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HÖLDER CONTINUITY OF THE INTEGRATED DENSITY OF STATES FOR MATRIX-VALUED ANDERSON MODELS

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We study a class of continuous matrix-valued Anderson models acting on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^N$. We prove the existence of their Integrated Density of States for any $d \geq 1$ and $N \geq 1$. Then, for $d = 1$ and for arbitrary N, we prove the Hölder continuity of the Integrated Density of States under some assumption on the group G_{μ_E} generated by the transfer matrices associated to our models. This regularity result is based upon the analoguous regularity of the Lyapounov exponents associated to our model, and a new Thouless formula which relates the sum of the positive Lyapounov exponents to the Integrated Density of States. In the final section, we present an example of matrix-valued Anderson model for which we have already proved, in a previous article, that the assumption on the group G_{μ_E} is verified. Therefore, the general results developed here can be applied to this model.

Keywords: Integrated Density of States; Lyapounov exponents; Anderson model; Thouless formula.

Mathematics Subject Classification 2000: 47B80, 37H15

1. Introduction

We will study the question of the existence of the Integrated Density of States and its regularity for continuous matrix-valued Anderson models of the form:

$$H_A(\omega) = -\Delta_d \otimes I_N + \sum_{n\in\mathbb{Z}^d} V_\omega^{(n)}(x-n) \tag{1.1}$$

acting on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^N$, where d and N are non-negative integers, I_N is the identity matrix of order N and Δ_d denotes the d-dimensional continuous Laplacian. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a complete probability space and $\omega \in \Omega$. For every $n \in \mathbb{Z}$, the functions $x \mapsto V_\omega^{(n)}(x)$ will be symmetric matrix-valued functions, supported in $[0,1]^d$, and bounded uniformly on x, n and ω. We also set:

$$\forall x \in \mathbb{R}^d, \quad V_\omega(x) = \sum_{n \in \mathbb{Z}^d} V_\omega^{(n)}(x - n)$$

and denote by V_ω the maximal multiplication operator by $x \mapsto V_\omega(x)$. The function $x \mapsto V_\omega(x)$ is uniformly bounded on $\mathbb{R}$ in x and in ω. The potential V_ω will also be such that the operator $H_A(\omega)$ is $\mathbb{Z}^d$-ergodic. As a bounded perturbation of $-\Delta_d \otimes I_N$, the operator $H_A(\omega)$ is self-adjoint on the Sobolev space $H^2(\mathbb{R}^d) \otimes \mathbb{C}^N$.

We want to define a function of the real variable which will count the number of proper energy states of $H_A(\omega)$ below a fixed energy E. For systems like (1.1), such a definition will usually lead to an infinite function as the operators we study act on an infinite-dimensional Hilbert space and thus have infinitely many spectral values. To avoid this problem, we will define our function, the Integrated Density of States or IDS, as a thermodynamical limit as explained in Sec. 2. It will lead to a problem of existence of such a thermodynamical limit. We will prove the existence of the IDS in Sec. 2 for any d and any N. This existence proof will be based upon a matrix-valued Feynman–Kac formula proven in [2] and the adaptation of the argument of Carmona in [7] to matrix-valued operators. Once we have proven the existence of the IDS, we will study its regularity as a function of the energy parameter E. For this second step, we will restrict ourselves to the case where $d = 1$ and N is arbitrary, and to be able to use the tools coming from the theory of ODEs such as the notion of transfer matrix. We will prove in Sec. 4 that under some assumption on V_ω, or, more precisely, on the group generated by the transfer matrices associated to $H_A(\omega)$, the IDS is locally Hölder continuous. This result will come from the analoguous regularity result on Lyapounov exponents proved in Sec. 3, and from a Thouless formula proven in Sec. 4 which relates the IDS to the Lyapounov exponents. To prove this Thouless formula, we use results of Kotani and Simon in [20] and Kotani in [19]. The regularity result on Lyapounov exponents is based upon the results of Carmona and Lacroix in [9] and Lacroix, Klein and Speis in [17]. We also need to prove estimates on the transfer matrices for our model (1.1) (for $d = 1$) similar to those proven in [11] in the scalar-valued case. In a final section, we present an example of continuous matrix-valued Anderson model for which the needed assumption on the group generated by the transfer matrices is verified. This example is the following matrix-valued Anderson–Bernoulli model:

$$H_{AB}(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{n \in \mathbb{Z}} \begin{pmatrix} \omega_1^{(n)} \chi_{[0,1]}(x - n) & 0 \\ 0 & \omega_2^{(n)} \chi_{[0,1]}(x - n) \end{pmatrix} \tag{1.2}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, with $(\omega_1^{(n)})_{n \in \mathbb{Z}}$ and $(\omega_2^{(n)})_{n \in \mathbb{Z}}$ two independent sequences of independent and identically distributed (i.i.d.) random variables with common law ν such that $\{0, 1\} \subset \operatorname{supp} \nu$. This model has already been studied by the author in [3] as an improvement of a result by Stolz and the author in [5]. We proved in [5] the absence of absolutely continuous spectrum and pointed out that the improvement made in [3] was necessary to be able to prove local Hölder continuity of the IDS.

The study of the regularity of the IDS is an important step to prove Anderson localization by using a multiscale analysis scheme. It is the key ingredient to prove a Wegner estimate as was done in [8] and to adapt it to the case of scalar-valued continuous Anderson model in [11]. We believe that once we have proved a Wegner estimate and an Initial Length Scale Estimate for model (1.1) for $d = 1$ and arbitrary N, it will be possible to adapt existing multiscale analysis schemes to the case of matrix-valued operators. We will then be able to prove Anderson and dynamical localization for this model as explained in [24].

The question of localization for one-dimensional continuous matrix-valued Anderson model is based on a more general problem on Anderson models. Localization for continuous Anderson models in dimension $d \geq 2$ at all energies is still an open problem if one looks for arbitrary disorder, including Bernoulli randomness. A possible approach to the localization for $d = 2$ is to discretize one direction, which leads to considering a one-dimensional Anderson model, that is no longer scalar-valued, but $N \times N$ matrix-valued as we have here for $d = 1$. What is already well understood is the case of dimension one scalar-valued continuous Schrödinger operators with arbitrary randomness (see [11]) and discrete matrix-valued Schrödinger operators, also for arbitrary randomness (see [14, 17]). We want to combine here techniques of [11, 17] to get the local Hölder continuity of the IDS for continuous matrix-valued models.

We finish by mentioning that different methods have been used in [16] to prove localization properties for random operators on discrete strips. They are based upon the use of spectral averaging techniques which did not allow us to handle with singular distributions of the random parameters like in our model (1.2).

2. Existence of the IDS

In this section, we will define the IDS associated to the operator $H_A(\omega)$ and prove its existence. The proof of the existence for the IDS will strongly rely on a matrix-valued Feynman–Kac formula which we will present after the definition of the IDS.

As we have already noticed in the introduction, the operator $H_A(\omega)$ is self-adjoint and $\mathbb{Z}^d$-ergodic. But, in some parts of the following proofs, and also in Sec. 4, we will need a stronger assumption of $\mathbb{R}^d$-ergodicity for $H_A(\omega)$ instead of only $\mathbb{Z}^d$-ergodicity. To avoid this lack of $\mathbb{R}^d$-ergodicity in general, we can refer to the suspension procedure developed by Kirsch in [15]. This procedure allows us to construct from $H_A(\omega)$ an operator $\tilde{H}_A(\tilde{\omega})$, defined on a bigger probability space, which is $\mathbb{R}^d$-ergodic. $\tilde{H}_A(\tilde{\omega})$ is also constructed in a way such that its IDS and Lyapounov exponents exist if and only if those of $H_A(\omega)$ exist, and in this case they are equal for both operators. Considering the use of this suspension procedure we will work in the following with $H_A(\omega)$ as if it is $\mathbb{R}^d$-ergodic instead of being only $\mathbb{Z}^d$-ergodic.

2.1. *Definition of the IDS*

We aim at defining a function that will give us the mean number per unit volume of spectral values of $H_A(\omega)$ situated below a fixed real number E. In order to define this function, we will first restrict $H_A(\omega)$ to cubes of finite volume of $\mathbb{R}^d$. Let L be a strictly positive integer and $D = [-L, L]^d \subset \mathbb{R}^d$ be the cube centered at 0 and of length $2L$. We set:

$$H_A^{(D)}(\omega) = -\Delta_d^{(D)} \otimes I_N + \sum_{n \in \mathbb{Z}^d} V_\omega^{(n)}(x - n) \tag{2.1}$$

the restriction of $H_A(\omega)$ acting on $L^2(D) \otimes \mathbb{C}^N$ with Dirichlet boundary conditions on D.

Definition 1. The Integrated Density of States, or IDS, associated to $H_A(\omega)$ is the function from $\mathbb{R}$ to $\mathbb{R}_+$, $E \mapsto N(E)$ where $N(E)$ for $E \in \mathbb{R}$ is defined as the following thermodynamical limit:

$$N(E) = \lim_{L \to +\infty} \frac{1}{|D|} \#\{\lambda \leq E \mid \lambda \in \sigma(H_A^{(D)}(\omega))\} \tag{2.2}$$

where $|D|$ is the volume of D.

Here we have a double problem of existence in the expression (2.2). First, we have to prove that the cardinal $\#\{\lambda \leq E \mid \lambda \in \sigma(H_A^{(D)}(\omega))\}$ is finite for each fixed E and then we have to show the existence of the limit. The answer to each one of these problems relies on the existence of an L^2-kernel for the one-parameter semigroup $(\mathrm{e}^{-tH_A^{(D)}(\omega)})_{t>0}$.

2.2. *A matrix-valued Feynman–Kac formula*

We will first present a matrix-valued Feynman–Kac formula for the one-parameter semigroup $(\mathrm{e}^{-tH_A(\omega)})_{t>0}$ due to Boulton and Restuccia ([2]). We will then deduce a Feynman–Kac formula for $(\mathrm{e}^{-tH_A^{(D)}(\omega)})_{t>0}$.

Let $\mathsf{W} = C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. For every $t \geq 0$ we consider the coordinate function:

$$X_t : \begin{array}{l} \mathsf{W} \to \mathbb{R} \\ \mathsf{w} \mapsto X_t(\mathsf{w}) = \mathsf{w}(t). \end{array}$$

Let $\mathcal{W}$ be the smallest σ-algebra on W for which all the applications X_t are measurable. For $s, t \geq 0$ and $x, y \in \mathbb{R}^d$ we denote by $W_{s,x,t,y}$ the conditional Wiener measure, defined on $(\mathsf{W}, \mathcal{W})$, associated to the Brownian motion starting from x at the time s and arriving on y at the time t. We also denote by $\mathbb{E}_{s,x,t,y}$ the expectation value associated to the measure $W_{s,x,t,y}$. For a construction of such conditional Wiener measure and for a construction of the path integral associated to, we refer to [22, Chap. 2].

We now study the one-parameter semigroup $(\mathrm{e}^{-tH_A(\omega)})_{t>0}$. We fix $t > 0$ and $\omega \in \Omega$. By the Lie–Trotter formula we have:

$$\forall f \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^N, \quad \mathrm{e}^{-tH_A(\omega)} f = \lim_{n\to+\infty} (\mathrm{e}^{-(-\Delta_d \otimes I_N)\frac{t}{n}} \mathrm{e}^{-V_\omega \frac{t}{n}})^n f. \tag{2.3}$$

For a fixed $n \in \mathbb{N}$, we can use [13, Corollary 3.1.2, p. 47] to get that the operator:

$$(\mathrm{e}^{-(-\Delta_d \otimes I_N)\frac{t}{n}} \mathrm{e}^{-V_\omega \frac{t}{n}})^n$$

has an integral kernel given by the following path integral:

$$\int \prod_{j=1}^{n} \mathrm{e}^{-(\frac{jt}{n}) \cdot V_\omega(\mathsf{w}(\frac{jt}{n}))} \, \mathrm{d}W_{0,x,t,y}(\mathsf{w}). \tag{2.4}$$

But when n tends to infinity we find, by definition of the time-ordered exponential (see [12]):

$$\lim_{n\to+\infty} \prod_{j=1}^{n} \mathrm{e}^{-(\frac{jt}{n}) \cdot V_\omega(\mathsf{w}(\frac{jt}{n}))} = \exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(\mathsf{w}(s))\,\mathrm{d}s\right). \tag{2.5}$$

Then by Lebesgue's dominated convergence theorem, we have that:

$$\forall f \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^N, \quad \forall x \in \mathbb{R}^d, \quad \mathrm{e}^{-tH_A(\omega)} f(x) = \int_{\mathbb{R}^d} K_t(x,y) f(y)\,\mathrm{d}x \tag{2.6}$$

where:

$$\forall x,y \in \mathbb{R}^d, \quad \forall t > 0, \quad K_t(x,y) = \int \exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(\mathsf{w}(s))\,\mathrm{d}s\right) \mathrm{d}W_{0,x,t,y}(\mathsf{w}). \tag{2.7}$$

So we have just proven that $\mathrm{e}^{-tH_A(\omega)}$ has an integral kernel, $K_t(x,y)$. Let us see how to deduce from this integral kernel, the existence of an integral kernel for $\mathrm{e}^{-tH_A^{(D)}(\omega)}$. We denote by $T_D(\mathsf{w})$ the time of the first exit from D of the path $\mathsf{w} \in \mathsf{W}$:

$$T_D(\mathsf{w}) = \inf\{t > 0,\ X_t(\mathsf{w}) \notin D\}. \tag{2.8}$$

Then the fact that we used Dirichlet boundary conditions to define $H_A^{(D)}(\omega)$ allows us to use results on killed Brownian motions (see [18]) which lead us to the following formula:

$$\begin{aligned} &\forall t > 0, \quad \forall f \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^N, \quad \forall x \in \mathbb{R}^d, \quad \mathrm{e}^{-tH_A^{(D)}(\omega)} f(x) \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}^d} \int \chi_{\{t<T_D(\mathsf{w})\}}(\mathsf{w}) \exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(X_s(\mathsf{w}))\,\mathrm{d}s\right) \\ &\quad \times \mathrm{d}W_{0,x,t,y}(\mathsf{w}) \mathrm{e}^{-\frac{|x-y|^2}{2t}} f(y)\,\mathrm{d}y. \end{aligned} \tag{2.9}$$

So we have the following proposition:

Proposition 1. *For every $t > 0$, $\mathrm{e}^{-tH_A^{(D)}(\omega)}$ has an integral kernel given by the formula*:

$$\forall x, y \in \mathbb{R}^d, \quad \forall t > 0,\ K_t^{(D)}(x, y)$$
$$= \frac{1}{\sqrt{2\pi t}} \left(\int \chi_{\{t<T_D(\mathsf{w})\}}(\mathsf{w}) \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(X_s(\mathsf{w}))\, \mathrm{d}s \right) \mathrm{d}W_{0,x,t,y}(\mathsf{w}) \mathrm{e}^{-\frac{|x-y|^2}{2t}} \right) \tag{2.10}$$

and $K_t^{(D)}$ is in $L^2(D^2) \otimes \mathcal{M}_{\mathrm{N}}(\mathbb{C})$ for every $t > 0$.

Proof. The first assertion and the formula (2.10) come from (2.9). Then D is a compact domain in $\mathbb{R}^d$ and for a fixed $t > 0$, $(x, y) \mapsto K_t^{(D)}(x, y)$ is continuous. As in (2.10), t is bounded by $T_D(\mathsf{w})$, we have that $K_t^{(D)}$ is in $L^2(D^2) \otimes \mathcal{M}_{\mathrm{N}}(\mathbb{C})$ as it is a bounded continuous function on D^2. □

This proposition will be the main ingredient to prove the existence of the IDS associated to $H_A(\omega)$.

2.3. *Existence of the IDS*

From Proposition 2.10, we deduce that for every $t > 0$, the operator $\mathrm{e}^{-tH_A^{(D)}(\omega)}$ is Hilbert–Schmidt on $L^2(D) \otimes \mathbb{C}^N$. Thus, its spectrum is of the form:

$$\{\mathrm{e}^{-t\lambda_j^{(D)}(\omega)},\ j \geq 0\}$$

where $(\lambda_j^{(D)}(\omega))_{j\geq 0}$ is an increasing sequence of real numbers, bounded from below and tending to $+\infty$. This sequence is the spectrum of $H_A^{(D)}(\omega)$. In particular, for a fixed $E \in \mathbb{R}$:

$$\#\{\lambda \leq E \mid \lambda \in \sigma(H_A^{(D)}(\omega))\} = \#\{\lambda_j^{(D)}(\omega) \leq E\} < +\infty.$$

This answers the first part of the problem of existence of $N(E)$. It remains for us to prove that the sequence $\left(\frac{1}{|D|}\#\{\lambda_j^{(D)}(\omega) \leq E\}\right)_{L\geq 1}$ converges to a real number independent of $\omega : N(E)$. To that end, we introduce the counting measure of the eigenvalues of $H_A^{(D)}(\omega)$:

$$\mathfrak{n}_{D,\omega} = \frac{1}{|D|} \sum_{j\geq 0} \delta_{\lambda_j^{(D)}(\omega)} \tag{2.11}$$

where $\delta_{\lambda_j^{(D)}(\omega)}$ is the Dirac measure at $\lambda_j^{(D)}(\omega)$. Then we have:

Proposition 2. *The sequence of measures $(\mathfrak{n}_{D,\omega})_{L\geq 1}$ converges vaguely to a measure $\mathfrak{n}$ independent of ω as L tends to infinity for P-almost every ω in Ω. Moreover,* *the Laplace transform of this measure* $\mathfrak{n}$ *is given by*: $\forall t > 0$,

$$L(\mathfrak{n})(t) = \frac{1}{\sqrt{2\pi t}} \iint_{\Omega} \mathrm{Tr}_{\mathbb{C}^N} \exp_{\mathrm{ord}} \left(- \int_0^t V_\omega(X_s(\mathsf{w}))\, \mathrm{d}s \right) \mathrm{d}\omega \mathrm{d}W_{0,0,t,0}(\mathsf{w}). \tag{2.12}$$

Corollary 1. *For every* $E \in \mathbb{R}$, *the limit*:

$$N(E) = \lim_{L \to +\infty} \frac{1}{|D|} \#\{\lambda \le E \mid \lambda \in \sigma(H_A^{(D)}(\omega))\}$$

exists and is P-*almost surely independent of* ω. *The function* $E \mapsto N(E)$ *is the distribution function of* $\mathfrak{n}$:

$$\forall E \in \mathbb{R}, \quad N(E) = \mathfrak{n}((-\infty, E]).$$

Before proving this proposition, we need to prove a lemma which gives the expression of the trace of an operator with matrix-valued integral kernel. We adapt here a result of Simon proven in [23, Theorem 3.9, p. 35].

Lemma 1. *Let* H *be a self-adjoint operator acting on* $L^2(D) \otimes \mathbb{C}^N$ *where* $D \subset \mathbb{R}^d$ *is a compact set. We assume that for all* $t > 0$ *the operator* e^{-tH} *is class-trace and has a matrix-valued integral kernel* K_t. *Then*:

$$\mathrm{Tr}(\mathrm{e}^{-tH}) = \int_D \mathrm{Tr}_{\mathbb{C}^N} K_t(x,x)\, \mathrm{d}x$$

where $\mathrm{Tr}_{\mathbb{C}^N}$ *denotes the usual trace on* $N \times N$ *matrices.*

Proof. Let $n \in \mathbb{N}$, $m \in \{0, \dots, 2^n\}$ and $k \in \{1, \dots, N\}$. We set:

$$\phi_{n,m,k}(x) = \begin{cases} {}^t(0,\dots,0,2^{\frac{n}{2}},0,\dots,0) & \text{if } \forall i \in \{1,\dots N\},\ -L \cdot \dfrac{m-1}{2^n} \le x_i < L \cdot \dfrac{m}{2^n} \\ {}^t(0,\dots,0) & \text{otherwise} \end{cases}$$

where $2^{\frac{n}{2}}$ is at the kth position. Then the family $\{\phi_{n,m,k}\}_{n \in \mathbb{N}, 0 \le m \le 2^n}^{1 \le k \le N}$ is a Hilbert basis of the Hilbert space $L^2(D) \otimes \mathbb{C}^N$.

Let P_n be the projection on the subspace spanned by the $2^n N$ functions $\phi_{n,m,k}$ for n fixed and $m \in \{0, \dots, 2^n\}$, $k \in \{1, \dots, N\}$. Then one can construct a Hilbert basis $(\psi_1, \psi_2, \dots)$ of $L^2(D) \otimes \mathbb{C}^N$ such that:

$$\forall n \in \mathbb{N}, \quad \psi_1, \dots, \psi_{2^n N} \in \mathrm{Im}\ P_n.$$

Then we have:

$$\mathrm{Tr}(\mathrm{e}^{-tH}) = \lim_{n \to +\infty} \mathrm{Tr}(P_n \mathrm{e}^{-tH} P_n)$$

by [23, Theorem 3.1, p. 31]. But:

$$\begin{aligned}
&\forall n \in \mathbb{N},\\
\mathrm{Tr}(P_n \mathrm{e}^{-tH} P_n) &= \sum_{k=1}^{N} \sum_{m=1}^{2^n} \langle \phi_{n,m,k}, \mathrm{e}^{-tH} \phi_{n,m,k} \rangle \\
&= \sum_{k=1}^{N} \sum_{m=1}^{2^n} \int_D \int_D {}^t\overline{\phi_{n,m,k}(x)} K_t(x,y) \phi_{n,m,k}(y)\, \mathrm{d}x\mathrm{d}y \\
&= \sum_{m=1}^{2^n} \iint_{-L\cdot\frac{m-1}{2^n} \le x_i, y_i < L\cdot \frac{m}{2^n}} 2^{\frac{n}{2}} \cdot 2^{\frac{n}{2}} \\
&\quad \times \underbrace{\left(\sum_{k=1}^{N} (0,\ldots,1,\ldots,0) K_t(x,y)\, {}^t(0,\ldots,1,\ldots,0) \right)}_{\mathrm{Tr}_{\mathbb{C}^N}(K_t(x,y))} \mathrm{d}x\mathrm{d}y \\
&= 2^n \sum_{m=1}^{2^n} \iint_{-L\cdot\frac{m-1}{2^n} \le x_i, y_i < L\cdot \frac{m}{2^n}} \mathrm{Tr}_{\mathbb{C}^N}(K_t(x,y))\, \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Then by uniform continuity of K_t on the compact set D^2:

$$\begin{aligned}
&\lim_{n\to+\infty} 2^n \sum_{m=1}^{2^n} \iint_{-L\cdot\frac{m-1}{2^n} \le x_i, y_i < L\cdot \frac{m}{2^n}} \mathrm{Tr}_{\mathbb{C}^N}(K_t(x,y))\, \mathrm{d}x\mathrm{d}y \\
&= \int_D \mathrm{Tr}_{\mathbb{C}^N}(K_t(x,x))\, \mathrm{d}x.
\end{aligned}$$

□

Proof of Proposition 2. We fix $t > 0$. We have:

$$\begin{aligned}
L(\mathfrak{n}_{D,\omega})(t) &= \int_{\mathbb{R}} \mathrm{e}^{-Et} \mathfrak{n}_{D,\omega}(E) \\
&= \frac{1}{|D|} \sum_{j\ge 0} \mathrm{e}^{-\lambda_j^{(D)}(\omega)t} \\
&= \frac{1}{|D|} \mathrm{Tr}(\mathrm{e}^{-tH_A^{(D)}(\omega)}) \\
&= \frac{1}{|D|} \int_D \mathrm{Tr}_{\mathbb{C}^N}(K_t(x,x))\, \mathrm{d}x \\
&= \frac{1}{|D|} \frac{1}{\sqrt{2\pi t}} \int_D \int \chi_{\{t<T_D(\mathsf{w})\}}(\mathsf{w})\, \mathrm{Tr}_{\mathbb{C}^N} \\
&\quad \times \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(X_s(\mathsf{w}))\, \mathrm{d}s \right) \mathrm{d}W_{0,x,t,x}(\mathsf{w})\, \mathrm{d}x
\end{aligned}$$

by (2.10). We set:

$$A_D = \frac{1}{|D|}\frac{1}{\sqrt{2\pi t}}\int_D\int \mathrm{Tr}_{\mathbb{C}^N}\exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(X_s(\mathsf{w}))\,\mathrm{d}s\right)\mathrm{d}W_{0,x,t,x}(\mathsf{w})\,\mathrm{d}x \tag{2.13}$$

and:

$$\begin{aligned}B_D = \frac{1}{|D|}\frac{1}{\sqrt{2\pi t}}\int_D\int &\chi_{\{t\geq T_D(\mathsf{w})\}}(\mathsf{w})\,\mathrm{Tr}_{\mathbb{C}^N}\exp_{\mathrm{ord}}\\ &\times\left(-\int_0^t V_\omega(X_s(\mathsf{w}))\,\mathrm{d}s\right)\mathrm{d}W_{0,x,t,x}(\mathsf{w})\,\mathrm{d}x.\end{aligned} \tag{2.14}$$

Using Birkhoff's theorem when $L\to+\infty$ in A_D, we get:

$$\lim_{L\to+\infty} A_D = \frac{1}{\sqrt{2\pi t}}\iint_\Omega \mathrm{Tr}_{\mathbb{C}^N}\exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(X_s(\mathsf{w}))\,\mathrm{d}s\right)\mathrm{d}\omega\mathrm{d}W_{0,0,t,0}(\mathsf{w}). \tag{2.15}$$

Let $\mathfrak{n}$ be the measure on $\mathbb{R}$ (with the Borel σ-algebra) such that:

$$L(\mathfrak{n})(t) = \frac{1}{\sqrt{2\pi t}}\iint_\Omega \mathrm{Tr}_{\mathbb{C}^N}\exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(X_s(\mathsf{w}))\,\mathrm{d}s\right)\mathrm{d}\omega\mathrm{d}W_{0,0,t,0}(\mathsf{w}). \tag{2.16}$$

To prove that $\mathfrak{n}_{D,\omega}$ converges vaguely to $\mathfrak{n}$ as L tends to infinity, it remains to prove that $B_D\to 0$ and that the convergence of A_D and B_D happens on a set Ω_1 independent of t and of measure 1. Actually, for the rest of the proof, we can refer to the proof of Carmona in [7, Theorem V1, pp. 66 and 67]. Indeed, as V_ω is uniformly bounded on $\mathbb{R}$ in x and in ω, the function:

$$\begin{aligned}\Omega\times\mathsf{W}&\to\mathbb{C}\\ (\omega,\mathsf{w})&\mapsto \mathrm{Tr}_{\mathbb{C}^N}\exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(X_s(\mathsf{w}))\,\mathrm{d}s\right)\end{aligned} \tag{2.17}$$

for $t>0$ fixed is in every $L^r(\Omega\times\mathsf{W},\mathsf{P}\otimes W_{0,0})$ for all $r>1$. Here $W_{0,0}$ is the Wiener measure defined on $(\mathsf{W},\mathcal{W})$ associated to the Brownian motion starting from 0 at time 0. Thus, function (2.17) has the same properties as the function:

$$\begin{aligned}\Omega\times\mathsf{W}&\to\mathbb{C}\\ (\omega,\mathsf{w})&\mapsto \exp\left(-\int_0^t q^-(X_s(\mathsf{w}),\omega)\,\mathrm{d}s\right)\end{aligned} \tag{2.18}$$

in [7, Theorem V1]. Then, changing (2.18) by (2.17), one can rewrite the proof of [7]. □

Remark. In the proof of Proposition 2, we did not verify that the limit measure $\mathfrak{n}$ does not depend on the choice of boundary conditions for $H_A^{(D)}(\omega)$. This choice appears in formula (2.9) by introducing the characteristic function $\chi_{\{t<T_D(\omega)\}}$ corresponding to a killed Brownian motion. If, by example, we had chosen Neumann boundary conditions instead of Dirichlet boundary conditions we should have to change this characteristic function to make it correspond to a reflected Brownian

motion (see [18, Chap. 4]). The rest of the proofs is unchanged and the expression of $\mathfrak{n}$ does not depend on $\chi_{\{t<T_D(\omega)\}}$.

We finish this section by proving a formula which relates the measure $\mathfrak{n}$ to the spectral measure $E_{H_A(\omega)}$ associated to the self-adjoint operator $H_A(\omega)$.

Proposition 3. *Let f be a continuous, positive, compactly supported function on $\mathbb{R}^d$, such that $\|f\|_{L^2(\mathbb{R}^d)} = 1$. We denote by M_f the maximal multiplication operator by f. Then for every bounded Borel set B of $\mathbb{R}$, the operator $M_f E_{H_A(\omega)}(B) M_f$ is trace-class P-almost surely in ω and:*

$$\mathfrak{n}(B) = \mathbb{E}(\mathrm{Tr}(M_f E_{H_A(\omega)}(B) M_f)) \tag{2.19}$$

where $\mathbb{E}$ is the expectation value associated to the probability measure P.

Proof. If $B \subset \mathbb{R}$ is a bounded Borel set of $\mathbb{R}$, then there exist strictly positives constants C and t such that:

$$\forall x \in \mathbb{R}, \quad \chi_B(x) \leq C\mathrm{e}^{-tx}. \tag{2.20}$$

Let $\{f_k\}_{k\geq 1}$ be a Hilbert basis of $L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$. Let f be a positive, continuous, compactly supported function on $\mathbb{R}^d$, such that $\|f\|_{L^2(\mathbb{R}^d)} = 1$. Then:

$$\mathbb{E}\left(\sum_{k\geq 1} \langle (M_f E_{H_A(\omega)}(B) M_f) f_k, f_k\rangle\right) \leq C\mathbb{E}\left(\sum_{k\geq 1} \langle \mathrm{e}^{-tH_A(\omega)}(f f_k), (f f_k)\rangle\right)$$

by the spectral theorem applicated to χ_B, the inequality (2.20) and the fact that M_f is self-adjoint as f is real-valued. But:

$$\mathbb{E}\left(\sum_{k\geq 1} \langle \mathrm{e}^{-tH_A(\omega)}(f f_k), (f f_k)\rangle\right) = \mathbb{E}(\mathrm{Tr}(M_f \mathrm{e}^{-tH_A(\omega)} M_f)).$$

Let L be large enough for $D = [-L, L]^d$ to contain the support of f. Then using Lemma 1:

$$\begin{aligned} \mathbb{E}(\mathrm{Tr}(M_f \mathrm{e}^{-tH_A(\omega)} M_f)) &= \mathbb{E}\left(\int_{\mathrm{supp}\, f} f(x)^2 \,\mathrm{Tr}_{\mathbb{C}^N} K_t(x,x)\,\mathrm{d}x\right) \\ &= \mathbb{E}\left(\int_{\mathbb{R}} f(x)^2 \,\mathrm{Tr}_{\mathbb{C}^N} K_t(x,x)\,\mathrm{d}x\right) \end{aligned} \tag{2.21}$$

with K_t given by (2.7). Then, using the $\mathbb{R}^d$-ergodicity of $H_A(\omega)$ at the second equality:

$$\begin{aligned} &\mathbb{E}\left(\int_{\mathbb{R}^d} f(x)^2 \,\mathrm{Tr}_{\mathbb{C}^N} K_t(x,x)\,\mathrm{d}x\right) \\ &= \frac{1}{\sqrt{2\pi t}} \mathbb{E}\left(\int_{\mathbb{R}^d} f(x)^2 \int \mathrm{Tr}_{\mathbb{C}^N} \exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(\mathsf{w}(s))\,\mathrm{d}s\right) \mathrm{d}W_{0,x,t,x}(\mathsf{w})\,\mathrm{d}x\right) \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi t}} \mathbb{E}\left(\int_{\mathbb{R}^d} f(x)^2 \int \mathrm{Tr}_{\mathbb{C}^N} \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(x+\mathsf{w}(s))\,\mathrm{d}s \right) \mathrm{d}W_{0,0,t,0}(\mathsf{w})\,\mathrm{d}x \right)$$

$$= \frac{1}{\sqrt{2\pi t}} \mathbb{E}\left(\int_{\mathbb{R}^d} f(x)^2 \int \mathrm{Tr}_{\mathbb{C}^N} \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(\mathsf{w}(s))\,\mathrm{d}s \right) \mathrm{d}W_{0,0,t,0}(\mathsf{w})\,\mathrm{d}x \right)$$

$$= \frac{1}{\sqrt{2\pi t}} \mathbb{E}\left(\int \mathrm{Tr}_{\mathbb{C}^N} \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(\mathsf{w}(s))\,\mathrm{d}s \right) \mathrm{d}W_{0,0,t,0}(\mathsf{w}) \right). \tag{2.22}$$

And this last expectation value is finite by Proposition 2. So we have proved that:

$$\mathbb{E}\left(\sum_{k\geq 1} \langle (M_f E_{H_A(\omega)}(B) M_f) f_k, f_k \rangle \right)$$
$$\leq C \frac{1}{\sqrt{2\pi t}} \mathbb{E}\left(\int \mathrm{Tr}_{\mathbb{C}^N} \exp_{\mathrm{ord}} \right.$$
$$\left. \times \left(-\int_0^t V_\omega(\mathsf{w}(s))\,\mathrm{d}s \right) \mathrm{d}W_{0,0,t,0}(\mathsf{w}) \right) < +\infty \tag{2.23}$$

which means that the operator $M_f E_{H_A(\omega)}(B) M_f$ is trace class P-almost surely on $\omega \in \Omega$. It also proves that $B \mapsto \mathbb{E}(\mathrm{Tr}(M_f E_{H_A(\omega)}(B) M_f))$ defines a Radon measure on $\mathbb{R}$ whose Laplace transform is:

$$L(\mathbb{E}(\mathrm{Tr}(M_f E_{H_A(\omega)}(\cdot) M_f)))(t) = \mathbb{E}(\mathrm{Tr}(M_f \mathrm{e}^{-tH_A(\omega)} M_f)) = L(\mathfrak{n})(t) \tag{2.24}$$

by (2.22), (2.21) and (2.12). By injectivity of the Laplace transform, we have that for every bounded Borel set $B \subset \mathbb{R}$:

$$\mathfrak{n}(B) = \mathbb{E}(\mathrm{Tr}(M_f E_{H_A(\omega)}(B) M_f)). \qquad \square$$

All the results of this section were valid for $H_A(\omega)$ acting on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$ for every d and every N. In the next few sections, we will restrict our presentation to the case of $d = 1$ and N arbitrary, $N \geq 1$. It will allow us to introduce the Lyapounov exponents associated to $H_A(\omega)$.

We want to study the regularity of the function $E \mapsto N(E)$. As an increasing function we already know that it has left and right limits at each point of the real line. We will actually prove that the IDS is locally Hölder continuous. To prove this, we will prove the same regularity property for the Lyapounov exponents associated to $H_A(\omega)$ and show that the IDS and the Lyapounov exponents are related to each other trough an harmonic analysis formula, a Thouless formula.

3. Lyapounov Exponents

3.1. *Definition and integral representation*

We start with a review of some results about Lyapounov exponents. These results holds for general sequences of independent and identically distributed (i.i.d.) random symplectic matrices. Let N be a positive integer. Let $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ denote the group of $2N \times 2N$ real symplectic matrices. It is the subgroup of $\mathrm{GL}_{2N}(\mathbb{R})$ of matrices M satisfying

$$^{t}MJM = J,$$

where J is the matrix of order $2N$ defined by $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$.

Definition 2. Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random matrices in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ with

$$\mathbb{E}(\log^+ \|A_0^\omega\|) < \infty.$$

The Lyapounov exponents $\gamma_1, \ldots, \gamma_{2N}$ associated with $(A_n^\omega)_{n\in\mathbb{N}}$ are defined inductively by

$$\sum_{i=1}^{p} \gamma_i = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log \| \wedge^p (A_{n-1}^\omega \cdots A_0^\omega) \|) \tag{3.1}$$

for every $p \in \{1, \ldots, 2N\}$.

Here, $\wedge^p M$ denotes the pth exterior power of the matrix M, acting on the pth exterior power of $\mathbb{R}^{2N}$. One has $\gamma_1 \geq \cdots \geq \gamma_{2N}$. Moreover, the random matrices $(A_n^\omega)_{n\in\mathbb{N}}$ being symplectic, we have the symmetry property $\gamma_{2N-i+1} = -\gamma_i$, for every $i \in \{1, \ldots, N\}$ (see [1, Proposition 8.2, p. 89].

Let μ be a probability measure on $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$. We denote by G_μ the smallest closed subgroup of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ which contains the topological support of μ, supp μ. We also define for every $p \in \{1, \ldots, 2N\}$, the p-Lagrangian submanifold L_p of $\mathbb{R}^{2N}$, as the subspace of $\wedge^p \mathbb{R}^{2N}$ spanned by $\{Me_1 \wedge \cdots \wedge Me_p \mid M \in \mathrm{Sp}_{\mathrm{N}}(\mathbb{R})\}$, where $(e_1, \ldots, e_{2N})$ is the canonical basis of $\mathbb{R}^{2N}$.

We can now give a generalization of Fürstenberg's theorem for $N > 1$. For the definitions of L_p-strong irreducibility and p-contractivity we refer to [1, Definitions *A.IV*.3.3 and *A.IV*.1.1], respectively.

Proposition 4. *Let* $(A_n^\omega)_{n\in\mathbb{N}}$ *be a sequence of* i.i.d. *random symplectic matrices of order* $2N$ *and* p *be an integer,* $p \in \{1, \ldots, 2N\}$. *Let* μ *be the common distribution of the* A_n^ω. *If*

(a) G_μ *is* p*-contracting and* L_p*-strongly irreducible*,
(b) $\mathbb{E}(\log \|A_0^\omega\|) < \infty$,

then the following holds:

(i) $\gamma_p > \gamma_{p+1}$.
(ii) *For any non zero* x *in* L_p:

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\big(\log \|(\wedge^p A_{n-1}^\omega \ldots A_0^\omega) x\|\big) = \sum_{i=1}^{p} \gamma_i .$$

(iii) *There exists a unique μ-invariant probability measure ν_p on $\mathbb{P}(L_p) = \{\bar{x} \in \mathbb{P}(\wedge^p \mathbb{R}^{2N}) \mid x \in L_p\}$ such that*:

$$\sum_{i=1}^{p} \gamma_i = \int_{\mathrm{Sp_N}(\mathbb{R}) \times \mathbb{P}(L_p)} \log \frac{\|(\wedge^p M)x\|}{\|x\|} \, \mathrm{d}\mu(M) \, \mathrm{d}\nu_p(\bar{x}).$$

Proof. This is [1, Proposition 3.4]. □

It remains to define the Lyapounov exponents associated to the operator $H_A(\omega)$ for $d = 1$ and $N \geq 1$. For $E \in \mathbb{R}$ we can consider the second order differential system:

$$H_A(\omega)u = Eu \;\Leftrightarrow\; -u'' + V_\omega u = Eu \tag{3.2}$$

with $u = (u_1, \ldots, u_N)$ a function taking values in $\mathbb{C}^N$. We introduce the transfer matrix $A_n^\omega(E)$ from n to $n+1$, defined by the relation:

$$\begin{pmatrix} u(n+1, E) \\ u'(n+1, E) \end{pmatrix} = A_n^\omega(E) \begin{pmatrix} u(n, E) \\ u'(n, E) \end{pmatrix}. \tag{3.3}$$

Then one can verify that $(A_n^\omega(E))_{n \in \mathbb{N}}$ is a sequence of i.i.d. random symplectic matrices because the system (3.2) is Hamiltonian. So we can define the Lyapounov exponents associated to the operator $H_A(\omega)$ as the Lyapounov exponents of the sequence of transfer matrices $(A_n^\omega(E))_{n \in \mathbb{N}}$. Since the transfer matrices depend on a real parameter E, so will the Lyapounov exponents of $H_A(\omega)$ and so do the measure μ_E (the common law of the $A_n^\omega(E)$), the group G_{μ_E} and the μ_E-invariant probability measure $\nu_{p,E}$ of Proposition 4.

3.2. *Regularity of the Lyapounov exponents*

We want to study the regularity of the function $E \mapsto \gamma_p(E)$ for $p \in \{1, \ldots, N\}$. According to the integral representation obtained at Proposition 4, we have to understand the regularity of $E \mapsto \nu_{p,E}$ for any $p \in \{1, \ldots, N\}$ and to control the term $\| \wedge^p M\|$ in the integral, which depends on E as μ_E depends on E. We will now give a general theorem for the regularity of the Lyapounov exponents of sequences of i.i.d. random symplectic matrices depending on a real parameter.

Theorem 1. *Let $(A_n^\omega(E))_{n \in \mathbb{N}}$ be a sequence of* i.i.d. *random symplectic matrices depending on a real parameter E. Let μ_E be the common distribution of the $A_n^\omega(E)$. We fix a compact interval I in $\mathbb{R}$ and we assume that for $E \in I$ we have*:

(i) *G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1, \ldots, N\}$.*
(ii) *There exist $C_1 > 0$, $C_2 > 0$ independent of n, ω, E such that for every $p \in \{1, \ldots, N\}$*:

$$\| \wedge^p A_n^\omega(E)\|^2 \leq \exp(pC_1 + p|E| + p) \leq C_2. \tag{3.4}$$

(iii) *There exists* $C_3 > 0$ *independent of* n, ω, E *such that for every* $E, E' \in I$ *and every* $p \in \{1, \dots, N\}$:

$$\|\wedge^p A_n^\omega(E) - \wedge^p A_n^\omega(E')\| \leq C_3 |E - E'|. \tag{3.5}$$

Then there exist two real numbers $\alpha > 0$ *and* $0 < C < +\infty$ *such that*:

$$\forall p \in \{1, \dots, N\}, \quad \forall E, E' \in I, \quad |\gamma_p(E) - \gamma_p(E')| \leq C|E - E'|^\alpha.$$

Proof. The methods to prove this theorem can be found in [9, Chap. V]. In this reference this regularity result is written for transfer matrices associated to matrix-valued discrete Schrödinger operators. But this restriction to discrete operators only concerns the estimates (3.4) and (3.5). They are obviously verified in the case of transfer matrices of discrete Schrödinger operators as it is explained in [9, p. 279]. For a presentation using estimates (3.4) and (3.5), one can read [11] where it is done in the case of transfer matrices associated to scalar-valued continuous Schrödinger operators.

The main steps of the proof are the following. First we prove continuity of the Lyapounov exponents on I by proving continuity of the function:

$$\Phi_{p,E} \; : \quad \begin{array}{c} I \times \mathbb{P}(L_p) \to \mathbb{R} \\ (E, \bar{x}) \mapsto \Phi_{p,E}(\bar{x}) = \mathbb{E}\left(\log \dfrac{\|(\wedge^p A_n^\omega(E))x\|}{\|x\|} \right) \end{array}$$

for every $p \in \{1, \dots, N\}$. We only use estimates (3.4) and (3.5) to prove this continuity. Then we prove weak continuity of the function $E \mapsto \nu_{p,E}$ using Banach–Alaoglu theorem and the unicity of the μ_E-invariant measure $\nu_{p,E}$ as stated in point (iii) of Proposition 4. Combining these two continuity properties and noting that:

$$\gamma_1(E) + \cdots + \gamma_p(E) = \nu_{p,E}(\Phi_{p,E})$$

we get the continuity of the Lyapounov exponents.

To prove the Hölder continuity of the Lyapounov exponents we need a result on negative cocyles as stated in [9, Proposition IV 3.5, p. 187]. We also need estimates on Laplace operators on Hölder spaces like [9, Proposition V 4.13, p. 277] which relies on estimates (3.4) and (3.5). Finally using the decomposition given in [9, Proposition IV 3.12, p. 192] one can prove the Hölder continuity of $E \mapsto \nu_{p,E}$ on I.

For a complete presentation of this proof in the case of transfer matrices for continuous matrix-valued Schrödinger operators, with proofs showing the role of the pth exterior powers, we refer to [4, Chap. 6]. □

We will now use this general result to prove the following theorem:

Theorem 2. *Let* I *be a compact interval in* $\mathbb{R}$. *We assume that the potential* V_ω *in* $H_A(\omega)$ *for* $d = 1$ *and* $N \geq 1$ *is such that the group* G_{μ_E} *associated to the transfer matrices of* $H_A(\omega)$ *is* p*-contracting and* L_p*-strongly irreducible for every* $p \in \{1, \dots, N\}$ *and all* $E \in I$. *Then the Lyapounov exponents associated to* $H_A(\omega)$

are Hölder continuous on I, i.e. *there exist two real numbers $\alpha > 0$ and $0 < C < +\infty$ such that*:

$$\forall p \in \{1, \dots N\}, \quad \forall E, E' \in I, \quad |\gamma_p(E) - \gamma_p(E')| \leq C|E - E'|^{\alpha}.$$

According to Theorem 1 we only have to show that the transfer matrices $A_n^{\omega}(E)$ associated to $H_A(\omega)$ verify estimates (3.4) and (3.5). They already verify point (i) of Theorem 1 by assumption. Before proving (3.4) and (3.5) we will give two lemmas which are the analog for matrix-valued operators of [11, Lemmas A.1 and A.2].

Lemma 2. *Let V be a matrix-valued function in $L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_{\mathrm{N}}(\mathbb{R}))$ and u a solution of $-u'' + Vu = 0$. Then for every $x, y \in \mathbb{R}$*:

$$\|u(x)\|^2 + \|u'(x)\|^2 \leq (\|u(y)\|^2 + \|u'(y)\|^2) \exp\left(\int_{\min(x,y)}^{\max(x,y)} \|V(t) + 1\| \, \mathrm{d}t \right).$$

Proof. Let $R(t) = \|u(t)\|^2 + \|u'(t)\|^2$. We have:

$$\begin{aligned}
R'(t) &= \langle u(t), u'(t)\rangle + \langle u'(t), u(t)\rangle + \langle u''(t), u'(t)\rangle + \langle u'(t), u''(t)\rangle \\
&= 2\,\mathrm{Re}(\langle u(t), u'(t)\rangle) + 2\,\mathrm{Re}(\langle u'(t), V(t)u(t)\rangle) \\
&= 2\,\mathrm{Re}(\langle u'(t), (V(t)+1)u(t)\rangle) \\
&\leq 2\,\mathrm{Re}(\|u'(t)\| \; \|V(t)+1\| \; \|u(t)\|) \\
&\leq 2\|V(t)+1\| \left(\frac{\|u(t)\|^2 + \|u'(t)\|^2}{2} \right) \\
&= \|V(t)+1\| R(t).
\end{aligned}$$

We have used the Cauchy–Schwarz inequality and the arithmetico-geometric inequality. Finally, we have the inequality:

$$R'(t) \leq \|V(t)+1\| R(t)$$

which by integration gives us the expected inequality. ∎

Lemma 3. *For $i = 1, 2$, let $V_i \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{M}_{\mathrm{N}}(\mathbb{R}))$ and u_i a solution of $-u'' + V_i u = 0$ such that*:

$$\exists y \in \mathbb{R}, \quad u_1(y) = u_2(y) \quad \textit{and} \quad u_1'(y) = u_2'(y).$$

Then, for every $x \in \mathbb{R}$:

$$\begin{aligned}
&\left(\|u_1(x) - u_2(x)\|^2 + \|u_1'(x) - u_2'(x)\|^2\right)^{\frac{1}{2}} \\
&\quad \leq \left(\|u_1(y)\|^2 + \|u_1'(y)\|^2\right)^{\frac{1}{2}} \\
&\qquad \times \exp\left(\int_{\min(x,y)}^{\max(x,y)} \|V_1(t)\| + \|V_2(t)\| + 2 \, \mathrm{d}t \right) \times \int_{\min(x,y)}^{\max(x,y)} \|V_1(t) - V_2(t)\| \, \mathrm{d}t.
\end{aligned}$$

Proof. Without loss of generality, we can assume that $y \leq x$. We have, because of the assumptions made on the solutions u_1 and u_2:

$$\begin{aligned}\begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} &= \int_x^y \begin{pmatrix} 0 \\ (V_1(t) - V_2(t))u_1(t) \end{pmatrix} \mathrm{d}t \\ &\quad + \int_x^y \begin{pmatrix} 0 & I \\ V_2(t) & 0 \end{pmatrix} \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} \mathrm{d}t.\end{aligned}$$

We take the norm of the two sides of the equality:

$$\begin{aligned}\left\| \begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} \right\| &\leq \int_x^y \|V_1(t) - V_2(t)\| \ \|u_1(t)\| \, \mathrm{d}t \\ &\quad + \int_x^y (\|V_2(t)\| + 1) \left\| \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} \right\| \mathrm{d}t.\end{aligned}$$

Then by Gronwall lemma:

$$\begin{aligned}\left\| \begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} \right\| &\leq \left(\int_x^y \|V_1(t) - V_2(t)\| \ \|u_1(t)\| \, \mathrm{d}t \right) \\ &\quad \times \exp\left(\int_x^y (\|V_2(t)\| + 1) \, \mathrm{d}t \right).\end{aligned} \tag{3.6}$$

But by Lemma 2, for every $t \in [y, x]$:

$$\begin{aligned}\|u_1(t)\|^2 \leq \|u_1(t)\|^2 + \|u_1'(t)\|^2 &\leq \left(\|u_1(y)\|^2 + \|u_1'(y)\|^2 \right) \\ &\quad \times \exp\left(\int_x^y (\|V_1(s)\| + 1) \, \mathrm{d}s \right).\end{aligned}$$

So:

$$\|u_1(t)\| \leq \left(\|u_1(y)\|^2 + \|u_1'(y)\|^2 \right)^{\frac{1}{2}} \exp\left(\frac{1}{2} \int_x^y (\|V_1(s)\| + 1) \, \mathrm{d}s \right).$$

We put this in (3.6):

$$\begin{aligned}&\left(\|u_1(x) - u_2(x)\|^2 + \|u_1'(x) - u_2'(x)\|^2 \right)^{\frac{1}{2}} \\ &\quad \leq \left(\|u_1(y)\|^2 + \|u_1'(y)\|^2 \right)^{\frac{1}{2}} \exp\left(\int_{\min(x,y)}^{\max(x,y)} \frac{1}{2}\|V_1(t)\| + \frac{1}{2} + \|V_2(t)\| + 1 \, \mathrm{d}t \right) \\ &\qquad \times \int_{\min(x,y)}^{\max(x,y)} \|V_1(t) - V_2(t)\| \, \mathrm{d}t.\end{aligned}$$

And we have finished the proof because: $\frac{1}{2}\|V_1(t)\| + \frac{1}{2} \leq \|V_1(t)\| + 1$. □

Notation. Let $u^1, \dots, u^{2N}$ be solutions of (3.2) with initial conditions:

$$\begin{pmatrix} u^1(n,E) \\ (u^1)'(n,E) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u^{2N}(n,E) \\ (u^{2N})'(n,E) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3.7}$$

Then the transfer matrix $A_n^\omega(E)$ has the expression:

$$A_n^\omega(E) = \begin{pmatrix} u^1(n+1,E) \cdots u^{2N}(n+1,E) \\ (u^1)'(n+1,E) \cdots (u^{2N})'(n+1,E) \end{pmatrix}. \tag{3.8}$$

Proof of Theorem 2. We start by proving (3.4). Let ${}^t(u^i(n+1,E)\ (u^i)'(n+1,E))$ be the column of $A_n^\omega(E)$ of maximal norm. Then:

$$\|A_n^\omega(E)\|^2 = \|u^i(n+1,E)\|^2 + \|(u^i)'(n+1,E)\|^2.$$

Applying Lemma 2 with $x = n+1$ and $y = n$ one gets:

$$\begin{aligned} &\|u^i(n+1,E)\|^2 + \|(u^i)'(n+1,E)\|^2 \\ &\quad \le \left(\|u^i(n,E)\|^2 + \|(u^i)'(n,E)\|^2\right) \exp\left(\int_n^{n+1} \|V_\omega(t) - E\| + 1\, \mathrm{d}t\right). \end{aligned}$$

But due to (3.7) we have: $\|u^i(n,E)\|^2 + \|(u^i)'(n,E)\|^2 = 1$. We also have that $x \mapsto V_\omega(x)$ is 1-periodic. Thus:

$$\begin{aligned} \int_n^{n+1} \|V_\omega(t) - E\| + 1\, \mathrm{d}t &= \int_0^1 \|V_\omega(t) - E\| + 1\, \mathrm{d}t \\ &\le \left(\sup_{t\in[0,1]} \|V_\omega(t)\|\right) + |E| + 1. \end{aligned}$$

But V_ω being uniformly bounded on x and ω, there exists $C_1 > 0$ independent of ω, n and E such that:

$$\left(\sup_{t\in[0,1]} \|V_\omega(t)\|\right) \le C_1.$$

Then:

$$\|A_n^\omega(E)\|^2 \le \exp(C_1 + |E| + 1).$$

As I is compact, $|E|$ is also bounded and so there exists $\tilde{C}_2 > 0$ independent of ω, n and E such that: $\exp(C_1 + |E| + 1) \le \tilde{C}_2$. Finally, we use that for every $p \in \{1, \dots, 2N\}$ and for every $M \in \mathrm{GL_{2N}}(\mathbb{R})$, $\| \wedge^p M\| \le \|M\|^p$. Applying it to $M = A_n^\omega(E)$, we obtain (3.4).

To prove (3.5) we first prove it for $p = 1$. Let $E, E' \in I$. First there exists $i \in \{1, \dots, 2N\}$ such that:

$$\begin{aligned}
\|A_n^\omega(E) - A_n^\omega(E')\| &= \left\| \begin{pmatrix} u^i(n+1,E) \\ (u^i)'(n+1,E) \end{pmatrix} - \begin{pmatrix} u^i(n+1,E') \\ (u^i)'(n+1,E') \end{pmatrix} \right\| \\
&\leq \left\| \begin{pmatrix} u^i(n,E) \\ (u^i)'(n,E) \end{pmatrix} \right\| \left(\int_n^{n+1} \|V_\omega(t) - E - (V_\omega(t) - E')\| \,\mathrm{d}t \right) \\
&\quad \times \exp\left(\int_n^{n+1} \|V_\omega(t) - E\| + \|(V_\omega(t) - E')\| + 2 \,\mathrm{d}t \right).
\end{aligned}$$

by Lemma 3. Thus:

$$\begin{aligned}
\|A_n^\omega(E) - A_n^\omega(E')\| &\leq |E - E'| \exp\left(\int_0^1 2\|V_\omega(t)\| + |E| + |E'| + 2 \,\mathrm{d}t \right) \\
&\leq |E - E'| \exp(2C_1 + 2 + 2\ell(I)) \\
&\leq \tilde{C}_3 |E - E'|
\end{aligned}$$

with $\tilde{C}_3$ independent of n, ω and E. Now for $p \geq 1$ we use the following estimate valid for $M, N \in \mathrm{GL}_{2\mathrm{N}}(\mathbb{R})$ and $p \in \{1, \dots, 2N\}$:

$$\|\wedge^p M - \wedge^p N\| \leq \|N - M\|(\|N\|^{p-1} + \|M\| \cdot \|N\|^{p-2} + \cdots + \|M\|^{p-1}).$$

It is a direct computation (see [4, p. 118] for details). Applying it to $M = A_n^\omega(E)$ and $N = A_n^\omega(E')$ one gets:

$$\|\wedge^p A_n^\omega(E) - \wedge^p A_n^\omega(E')\| \leq p C_2^{p-1} \tilde{C}_3 |E - E'|$$

and $C_3 = p C_2^{p-1} \tilde{C}_3$ is independent of n, ω, E and E'.

We have checked (ii) and (iii) in Theorem 1 and (i) is an assumption in Theorem 2. Therefore we can apply Theorem 1 to have the Hölder continuity on I of the Lyapounov exponents associated to $H_A(\omega)$. □

4. Hölder Continuity of the IDS

4.1. *Kotani's w function*

We start by introducing the w function of Kotani as defined in [20] for matrix-valued Schrödinger operators. For this, we first have to define the m-functions associated to such operators. We follow [20] and we will refer to this article for all proofs of this paragraph. Let $\mathbb{C}_+$ denote the half upper plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ and $\mathbb{C}_-$ the lower half plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) < 0\}$.

Proposition 5. *Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. We fix $\omega \in \Omega$. Then there exists a unique function $x \mapsto F_+(x, E)$ with values in $\mathcal{M}_\mathrm{N}(\mathbb{C})$ (respectively $x \mapsto F_-(x, E)$) satisfying:*

$$-F_+'' + V_\omega F_+ = EF_+, \quad F_+(0, E) = I \quad \textit{and} \quad \int_0^\infty \|F_+(x, E)\|^2 dx < +\infty$$

respectively:

$$-F_-'' + V_\omega F_- = EF_-, \quad F_-(0,E) = I \quad \textit{and} \quad \int_{-\infty}^0 \|F_-(x,E)\|^2 dx < +\infty.$$

Proof. See [20, Corollary 2.2]. □

Definition 3. For $E \in \mathbb{C}_+ \cup \mathbb{C}_-$ we define the m-functions M_+ and M_- associated to $H_A(\omega)$ by:

$$M_+(E) = \frac{d}{dx} F_+(x,E)|_{x=0} \quad \text{and} \quad M_-(E) = -\frac{d}{dx} F_-(x,E)|_{x=0}.$$

With these functions we can compute the Green kernel of the resolvant of $H_A(\omega)$.

Proposition 6. *Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. Then $(H_A(\omega) - E)^{-1}$ has a continuous integral kernel $G_E(x,y,\omega)$ given by*:

$$G_E(x,y,\omega) = \begin{cases} -F_-(x)(M_+ + M_-)^{-1}\ {}^tF_+(y) & if\ x \le y \\ -F_+(x)(M_+ + M_-)^{-1}\ {}^tF_-(y) & if\ y \le x. \end{cases}$$

Proof. See [20, Theorem 3.2]. □

We can now define the w function of Kotani. This function will be the link between the Lyapounov exponents and the IDS. Indeed, its real part will be the sum of the N positive Lyapounov exponents while its imaginary part will tend to $\pi N(E)$ when E tends to the real line.

Definition 4. Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. We define the w function of Kotani by:

$$w(E) = \frac{1}{2}\mathbb{E}(\mathrm{Tr}(M_+(E) + M_-(E))).$$

Then the w function has the following properties:

Proposition 7. *For $E \in \mathbb{C}_+ \cup \mathbb{C}_-$*:

(i) $w(E) = \mathbb{E}(\mathrm{Tr}(M_+(E))) = \mathbb{E}(\mathrm{Tr}(M_-(E)))$.
(ii) $\frac{\mathrm{d}}{\mathrm{d}E} w(E) = \mathbb{E}(\mathrm{Tr}(G_E(0,0,\omega)))$.
(iii) $-\mathrm{Re}\ w(E) = (\gamma_1 + \cdots + \gamma_N)(E)$.
(iv) $\mathbb{E}\left(\mathrm{Tr}(\mathrm{Im}\ M_\pm(E,\omega)^{-1})\right) = -\frac{2\,\mathrm{Re}\ w(E)}{\mathrm{Im}\,E} = \frac{2(\gamma_1 + \cdots + \gamma_N)(E)}{\mathrm{Im}\,E}$.

Proof. See [20, Theorem 6.2C]. □

In point (iii) we have to precise that the formula:

$$\gamma_1(E) + \cdots + \gamma_N(E) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log \| \wedge^N (A_{n-1}^\omega(E) \dots A_0^\omega(E))\|)$$

makes sense for every $E \in \mathbb{C}$.

We can now generalize results of harmonic analysis of the w function presented in the case of scalar-valued Schrödinger operators by Kotani in [19] to the case of matrix-valued Schrödinger operators. First we introduce the space of Herglotz functions:

$$\mathcal{H} = \{h \mid h \text{ is holomorphic on } \mathbb{C}_+ \text{ and } h : \mathbb{C}_+ \to \mathbb{C}_+\}.$$

Then we define a subspace of $\mathcal{H}$:

$$\mathcal{W} = \{w \in \mathcal{H} \mid w,\ w',\ -\mathrm{i}w \in \mathcal{H}\}.$$

Proposition 8. *The Kotani's function w is in $\mathcal{W}$.*

Proof. First, as $H_A(\omega)$ is self-adjoint, its spectrum is included in $\mathbb{R}$ and $E \mapsto M_+(E)$ is holomorphic on $\mathbb{C}\backslash\mathbb{R}$ and so is $E \mapsto \mathrm{Tr}(M_+(E))$. If $\mathrm{Im}\, E > 0$, by [20, Proposition 2.3(a)], one has:

$$\mathrm{Im}\ M_+(E) = (\mathrm{Im}\, E) \int_0^{+\infty} F_+(x,E)^* F_+(x,E) > 0.$$

Thus, $E \mapsto \mathrm{Tr}(M_+(E))$ is in $\mathcal{H}$ and $w \in \mathcal{H}$.

Then by Proposition 7(ii), $w'(E) = \mathbb{E}(\mathrm{Tr}(G_E(0,0,\omega)))$. But $G_E(0,0,\omega)$ is holomorphic away from the spectrum of $H_A(\omega)$ and so is $\mathrm{Tr}(G_E(0,0,\omega))$. If $\mathrm{Im}\, E > 0$, then the operator $\mathrm{Im}(H_A(\omega) - E)^{-1}$ is a positive definite operator and $\mathrm{Im}\,\mathrm{Tr}(G_E(0,0,\omega)) > 0$. Then $\mathrm{Im}\, w'(E) = \mathrm{Im}\,\mathrm{Tr}(G_E(0,0,\omega)) > 0$ and $w' \in \mathcal{H}$.

Finally, $-\mathrm{i}w$ is holomorphic on $\mathbb{C}_+$ as w is. If $E \in \mathbb{C}_+$:

$$\mathrm{Im}(-\mathrm{i}w(E)) = -\mathrm{Re}\ w(E) = (\mathrm{Im}\, E)\mathbb{E}(\mathrm{Tr}(\mathrm{Im}\, M_+(E,\omega)^{-1}))$$

by Proposition 7(iv). But if $E \in \mathbb{C}_+$, $\mathrm{Tr}(\mathrm{Im}\ M_+(E,\omega)^{-1}) > 0$ and then $\mathrm{Im}(-\mathrm{i}w(E)) > 0$. Therefore, $-\mathrm{i}w \in \mathcal{H}$. □

4.2. *A Thouless formula*

Let $\mathfrak{n}$ be the measure defined in Proposition 2.

Proposition 9.

$$\forall E \in \mathbb{C}\backslash\mathbb{R}, \quad \mathbb{E}(\mathrm{Tr}\ G_E(0,0,\omega)) = \int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-E}. \tag{4.1}$$

Proof. As $\mathbb{R}$ is a limit of bounded Borel sets and the Dirac distribution at 0, δ_0, can be approached by compactly supported continuous functions, positives and of L^2-norm equal to 1, using Proposition 3 we have:

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-E} = \int_{\mathbb{R}} \frac{1}{E'-E}\, \mathrm{d}\mathbb{E}\big(\mathrm{Tr}(\langle \delta_0, E_{H_A(\omega)}((-\infty,E'])\delta_0\rangle)\big)\,.$$

Then applying the spectral theorem to the self-adjoint operator $H_A(\omega)$:

$$\begin{aligned}\int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-E} &= \mathbb{E}\left(\mathrm{Tr}\left(\int_{\mathbb{R}} \frac{1}{E'-E}\,\mathrm{d}\langle \delta_0, E_{H_A(\omega)}((-\infty,E'])\delta_0\rangle\right)\right)\\ &= \mathbb{E}\left(\mathrm{Tr}\left(\left\langle \delta_0, \left(\int_{\mathbb{R}} \frac{1}{E'-E}\,\mathrm{d}E_{H_A(\omega)}((-\infty,E'])\right)\delta_0\right\rangle\right)\right)\\ &= \mathbb{E}\big(\mathrm{Tr}(\langle \delta_0, (H_A(\omega)-E)^{-1}\delta_0\rangle)\big)\\ &= \mathbb{E}(\mathrm{Tr}(G_E(0,0,\omega)))\,.\end{aligned}$$

□

With this proposition, we can express the imaginary part of w in terms of the IDS, $E \mapsto N(E)$.

Proposition 10.

$$\forall E \in \mathbb{R}, \quad \lim_{a\to 0^+} \mathrm{Im}\, w(E+\mathrm{i}a) = \pi N(E). \tag{4.2}$$

Proof. First, by Proposition 7(ii):

$$\forall z \in \mathbb{C}\backslash\mathbb{R}, \quad w'(z) = \mathbb{E}(\mathrm{Tr}(G_z(0,0,\omega))).$$

Then, we can apply Proposition 9:

$$\begin{aligned}\forall z \in \mathbb{C}\backslash\mathbb{R}, \quad w'(z) &= \int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-z}\\ &= \int_{\mathbb{R}} \frac{N(E')}{(E'-z)^2}\,\mathrm{d}E'\end{aligned}$$

by integrating by parts. Then by integrating this expression, there exists a constant $c \in \mathbb{C}$ such that:

$$w(z) = c + \int_{\mathbb{R}} \frac{1+E'z}{(E'-z)(1+E'^2)} N(E')\,\mathrm{d}E'. \tag{4.3}$$

But if $z \in \mathbb{R}$ is not in the spectrum of $H_A(\omega)$ then $w(z) \in \mathbb{R}$ (see [7, Lemma 5.10, p. 84]). Thus we must have $c \in \mathbb{R}$. Then, taking imaginary part in (4.3) and writing for $z \in \mathbb{C}_+$, $z = E + \mathrm{i}a$, $E \in \mathbb{R}$, $a > 0$:

$$\begin{aligned}\mathrm{Im}\ w(E+\mathrm{i}a) &= a\int_{\mathbb{R}} \frac{N(E')}{(E'-E)^2+a^2}\,\mathrm{d}E'\\ &= \int_{\mathbb{R}} \frac{N(E+au)}{1+u^2}\,\mathrm{d}u\end{aligned}$$

where $u = \frac{E'-E}{a}$. But $N(E)$ being a distribution function, it is right continuous and so:

$$\forall E \in \mathbb{R}, \quad \lim_{a\to 0^+} \mathrm{Im}\, w(E+\mathrm{i}a) = N(E)\int_{\mathbb{R}} \frac{1}{1+u^2}\,\mathrm{d}u = \pi N(E).$$

□

We have an analoguous proposition for the real part of $w(E)$.

Proposition 11. *For Lebesgue-almost every E in $\mathbb{R}$, we have*:

$$\lim_{a\to 0^+} \operatorname{Re} w(E+\mathrm{i}a) = -(\gamma_1+\cdots+\gamma_N)(E). \tag{4.4}$$

Moreover, if $I\subset\mathbb{R}$ is an interval on which $E\mapsto -(\gamma_1+\cdots+\gamma_N)(E)$ is continuous then (4.4) *holds for every $E\in I$.*

Proof. First by Proposition 7(iii), we have:

$$\forall z\in\mathbb{C}\backslash\mathbb{R},\quad \operatorname{Re} w(z) = -(\gamma_1+\cdots+\gamma_N)(z). \tag{4.5}$$

The function $z\mapsto -(\gamma_1+\cdots+\gamma_N)(z)$ is subharmonic (see [10]) and so for almost every E in $\mathbb{R}$ the following limit exists:

$$\lim_{a\to 0}(\gamma_1+\cdots+\gamma_N)(E+\mathrm{i}a) = (\gamma_1+\cdots+\gamma_N)(E). \tag{4.6}$$

Let E be a real number such that (4.6) holds. Then setting $z=E+\mathrm{i}a$ with $a>0$ in (4.5) one gets the existence of the following limit:

$$\lim_{a\to 0^+} \operatorname{Re} w(E+\mathrm{i}a) = -(\gamma_1+\cdots+\gamma_N)(E). \tag{4.7}$$

Moreover, if I is an interval on which $E\mapsto(\gamma_1+\cdots+\gamma_N)(E)$ is continuous, the relation (4.7) holds for every E in I as it holds for almost every $E\in I$. □

Now we can prove a Thouless formula adapted to matrix-valued continuous Schrödinger operators. As $(\gamma_1+\cdots+\gamma_N)(E)$ and $N(E)$ are respectively the real and imaginary part of the function w which lies in $\mathcal{W}$, the harmonic analysis developed in [19] says that these two functions are linked by an integral relation.

Theorem 3 (Thouless Formula). *For Lebesgue-almost every $E\in\mathbb{R}$ we have*:

$$(\gamma_1+\cdots+\gamma_N)(E) = -\alpha + \int_{\mathbb{R}} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,\mathrm{d}\mathfrak{n}(E') \tag{4.8}$$

where α is a real number independent of E and $\mathfrak{n}$ is the measure of which the IDS $E\mapsto N(E)$ is the distribution function. Moreover, if $I\subset\mathbb{R}$ is an interval on which $E\mapsto -(\gamma_1+\cdots+\gamma_N)(E)$ is continuous then (4.8) *holds for every $E\in I$.*

Proof. As $w\in\mathcal{W}$, we can apply to w the [19, Lemma 7.7]. In particular, using also Proposition 10, we have:

$$\forall z\in\mathbb{C}\backslash\mathbb{R},\quad w(z) = w(\mathrm{i}) + \int_{\mathbb{R}} \log\left(\frac{E'-\mathrm{i}}{E'-z}\right)\mathrm{d}\mathfrak{n}(E'). \tag{4.9}$$

Then:

$$\operatorname{Re} w(z) = \operatorname{Re} w(\mathrm{i}) + \int_{\mathbb{R}} \log\left(\left|\frac{E'-\mathrm{i}}{E'-z}\right|\right)\mathrm{d}\mathfrak{n}(E'). \tag{4.10}$$

Let $z = E + \mathrm{i}a$ with $E \in \mathbb{R}$ such that (4.4) holds and $a > 0$. Then when a goes to 0, by Proposition 11 we have:

$$-(\gamma_1 + \cdots + \gamma_N)(E) = \mathrm{Re}\, w(\mathrm{i}) + \int_{\mathbb{R}} \log\left(\left| \frac{E' - \mathrm{i}}{E' - E} \right| \right) \mathrm{d}\mathfrak{n}(E'). \tag{4.11}$$

If we set $\alpha = \mathrm{Re}\ w(\mathrm{i})$ we finally get (4.8) for every E in $\mathbb{R}$ such that (4.4) holds, i.e. for almost every E in $\mathbb{R}$. Then if I is an interval on which $E \mapsto (\gamma_1 + \cdots + \gamma_N)(E)$ is continuous, by Proposition 11, (4.8) will hold for every E in I. □

We can now use this Thouless formula to prove that the IDS, $E \mapsto N(E)$, has the same regularity as the Lyapounov exponents.

4.3. *Local Hölder continuity of the IDS*

We start by a quick review of the Hilbert transform and its main properties. For the proofs we refer to [21, Chap. 3].

Definition 5. If $\psi \in L^2(\mathbb{R})$, its Hilbert transform is the function defined on $\mathbb{R}$ by:

$$(T\psi)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{\psi(t)}{x - t}\, \mathrm{d}t.$$

Proposition 12. *Let $\psi \in L^2(\mathbb{R})$.*

(i) *Then $T^2\psi(x) = -\psi(x)$ for Lebesgue-almost every x in $\mathbb{R}$.*
(ii) *If ψ is Hölder continuous on the interval $[x_0 - a, x_0 + a], a > 0$, then $T\psi$ is Hölder continuous on the interval $[x_0 - \frac{a}{2}, x_0 + \frac{a}{2}]$.*

Now we can prove the following result of regularity of the IDS.

Theorem 4. *Let I be a compact interval in $\mathbb{R}$ and $\tilde{I}$ be an open interval, $I \subset \tilde{I}$. We assume that the potential V_ω in $H_A(\omega)$ for $d = 1$ and $N \geq 1$ is such that the group G_{μ_E} associated to the transfer matrices of $H_A(\omega)$ is p-contracting and L_p-strongly irreducible for every $p \in \{1, \ldots, N\}$ and every $E \in \tilde{I}$. Then the IDS associated to $H_A(\omega)$ is Hölder continuous on I.*

Proof. First, the application $E' \mapsto \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)$ is $\mathfrak{n}$-integrable on $\mathbb{R}$. Indeed the renormalization term $E' - \mathrm{i}$ at the denominator balances the fact that the support of $\mathfrak{n}$ is non-compact. Thus, we have:

$$\forall E \in \mathbb{R}, \quad \lim_{\varepsilon \to 0^+} \int_{E-\varepsilon}^{E+\varepsilon} \left| \log\left(\left| \frac{E' - E}{E' - \mathrm{i}} \right| \right) \right| \mathrm{d}\mathfrak{n}(E') = 0 \tag{4.12}$$

from which we deduce that:

$$\forall E \in \mathbb{R}, \quad \lim_{\varepsilon \to 0^+} |\log(\varepsilon)|(N(E + \varepsilon) - N(E - \varepsilon)) = 0. \tag{4.13}$$

It implies that $E \mapsto N(E)$ is continuous on $\mathbb{R}$. Let $E_0 \in I$ be fixed and $a > 0$ such that $[E_0 - 4a, E_0 + 4a] \subset \tilde{I}$. Then, by Theorem 3, for $E \in]\, E_0 - 4a, E_0 + 4a[$:

$$(\gamma_1 + \cdots + \gamma_N)(E) + \alpha - \int_{|E'-E_0|>4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E')$$
$$= \int_{E_0-4a}^{E_0+4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E').$$

Then:

$$\int_{E_0-4a}^{E_0+4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E')$$
$$= \lim_{\varepsilon\to 0^+} \left(\int_{E_0-4a}^{E-\varepsilon} \log|E'-E| \mathrm{d}\mathfrak{n}(E') \right.$$
$$\left. + \int_{E+\varepsilon}^{E_0+4a} \log|E'-E| \mathrm{d}\mathfrak{n}(E')\right) - \frac{1}{2}\int_{E_0-4a}^{E_0+4a} \log(1+(E')^2)\, \mathrm{d}\mathfrak{n}(E').$$

We set:

$$\mathcal{I}(E_0) = \frac{1}{2}\int_{E_0-4a}^{E_0+4a} \log(1+(E')^2)\, \mathrm{d}\mathfrak{n}(E').$$

Then, integrating by parts the first two integrals leads to:

$$\int_{E_0-4a}^{E_0+4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E')$$
$$= \lim_{\varepsilon\to 0^+} \Bigg[[N(E')\log|E'-E|]_{E_0-4a}^{E-\varepsilon}$$
$$- \int_{E_0-4a}^{E-\varepsilon} \frac{N(E')}{E-E'} dE' + [N(E')\log|E'-E|]_{E+\varepsilon}^{E_0+4a}$$
$$- \int_{E+\varepsilon}^{E_0+4a} \frac{N(E')}{E'-E} dE' \Bigg] - \mathcal{I}(E_0).$$

We set $\psi(E) = N(E)\chi_{\{|E-E_0|\leq 4a\}} \in L^2(\mathbb{R})$. By definition of the Hilbert transform:

$$\int_{E_0-4a}^{E_0+4a} \log|E'-E|\, \mathrm{d}\mathfrak{n}(E')$$
$$= \pi(T\psi)(E) + \lim_{\varepsilon\to 0^+} [(N(E-\varepsilon) - N(E+\varepsilon))\log\varepsilon + N(E_0+4a)\log|E_0 - E + 4a|$$
$$- N(E_0-4a)\log|E_0-E-4a|] - \mathcal{I}(E_0)$$
$$= \pi(T\psi)(E) + N(E_0+4a)\log|E_0-E+4a|$$
$$- N(E_0-4a)\log|E_0-E-4a| - \mathcal{I}(E_0)$$

by (4.13). We finally get:

$$\pi(T\psi)(E) = (\gamma_1 + \cdots + \gamma_N)(E) + \alpha - \int_{|E'-E_0|>4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E')$$
$$- N(E_0+4a)\log|E_0 - E + 4a| + N(E_0-4a)\log|E_0 - E - 4a| + \mathcal{I}(E_0)$$
$$= (\gamma_1 + \cdots + \gamma_N)(E) + \alpha - \int_{|E'-E_0|\geq 4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E') + \mathcal{I}(E_0).$$

But as $[E_0-4a, E_0+4a] \subset I \subset \tilde{I}$, $E \mapsto (\gamma_1 + \cdots + \gamma_N)(E)$ is Hölder continuous on $[E_0-4a, E_0+4a]$ by Theorem 2. Moreover, $E \mapsto \int_{|E'-E_0|\geq 4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E')$ is Hölder continuous of order 1 on the interval $]E_0-4a, E_0+4a[$.

Then $T\psi$ is Hölder continuous on every compact interval included in $]E_0 - 4a, E_0+4a[$, in particular it is Hölder continuous on $[E_0-2a, E_0+2a]$. Thus by Proposition 12(ii), $T^2\psi$ is Hölder continuous on $[E_0-a, E_0+a]$. But by Proposition 12(i) and by continuity of $E \mapsto N(E)$ (by (4.13)), we have:

$$\forall E \in [E_0-a, E_0+a], \quad (T^2\psi)(E) = -N(E).$$

Then $E \mapsto N(E)$ is Hölder continuous on $[E_0-a, E_0+a]$. But I being compact, it can be covered by a finite number of intervals $]E_0-a, E_0+a[\subset \tilde{I}$ with $E_0 \in I$. Thus, $E \mapsto N(E)$ is Hölder continuous on I. □

The Hölder continuity of the Lyapounov exponents and of the IDS relies on the assumptions of p-contractivity and L_p-strong irreducibility for every $p \in \{1, \ldots, N\}$ made on G_{μ_E}. But, for arbitrary potential V_ω, we do not know if these assumptions are verified or not. In the next section we will present a first example of continuous matrix-valued Anderson model for which these assumptions are verified.

5. Anderson Model on Two Coupled Strings

We will now see how to apply Theorem 4 to a particular case of $H_A(\omega)$, which is the following operator:

$$H_{AB}(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{n\in\mathbb{Z}} \begin{pmatrix} \omega_1^{(n)}\chi_{[0,1]}(x-n) & 0 \\ 0 & \omega_2^{(n)}\chi_{[0,1]}(x-n) \end{pmatrix}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. Here, $\chi_{[0,1]}$ denotes the characteristic function of the interval $[0,1]$ and $(\omega_1^{(n)})_{n\in\mathbb{Z}}$ and $(\omega_2^{(n)})_{n\in\mathbb{Z}}$ are two independent sequences of i.i.d. random variables with common law ν such that $\{0,1\} \subset \operatorname{supp}\nu$. This operator is a bounded perturbation of $(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}) \otimes I_2$ and thus self-adjoint on the Sobolev space $H^2(\mathbb{R}) \otimes \mathbb{C}^2$.

For the operator $H_{AB}(\omega)$, we have the following result:

Theorem 5. *The Integrated Density of States $N(E)$ associated to $H_{AB}(\omega)$ exists for every $E \in \mathbb{R}$. Moreover, there exists a discrete subset $\mathcal{S}_B \subset \mathbb{R}$ such that for every compact interval $I \subset (2, +\infty)\backslash\mathcal{S}_B$, the function $E \mapsto N(E)$ is Hölder continuous on I.*

According to Theorem 4, we only have to prove that there exists a discrete subset $\mathcal{S}_B \subset \mathbb{R}$ such that for every $E \in (2, +\infty)\backslash\mathcal{S}_B$, the group G_{μ_E} associated to the transfer matrices of $H_{AB}(\omega)$ is p-contracting and L_p-strongly irreducible for $p \in \{1, 2\}$. It has already been proved in a previous article of the author, [3], and we will only give here the outlines of the proof and some comments.

To prove that an explicit group is p-contracting and L_p-strongly irreducible can be very complicated. It has been done in [11] for the case of a scalar-valued continuous Anderson model, but their proof relies on properties of reflection and transmission coefficients which no longer holds in the matrix-valued case. In the case of a discrete matrix-valued Anderson model, a more algebraic approach has been successfully used by Gol'dsheid and Margulis in [14]. We follow here this approach and adapt it to the case of continuous matrix-valued Anderson models. It is based on the following criterion:

Theorem 6 ([14]). *If a subgroup G of $\mathrm{Sp_N}(\mathbb{R})$ is dense for the Zariski topology in $\mathrm{Sp_N}(\mathbb{R})$ then it is p-contracting and L_p-strongly irreducible for every $p \in \{1, \dots, N\}$.*

In the case of a discrete matrix-valued Anderson model, the transfer matrices have a simple enough expression to make possible a direct construction of the Zariski closure of the group G_{μ_E} generated by these transfer matrices. And so it can be proved that for every $E \in \mathbb{R}$, G_{μ_E} is Zariski dense in $\mathrm{Sp_N}(\mathbb{R})$.

In our case, the transfer matrices associated to $H_{AB}(\omega)$, even if they are still explicit, are complicated enough to not allow a direct reconstruction of the Zariski closure of $G_{\mu E}$ for every E except those in a discrete set. It is due to the fact that E and the ω_i's are not separated in the expressions of these transfer matrices. A direct reconstruction of the Zariski closure of G_{μ_E} is in fact possible, but only for values of E away from a dense countable subset of $\mathbb{R}$, as shown in [5]. It leads to the impossibility to find an interval of values of E such that G_{μ_E} is p-contracting and L_p-strongly irreducible and makes it impossible to apply Theorem 4.

The idea in [3], to improve the result of [5], is to combine the criterion of Gol'dsheid and Margulis to a recent result of Breuillard and Gelander on Lie groups:

Theorem 7 ([6]). *Let G be a real, connected, semisimple Lie group, whose Lie algebra is $\mathfrak{g}$.*

Then there exists a neighborhood $\mathcal{O}$ of 1 in G, on which $\log = \exp^{-1}$ is a well defined diffeomorphism, such that $g_1, \dots, g_m \in \mathcal{O}$ generate a dense subgroup whenever $\log g_1, \dots, \log g_m$ generate $\mathfrak{g}$.

Using this theorem leads us to:

(i) Prove that we can find suitable powers of the transfer matrices which lies in an arbitrary neighborhood of the identity in $\mathrm{Sp_2}(\mathbb{R})$. These powers will be our "$g_1, \dots, g_m$". To construct these powers we use simultaneous diophantine approximation which can be used only for $E > 2$ in our model, as explained in [3, Sec. 4.1].

(ii) Compute the logarithms of these powers of transfer matrices. It leads to a first discrete set of E's in $\mathbb{R}$ on which these logarithms are not defined.

(iii) Out of this discrete set of E's, prove that these logarithms generates the Lie algebra $\mathfrak{sp}_2(\mathbb{R})$ of $\mathrm{Sp}_2(\mathbb{R})$, except for E's in an other discrete subset of $\mathbb{R}$ which corresponds to zeros of some determinants (see [3, Sec. 4.3]). This part of the proof is constructive and for the moment it was not possible to do it for N strictly larger than 2.

So finally, in [3], we were able to prove that there exists a discrete set $\mathcal{S}_B \subset \mathbb{R}$ such that for every E in $\mathcal{S}_B$, $E > 2$, the closed group G_{μ_E} is dense and, therefore, equal to $\mathrm{Sp}_2(\mathbb{R})$. So we can apply Theorem 4, because any compact interval $I \subset (2, +\infty)\backslash\mathcal{S}_B$ is also included in an interval $\tilde{I} \subset (2, +\infty)\backslash\mathcal{S}_B$ on which G_{μ_E} is p-contracting and L_p-strongly irreducible for $p \in \{1, 2\}$. This finishes the proof of Theorem 5.

To conclude the study of the operator $H_{AB}(\omega)$, we would like to precise that the approach used here to prove the density of G_{μ_E} in $\mathrm{Sp}_2(\mathbb{R})$, and based on Breuillard and Gelander's result, relies on the fact that the transfer matrices can be expressed as exponentials of matrices. This fact is no longer true if instead of considering characteristic functions in the definition of $H_{AB}(\omega)$, we consider more general functions. If we do, the transfer matrices becomes time-ordered exponentials instead of exponentials and the Breuillard and Gelander's result no longer applies directly. But we hope that, by some perturbation argument, the case of more general functions could still be handled.

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