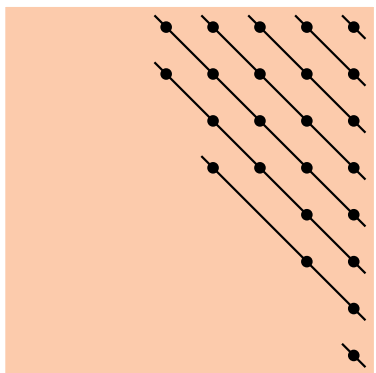
Remark

If $\mathcal{T}_s$ is the triangular diagram of order s and $\mathcal{T}^B_{s,m}$ is its m-block version, then $(\mathcal{T}_s, \delta)$ is MDS-constructible for every $\delta \in [s]$. Therefore, the "$\Longleftrightarrow$" holds in this case.

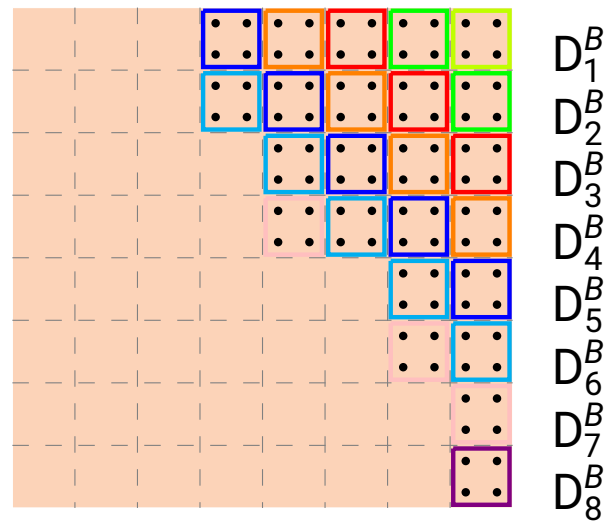
Connection-MDS/MSRD

Theorem

Let $\mathcal{F}$ be a Ferrers diagram of order s and $\mathcal{F}^m$ be its m-block version, let $\delta \in [s], d \in [sm]$ be such that $d-1 = m(\delta-1)$. Then $(\mathcal{F}, \delta)$ is MDS-constructible if and only if $(\mathcal{F}^m, d)$ is MSRD-constructible.



$$\sum_{i=1}^{8} \max\{|D_i \cap \mathcal{F}| - 2, 0\} = 10$$
$$\nu_{\min}(\mathcal{F}, 3) = 10$$
$$\sum_{i=1}^{8} \max\{|D_i \cap \mathcal{F}| - 3, 0\} = 5$$
$$\nu_{\min}(\mathcal{F}, 4) = 6$$



$$2\sum_{i=1}^{8} \max\{2|D_i^B \cap \mathbf{B}(\mathcal{F}^2)| - 4, 0\} = 40$$
$$\nu_{\min}(\mathcal{F}^2, 5) = 40$$
$$2\sum_{i=1}^{8} \max\{2|D_i^B \cap \mathbf{B}(\mathcal{F}^2)| - 6, 0\} = 20$$
$$\nu_{\min}(\mathcal{F}^2, 7) = 24.$$

Connection-MDS/MSRD

Theorem

Let $\mathcal{F}$ be a Ferrers diagram of order s and $\mathcal{F}^m$ be its m-block version, let $\delta \in [s], d \in [sm]$ such that $d-1 = m(\delta-1)+r$, with $r \in [m-1]$. Suppose that $(\mathcal{F}, \delta)$ is MDS-constructible, and there exists some $j \in \{0, \ldots, \delta-1\}$ such that

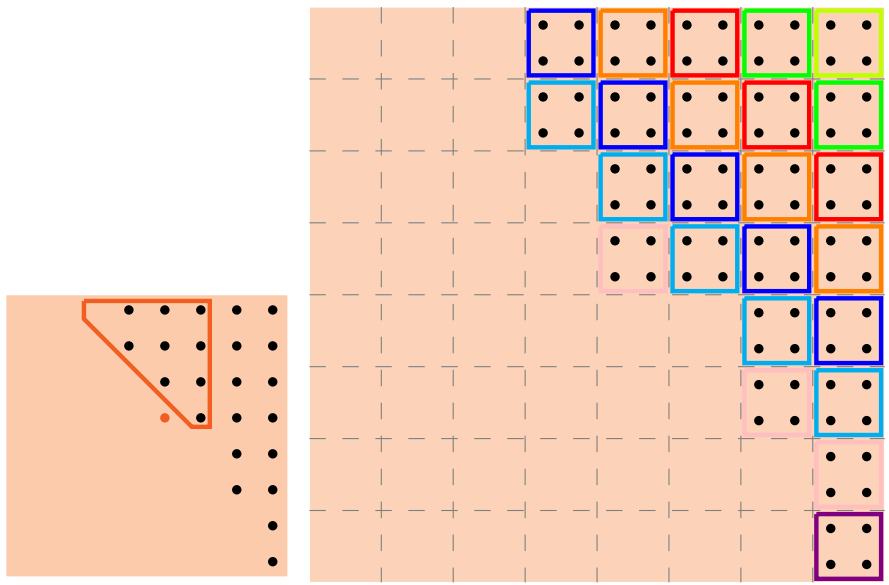
- $(\mathcal{F}, \delta)$ is j-Singleton and $(\mathcal{F}, \delta+1)$ is j-Singleton or $(j+1)$-Singleton,
- $\mathcal{F} \cap \mathcal{S}_{s,\delta,j}$ is triangular covered.

Then $(\mathcal{F}^m, d)$ is MSRD-constructible.

Connection-MDS/MSRD

Example

$$d-1=m(\delta-1)+r, r\in[m-1]$$



$d=6$ is linked to $\delta=3$.
$(\mathcal{F},3)$ is MDS-constructible but $\mathcal{F}\cap\mathcal{S}_{8,3,2}$ is not triangular covered
$(\mathcal{F}^2,6)$ is not MSRD-constructible.
$2\sum_{i=1}^{8}\max\{\,2|D_i^B\cap\mathbf{B}(\mathcal{F}^2)|-5,0\,\}=30\neq\nu_{\min}(\mathcal{F}^2,6)=32.$

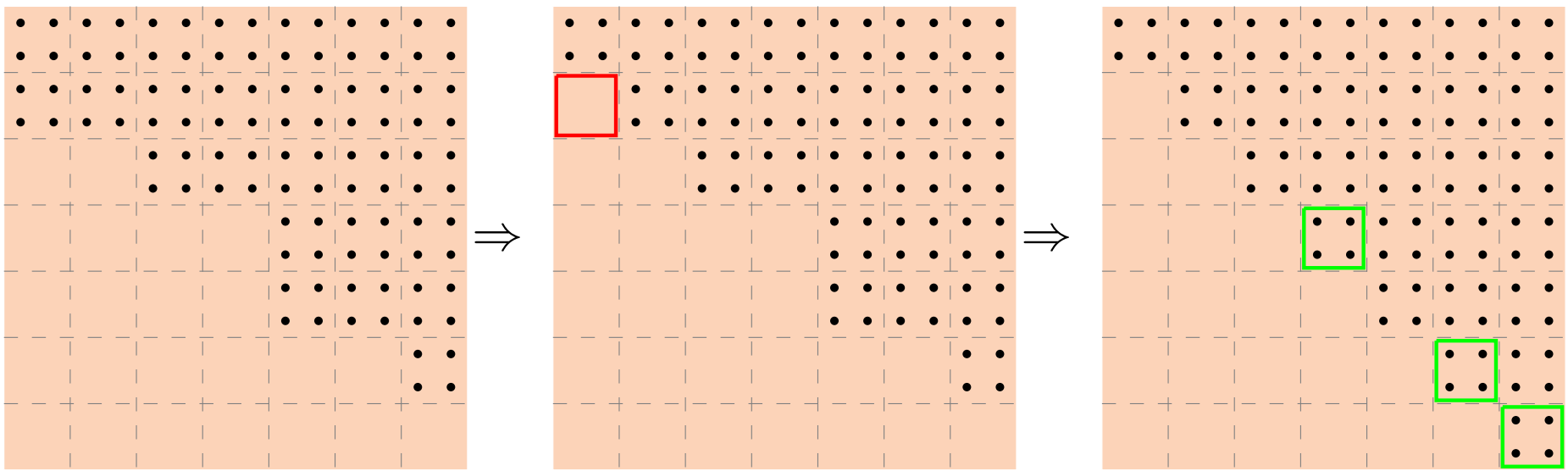
The field size

Theorem

Let $\mathcal{F}$ be an m-block Ferrers diagram of order sm, $d \in \{2, \ldots, sm\}$ and $\mathbb{F}_q$ a finite field. If

1. there exists an MFD $[\mathcal{T}^B_{l,m}, d]_q$ code for every $\lceil \frac{d}{m} \rceil \leq l \leq s$,
2. $(\mathcal{F}, d)$ is MSRD-constructible,

then there exists an MFD $[\mathcal{F}, d]_q$ code.



MFD codes on block triangular diagrams

Corollary

Let $\mathcal{F}$ be a p^h-block Ferrers diagram of order sp^h, $d \in \{2, \ldots, sp^h\}$ and $\mathbb{F}_q$ a field of characteristic p. If $(\mathcal{F}, d)$ is MSRD-constructible, then there exists an MFD $[\mathcal{F}, d]_q$ code.

Theorem

Let $m, s \geq 2$. Then, for any prime power q there exists an MFD $[\mathcal{T}^B_{s,m}, d]_q$ code for $d = sm$ and $d = sm - 1$.

Corollary

Let $m, s \geq 2$ and $\mathcal{F}$ be an m-block Ferrers diagram of order sm and $d \in \{ms - 1, ms\}$. If $(\mathcal{F}, d)$ is MSRD-constructible, then there exists an $[\mathcal{F}, d]_q$ MFD code over any finite field $\mathbb{F}_q$.

Conclusions

Main results

1. Construction
2. Connection - MDS/MSRD
3. The field size - Reduction triangular

Future directions

1. Investigate further the connection in the general case.
2. Prove the Etzion-Silberstein conjecture in its generality for block triangular diagrams.

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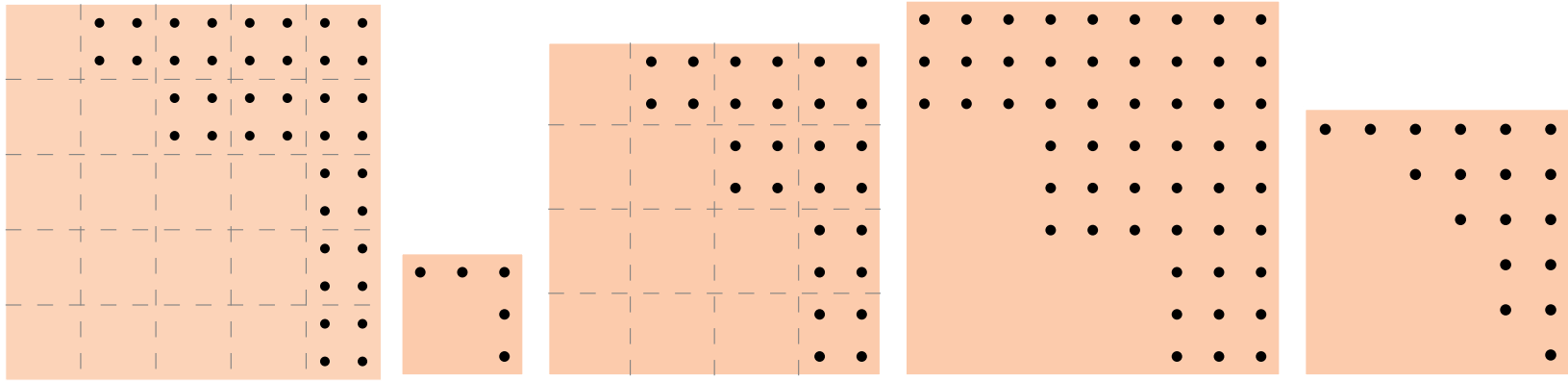

Thank you for your attention!