# Delsarte duality on subspaces and applications to rank-metric codes

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*OpeRa 2026*
*Bordeaux, France*

**Borello, P., Zullo**: *Delsarte duality on subspaces and applications to rank-metric codes and q-matroids*
arXiv:2509.24409, **2025**.

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$\exists!$ *$n$-dimensional $\mathbb{F}_q$-subgeometry in $\mathbb{V}$ up to the action of $GL(k,q^m)$*

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$s$ is the length of $(**)$

# MD-Sequence of *h*-scattered spaces

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# MD-Sequence of $h$-scattered spaces

## *MD-Sequence*

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$\mathcal{C}$ nondegenerate $[n, k]_{q^m/q}$ linear rank-metric code,

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**Randrianarisoa**: *A geometric approach to rank-metric codes and classification of constant weight codes*
Designs, Codes and Cryptography, **2020**.

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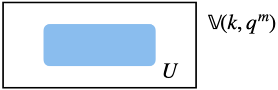
# $\mathbb{F}_q$-subspaces as projections



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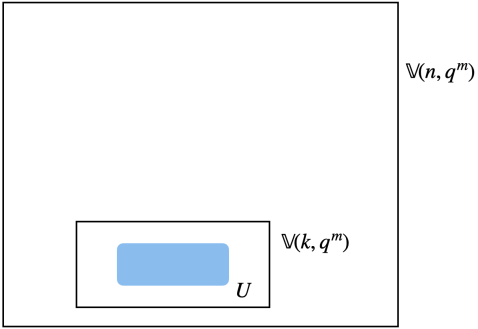

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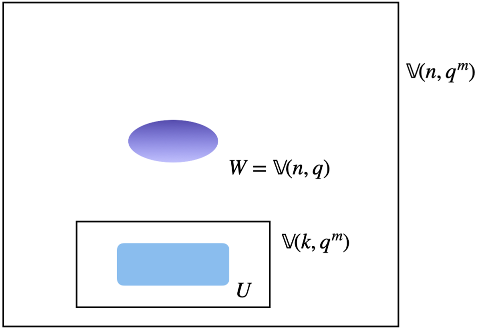

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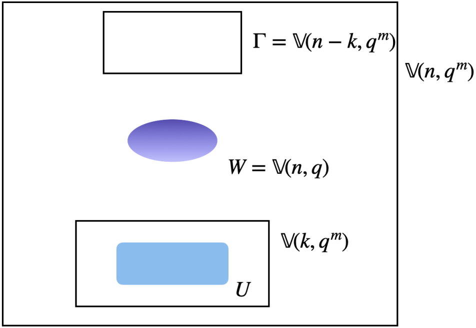


**Lunardon, P.**: *Translation ovoids of orthogonal polar spaces* Forum Mathematicum, **2004**.

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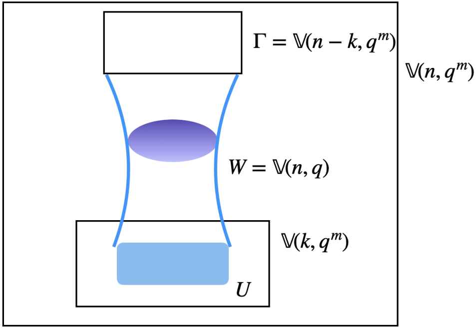


$$U \cong \frac{W+\Gamma}{\Gamma} \leq_{\mathbb{F}_q} \frac{\mathbb{V}(n,q^m)}{\Gamma}$$

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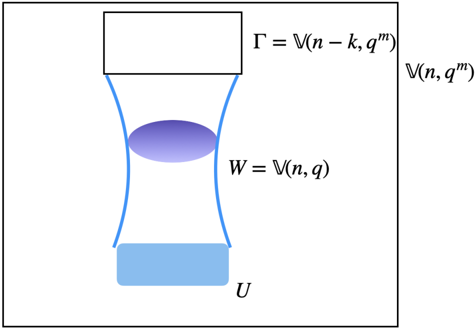


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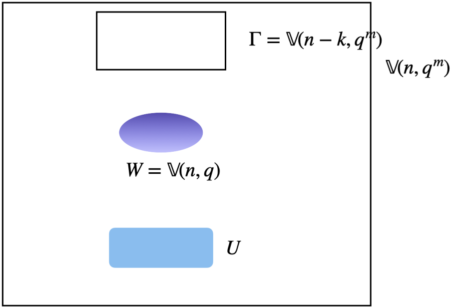

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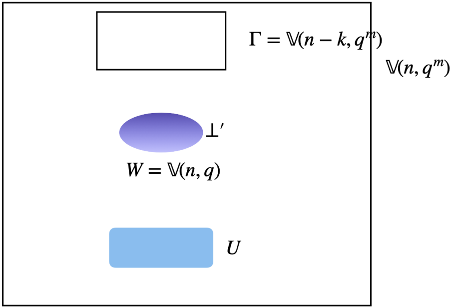

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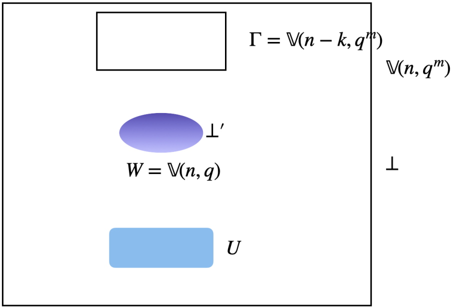

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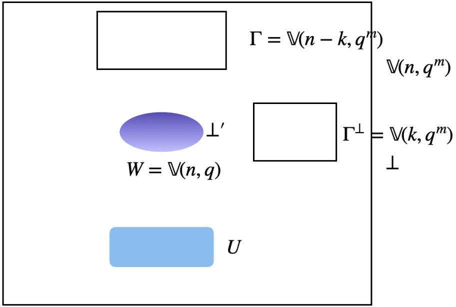

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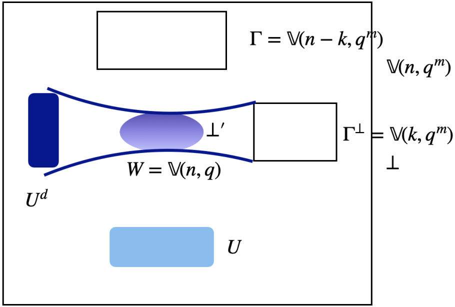

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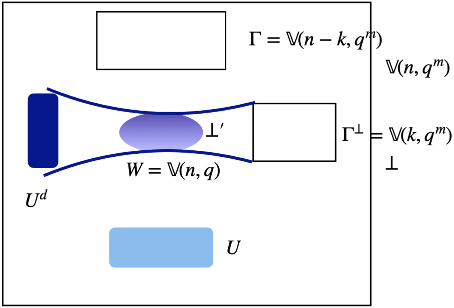
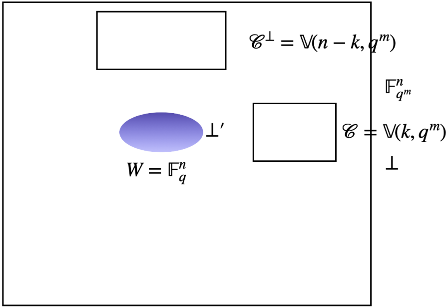



$$U^d := \frac{W+\Gamma^\perp}{\Gamma^\perp} \leq_{\mathbb{F}_q} \frac{\mathbb{V}(n,q^m)}{\Gamma^\perp} = \mathbb{V}(n-k, q^m)$$

$$\Gamma^\perp \cap W = \{0\} \iff \dim_{\mathbb{F}_q} U^d = n$$

$U^d$ Delsarte dual of $U$

# Delsarte dual of a system



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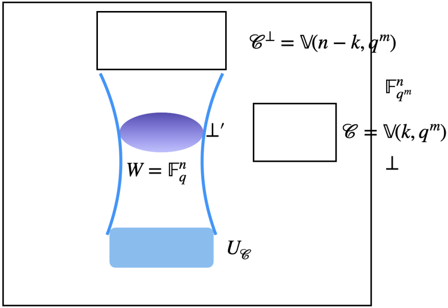


$\mathcal{C}$ nondegenerate $[n,k]_{q^m/q} \longrightarrow U_{\mathcal{C}} \cong U = \frac{\mathbb{F}_q^n + \mathcal{C}^\perp}{\mathcal{C}^\perp} \leq_{\mathbb{F}_q} \frac{\mathbb{F}_{q^m}^n}{\mathcal{C}^\perp}$

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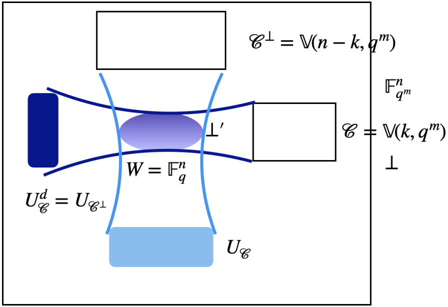


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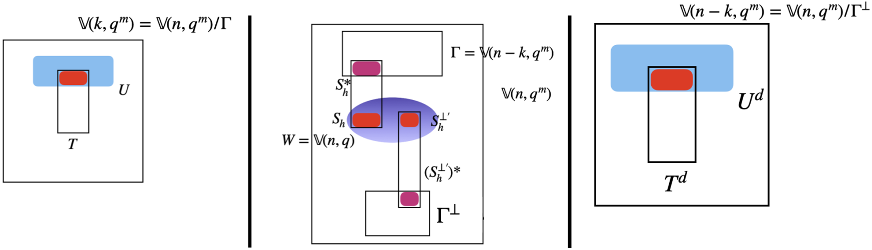
# Delsarte dual of $T$




$$T^d := \frac{(S_h^*)^\perp + \Gamma^\perp}{\Gamma^\perp} \leq_{\mathbb{F}_{q^m}} \frac{\mathbb{V}(n,q^m)}{\Gamma^\perp} = \mathbb{V}(n-k, q^m).$$

$$\varepsilon_U(T) = \dim_{\mathbb{F}_{q^m}}(S_h^* \cap \Gamma)$$

# Delsarte duality

**Theorem**

$T \in \mathcal{L}_{\mathbb{F}_{q^m}}(\mathbb{V}(k, q^m))$ and $\varepsilon_U(T) \geq 0 \implies T^d \in \mathcal{L}_{\mathbb{F}_{q^m}}(\mathbb{V}(n-k, q^m))$

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$$\mathcal{E}_U(\mathbb{V}(k,q^m)) = \{T \leq_{\mathbb{F}_{q^m}} \mathbb{V}(k,q^m) \colon T \text{proper, minimal, } \varepsilon_U(T) > 0\}$$

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**Corollary**

$$\varphi : T \in \mathcal{E}_U(\mathbb{V}(k,q^m)) \xrightarrow{\text{bijective}} T^d \in \mathcal{E}_{U^d}(\mathbb{V}(n-k,q^m)).$$

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$$T' \subseteq T \iff T^d \subseteq T'^d$$

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$$t'_1 = n-k-\varepsilon_U(t_{s-1})$$

# Linear rank-metric codes: generalized weights

**Theorem**

*$\mathcal{C}$ non-degenerate $[n,k]_{q^m/q}$ code such that $\mathcal{C}^\perp$ non-degenerate*

$$0 = \varepsilon_{U_\mathcal{C}}(0) < \varepsilon_{U_\mathcal{C}}(t_1) < \cdots < \varepsilon_{U_\mathcal{C}}(t_{s-1}) < \varepsilon_{U_\mathcal{C}}(k) = n-k$$

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$$r \in \{1, \ldots, k\}\ ,\ d_r(\mathcal{C}) = n - (k - r) - \varepsilon_{U_\mathcal{C}}(t_i)$$
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$$r \in \{1, \ldots, n-k\}\ ,\ d_r(\mathcal{C}^\perp) = r + t_j$$
$$\varepsilon_{U_\mathcal{C}}(t_{j-1}) < r \leq \varepsilon_{U_\mathcal{C}}(t_j)$$

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*Wei-type duality*

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*Wei-type duality*

**Ducoat**: *Generalized rank weights:A duality statement* Topics in Finite Fields, **2015**.

# Families of RD-codes closed under Delsarte duality

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**Theorem**

*$\mathcal{C}$ non-degenerate $[m+\rho,k]_{q^m/q}$ code, $1 \leq \rho < m/(k-1)$, such that $\mathcal{C}^\perp$ non-degenerate*

$$\mathcal{C} \text{ is quasi-MRD} \iff \varepsilon_{U_\mathcal{C}}(k-1) = \rho$$

$$\mathcal{C}^\perp \text{ is quasi-MRD} \iff U_\mathcal{C} \text{ is a } (k-2)\text{-scattered space}$$

$$\mathcal{C} \text{ is dually quasi-MRD} \iff U_\mathcal{C} \text{ is a } (k-2)\text{-scattered space and } \varepsilon_{U_\mathcal{C}}(k-1) = \rho$$

## Thank you

***Thank you for your attention!***