New structural insights and a syndrome-based decoding approach for Trombetti–Zhou codes

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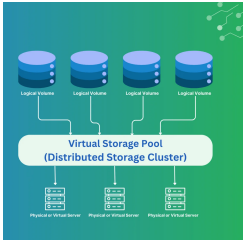
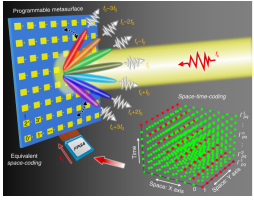
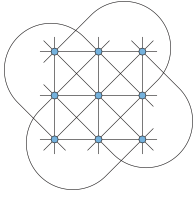
Code-based cryptography



Distributed storage



Space-time coding



Finite Geometry

D. Silva, F. R. Kschischang and R. Kötter
A Rank-Metric Approach to Error Control in Random Network Coding
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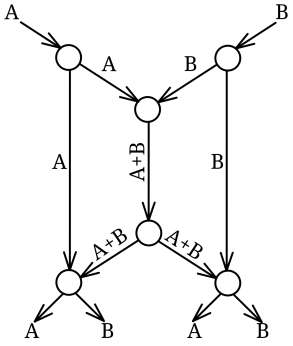

Network coding

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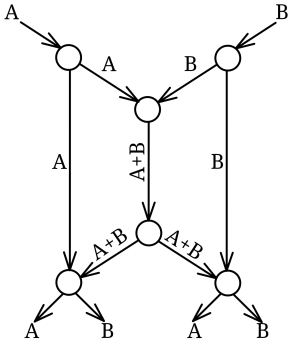

Network coding

Gabidulin codes

- $k \leq n \leq m$
- $\mathcal{L}_k(\mathbb{F}_{q^m}) = \left\{ g_0 x + g_1 x^q + \ldots + g_{k-1} x^{q^{k-1}} \mid g_i \in \mathbb{F}_{q^m} \right\}$
- $\mathbb{F}_q$-linearly independent $\beta_0, \beta_1, \ldots, \beta_{n-1} \in \mathbb{F}_{q^m}$

$$\mathcal{G}_{n,k}(\underline{\beta}) = \left\{ (g(\beta_0), \ldots, g(\beta_{n-1})) \mid g(x) \in \mathcal{L}_k(\mathbb{F}_{q^m}) \right\} \subseteq \mathbb{F}_{q^m}^n$$

Readapt the Generalized decoding algorithm for Gabidulin codes arising from Random Network coding to Trombetti-Zhou codes.

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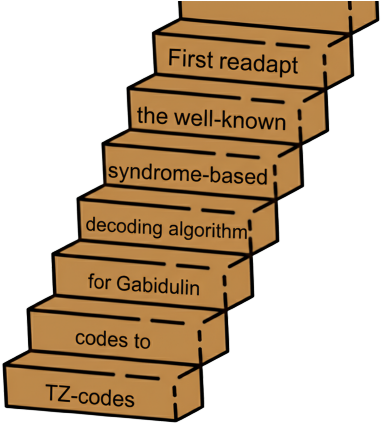

Trombetti-Zhou (TZ) codes

$$\mathbb{F}_{q^{2n}}^{2n} = \left\{\underline{x} = (x_0, \ldots, x_{2n-1}) : x_i \in \mathbb{F}_{q^{2n}}\right\}$$

R. Trombetti and Y. Zhou

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is an MRD q^n-$\left[2n, 2k, 2n-k+1\right]_{q^{2n}/q}$ code.

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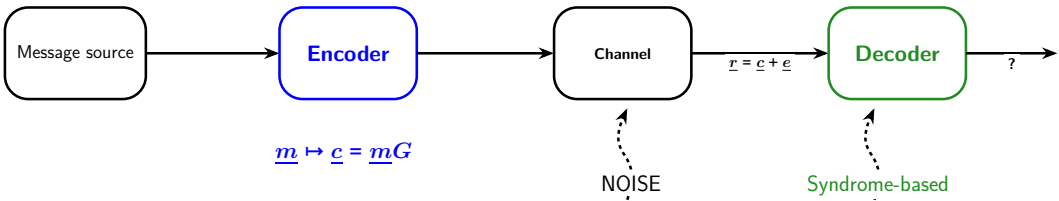

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$\mathbb{F}_{q^n}$-Generator Matrix and encoding of TZ-codes

$\mathcal{TZ}_k(\gamma) \subseteq \mathbb{F}_{q^{2n}}^{2n}$ are $\mathbb{F}_{q^n}$-linear codes of $\mathbb{F}_{q^{2n}}^{2n}$ of $\mathbb{F}_{q^n}$-dimension $2k$

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$$G = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{2n-1} \\ \lambda_0^q & \lambda_1^q & \lambda_2^q & \cdots & \lambda_{2n-1}^q \\ \gamma\lambda_0^q & \gamma\lambda_1^q & \gamma\lambda_2^q & \cdots & \gamma\lambda_{2n-1}^q \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_0^{q^{k-1}} & \lambda_1^{q^{k-1}} & \lambda_2^{q^{k-1}} & \cdots & \lambda_{2n-1}^{q^{k-1}} \\ \gamma\lambda_0^{q^{k-1}} & \gamma\lambda_1^{q^{k-1}} & \gamma\lambda_2^{q^{k-1}} & \cdots & \gamma\lambda_{2n-1}^{q^{k-1}} \\ \gamma\lambda_0^{q^k} & \gamma\lambda_1^{q^k} & \gamma\lambda_2^{q^k} & \cdots & \gamma\lambda_{2n-1}^{q^k} \end{pmatrix} \in \mathbb{F}_{q^{2n}}^{2k \times 2n} \tag{1}$$

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Encoding process:

$$\underline{\tilde{f}} = (a, f_{1,1}, f_{1,2}, \dots, f_{k-1,1}, f_{k-1,2}, b) \in \mathbb{F}_{q^n}^{2k} \mapsto \underline{c} = \underline{\tilde{f}} G \in \mathcal{TZ}_k(\gamma) \subseteq \mathbb{F}_{q^{2n}}^{2n}$$

The *syndrome* of $\underline{a} \in \mathbb{F}_{q^{2n}}^{2n}$ is zero iff $\underline{a} \in \mathcal{C}$

$\mathbb{F}_{q^n}$-Parity-Check Matrix and $\mathbb{F}_{q^n}$-syndrome

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$$Tr_{q^{2n}/q^n}(\underline{a}H^\top) = \underline{a}H^\top + \left(\underline{a}H^\top\right)^{q^n} \text{ is zero iff } \underline{a} \in \mathcal{TZ}_k(\gamma)$$

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If $\underline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{2n-1})$ is an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^{2n}}$

$$\Downarrow$$

there exists a unique $\mathbb{F}_q$-basis $\underline{\mu} = (\mu_0, \mu_1, \ldots, \mu_{2n-1})$ of $\mathbb{F}_{q^{2n}}$

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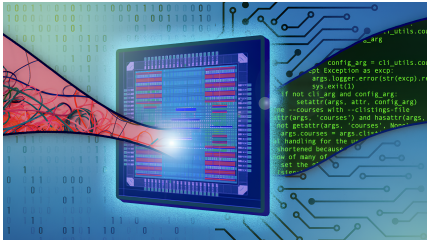
$$H = \begin{pmatrix} \gamma^{q^{2n-k}}\mu_0 & \gamma^{q^{2n-k}}\mu_1 & \gamma^{q^{2n-k}}\mu_2 & \cdots & \gamma^{q^{2n-k}}\mu_{2n-1} \\ \mu_0^{q^{k+1}} & \mu_1^{q^{k+1}} & \mu_2^{q^{k+1}} & \cdots & \mu_{2n-1}^{q^{k+1}} \\ \gamma\mu_0^{q^{k+1}} & \gamma\mu_1^{q^{k+1}} & \gamma\mu_2^{q^{k+1}} & \cdots & \gamma\mu_{2n-1}^{q^{k+1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_0^{q^{2n-1}} & \mu_1^{q^{2n-1}} & \mu_2^{q^{2n-1}} & \cdots & \mu_{2n-1}^{q^{2n-1}} \\ \gamma\mu_0^{q^{2n-1}} & \gamma\mu_1^{q^{2n-1}} & \gamma\mu_2^{q^{2n-1}} & \cdots & \gamma\mu_{2n-1}^{q^{2n-1}} \\ \mu_0^{q^k} & \mu_1^{q^k} & \mu_2^{q^k} & \cdots & \mu_{2n-1}^{q^k} \end{pmatrix} \in \mathbb{F}_{q^{2n}}^{(4n-2k)\times 2n} \tag{3}$$



E. M. Gabidulin
Theory of codes with maximum rank distance
In: *P.P.I.* 21.1 (1985), pp. 3-16.

A. Wachter-Zeh
Decoding of block and convolutional codes in rank metric
General Mathematics [math.GM]. Université de Rennes; Universität Ulm, 2013.

Decoding of Trombetti-Zhou codes $\mathcal{TZ}_k(\gamma)$: a new syndrome-based decoding approach

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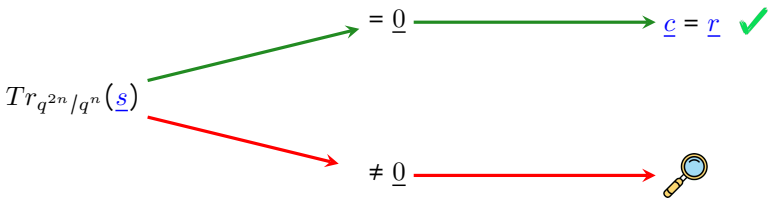

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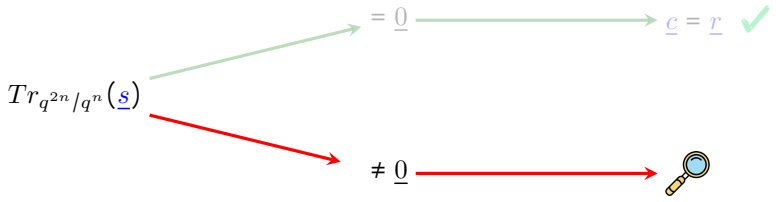

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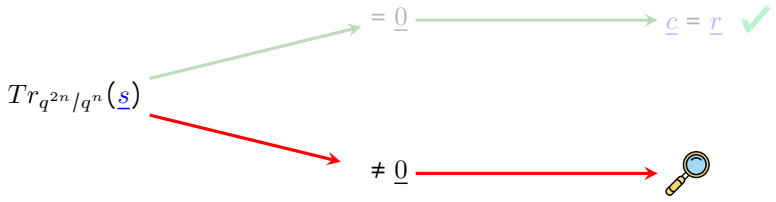

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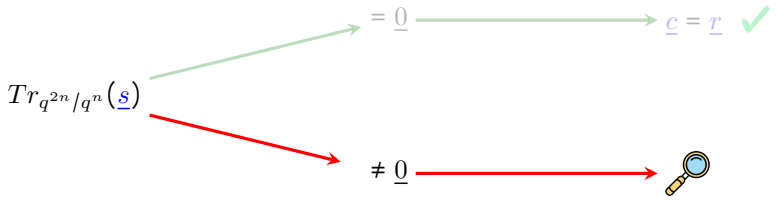


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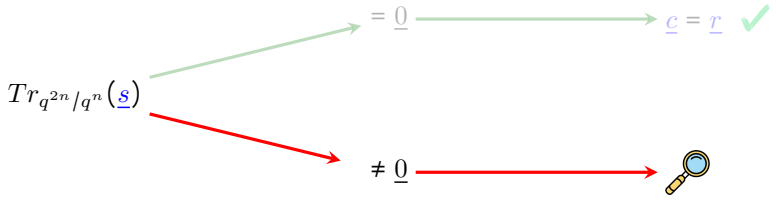


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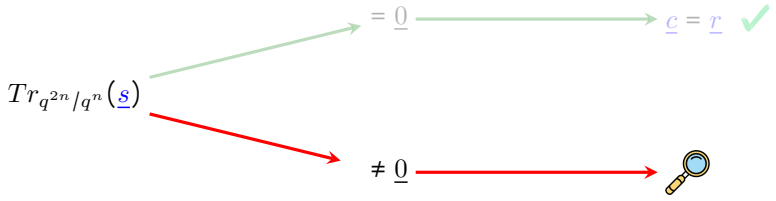


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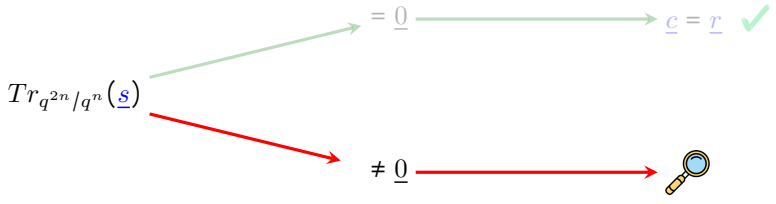


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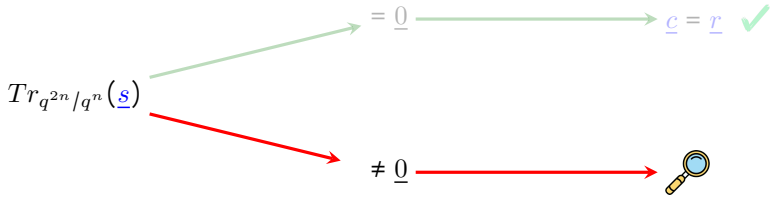


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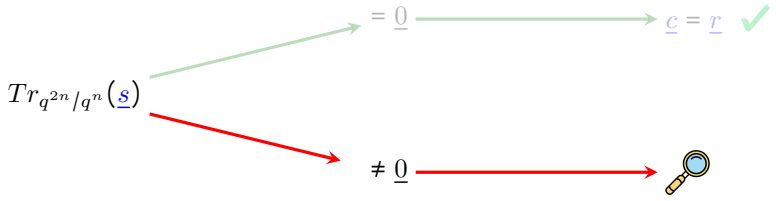


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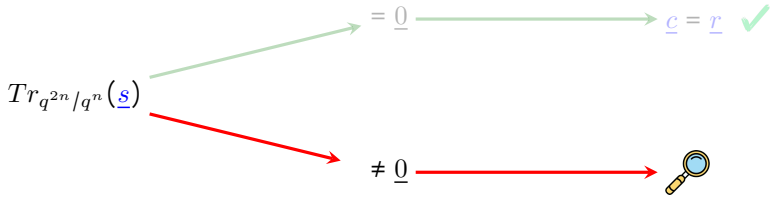

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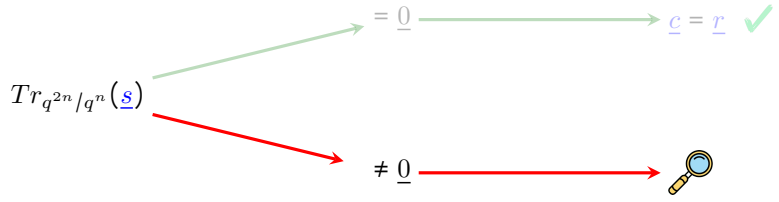

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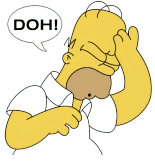

More information is needed!

System of $2t$ equations in $t+1$ unknowns:

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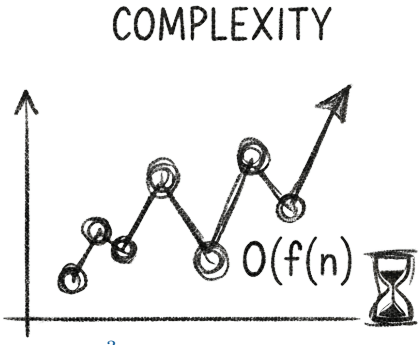
Is there anything under the rug?

Algorithm 1: Decoding of Trombetti-Zhou codes: a new syndrome-based decoding approach.

Input: $\underline{r} = (r_0, r_1, \ldots, r_{2n-1}) \in \mathbb{F}_{q^{2n}}^{2n}$;
Set H as in (3);
$\mathbb{F}_{q^n}$-*syndrome* calculation: $\underline{s} \leftarrow \underline{r} \cdot H^\top \in \mathbb{F}_{q^{2n}}^{4n-2k}$;
if $Tr_{q^{2n}/q^n}(\underline{s}) = \underline{0}$ **then**
    Estimated codeword: $\underline{c} \leftarrow \underline{r}$
end
else
    if k *is even and* $\mathrm{rank}(S_{\exp}) = n - \frac{k}{2}$ **then**
        Set up $t = n - \frac{k}{2}$;
        Solve $S_{\exp} \cdot \underline{\Lambda}^\top = \underline{0}^\top$ for $\underline{\Lambda} = (\Lambda_0, \Lambda_1, \ldots, \Lambda_t) \in \mathbb{F}_{q^{2n}}^{t+1}$;
        if $\left(\frac{\Lambda_0}{\Lambda_t}, \frac{\Lambda_1}{\Lambda_t}, \ldots, \frac{\Lambda_{t-1}}{\Lambda_t}, 1\right) \notin \mathbb{F}_{q^n}^{t+1}$ **then**
            Declare decoding failure
        end
    end
    else
        Set up $S^{(t)}$ as in (5) for $t = \lfloor (2n - (k+1))/2 \rfloor$;
        while $\mathrm{rank}(S^{(t)}) < t$ **do**
            $t \leftarrow t - 1$;
            Set up $S^{(t)}$ as in (5)
        end
        Solve $S^{(t)} \cdot \underline{\Lambda}^T = \underline{0}$ for $\underline{\Lambda} = (\Lambda_0, \Lambda_1, \ldots, \Lambda_t) \in \mathbb{F}_{q^{2n}}^{t+1}$
    end
    Find basis $(a_0, a_1, \ldots, a_{\varepsilon-1}) \in \mathbb{F}_{q^{2n}}^{\varepsilon}$ of the root space of $\Lambda(x) = \sum_{i=0}^{t} \Lambda_i x^{[i]}$ over $\mathbb{F}_{q^{2n}}$;
    if $\varepsilon = t$ **then**
        Find $(d_0, d_1, \ldots, d_{t-1}) \in \mathbb{F}_{q^{2n}}^{t}$ by solving (6);
        Find $B = (B_{i,j})_{j \in \{0,\ldots,2n-1\}}^{i \in \{0,\ldots,t-1\}} \in \mathbb{F}_q^{t \times 2n}$ such that $d_i = \sum_{j=0}^{2n-1} B_{i,j} \mu_j^{[k]}$;
        Estimated codeword: $\underline{c} \leftarrow \underline{r} - \underline{a} \cdot B$
    end
    else
        Declare decoding failure
    end
end
Output: Estimated codeword $\underline{c} \in \mathbb{F}_{q^{2n}}^{2n}$ or "decoding failure"

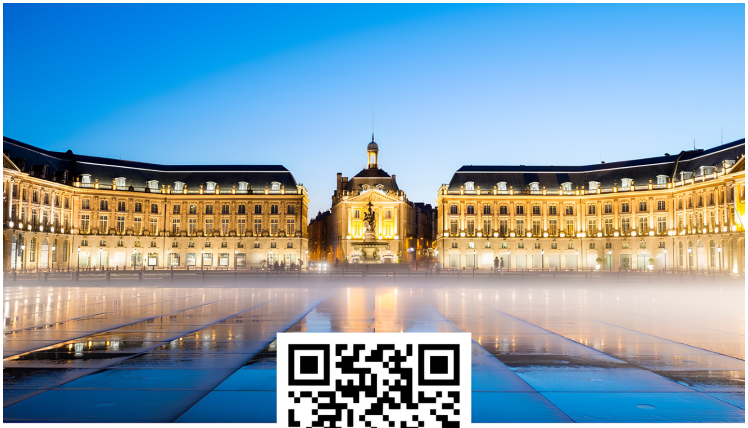
Algorithm 2: Decoding of Trombetti-Zhou codes: a new syndrome-based decoding approach.

Input: $\underline{r} = (r_0, r_1, \ldots, r_{2n-1}) \in \mathbb{F}_{q^{2n}}^{2n}$;
Set H as in (3);
$\mathbb{F}_{q^n}$-*syndrome* calculation: $\underline{s} \leftarrow \underline{r} \cdot H^\top \in \mathbb{F}_{q^{2n}}^{4n-2k}$;
if $Tr_{q^{2n}/q^n}(\underline{s}) = \underline{0}$ **then**
    Estimated codeword: $\underline{c} \leftarrow \underline{r}$
end
else
    if k *is even and* $\mathrm{rank}(S_{\exp}) = n - \frac{k}{2}$ **then**
        Set up $t = n - \frac{k}{2}$;
        Solve $S_{\exp} \cdot \underline{\Lambda}^\top = \underline{0}^\top$ for $\underline{\Lambda} = (\Lambda_0, \Lambda_1, \ldots, \Lambda_t) \in \mathbb{F}_{q^{2n}}^{t+1}$;
        if $\left(\frac{\Lambda_0}{\Lambda_t}, \frac{\Lambda_1}{\Lambda_t}, \ldots, \frac{\Lambda_{t-1}}{\Lambda_t}, 1\right) \notin \mathbb{F}_{q^n}^{t+1}$ **then**
            Declare decoding failure
        end
    end
    else
        Set up $S^{(t)}$ as in (5) for $t = \lfloor (2n - (k+1))/2 \rfloor$;
        while $\mathrm{rank}(S^{(t)}) < t$ **do**
            $t \leftarrow t - 1$;
            Set up $S^{(t)}$ as in (5)
        end
        Solve $S^{(t)} \cdot \underline{\Lambda}^\top = \underline{0}$ for $\underline{\Lambda} = (\Lambda_0, \Lambda_1, \ldots, \Lambda_t) \in \mathbb{F}_{q^{2n}}^{t+1}$
    end
    Find basis $(a_0, a_1, \ldots, a_{\varepsilon-1}) \in \mathbb{F}_{q^{2n}}^{\varepsilon}$ of the root space of $\Lambda(x) = \sum_{i=0}^{t} \Lambda_i x^{[i]}$ over $\mathbb{F}_{q^{2n}}$;
    if $\varepsilon = t$ **then**
        Find $(d_0, d_1, \ldots, d_{t-1}) \in \mathbb{F}_{q^{2n}}^{t}$ by solving (6);
        Find $B = (B_{i,j})_{j\in\{0,\ldots,t-1\}}^{i\in\{0,\ldots,2n-1\}} \in \mathbb{F}_q^{t\times 2n}$ such that $d_i = \sum_{j=0}^{2n-1} B_{i,j}\, \mu_j^{[k]}$;
        Estimated codeword: $\underline{c} \leftarrow \underline{r} - \underline{a} \cdot B$
    end
    else
        Declare decoding failure
    end
end
Output: Estimated codeword $\underline{c} \in \mathbb{F}_{q^{2n}}^{2n}$ or "decoding failure"



$\mathcal{O}(n^3)$ operations over $\mathbb{F}_{q^{2n}}$

Thanks for your attention!



Appendix

Let $[i] \coloneqq q^i$,

$$S^{(t)} = \begin{pmatrix} s_{2(t+1)-1}^{[0]} & \cdots & s_3^{[t-1]} & s_1^{[t]} \\ s_{2(t+2)-1}^{[0]} & \cdots & s_5^{[t-1]} & s_3^{[t]} \\ \vdots & \ddots & \vdots & \vdots \\ s_{2(2t)-1}^{[0]} & \cdots & s_{2t+1}^{[t-1]} & s_{2t-1}^{[t]} \end{pmatrix} \tag{5}$$

$$\begin{pmatrix} a_0^{[-1]} & a_1^{[-1]} & \cdots & a_{t-1}^{[-1]} \\ a_0^{[-2]} & a_1^{[-2]} & \cdots & a_{t-1}^{[-2]} \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{[-(2n-k-1)]} & a_1^{[-(2n-k-1)]} & \cdots & a_{t-1}^{[-(2n-k-1)]} \end{pmatrix} \cdot \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{t-1} \end{pmatrix} = \begin{pmatrix} s_1^{[-1]} \\ s_3^{[-2]} \\ \vdots \\ s_{4n-2k-3}^{[-(2n-k-1)]} \end{pmatrix} \tag{6}$$