Regular fat linear sets

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Rank-metric codes

Definitions

Definition

A linear vector $[m,k]_{q^n/q}$-rank-metric linear code $\mathcal{C}$ is a $\mathbb{F}_{q^n}$-subspace of $\mathbb{F}_{q^n}^m$ of dimension k.

Definition

- *The rank weight of a vector $v=(v_1,v_2,\ldots,v_n)\in\mathbb{F}_{q^n}$ is $w(v)=dim_{\mathbb{F}_q}(\langle v_1,v_2,\ldots,v_n\rangle)$.*
- *The minimum distance of $\mathcal{C}$ is $d=d(\mathcal{C})=min\{w(v)|0\neq v\in\mathcal{C}\}$.*

In this case we say $\mathcal{C}$ is a $[m,k,d]_{q^n/q}$-rank-metric linear code.

Rank-metric codes
Definitions

Theorem (Singleton-like bound)

$log_q|\mathcal{C}| \leq max\{m,n\}(min\{m,n\} - d + 1).$

Definition

A code attaining the Singleton-like bound is called MRD-code.

Regular fat linear sets

Linear sets

Definition

A linear set of rank ρ in $PG(k-1,q^n)$ associated with an $\mathbb{F}_q$-subspace U of $\mathbb{F}_{q^n}^k$, $dim_{\mathbb{F}_q}(U)=\rho$ is

$$L_U = \{\langle v\rangle_{\mathbb{F}_{q^n}} \mid v \in U, v \neq \mathbf{0}\}.$$

Definition

- *The weight of a point $P=\langle v\rangle_{\mathbb{F}_{q^n}} \in PG(k-1,q^n)$ w.r.t. L_U is*

$$w_{L_U}(P) = dim_{\mathbb{F}_q}(\langle v\rangle_{\mathbb{F}_{q^n}} \cap U).$$

- *L_U is said to be scattered if $dim_{\mathbb{F}_q}(\langle v\rangle_{\mathbb{F}_{q^n}} \cap U) \leq 1\ \forall v \in \mathbb{F}_{q^n}^k$.*

Regular fat linear sets

(r, i)-RFLSs

Definition

An (r, i)-regular fat linear set ((r, i)-RFLS) is one that has precisely r points with weight greater than one, and all of these points have weight i ($r \geq 0$, $i \geq 2$).

Example

Any $(0, i)$-RFLS is a scattered linear set, and vice versa.

Proposition

If L_U is an (r, i)-RFLS of rank ρ then

$$|L_U| = \frac{q^\rho - 1 - r(q^i - q)}{q - 1}.$$

Regular fat linear sets

Examples for $r = 1$ or $i = 2$

- The $(1, i)$-RFLS are called i-clubs (Sz. Fancsali, P. Sziklai, 2006) and have been widely studied.
- (M. De Boeck, G. Van de Voorde, 2022) deal with linear sets or rank $\rho \leq 4$ in $PG(1, q^n)$ and rank 5 in $PG(1, q^5)$. In particular, for $\rho = 4$ all (r, i)-RFLS have parameters $(1, 3)$, $(q + 1, 2)$, $(1, 2)$ or $(2, 2)$.
- $(r, 2)$-RFLSs are also investigated in: (D. Bartoli, G. Micheli, G. Zini, F. Zullo, 2022), (B. Csajbók, G. Marino, O. Polverino, C. Zanella, 2018), (C. Zanella, 2019), (O. Polverino, F. Zullo, 2020), (D. Bartoli, B. Csajbók, M. Montanucci, 2021), . . .

Regular fat linear sets

Points with complementary weights and $(2, i)$-RFLSs

Theorem (V. Napolitano, O. Polverino, P. Santonastaso, F. Zullo, 2022)

Let $1 < t < n$ and $n = \ell t$. There exist $\mathbb{F}_q$-linear sets of rank ρ in $PG(1, q^n)$ with one point of weight t, one point of weight s and all others of weight one for the following values of n, k and s:

- *n even, $\rho = t + s$ and any $s \in \{1, \dots, \frac{n}{2}\}$;*
- *n odd, $\rho = t + s$ and any $s \in \{1, \dots, \frac{n-t}{2}\}$.*

Corollary

If t divides n, there is a $(2, t)$-RFLS in $PG(1, q^n)$.

Regular fat linear sets

Points with complementary weights and $(2,i)$-RFLSs

Theorem (G. N. Alfarano, R Jurrius, A. Neri, F. Zullo, 202X)

There exist $(2,i)$-regular fat linear sets in $PG(1,q^n)$ in the following cases:

- n *even and* $n \geq 2i$;
- $n \geq i^2$;
- $n = t_1 t_2$ *with* $t_1 \geq i$ *and* $i \leq 1 + t_1(t_2-1)/2$;
- $n = t_1 t_2$ *with* $t_1 \geq i$ *and* $t_2 \geq 2$;
- $q = p^h$, $n = p^r$, $2i - 1 \leq n/2$.

Associated rank-metric codes

Definition

Let L_U be an (r, i)-RFLS of rank ρ in $PG(k-1, q^n)$. Take $G \in \mathbb{F}_{q^n}^{k\times(nk-\rho)}$ having as columns an $\mathbb{F}_q$-basis of

$$U^{\perp'} = \{x \in \mathbb{F}_{q^n}^k \mid Tr_{q^n/q}(x \cdot u) = 0, \forall u \in U\}.$$

Define the code $\mathcal{C} \leq \mathbb{F}_{q^n}^{nk-\rho}$ associated with the linear set $L_{U^{\perp'}}$ as the rowspace of G.

Associated rank-metric codes

Proposition

If $i < n$, the rank-metric code $\mathcal{C}$ associated with $L_{U^{\perp'}}$, where U is a (r, i)-RFLS of rank ρ in $PG(k-1, q^n)$, is an $[nk-\rho, k, n-i]_{q^n/q}$-code with

- $r(q^n - 1)$ *codewords of weight* $n - i$
- $(|L_U| - r)(q^n - 1)$ *codewords of weight* $n - 1$
- $(q^{nk} - 1) - |L_U|(q^n - 1)$ *codewords of weight* n

recall $|L_U| = \frac{q^\rho - 1 - r(q^i - q)}{q-1}$

Associated rank-metric codes

Theorem (A. Ravagnani, 2016)

Let $\mathcal{C}$ be a rank-metric code in $\mathbb{F}_{q^n}^m$. Let $(A_i)_{i\in\mathbb{N}}$ and $(B_j)_{j\in\mathbb{N}}$ be the rank distributions of $\mathcal{C}$ and $\mathcal{C}^\perp$, respectively. For any integer $0 \le \nu \le n$ we have

$$\sum_{j=0}^{n-\nu} A_j \begin{bmatrix} n-j \\ \nu \end{bmatrix}_q = \frac{|\mathcal{C}|}{q^{m\nu}} \sum_{j=0}^{\nu} B_j \begin{bmatrix} n-j \\ \nu-j \end{bmatrix}_q .$$

Theorem

Let $U \le_{\mathbb{F}_q} \mathbb{F}_{q^n}^k$, such that L_U is a (r,i)-RLFS of rank ρ, $i < n$.

$$r \ge \frac{(q^{2\rho-nk}-1)\begin{bmatrix} n \\ 2 \end{bmatrix}_q}{(q^n-1)\begin{bmatrix} i \\ 2 \end{bmatrix}_q}.$$

Examples with $r, i > 2$

A construction in $PG(k-1, q^{2t})$

Theorem

Let q be odd, $t \geq 3$, $(s,t) = 1$, $w \in E = \{x \in \mathbb{F}_{q^{2t}} \mid Tr_{q^{2t}/q^t}(x) = 0\}$, $w \neq 0$, $N_{q^t/q}(w^2) \neq (-1)^t$, and I be an i-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^t}$, $i \geq 2$. Define

$$T = T_{s,w,I} = \{x + wx^{q^s} \mid x \in I\}.$$

Then for any $k > 1$, L_{T^k} is a $((q^k-1)/(q-1), i)$-regular fat linear set of rank ki in $PG(k-1, q^{2t})$. The points of weight i are precisely the elements of $PG(k-1, q)$, i.e. $(a_1 : a_2 : \ldots : a_k)$ with $(0, 0, \ldots, 0) \neq (a_1, a_2, \ldots, a_k) \in \mathbb{F}_q^k$.

Examples with $r, i > 2$

A construction in $PG(k-1, q^{\ell t})$, $\ell > 2$

$\ell | q^t - 1$. Let $\varepsilon \in \mathbb{F}_{q^t}$ such that $\{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{\ell-1}\}$ is the set of all roots of $X^\ell - 1$. $E_j = \{x \in \mathbb{F}_{q^{\ell t}} \mid x^{q^t} - \varepsilon^j x = 0\}, \quad j = 0, \ldots, \ell - 1$. There exist $\eta \in E_1$ such that $E_j = \langle \eta^j \rangle_{\mathbb{F}_{q^t}}, \quad j = 0, \ldots, \ell - 1$.

Theorem

Let $(s, t) = 1$, $\ell > 2$, $\ell \mid q^t - 1$, η as before and I be an i-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^t}$, $i \geq 2$. Define $T = T_{s,\eta,I} = \{x + \eta x^{q^s} \mid x \in I\}$. Then for any $k > 1$, L_{T^k} is a $((q^k - 1)/(q - 1), i)$-regular fat linear set of rank ki in $PG(k-1, q^{\ell t})$. The points of weight i are precisely the elements of $PG(k-1, q)$, i.e. $(a_1 : a_2 : \ldots : a_k)$ with $(0, \ldots, 0) \neq (a_1, \ldots, a_k) \in \mathbb{F}_q^k$.

Examples with $r, i > 2$

Linearized polynomials

Definition

- *A linearized polynomial over* $\mathbb{F}_{q^n}$ *is* $f = \sum_{i=0}^{k} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$.
- *If* $a_k \neq 0$ *then the q-degree of f is k.*
- $L_{n,q}$ *will denote the set of linearized polynomials over* $\mathbb{F}_{q^n}$.
- $\mathcal{L}_{n,q} = L_{n,q}/(x^{q^n} - x)$.
- *The graph of f is* $\mathcal{G}_f = \{(y, f(y)) \mid y \in \mathbb{F}_q\} \subseteq AG(2, q)$.

Definition

The linear set associated to f is
$L_f = L_{\mathcal{G}_f} = \{\langle (y, f(y)) \rangle_{q^n} \mid y \in \mathbb{F}_{q^n}^*\}$.

Examples with $r, i > 2$

The polynomial $\varphi_{m,\sigma}$

Theorem

Let $\varphi_{m,\sigma} = X^{\sigma^{t-1}} + X^{\sigma^{2t-1}} + m\left(X^{\sigma} - X^{\sigma^{t+1}}\right) \in \mathbb{F}_{q^{2t}}[X]$, $t \geq 3$, q *odd*, $\sigma = q^J$, $(J, 2t) = 1$, *and* $m \in \mathbb{F}^*_{q^t}$.

- *If* m *is a* $(\sigma - 1)$*-power of an element of* $E = \{x \in \mathbb{F}_{q^{2t}} \mid x^{q^t} + x = 0\}$, *then* $L_{\varphi_{m,\sigma}}$ *is an* $(r, 2)$*-regular fat linear set for some integer* r.
- *If* m *is a* $(\sigma + 1)$*-power of an element of* E *and* t *is odd, then* $L_{\varphi_{m,\sigma}}$ *is a* $(2, t)$*-regular fat linear set.*
- *If* m *is a* $(\sigma + 1)$*-power of an element of* E *and* t *is even, then* $L_{\varphi_{m,\sigma}}$ *is a* $(q + 1, t)$*-regular fat linear set.*
- *Otherwise,* $L_{\varphi_{m,\sigma}}$ *is a* $(0, -)$*-regular fat linear set, i.e., a scattered linear set.*