Schubert Subspace Codes

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25.03.2026

Open Problems on Rank-metric Codes (OpeRa 2026)

Subspace Codes

alphabet $\mathbb{F}_q$, $k \leq n \in \mathbb{N}$

FINITE FIELD

$Gr_q(k,n) = \{ k\text{-dimensional subspaces in } \mathbb{F}_q^n \}$

GRASSMANNIAN

Kötter and Kschischang: "Coding for errors and erasures in random network coding", 2008

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(constant dimension) subspace code $\quad \mathcal{C} \subseteq Gr_q(k,n)$

<u>def</u> (Subspace distance) $\forall U, V \leq \mathbb{F}_q^n$

$$d_S(U,V) = \dim(U+V) - \dim(U \cap V) = 2[k - \dim(U \cap V)]$$

if $U, V \in Gr_q(k,n)$

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Rank-metric Codes

$\mathcal{C} \subseteq \mathbb{F}_q^{k \times m}$ ← $k \times m$ dimensional matrices

<u>def</u> (rank distance) $\forall A, B \in \mathbb{F}_q^{k \times m}$

$$d_{rk}(A, B) = \text{rank}(A - B)$$

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row space

<u>LEMMA</u> $A, B \in \mathbb{F}_q^{k \times m}$. $U = rs[I_k \; A]$, $V = rs[I_k \; B]$

then $d_s(U,V) = 2 \cdot d_{rk}(A,B)$

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<u>proof</u> $\dim(U+V) = \text{rank}\begin{bmatrix} I_k & A \\ I_k & B \end{bmatrix} = \text{rank}\begin{bmatrix} I_k & A \\ O_k & B-A \end{bmatrix}$

$= k + \text{rank}(B-A) = k + d_{rk}(A,B)$

$d_s(U,V) = 2(\dim(U+V) - k) = 2 \cdot d_{rk}(A,B)$ ✓

Rank-metric Codes

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$d_s(U,V) = 2(\dim(U+V) - k) = 2\cdot d_{rk}(A,B)$ ✓

$\Rightarrow \left(Gr_q(k, \underbrace{k+m}_{n}), d_s\right) \cong \left(\mathbb{F}_q^{k\times m}, 2\cdot d_{rk}\right)$

$\{rs[I_k \; A_{k\times m}]\}$

Ferrers diagram Rank-metric Codes

More in general ...

$$U = rs\left[\begin{array}{c|c} 1 & \\ 1 & \\ \begin{array}{c} a_{11} \\ a_{21} \\ 0 \end{array} & \begin{array}{cc} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{array} \\ 1 & \end{array}\right] \qquad V = rs\left[\begin{array}{c|c} 1 & \\ 1 & \\ \begin{array}{c} b_{11} \\ b_{21} \\ 0 \end{array} & \begin{array}{cc} b_{12} & b_{13} \\ b_{22} & b_{23} \\ b_{32} & b_{33} \end{array} \\ 1 & \end{array}\right]$$

$$\Longrightarrow d_s(U,V) = 2 \cdot d_{rk}(A,B)$$

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Schubert cell

$$rs\begin{bmatrix} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

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pivot positions

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$(1\,1\,0\,1\,0\,0)$ pivot positions $\longleftrightarrow$

$$\begin{matrix} * & * & * \\ * & * & * \\ & * & * \end{matrix}$$

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Ferrers diagram

Ferrers diagram rank-metric code

$$\left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & g & h \end{pmatrix} \in \mathbb{F}_q^{3\times 3} \right\}$$

Multilevel Construction

(?) Construct $\mathcal{S} \subseteq G_q(k,n)$ with fixed $d_S(\mathcal{S}) = 2\delta$ and big $|\mathcal{S}|$?

1. choose a $(n, \overset{\text{WEIGHT}}{k}, 2\delta)_2$ constant-weight Hamming-metric code $\mathbb{C}$ (skeleton code)
2. $\forall c \in \mathbb{C}$ construct a Ferrers diagram $\mathcal{F}$
3. $\forall \mathcal{F}$ construct a Ferrers-diagram rank-metric code $\mathbb{C}_{\mathcal{F}}$ with $d_{rk}(\mathbb{C}_{\mathcal{F}}) = \delta$ (E-S conjecture)
4. lift $\forall \mathbb{C}_{\mathcal{F}}$ to a constant-dimension subspace code $\mathcal{S}_c$ inside the corresponding Schubert cell
5. $\mathcal{S} := \bigcup_{c \in \mathbb{C}} \mathcal{S}_c$ $\leftarrow$ desired subspace code

Etzion and Silberstein: "Error-correcting codes in projective spaces via rank-metric codes and Ferrers diagrams", 2009

Cellular decomposition of $Gr_q(k,n)$

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix} \qquad \begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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<u>Example</u>

$Gr_q(2,4)$

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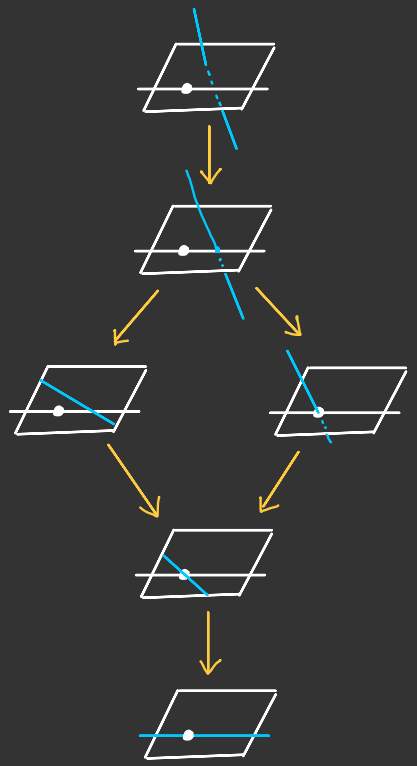

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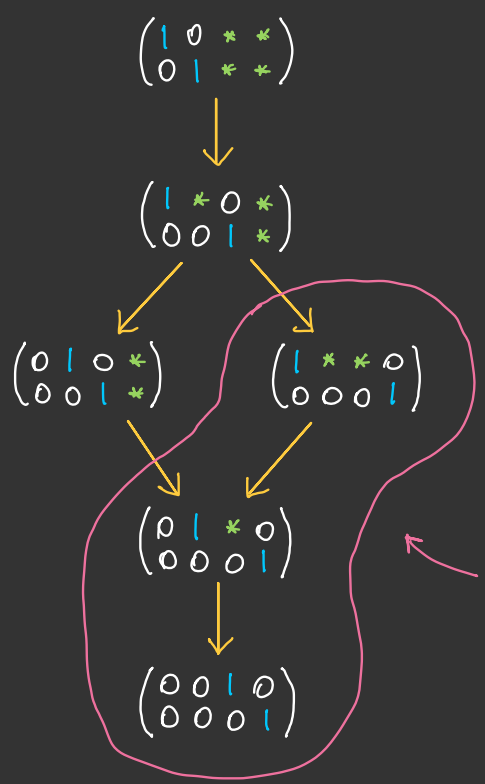
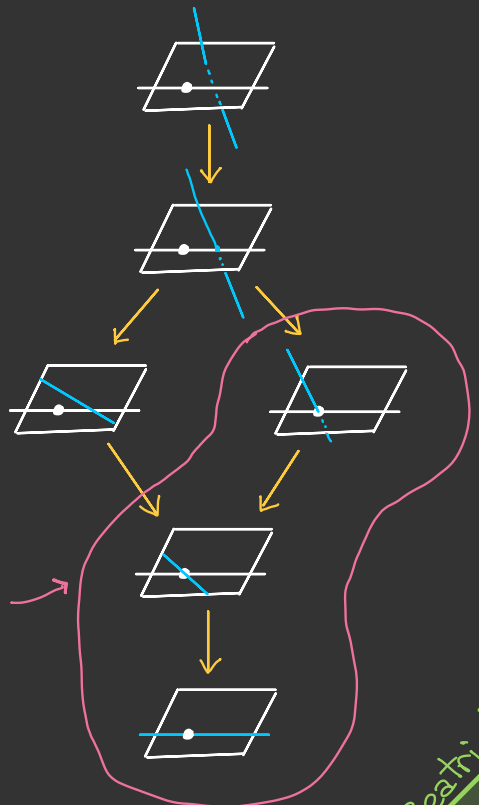


Example

$Gr_q(2,4)$

Schubert Variety



Schubert Subspace Codes

Let V be the flag:

$$\{0\} \subset V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{F}_q^n \text{ with } \dim(V_i) = i \quad \forall i$$

def Let $1 \leq a_1 < a_2 < \dots < a_k \leq n$ $\quad a = (a_1, \dots a_k)$ Schubert condition

The Schubert variety Ω_a is the set

$$\Omega_a = \{ W \in Gr_q(k,n) \text{ s.t. } \dim(W \cap V_{a_i}) \geq i \quad \forall i = 1, \dots, k \}$$

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$a = (a_1, \dots a_k)$
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A Schubert subspace code w.r.t. Ω_a is a family $\mathcal{S} \subseteq \Omega_a$ equipped with the subspace distance.

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$\boxed{\text{(?) Given } \Omega_a \text{ and } d_S(\mathcal{S}), \text{ maximize } |\mathcal{S}|.}$

Some Extra Assumptions...

$\mathbb{F}_q^{rk} \cong \mathbb{F}_{q^k}^r$

* $\underline{n = r \cdot k}$, with $r, k \geq 2$

k-dim spaces in $\mathbb{F}_q^{rk}$

$Gr_q(k, rk)$

$a_1 = u$

* $U \leq \mathbb{F}_q^{r \cdot k}$, $\dim(U) = u$ $\quad \Omega_U = \{ W \in Gr_q(k, rk) \text{ s.t. } \dim(W \cap U) \geq 1 \}$

  $\underline{\mathcal{S} \subseteq \Omega_U}$

* $\underline{d_S(\mathcal{S}) = 2k}$

  $\hookrightarrow$ $\forall S_i, S_j \in \mathcal{S}$ s.t. $S_i \neq S_j$ it holds $S_i \cap S_j = \{0\}$

Memo: $d_S(U, V) = 2k - 2\dim(U \cap V)$

Alfarano, Rosenthal, B.T.: "Schubert subspace codes", 2025

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  ↳ $\forall S_i, S_j \in \mathcal{S}$ s.t. $S_i \neq S_j$ it holds $S_i \cap S_j = \{0\}$

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PROP Let $k, r \geq 2$, $u < rk$. Then $|\mathcal{S}| \leq \begin{pmatrix} u \\ 1 \end{pmatrix}_q$

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PROP Let $k, r \geq 2$, $u < rk$. Then $|\mathcal{S}| \leq \begin{bmatrix} u \\ 1 \end{bmatrix}_q$

↳ for $u \leq \frac{rk}{2}$ there are cases when it is tight!

Alfarano, Rosenthal, B.T.: "Schubert subspace codes", 2025

1st Construction $\left(u \leq \frac{rk}{2}\right)$

* Let $\mathcal{U} \leq_{\mathbb{F}_q} \mathbb{F}_{q^k}^r$, $\dim_{\mathbb{F}_q}(\mathcal{U}) = u$ s.t. the linear set

$L_{\mathcal{U}} = \{ \langle v \rangle_{\mathbb{F}_{q^k}} \,/\, v \in \mathcal{U} \setminus \{0\} \}$ is maximum scattered ⊛

↳ $u = \frac{rk}{2}$ ↳ $|L_{\mathcal{U}}| = \begin{bmatrix} u \\ 1 \end{bmatrix}_q$

⊛ many existence results and constructions for some r, k
(Blokhuis, Lavrauw, Bartoli, Giulietti, Marino, Polverino, Csájbok, Zullo ...)

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* $\varphi : \mathbb{P}(\mathbb{F}_{q^k}^r) \longrightarrow \mathbb{P}(\mathbb{F}_q^{rk})$ field reduction map

points $\longmapsto$ $(k-1)$-dim spaces

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points $\longmapsto$ $(k-1)$-dim spaces

PROP $\mathcal{S} = \{ \varphi(P) \,/\, P \in L_{\mathcal{U}} \}$ is a Schubert subspace code s.t. $d_S(\mathcal{S}) = 2k$ and $|\mathcal{S}| = \begin{pmatrix} u \\ 1 \end{pmatrix}_q$

⊛ many existence results and constructions for some r, k
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2nd Construction (Special case $u=k$)

* $\mathcal{U} = \{ (s, s^q, s^{q^2}, \dots, s^{q^{r-1}}) \,/\, s \in \mathbb{F}_{q^k} \} \longrightarrow \dim_{\mathbb{F}_q}(\mathcal{U}) = k$
* $\forall a \in \mathbb{F}_{q^k}$ with field norm $N_{\mathbb{F}_{q^k}/\mathbb{F}_q}(a) = 1$

$$\mathcal{C}_a = \{ (s, as, s^{q^2}, \dots, s^{q^{r-1}}) \,/\, s \in \mathbb{F}_{q^k} \}$$

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$$\sigma_a = \{(s, as, s^{q^2}, \dots, s^{q^{r-1}}) \,/\, s \in \mathbb{F}_{q^k}\}$$

PROP $\mathcal{S} = \{\varphi(\sigma_a) \,/\, a \in \mathbb{F}_{q^k},\ N_{\mathbb{F}_{q^k}/\mathbb{F}_q}(a) = 1\}$ is a Schubert subspace code, $d_S(\mathcal{S}) = 2k$ and $|\mathcal{S}| = \begin{pmatrix} k \\ 1 \end{pmatrix}_q$ ($k \to u$)

We reached the upper bound!

Comparison with Multilevel Construction

PROP Let:
- $s = \#$ of cells selected
- j_i = position of last pivot in cell i

Then, if $\mathcal{S}$ is constructed via the multilevel construction:

$$|\mathcal{S}| \leq \sum_{1 \leq i \leq s} q^{j_i - 1}$$

Singleton-like bound inside one cell

COROLL if $r < u$, then $|\mathcal{S}| < \binom{u}{1}_q = q^{u-1} + q^{u-2} + \dots + q + 1$

$$\left(j_i \leq u, \quad s \leq \frac{kr}{k} = r \right)$$

↳ Our constructions have larger size!

A More General Problem

Fix U subspace of $\mathbb{F}_q^{rk}$. $\quad \ell, t \in \{0, \dots, k\}$

$\mathcal{S} \subseteq Gr_q(k, rk)$ such that $\forall S_i, S_j \in \mathcal{S}$:

- $\dim(S_i \cap U) \geq \ell$ $\quad \rightsquigarrow$ SCHUBERT VARIETY
- $\dim(S_i \cap S_j) \leq t$ $\quad \rightsquigarrow d_S(S_i, S_j) \geq 2(k-t)$

(?) study $\max|\mathcal{S}|$

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- $\ell = 1,\ t = 0$ $\rightsquigarrow$ previous case ✓ Alfarano, Rosenthal, B.T.: Schubert subspace codes
- $\ell = k-1,\ t = k-2$ $\rightsquigarrow$ ongoing work with G. Alfarano and A. Neri
  ($t = \ell - 1$?)

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