Improved Gilbert-Varshamov bound for sum-rank-metric codes via graph theory

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The sum-rank metric

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- Example: $\mathbf{n} = (2, 1), \mathbf{m} = (2, 1)$:

$$\mathbf{X} = \left(\underbrace{\begin{pmatrix} & \\ & \end{pmatrix}}_{2\times 2}, \underbrace{\begin{pmatrix} \end{pmatrix}}_{1\times 1} \right).$$

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- $A_q^{SRK}(\mathbf{n},\mathbf{m},d) = \max\{|C| : C \in \mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}, \mathrm{srk}(C) \geq d\}$.

Special cases of the sum-rank metric

- Hamming metric: $\mathbf{n} = (1, 1, \ldots, 1), \mathbf{m} = (1, 1, \ldots, 1)$:

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- Rank metric: $\mathbf{n} = (n), \mathbf{m} = (m)$:

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The Gilbert-Varshamov bound for the sum-rank metric

Gilbert-Varshamov bound [Byrne, Gluesing-Luerssen, Ravagnani, 2021]

Let D be the volume of the ball of radius $d-1$ around a codeword. Then

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Main goal: improve the GV bound.

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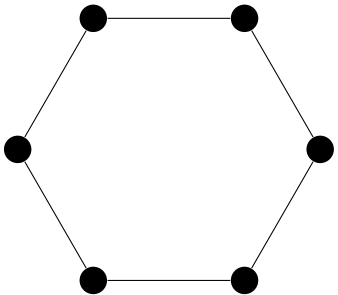

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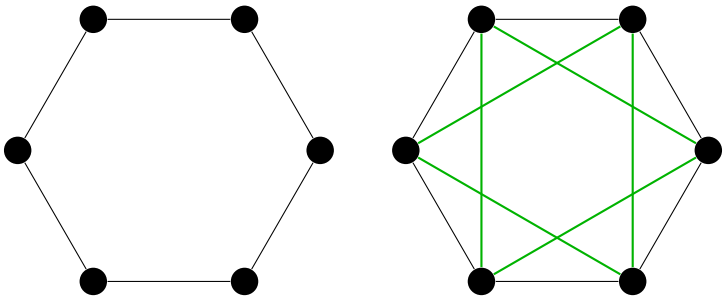

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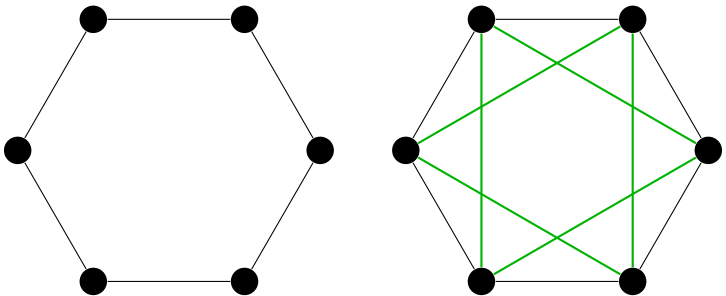

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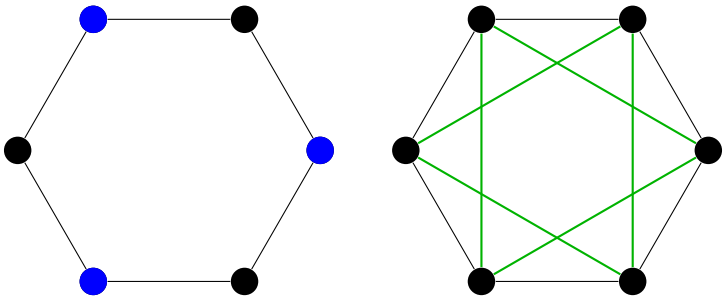

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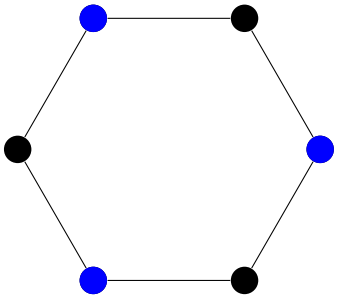

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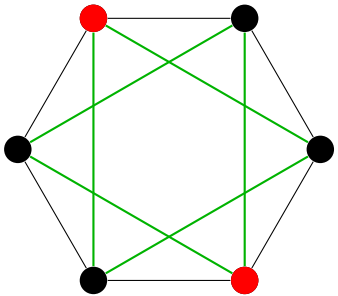

$\alpha(G^2) = 2$

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Proposition [Abiad, Khramova, Ravagnani, 2024]

$$\alpha(\Gamma(\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}})^k) = A_q^{SRK}(\mathbf{n}, \mathbf{m}, k+1).$$

The graph-theoretic approach

Theorem [Bollobás, 1985]

Let $G = (V, E)$ be a graph with maximum degree D and fewer than Δ triangles. Then,

$$\alpha(G) \geq \frac{|V|}{10D}\left(\log_2 D - \frac{1}{2}\log_2\left(\frac{\Delta}{|V|}\right)\right).$$

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Corollary

Let $G = (V, E)$ be a graph with maximum degree D and fewer than $|V|D^{2-\varepsilon}$ triangles. Then,

$$\alpha(G) \geq \varepsilon\frac{|V|}{20D}\log_2 D.$$

Usage of the graph theoretic approach

- Hamming metric codes [Jiang, Vardy, 2004], [Vu, Wu, 2005].
- Constant-weight codes [Kim, Liu, Tran, 2022].
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- Sum-rank-metric codes?

The graph-theoretic approach

Corollary [Abiad, R., Tait, 2025+]

Let $\Gamma(\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}) = (V, E)$ be a sum-rank-metric graph with maximum degree D and fewer than $|V|D^{2-\varepsilon}$ triangles. Then,

$$A_q^{SRK}(\mathbf{n}, \mathbf{m}, k+1) = \alpha(\Gamma(\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}})^k) \geq \varepsilon \frac{|V|}{20D} \log_2 D.$$

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Lemma

Let G be walk-regular with Δ triangles, and denote by T the amount of edges in a neighborhood of a vertex, then

$$T = \frac{3\Delta}{|V|}.$$

Triangles in walk-regular graphs

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Proposition [Abiad, R., Tait, 2025+]

The graph $\Gamma(\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}})^k$ is walk-regular.

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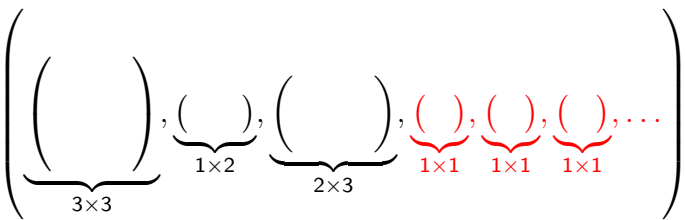
Corollary [Abiad, R., Tait, 2025+]

Let $\Gamma(\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}) = (V, E)$ be a sum-rank-metric graph with maximum degree D and T edges in a neighborhood. If $T \leq D^{2-\varepsilon}$, then,

$$A_q^{SRK}(\mathbf{n}, \mathbf{m}, k+1) \geq \varepsilon \frac{|V|}{20D} \log_2 D.$$

The main objective: Show $T \leq D^{2-\varepsilon}$

Asymptotic in the number of components



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- For technical reasons, we need $k = \alpha t$ for $0 < \alpha < \frac{q-1}{q}$.

Asymptotic in the number of components

The asymptotic regime:

$$t = 1 \left(\left(\quad \right) \right)$$

$$t = 2 \left(\left(\quad \right), \left(\quad \right) \right)$$

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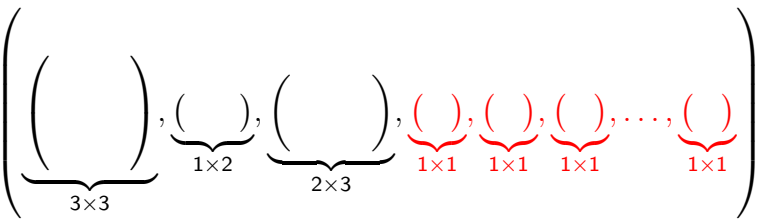

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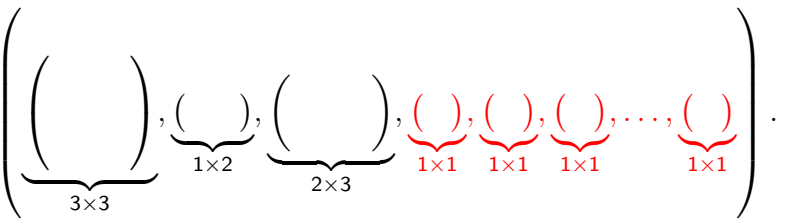

- The sum-rank metric is "dominated" by the Hamming components. $D \sim D_H$, $T \sim T_H$.

Asymptotic in the number of components

Lemma [Vu, Wu, 2005]

Let D_H denote the regularity of $H(t,q)^k$, and let T_H be the number of edges in a neighborhood of $H(t,q)^k$. Then, **under some technical assumptions**, for sufficiently large t:

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For a sum-rank-metric space $\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}$ which is "dominated" by Hamming components, **under some technical assumptions**, for sufficiently large t:

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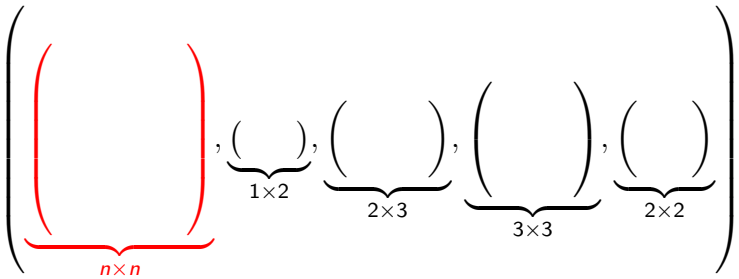
Asymptotic in the size of the matrix



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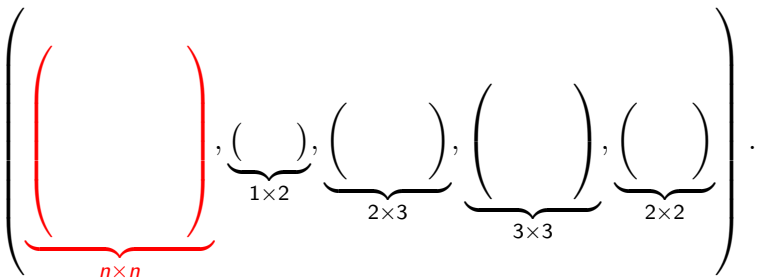

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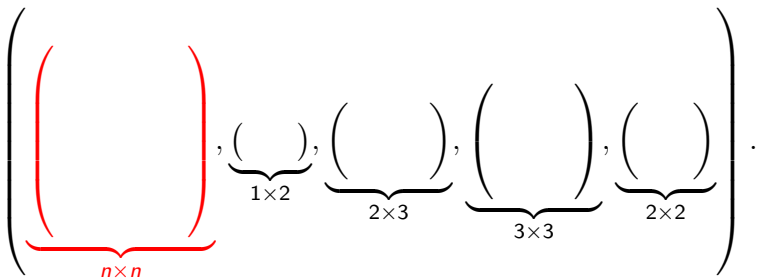

- In this case, D and T is dominated by contributions from the first component. $D \sim D_R$, $T \sim T_R$.

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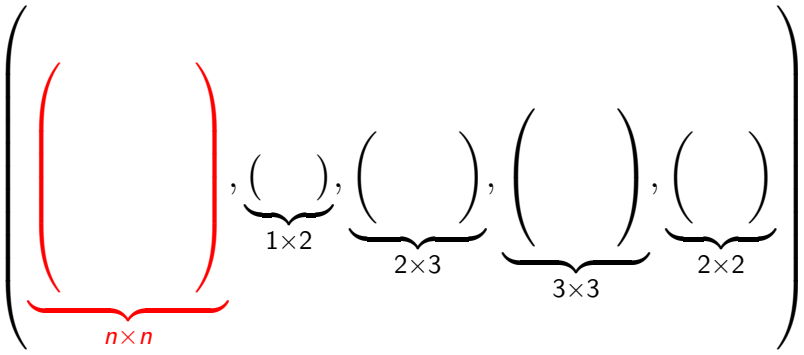

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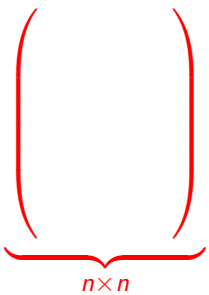

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Lemma [Marsaglia, 1964]

Let X and Y be $n \times n$ matrices with rank i and j respectively. Let the dimension of $\mathrm{col}(X) \cap \mathrm{col}(Y)$ be c and the dimension of $\mathrm{row}(X) \cap \mathrm{row}(Y)$ be r. Then

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- Matrices $X, Y \in \mathbb{F}_q^{n \times n}$ with $\mathrm{rk}(X - Y) \leq k$.
- Column spaces $\mathrm{col}(X), \mathrm{col}(Y)$ with $\dim(\mathrm{col}(X) \cap \mathrm{col}(Y)) \geq \frac{k-i-j}{2}$.

Asymptotic in the size of a matrix

Lemma [Abiad, R., Tait, 2025+]

For a sum-rank-metric space $\mathbb{F}_q^{\mathbf{n}\times\mathbf{m}}$ which is "dominated" by a single $n \times n$ component, **under some technical assumptions**, for sufficiently large n:

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Theorem [Abiad, R., Tait, 2025+]

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$$A_q^{SRK}(\mathbf{n_n}, \mathbf{m_n}, k+1) \geq \varepsilon \frac{|V|}{20D} \log_2 D.$$

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Future directions

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- Possibly loosen restriction on α.

Thank you for listening!

For further details:

A. Abiad, H. Reijnders, M. Tait.

Improved Gilbert-Varshamov bound for sum-rank-metric codes via graph theory.

arXiv:2510.21298