



Dynamical localization for a random scattering zipper

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Abstract

This article establishes a proof of dynamical localization for a random scattering zipper model. The scattering zipper operator is the product of two unitary by blocks operators, multiplicatively perturbed on the left and right by random unitary phases. One of the operator is shifted so that this configuration produces a random 5-diagonal by blocks unitary operator. To prove the dynamical localization for this operator, we use the fractional moments method. We first prove the continuity and strict positivity of the Lyapunov exponents in an annulus around the unit circle, which leads to the exponential decay of a power of the norm of the products of transfer matrices. We then establish an explicit formula of the coefficients of the finite resolvent in terms of the coefficients of the transfer matrices using Schur's complement. From this, we deduce, through two reduction results, the exponential decay of the resolvent, from which we get the dynamical localization.

Keywords Anderson localization · Lyapunov exponents · Fractional Moments Methods · Unitary operators · Spectral theory · Random operators

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1 Introduction: model and main results

In this paper, we aim at proving a result of dynamical localization for a random scattering zipper of arbitrary size. Let $L \geq 1$ an integer. A scattering zipper is a system obtained by concatenation of elementary unitary diffusion with a fixed number $2L$ of incoming and outgoing channels each.

Throughout this article, $\|\cdot\|$ denotes any subordinate norm on $\mathcal{M}_L(\mathbb{C})$, the vector space of $L \times L$ complex-valued matrices. Let $U(L)$ be the unitary subgroup of the

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We then define the product probability space:

$$(\Omega, \mathcal{B}, \mathbb{P}) = \left(\tilde{\Omega}^{\mathbb{Z}}, \bigotimes_{n \in \mathbb{Z}} \tilde{\mathcal{B}}, \bigotimes_{n \in \mathbb{Z}} \tilde{\mathbb{P}} \right). \tag{1.8}$$

Let $\omega \in \Omega$ and $n \in \mathbb{Z}$. Then $\omega_n \in \tilde{\Omega}$ and we set

$$\omega_n = (\tilde{V}_\omega^{(n)}, d_\omega^{(n)}, \theta_\omega^{(n)}, \tilde{U}_\omega^{(n)}, D_\omega^{(n)}, \Theta_\omega^{(n)}) \tag{1.9}$$

with :

- (i) $\tilde{U}_\omega^{(n)} \in U(L)$ and $\tilde{V}_\omega^{(n)} \in U(L)$;
- (ii) $d_\omega^{(n)} = (d_{\omega,1}^{(n)}, \dots, d_{\omega,L}^{(n)}) \in \{-1, 1\}^L$ which will represent indifferently the L -uple or the diagonal matrix whose diagonal elements are the $d_{\omega,1}^{(n)}, \dots, d_{\omega,L}^{(n)}$ and the same for $D_\omega^{(n)} = (D_{\omega,1}^{(n)}, \dots, D_{\omega,L}^{(n)}) \in \{-1, 1\}^L$;
- (iii) $\theta_\omega^{(n)} = (\theta_{\omega,1}^{(n)}, \dots, \theta_{\omega,L}^{(n)}) \in [0, 2\pi]^L$ which will represent indifferently the L -uple or the diagonal matrix whose diagonal elements are the $\theta_{\omega,1}^{(n)}, \dots, \theta_{\omega,L}^{(n)}$ and the same for $\Theta_\omega^{(n)} = (\Theta_{\omega,1}^{(n)}, \dots, \Theta_{\omega,L}^{(n)}) \in [0, 2\pi]^L$.

With these notations we define, for every $\omega \in \Omega$ and every $n \in \mathbb{Z}$, the following phases

$$\begin{aligned} U_\omega^{(n)} &= e^{i\Theta_\omega^{(n)}} \tilde{U}_\omega^{(n)} D_\omega^{(n)} (\tilde{U}_\omega^{(n)})^* := e^{i\Theta_\omega^{(n)}} \widehat{U}_\omega^{(n)} \quad \text{and} \\ V_\omega^{(n)} &= \tilde{V}_\omega^{(n)} d_\omega^{(n)} (\tilde{V}_\omega^{(n)})^* e^{i\theta_\omega^{(n)}} := \widehat{V}_\omega^{(n)} e^{i\theta_\omega^{(n)}} \end{aligned} \tag{1.10}$$

where the reduced notations $\widehat{U}_\omega^{(n)}$ and $\widehat{V}_\omega^{(n)}$ will be used in the proofs of Theorem 1.4, Lemma 1.7 and Theorem 1.10.

Remark 1.1 In view of the definition of the probability space $(\Omega, \mathcal{B}, \mathbb{P})$, one can see $(\tilde{U}_\omega^{(n)})_{n \in \mathbb{Z}}$ and $(\tilde{V}_\omega^{(n)})_{n \in \mathbb{Z}}$ as sequences of independent and identically distributed (*i.i.d.* for short) random variables in $U(L)$ of uniform law according to the Haar measure on $U(L)$. The sequences $(d_{\omega,i}^{(n)})_{n \in \mathbb{Z}}$ and $(D_{\omega,j}^{(n)})_{n \in \mathbb{Z}}$, for $i, j \in \{1, \dots, L\}$, can be considered as sequences of *i.i.d.* random variables in $\{-1, 1\}$ of common law the Bernoulli law $\mathcal{B}(p)$ of parameter $p \in (0, 1)$ and the sequences $(\theta_{\omega,i}^{(n)})_{n \in \mathbb{Z}}$ and $(\Theta_{\omega,j}^{(n)})_{n \in \mathbb{Z}}$ for $i, j \in \{1, \dots, L\}$ can be considered as sequences of *i.i.d.* random variables in $[0, 2\pi]$ of common law the uniform law on $[0, 2\pi]$. Moreover, all these sequences of random variables are independent from each others.

Remark 1.2 Note that for every $\omega \in \Omega$ and every $n \in \mathbb{Z}$, $\widehat{U}_\omega^{(n)}$ and $\widehat{V}_\omega^{(n)}$ are elements of $U(L) \cap H_L(\mathbb{C})$ where $H_L(\mathbb{C})$ is the vector space of the Hermitian matrices of size $L \times L$. This property will be used in the proof of Lemma 1.7 where will need that some square roots of unitary matrices are also Hermitian (see (3.19) and (3.20)). This property is the reason why we introduce the randomness in this particular form. The

presence of the Bernoulli variables as diagonal elements of the matrices $D_\omega^{(n)}$ and $d_\omega^{(n)}$ is therefore necessary in order to have that $\widehat{U}_\omega^{(n)}$ and $\widehat{V}_\omega^{(n)}$ can take all the possible values in $U(L) \cap H_L(\mathbb{C})$. Then we add the diagonal matrices $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$ to ensure that the random phases $U_\omega^{(n)}$ and $V_\omega^{(n)}$ can take all the possible values in $U(L)$. This fact is used in Section 2.

The presence of separates terms $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$ also simplifies the formulation of the finite-rank perturbation argument in the proofs of Theorem 1.4 and Theorem 1.10.

With these notations, we now introduce our model of random scattering zipper. Consider the random family of unitary operators $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ where, for every $\omega \in \Omega$, the operator \mathbb{U}_ω acts on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ and is the scattering zipper associated with the sequence of random scattering matrices $(S_\omega^{(n)})_{n \in \mathbb{Z}}$ defined by

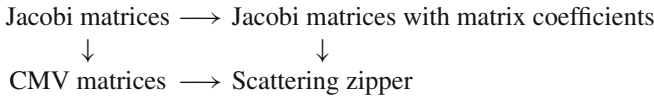
$$S_\omega^{(n)} = S(\alpha, U_\omega^{(n)}, V_\omega^{(n)}) = \begin{pmatrix} \alpha & \rho(\alpha)U_\omega^{(n)} \\ V_\omega^{(n)}\tilde{\rho}(\alpha) & -V_\omega^{(n)}\alpha^*U_\omega^{(n)} \end{pmatrix}, \quad (1.11)$$

for a fixed $\alpha \in \mathcal{M}_L(\mathbb{C})$ such that $\|\alpha\| < 1$.

Note that the family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ has a very important property of $2\mathbb{Z}$ -ergodicity which implies the existence of an almost-sure spectrum for this family. See [3] or [4] for a definition of this ergodicity property and for the existence of the almost-sure spectrum. Note that if Σ denotes the almost-sure spectrum of $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$, $\Sigma \subset \mathbb{S}^1$ since the operators \mathbb{U}_ω are unitary. The $2\mathbb{Z}$ -ergodicity also implies the existence of almost-sure pure point, absolutely continuous and singular continuous spectra.

The block shifting between \mathbb{V}_ω and \mathbb{W}_ω in the product defining \mathbb{U}_ω allows us to consider the scattering zipper model as a matrix-valued version of the Blatter–Browne model introduced in [5] to understand Zener tunneling effect in diodes. It can also be considered as a matrix-valued version of the Chalker–Coddington model on a band studied in [6]. Indeed, \mathbb{U}_ω has the same band structure as that found in the Blatter–Browne and quasi-one-dimensional Chalker–Coddington models, but with an arbitrarily large diagonal bandwidth. Note that since \mathbb{U}_ω is a unitary band operator, the general results of [7] apply to it. To avoid any confusion, note that the scattering zipper model is very different from the Chalker–Coddington model over the entire \mathbb{Z}^2 network, since it is one-dimensional rather than two-dimensional. The scattering zipper model can also be seen as a version of CMV matrices with matrix coefficients. CMV matrices are the unitary analog of Jacobi matrices and were originally introduced in the study of orthogonal polynomials on the unit circle (see [8] for a comprehensive review of this vast subject). On the other hand, Schulz-Baldes studied in [9] a generalization of Jacobi matrices with matrix coefficients. The scattering zipper was introduced by Marin and Schulz-Baldes in [1] with the aim of defining a unitary analog to Jacobi matrices with matrix coefficients that shares with CMV matrices their spectral properties and a simple way of representing their coefficients. In particular, the block factorization of CMV matrices into two diagonal operators (see [10]) is used to define the scattering zipper, without having to resort to an interpretation in terms of orthogonal polynomials on the unit disk.

We can summarize this discussion in a diagram in which the horizontal arrows represent the transition from a scalar-valued operator to a matrix-valued operator, and the vertical arrows represent the transition from a self-adjoint model to its unitary analogue.



Notation. Let $\{e_k\}_{k \in \mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^L$. For $i, j \in \mathbb{Z}$, we set $e_{\{i,j\}} := e_{iL+j}$, which corresponds to the j th component in the i th L -block.

We now state the main result of this article.

Theorem 1.3 *There exists $r_0 > 0$ such that, for every $\alpha \in \text{GL}_L(\mathbb{C})$ with $\|\alpha\| \leq r_0$, there exists $C_{r_0} > 0$ and $b > 0$ such that for all $\{k, p\}$ and $\{l, q\}$ in $\mathbb{Z} \times \{1, \dots, L\}$,*

$$\mathbb{E} \left[\sup_{n \in \mathbb{Z}} |\langle e_{\{k,p\}}, (\mathbb{U}_\omega)^n e_{\{l,q\}} \rangle| \right] \leq C_{r_0} e^{-b|k-l|}. \tag{1.12}$$

The estimate (1.12) means that the family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ satisfies the condition of dynamical localization as defined for example in [11]. This property is dynamic in nature and follows the evolution of wave packets over discrete time $n \in \mathbb{Z}$. It tells us that the solutions of the Schrödinger equation are localized in space in the vicinity of their initial position and this, uniformly over time. This reflects the absence of quantum transport.

The proof of Theorem 1.3 is based upon a number of intermediate results. First, following the result of positivity of the Lyapunov exponents on the unit circle \mathbb{S}^1 obtained by Boumaza and Marin in [2], we prove their strict positivity on an annulus

$$\mathbb{S}_\epsilon := \{z \in \mathbb{C}; 1 - \epsilon < |z| < 1 + \epsilon\} \tag{1.13}$$

for some $\epsilon \in (0, 1]$. The proof is done by combining the positivity of the Lyapunov exponents on the unit circle and their continuity on $\mathbb{C} \setminus \{0\}$.

Notation. For $j \in \mathbb{Z}$, let $E_j = (\dots, 0, I_L, 0, \dots)$ where I_L is in the j^{th} position.

For $z \in \mathbb{C} \setminus \mathbb{S}^1$ let $G_\omega(z)$ be the resolvent at z of the operator \mathbb{U}_ω and let $G_\omega(z, \cdot, \cdot)$ be its block Green kernel defined for every $k, l \in \mathbb{Z}$ by:

$$G_\omega(z, k, l) = \langle E_k, (\mathbb{U}_\omega - z)^{-1} E_l \rangle. \tag{1.14}$$

For $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a < b$, we denote by $G_\omega^{[a,b]}(z)$ the resolvent of the restriction of the operator \mathbb{U}_ω to the interval $[a, b]$ and by $G_\omega^{[a,b]}(z, \cdot, \cdot)$ its block Green kernel :

$$G_\omega^{[a,b]}(z) = \left(\mathbb{U}_\omega^{[a,b]} - z \right)^{-1}, \tag{1.15}$$

and for $k, l \in \mathbb{Z}$,

$$G_\omega^{[a,b]}(z, k, l) = \langle E_k, \left(\mathbb{U}_\omega^{[a,b]} - z \right)^{-1} E_l \rangle. \tag{1.16}$$

The precise definition of the restricted operator $\mathbb{U}_\omega^{[a,b]}$ is given at Section 3.1.

We study the fractional moments of $G_\omega^{[a,b]}(z, \cdot, \cdot)$ for $z \in \mathbb{C} \setminus \mathbb{S}^1$ and we first prove that they are uniformly bounded.

Theorem 1.4 *For every $s \in (0, \frac{1}{4})$ there exists $C(s) > 0$ such that, for every $z \in \mathbb{C} \setminus \mathbb{S}^1$, every $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a < b$, and every $k, l \in \mathbb{Z}$ such that $|k - l| > 4$,*

$$\mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l)\|^s \right) \leq C(s). \tag{1.17}$$

Once we have obtained this uniform bound, combining it with the positivity of the smallest Lyapunov exponent and proving estimates on the blocks of the products of transfer matrices, we prove the exponential decay of some blocks of the Green kernel restricted to suitable intervals.

Theorem 1.5 *There exist $r_0 > 0$, $\epsilon_0 > 0$, $s_0 \in (0, 1)$, $p_0 > 1$, $C_{s_0, r_0} > 0$ and $\gamma > 0$ such that, for every $\alpha \in \text{GL}_L(\mathbb{C})$ with $\|\alpha\| \leq r_0$, every $s \in (0, s_0]$ and every $\epsilon \in (0, \epsilon_0]$,*

$$\mathbb{E} \left(\|G_\omega^{[2n, 2m+1]}(z, 2n, 2m + 1)\|^s \right) \leq C e^{-\gamma|m-n|} \tag{1.18}$$

for every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and for every m and n in \mathbb{Z} such that $|m - n| > p_0$.

Theorem 1.5 is central in the proof of Theorem 1.3. Unlike traditional approaches, like the one used by Hamza, Joye and Stolz (see [11]) which relies on expressing the Green’s function in terms of eigenfunctions with particular boundary conditions to obtain exponential decay through transfer matrices, we adopt a different method. The block configuration of our model does not allow us to directly apply this strategy since the expression for the Green kernel obtained in our case is too complicated to estimate (see [12] for such an expression of the Green kernel). Therefore, we use the Schur complement as an alternative. This method allows us to identify a term whose norm expectation must be bounded to demonstrate the decay of the reduced case.

In order to be able to reduce our analysis to the blocks for which we have the exponential decay (1.18), we prove two reduction results. We start by showing that it suffices to deal with even-sized scattering zipper operators.

Proposition 1.6 *Assume $\alpha \in \text{GL}_L(\mathbb{C})$ is such that $\|\alpha\| < 1$. Let $s \in (0, \frac{1}{4})$, $\epsilon \in (0, 1)$ and let $k, l \in \mathbb{Z}$ such that $|k - l| > 4$. There exists $C(s, \epsilon) > 0$ such that:*

$$\mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l)\|^s \right)^2 \leq C(s, \epsilon) \sum_{i,j=0}^1 \mathbb{E}(\|G_\omega^{[a,b]}(z, 2n + 2i, 2m + 1 + 2j)\|^{4s})^{\frac{1}{2}}, \tag{1.19}$$

for all $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and n, m such that $k \in \{2n, 2n + 1\}$ and $l \in \{2m, 2m + 1\}$.

The proof involves bounding the norm of the “even” blocks $\|G_\omega^{[a,b]}(z, 2n, 2m)\|$ and $\|G_\omega^{[a,b]}(z, 2n + 1, 2m)\|$ by the norms of the “odd” blocks $\|G_\omega^{[a,b]}(z, 2n, 2m + 1)\|$ and $\|G_\omega^{[a,b]}(z, 2n + 1, 2m + 1)\|$. A necessary condition for establishing this is the invertibility of α . This invertibility is necessary to prove the following lemma.

Lemma 1.7 *If $\alpha \in GL_L(\mathbb{C})$, then for every $\epsilon > 0$, for every $n \in \mathbb{Z}$ and for every $z \in \mathbb{S}_\epsilon$, the matrix $\alpha + zV_\omega^{(n)}\alpha U_\omega^{(n)}$ is invertible almost surely, and for every $s \in (0, 1)$, there exists $C(s) > 0$ such that*

$$\forall n \in \mathbb{Z}, \mathbb{E} \left(\|(\alpha + zV_\omega^{(n)}\alpha U_\omega^{(n)})^{-1}\|^s \right) \leq C(s). \tag{1.20}$$

Once we get Lemma 1.7, using Hölder inequality and Theorem 1.4, one gets Proposition 1.6.

The second reduction result show that it suffices to control the Green kernel of the restricted operator to some finite interval in order to control the Green kernel of \mathbb{U}_ω .

Proposition 1.8 *Let $s \in (0, \frac{1}{2})$ and $\epsilon > 0$. There exists $C(s) > 0$ such that*

$$\mathbb{E} \left(\|G_\omega(z, k, l)\|^s \right)^2 \leq C(s)\mathbb{E} \left(\|G_\omega^{[k,l]}(z, k, l)^{2s} \right) \tag{1.21}$$

for every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and every $k, l \in \mathbb{Z}$ such that $|k - l| > 4$.

The proof of Lemma 1.8 relies on the geometric resolvent identity.

Combining Proposition 1.6 and Proposition 1.8, as well as Theorem 1.5 on the exponential decay of the reduced case, we obtain the exponential decay of the fractional moments.

Theorem 1.9 *There exist $r_0 > 0, s \in (0, 1), \epsilon_0 > 0, C_{s,r_0} > 0$ and $\gamma > 0$, such that for every $\alpha \in GL_L(\mathbb{C})$ with $\|\alpha\| < r_0$,*

$$\mathbb{E} \left(\|G_\omega(z, k, l)\|^s \right) \leq C_{s,r_0}e^{-\gamma|k-l|} \tag{1.22}$$

for every $\epsilon \in (0, \epsilon_0]$, every $k, l \in \mathbb{Z}$ and every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$.

The proof of Theorem 1.3 follows from Theorem 1.9 and from the following estimate on the moments of order two of the coefficients of the resolvent.

Theorem 1.10 *There exist $\epsilon_0 > 0, r_0 > 0, C_{r_0} > 0$ and $\gamma > 0$ such that, for every $\epsilon \in (0, \epsilon_0]$ and every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$, for every $\alpha \in GL_L(\mathbb{C})$ with $\|\alpha\| \leq r_0$, and every $\{k, p\}$ and $\{l, q\}$ in $\mathbb{Z} \times \{1, \dots, L\}$:*

$$\mathbb{E} \left((1 - |z|^2) \left| \langle e_{\{k,p\}} | (\mathbb{U}_\omega - z)^{-1} e_{\{l,q\}} \rangle \right|^2 \right) \leq C_{r_0}e^{-\gamma(k-l)}$$

Theorem 1.10 is a consequence of Theorem 1.9 using second order perturbation theory.

Remark 1.11 The parameter ϵ_0 is chosen to guarantee strictly positive Lyapunov exponents over the annulus \mathbb{S}_{ϵ_0} and is introduced in Corollary 2.11.

Results of strict positivity of the positive Lyapunov exponents are already known for models of unit-band operators (see [7, 13]). Bourget, Howland and Joye’s article [7] was one of the first to present the study of unitary models with one diffusion channel, and was followed by other articles by Joye on the subject, such as [14] and [15]. For a synthetic presentation of these results and the associated physical models, we refer to [16]. Note that the unitary models considered in these first articles are all with scalar coefficients, and not with matrix coefficients as in the case of the scattering zipper.

One can see the random scattering zipper model as a unitary version of the quasi-one-dimensional Anderson model. There is already a unitary version of the scalar-valued Anderson model for which Hamza, Joye and Stolz have proved dynamical localization in [11]. The random scattering zipper can therefore also be seen as a quasi-one-dimensional version of the unitary Anderson model. This is why, as we will see in the rest of the paper, it is possible to follow the strategy of the proof of the dynamical localization result of [11]. Based upon the fractional moments method (see [17] for an overview of this method), this strategy is also the one used in the proof of localization in [6, 18] for the Chalker–Coddington model.

However, it would be possible to consider a multiscale analysis approach to study this unitary model, as Cedzich and Werner did in [19] in the context of unitary quantum walks. This would then make it possible to address the question of localization for a random scattering zipper model in which the unitary phases would reveal a much more singular randomness than a uniform Haar law. A last possible approach would be to adapt to the quasi-one-dimensional case the proofs of localization results for CMV matrices in [20, 21] which are based on large deviations inequalities.

The rest of the paper is devoted to the proof of the results we have just stated in this introductory section. In Section 2 we introduce the transfer matrices associated with the quasi-one-dimensional model $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ of unitary type (see [22]). We also define the Lyapunov exponents associated with the sequence of transfer matrices and we prove their continuity, and hence their positivity on some annulus \mathbb{S}_ϵ . Section 3 is the core of the article. In this section we prove the exponential decay of the fractional moments of the Green kernel. First, we show that it suffices to prove this exponential decay for some particular blocks of the resolvent by proving Proposition 1.6 and Proposition 3.5. To prove the exponential decay of the fractional moments of these particular blocks we look precisely at the behavior of the four $L \times L$ blocks of the products of transfer matrices and prove boundedness of some “Wronskian” involving these blocks. Finally in Section 4, we prove the exponential decay of moments of order 2 and we deduce from the spectral theorem for unitary operators the proof of the dynamical localization for $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$.

2 Transfer matrices and Lyapunov exponents

2.1 Transfer matrices

Using the transfer matrix formalism, we reduce the study of the asymptotic behavior of a solution ϕ of

$$\mathbb{U}_\omega \phi = z\phi, \quad \text{for } z \in \mathbb{C}, \tag{2.1}$$

to the asymptotic behavior of a product of random matrices.

Definition 2.1 For $L \in \mathbb{N}^*$, we define by \mathcal{L} the bilinear form whose matrix in the canonical basis of \mathbb{C}^L is $\begin{pmatrix} I_L & 0 \\ 0 & -I_L \end{pmatrix}$. We also denote by \mathcal{L} the matrix $\begin{pmatrix} I_L & 0 \\ 0 & -I_L \end{pmatrix}$ itself.

In the unitary setting, the role played by the symplectic group for the quasi-one-dimensional models of Schrödinger type will be played by the group $U(L, L)$ of the matrices of size $2L \times 2L$ which preserve the form \mathcal{L} in the sense that T is in $U(L, L)$ if and only if $T^* \mathcal{L} T = \mathcal{L}$.

To compute the transfer matrices we proceed as follows. Instead of looking at the input–output relations of the scattering matrix $S_\omega^{(n)}$, we look for a new matrix which allows to express $\begin{pmatrix} \phi_{n+1} \\ \psi_{n+1} \end{pmatrix}$ in terms of $\begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix}$ for ϕ a solution of (2.1) and $\psi = \mathbb{W}_\omega \phi$. This is done by transforming the scattering matrices $S_\omega^{(n)}$ belonging to $U(2L)_{\text{inv}}$ into elements of $U(L, L)$ via the bijection:

$$\begin{aligned} U(2L)_{\text{inv}} &\rightarrow U(L, L) \\ \varphi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \begin{pmatrix} \gamma - \delta \beta^{-1} \alpha & \delta \beta^{-1} \\ -\beta^{-1} \alpha & \beta^{-1} \end{pmatrix} \end{aligned}$$

Let $z \in \mathbb{C}$. We have the following relations, proved in [1]:

$$\forall n \in \mathbb{Z}, \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix} = \varphi(z^{-1} S_\omega^{(2n)}) \begin{pmatrix} \psi_{2n-1} \\ \phi_{2n-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi_{2n+1} \\ \phi_{2n+1} \end{pmatrix} = \varphi(S_\omega^{(2n+1)}) \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix}. \tag{2.2}$$

These relations lead to introduce the application $T(z, \cdot) : \Omega \rightarrow \text{GL}_{2L}(\mathbb{C})$,

$$\forall \omega \in \Omega, T(z, \omega) = \begin{pmatrix} V_\omega^{(0)} & 0 \\ 0 & (U_\omega^{(0)})^* \end{pmatrix} T_0(z) \begin{pmatrix} V_\omega^{(1)} & 0 \\ 0 & (U_\omega^{(1)})^* \end{pmatrix} T_1 \tag{2.3}$$

with

$$T_0(z) = \begin{pmatrix} z^{-1}(\tilde{\rho}(\alpha))^{-1} & (\tilde{\rho}(\alpha))^{-1} \alpha^* \\ \alpha(\tilde{\rho}(\alpha))^{-1} & z(\rho(\alpha))^{-1} \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} (\tilde{\rho}(\alpha))^{-1} & (\tilde{\rho}(\alpha))^{-1} \alpha^* \\ \alpha(\tilde{\rho}(\alpha))^{-1} & (\rho(\alpha))^{-1} \end{pmatrix}.$$

We define the 2-shift transformation $\tau_2 : \Omega \mapsto \Omega$ by:

$$\forall \omega \in \Omega, \forall n \in \mathbb{Z}, (\tau_2(\omega))_n = \omega_{n+2}.$$

Then, τ_2 is ergodic on $(\Omega, \mathcal{B}, \mathbb{P})$ and

$$\forall \omega \in \Omega, \forall z \in \mathbb{C}, \forall n \in \mathbb{Z}, T(z, \tau_2^n(\omega)) = \varphi(z^{-1} S_\omega^{(2n)}) \cdot \varphi(S_\omega^{(2n-1)}). \tag{2.4}$$

The matrix $T(z, \tau^n(\omega))$ is the n th transfer matrix associated with $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$. The sequence $(T(z, \tau^n(\omega)))_{n \in \mathbb{Z}}$ is a sequence of *i.i.d.* random matrices in $\text{GL}_{2L}(\mathbb{C})$ because of the definition of the probability \mathbb{P} as a tensor product of probability measures.

Note that for $z \in \mathbb{S}^1, T(z, \tau^n(\omega)) \in U(L, L)$.

Notation. Let $\omega \in \Omega, n \in \mathbb{Z}$ and $z \in \mathbb{C}$. In order to simplify the further computations, we will denote by $T_\omega^{(n)}(z) := T(z, \tau_2^n(\omega))$ the n th transfer matrix.

2.2 Lyapunov exponents

Let $z \in \mathbb{C}$. The transfer matrices $T(z, \cdot)$ generate a cocycle $\Phi(z, \cdot, \cdot) : \Omega \times \mathbb{Z} \rightarrow \text{GL}_{2L}(\mathbb{C})$ on the ergodic dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, (\tau^n)_{n \in \mathbb{Z}})$ defined by

$$\forall \omega \in \Omega, \forall n \in \mathbb{Z}, \Phi(z, \omega, n) = \begin{cases} T_\omega^{(n-1)}(z) \dots T_\omega^{(0)}(z) & \text{if } n > 0 \\ I_{2L} & \text{if } n = 0 \\ (T_\omega^{(n)}(z))^{-1} \dots (T_\omega^{(-1)}(z))^{-1} & \text{if } n < 0. \end{cases}$$

From this cocycle we define the Lyapunov exponents associated with the ergodic family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$. For \mathbb{P} -almost every $\omega \in \Omega$, the following limits exists and are equal:

$$\begin{aligned} \Psi(z, \omega) &:= \lim_{n \rightarrow +\infty} ((\Phi(z, \omega, n))^* \Phi(z, \omega, n))^{1/2n} \\ &= \lim_{n \rightarrow -\infty} ((\Phi(z, \omega, n))^* \Phi(z, \omega, n))^{1/2|n|}. \end{aligned} \tag{2.5}$$

For every $k \in \{1, \dots, 2L\}$, let $\lambda_k(z, \omega)$ the k th eigenvalue of $\Psi(z, \omega)$, the eigenvalues being ordered in increasing order. There are real numbers $\lambda_k(z) \geq 0$ such that, for \mathbb{P} -almost every $\omega \in \Omega, \lambda_k(z, \omega) = \lambda_k(z)$. We then define the Lyapunov exponents associated with the ergodic family $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ as being the real numbers $\gamma_k(z)$ defined by :

$$\forall z \in \mathbb{C}, \forall k \in \{1, \dots, 2L\}, \gamma_k(z) := \log(\lambda_k(z)).$$

Note that when $z \in \mathbb{S}^1$, the fact that the transfer matrices belong to $U(L, L)$ implies a symmetry relation on the Lyapunov exponents which is the same as in the case where the transfer matrices are in the symplectic group:

$$\forall k \in \{1, \dots, L\}, \gamma_{2L-k+1}(z) = -\gamma_k(z). \tag{2.6}$$

To study the separability of the Lyapunov exponents, we study the so-called Furstenberg group associated with $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$. This is the group generated by the common law of the transfer matrices. If μ_z is the common law of all the transfer matrices $T_\omega^{(n)}(z)$, we set

$$G_{\mu_z} = \overline{\langle \text{supp} \mu_z \rangle} \subset \text{GL}_{2L}(\mathbb{C}). \tag{2.7}$$

The closure is taken for the topology on $GL_{2L}(\mathbb{C})$ which is induced by the usual topology on $\mathcal{M}_{2L}(\mathbb{C})$.

In [2] we proved that for every $z \in \mathbb{S}^1$, $G_{\mu_z} = U(L, L)$. Actually, the algebraic construction made in the proof of [2, Proposition 2] is valid for any $z \in \mathbb{C}$ and we get

$$\forall z \in \mathbb{C}, U(L, L) \subseteq G_{\mu_z}. \tag{2.8}$$

Indeed, with our particular choice of randomness in the definition of the random scattering zipper, and in particular with the introduction of the extra random terms $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$, one recovers Step 1 of the proof of [2, Proposition 2] not only for matrices U and V in $U(L) \cap H_L(\mathbb{C})$ but for any matrices U and V in $U(L)$. Then the rest of the proof of [2, Proposition 2] remains unchanged.

Remark 2.2 Actually, we conjecture that the algebraic result (2.8) remains true with only random terms $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$ and well-chosen fixed matrices \tilde{U} and \tilde{V} (independent of n and ω) in (1.10). But up to now, we where not able to prove (2.8) with such hypotheses. This is the only point where we need that $\tilde{U}_\omega^{(n)}$ and $\tilde{V}_\omega^{(n)}$ are random but this point is crucial. All the others proofs in the present article require only the randomness of $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$.

One gets, using a similar proof as the one of [2, Theorem 1], that

$$\forall z \in \mathbb{C}, \gamma_1(z) > \dots > \gamma_L(z) > \gamma_{L+1}(z) > \dots > \gamma_{2L}(z). \tag{2.9}$$

For $z \in \mathbb{S}^1$, combining (2.9) with (2.6) gives:

$$\forall z \in \mathbb{S}^1, \gamma_1(z) > \gamma_2(z) > \dots > \gamma_L(z) > 0. \tag{2.10}$$

Using Kotani’s theory (see [2] which adapts results of [23]), this implies the absence of almost-sure absolutely continuous spectrum of $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$.

But for $z \in \mathbb{C} \setminus \mathbb{S}^1$ the transfer matrices are no longer in $U(L, L)$ and we no longer have the symmetry relation (2.6). Hence, there is no reason for $\gamma_L(z)$ to be positive if z is not in \mathbb{S}^1 . Still, what remains true is the following equality:

$$\forall n \in \mathbb{Z}, \forall \omega \in \Omega, \forall z \in \mathbb{C}^*, (T_\omega^{(n)}(z))^{-1} = \mathcal{L} \left(T_\omega^{(n)}(\bar{z}^{-1}) \right)^* \mathcal{L}. \tag{2.11}$$

The proof of (2.9) is based upon the Cayley transform, whose matrix in the canonical basis of \mathbb{C}^{2L} is

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} I_L & -iI_L \\ I_L & iI_L \end{pmatrix} \in \mathcal{M}_{2L}(\mathbb{C}) \tag{2.12}$$

and which maps the group $U(L, L)$ onto the complex symplectic group $Sp_{\mathbb{N}}^*(\mathbb{C})$ defined by :

$$Sp_{\mathbb{L}}^*(\mathbb{C}) = \{M \in \mathcal{M}_{2L}(\mathbb{C}) \mid M^* J M = J\}, \quad \text{with } J = \begin{pmatrix} 0 & -I_L \\ I_L & 0 \end{pmatrix}. \tag{2.13}$$

It also uses the following transformation which separates the real and imaginary parts of a matrix with complex coefficients and place them in blocks:

$$\pi : \begin{matrix} \mathcal{M}_{2L}(\mathbb{C}) & \rightarrow & \mathcal{M}_{4L}(\mathbb{R}) \\ A + iB & \mapsto & \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \end{matrix} \tag{2.14}$$

It allows to apply directly the results of [24] to get separability of the Lyapunov exponents and their integral representation from the properties of p -contractivity and L_p -strong irreducibility of the Furstenberg group as defined in [24].

We recall here the definition of the p -Lagrangian space L_p .

Definition 2.3 For $p \in \{1, \dots, L\}$, let L_p be the vector subspace of $\wedge^p \mathbb{C}^{2L}$, the p th exterior power of \mathbb{C}^{2L} , defined by

$$L_p = \text{Span}(\{Me_1 \wedge \dots \wedge Me_p \mid M \in \text{Sp}_L^*(\mathbb{C})\}) \tag{2.15}$$

where (e_1, \dots, e_{2L}) is the canonical basis of \mathbb{C}^{2L} .

2.3 Continuity and strict positivity of the Lyapunov exponents

Among the key elements for the development of the fractional moments method, the continuity and positivity of Lyapunov exponents occupy a prominent place. As we have just explained, small changes in the proof of [2, Theorem 1] gives us the positivity of these exponents, but this assertion remains limited to the circle \mathbb{S}^1 .

In order to manipulate the resolvent efficiently, it is necessary to choose a spectral parameter z located outside of \mathbb{S}^1 . The objective of this subsection is therefore twofold. Firstly, we will demonstrate the continuity of Lyapunov exponents on $\mathbb{C} \setminus \{0\}$. Secondly, we will highlight the positivity of these exponents in some annulus \mathbb{S}_ε .

We start by giving upper and lower bounds on the norm of $\rho(\alpha)$ which will be used throughout the rest of the article.

Lemma 2.4 *Let $L \geq 1$. For $\alpha \in \mathcal{M}_L(\mathbb{C})$ such that $\|\alpha\| < 1$ and $\rho(\alpha) = (I_L - \alpha\alpha^*)^{\frac{1}{2}}$, the following inequalities hold:*

1. $\|\rho(\alpha)\| \geq \sqrt{1 - \|\alpha\|^2}$.
2. $\|\rho(\alpha)\| \leq 2 - \sqrt{1 - \|\alpha\|^2}$.
3. $\|\rho(\alpha)^{-1}\| \leq \frac{1}{\sqrt{1 - \|\alpha\|^2}}$.
4. $\|\rho(\alpha)^{-1}\| \geq \frac{1}{2 - \sqrt{1 - \|\alpha\|^2}}$.

and the same for $\tilde{\rho}(\alpha) = (I_L - \alpha^*\alpha)^{\frac{1}{2}}$.

Proof Point (1) follows from the triangle inequality, points (2) and (3) are direct consequences of the power series expansion of $z \mapsto \sqrt{1 - z}$ and $z \mapsto \frac{1}{\sqrt{1-z}}$ and the sub-multiplicativity of the matrix norm. Finally, to prove (4) it suffices to note that for a subordinate norm, we have $\frac{1}{\|A^{-1}\|} \leq \|A\|$. The proof is identical for $\tilde{\rho}(\alpha)$. \square

We now prove several estimates on the transfer matrices which are ingredients of the proof of the continuity of the Lyapunov exponents.

Lemma 2.5 *For any $\epsilon \in (0, 1)$, there exists $C_1 = C_1(\alpha, \epsilon) > 0$ such that:*

$$\forall z \in \mathbb{S}_\epsilon, \forall n \in \mathbb{Z}, \forall \omega \in \Omega, \|T_\omega^{(n)}(z)\| \leq C_1. \tag{2.16}$$

Proof For simplicity, we denote $\rho = \rho(\alpha)$ and $\tilde{\rho} = \tilde{\rho}(\alpha)$.

$$T_\omega^{(n)}(z) = \begin{pmatrix} V_\omega^{(2n)} & 0 \\ 0 & (U_\omega^{(2n)})^* \end{pmatrix} \begin{pmatrix} z^{-1}\tilde{\rho}^{-1} & -\tilde{\rho}^{-1}\alpha^* \\ -\alpha\tilde{\rho}^{-1} & z\rho^{-1} \end{pmatrix} \\ \begin{pmatrix} V_\omega^{(2n-1)} & 0 \\ 0 & (U_\omega^{(2n-1)})^* \end{pmatrix} \begin{pmatrix} \tilde{\rho}^{-1} & -\tilde{\rho}^{-1}\alpha^* \\ -\alpha\tilde{\rho}^{-1} & \rho^{-1} \end{pmatrix}.$$

Thus, for $z \in \mathbb{S}_\epsilon$,

$$\|T_\omega^{(n)}(z)\| \leq \left\| \begin{pmatrix} z^{-1}\tilde{\rho}^{-1} & -\tilde{\rho}^{-1}\alpha^* \\ -\alpha\tilde{\rho}^{-1} & z\rho^{-1} \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} \tilde{\rho}^{-1} & -\tilde{\rho}^{-1}\alpha^* \\ -\alpha\tilde{\rho}^{-1} & \rho^{-1} \end{pmatrix} \right\| \\ \leq (|z^{-1}|\|\tilde{\rho}^{-1}\| + \|\tilde{\rho}^{-1}\alpha^*\| + \|\alpha\tilde{\rho}^{-1}\| + |z|\|\rho^{-1}\|) \\ \cdot (\|\tilde{\rho}^{-1}\| + \|\tilde{\rho}^{-1}\alpha^*\| + \|\alpha\tilde{\rho}^{-1}\| + \|\rho^{-1}\|) \\ \leq \left(\frac{1}{1-\epsilon}\|\rho^{-1}\| + 2\|\rho^{-1}\| \cdot \|\alpha\| + (1+\epsilon)\|\rho^{-1}\| \right) \\ \left(\|\rho^{-1}\| + 2\|\rho^{-1}\| \cdot \|\alpha\| + \|\rho^{-1}\| \right).$$

We define $c_\epsilon := \max\{\frac{1}{1-\epsilon}; 1+\epsilon\} > 1$. Then, since $\frac{1}{c_\epsilon} < 1$,

$$\|T_\omega^{(n)}(z)\| \leq \left(c_\epsilon\|\rho^{-1}\| + 2\|\rho^{-1}\| \cdot \|\alpha\| + c_\epsilon\|\rho^{-1}\| \right)^2 \\ \leq c_\epsilon^2\|\rho^{-1}\|^2 (2 + 2\|\alpha\|)^2 \\ \leq 4c_\epsilon^2 \frac{1}{1-\|\alpha\|^2} (1 + \|\alpha\|)^2 := C_1(\alpha, \epsilon).$$

using Lemma 2.4 at the last inequality. □

Lemma 2.6 *Let $\epsilon \in (0, 1)$. Let $p \in \{1, \dots, L\}$. There exists $C_2 = C_2(p, \alpha, \epsilon) > 0$ such that:*

$$\forall z \in \mathbb{S}_\epsilon, \forall n \in \mathbb{Z}, \forall \omega \in \Omega, \left\| \wedge^p T_\omega^{(n)}(z) \right\| \leq C_2 \tag{2.17}$$

where \wedge^p denotes the p^{th} exterior power.

Proof The inequality (2.17) comes from (2.16) and the general fact: if $M \in GL_{2L}(\mathbb{C})$ then for any p , $\|\wedge^p M\| \leq \|M\|^p$ (see [24]). □

Lemma 2.7 *Let $\epsilon \in (0, 1)$. There exists $C_3 = C_3(\alpha, \epsilon) > 0$ such that:*

$$\forall n \in \mathbb{Z}, \forall \omega \in \Omega, \forall z_1, z_2 \in \mathbb{S}_\epsilon, \|T_\omega^{(n)}(z_1) - T_\omega^{(n)}(z_2)\| \leq C_3 |z_1 - z_2|. \tag{2.18}$$

Proof Let $n \in \mathbb{Z}, \omega \in \Omega$ and $z_1, z_2 \in \mathbb{S}_\epsilon$.

$$\begin{aligned} \|T_\omega^{(n)}(z_1) - T_\omega^{(n)}(z_2)\| &= \left\| \begin{pmatrix} V_\omega^{(2n)} & 0 \\ 0 & (U_\omega^{(2n)})^* \end{pmatrix} \begin{bmatrix} (z_1^{-1} \tilde{\rho}^{-1} - \tilde{\rho}^{-1} \alpha^*) & \\ & z_1 \rho^{-1} \end{bmatrix} - \begin{pmatrix} z_2^{-1} \tilde{\rho}^{-1} - \tilde{\rho}^{-1} \alpha^* & \\ -\alpha \tilde{\rho}^{-1} & z_2 \rho^{-1} \end{pmatrix} \right\| \\ &= \left\| \begin{pmatrix} V_\omega^{(2n-1)} & 0 \\ 0 & (U_\omega^{(2n-1)})^* \end{pmatrix} \begin{pmatrix} \tilde{\rho}^{-1} & -\tilde{\rho}^{-1} \alpha^* \\ -\alpha \tilde{\rho}^{-1} & \rho^{-1} \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} \tilde{\rho}^{-1} & -\tilde{\rho}^{-1} \alpha^* \\ -\alpha \tilde{\rho}^{-1} & \rho^{-1} \end{pmatrix} \right\| \cdot \left\| \begin{pmatrix} (z_1^{-1} - z_2^{-1}) \tilde{\rho}^{-1} & 0 \\ 0 & (z_1 - z_2) \rho^{-1} \end{pmatrix} \right\|. \end{aligned}$$

Since $\|\alpha\| < 1$ and using Lemma 2.4, there exists $C'_3 = C'_3(\alpha) > 0$ such that:

$$\begin{aligned} \|T_\omega^{(n)}(z_1) - T_\omega^{(n)}(z_2)\| &\leq C'_3 \left\| \begin{pmatrix} (z_1^{-1} - z_2^{-1}) \tilde{\rho}^{-1} & 0 \\ 0 & (z_1 - z_2) \rho^{-1} \end{pmatrix} \right\| \\ &\leq C'_3 \left(\|(z_1^{-1} - z_2^{-1}) \tilde{\rho}^{-1}\| + \|(z_1 - z_2) \rho^{-1}\| \right) \\ &\leq C'_3 \|\rho^{-1}\| \left(\frac{|z_1 - z_2|}{|z_1 z_2|} + |z_1 - z_2| \right). \end{aligned}$$

Since z_1 and z_2 are in $\mathbb{S}_\epsilon, |z_i| > 1 - \epsilon, \frac{1}{|z_1 z_2|} < \frac{1}{(1-\epsilon)^2}$ and using Lemma 2.4 we find,

$$\|T_\omega^{(n)}(z_1) - T_\omega^{(n)}(z_2)\| \leq C'_3 \frac{1}{\sqrt{1 - \|\alpha\|^2}} \left(1 + \frac{1}{(1 - \epsilon)^2} \right) |z_1 - z_2| \leq C_3 |z_1 - z_2|.$$

□

We now prove the last step before proving the continuity of the Lyapunov exponents.

Lemma 2.8 *Let $\epsilon \in (0, 1)$. Let $p \in \{1, \dots, L\}$. There exists $C_4 = C_4(\alpha, \epsilon, p) > 0$ such that:*

$$\forall n \in \mathbb{Z}, \forall \omega \in \Omega, \forall z_1, z_2 \in \mathbb{S}_\epsilon, \|\wedge^p T_\omega^{(n)}(z_1) - \wedge^p T_\omega^{(n)}(z_2)\| \leq C_4 |z_1 - z_2|. \tag{2.19}$$

Proof As the transfer matrices are invertible, we can reproduce the reasoning from the proof of [25, Lemma 6.2.6] and use the following inequality:

$$\begin{aligned} \forall M, N \in \mathcal{M}_L(\mathbb{C}), \|\wedge^p M - \wedge^p N\| \\ \leq \|N - M\| \left(\|N\|^{p-1} + \|M\| \cdot \|N\|^{p-2} + \dots + \|M\|^{p-1} \right). \end{aligned}$$

Then it suffices to apply Lemma 2.6 and Lemma 2.7 to conclude. □

Having these four lemmas, one can now deduce the continuity of the Lyapunov exponents.

For $p \in \{1, \dots, L\}, \epsilon \in (0, 1), \bar{x} \in P(\mathbb{C}^{2L})$ and $z \in \mathbb{S}_\epsilon$, we define:

$$\Phi(z, \bar{x}) = \mathbb{E} \left(\ln \frac{\| \wedge^p T_\omega^{(n)}(z) \bar{x} \|}{\| \bar{x} \|} \right) \tag{2.20}$$

where $P(\mathbb{C}^{2L})$ is the projective space associated with \mathbb{C}^{2L} .

Lemma 2.9 *The map $(z, \bar{x}) \mapsto \Phi(z, \bar{x})$ has the following properties:*

1. *The map $\bar{x} \mapsto \Phi(z, \bar{x})$ is continuous on $P(L_p)$.*
2. *There exists a constant $C > 0$ such that for all $z_1, z_2 \in \mathbb{S}_\epsilon$:*

$$\sup_{\bar{x} \in P(L_p)} |\Phi(z_1, \bar{x}) - \Phi(z_2, \bar{x})| < C|z_1 - z_2|.$$
3. *The function $\Phi : \mathbb{S}_\epsilon \times P(L_p) \rightarrow \mathbb{R}$ is continuous.*

$$(z, \bar{x}) \mapsto \Phi(z, \bar{x})$$

Proof We start by demonstrating the first point. According to Lemma 2.6, $\wedge^p T_\omega^{(n)}(z)$ is uniformly bounded in z and ω and

$$\forall z \in \mathbb{S}_\epsilon, \forall \omega \in \Omega, \ln \frac{\| \wedge^p T_\omega^{(n)}(z) \bar{x} \|}{\| \bar{x} \|} \leq \ln \| \wedge^p T_\omega^{(n)}(z) \| \leq \ln(C_2).$$

Hence, if $\bar{x}_m \rightarrow \bar{x}$ in $P(L_p)$, it suffices to apply the dominated convergence theorem to show that:

$$\Phi(z, \bar{x}_m) = \mathbb{E} \left(\frac{\| \wedge^p T_\omega^{(n)}(z) \bar{x}_m \|}{\| \bar{x}_m \|} \right) \xrightarrow{m \rightarrow +\infty} \mathbb{E} \left(\frac{\| \wedge^p T_\omega^{(n)}(z) \bar{x} \|}{\| \bar{x} \|} \right) = \Phi(z, \bar{x}).$$

For the second assertion, we begin by taking z_1 and z_2 in \mathbb{S}_ϵ . We have:

$$\Phi(z_1, \bar{x}) - \Phi(z_2, \bar{x}) = \mathbb{E} \left(\ln \frac{\| \wedge^p T_\omega^{(n)}(z_1) \bar{x} \|}{\| \wedge^p T_\omega^{(n)}(z_2) \bar{x} \|} \right).$$

Writing

$$\begin{aligned} \| \wedge^p T_\omega^{(n)}(z_1) \bar{x} \| &= \| \wedge^p T_\omega^{(n)}(z_1) \cdot \wedge^p (T_\omega^{(n)}(z_2))^{-1} \cdot \wedge^p T_\omega^{(n)}(z_2) \bar{x} \| \\ &\leq \| \wedge^p T_\omega^{(n)}(z_1) \cdot \wedge^p (T_\omega^{(n)}(z_2))^{-1} \| \cdot \| \wedge^p T_\omega^{(n)}(z_2) \bar{x} \| \end{aligned}$$

and the same exchanging z_1 and z_2 , one gets:

$$\begin{aligned} &\left| \ln \left(\frac{\| \wedge^p T_\omega^{(n)}(z_1) \bar{x} \|}{\| \wedge^p T_\omega^{(n)}(z_2) \bar{x} \|} \right) \right| \\ &\leq \ln (\max\{ \| \wedge^p T_\omega^{(n)}(z_1) (\wedge^p T_\omega^{(n)}(z_2))^{-1} \|; \| \wedge^p T_\omega^{(n)}(z_2) (\wedge^p T_\omega^{(n)}(z_1))^{-1} \| \}) \end{aligned}$$

Moreover,

$$\begin{aligned} \|T_\omega^{(n)}(z_1)(T_\omega^{(n)}(z_2))^{-1}\| &\leq \|(\wedge^p T_\omega^{(n)}(z_1) - \wedge^p T_\omega^{(n)}(z_2)) \cdot (\wedge^p T_\omega^{(n)}(z_2))^{-1} + I_{2L}\| \\ &\leq \|(\wedge^p T_\omega^{(n)}(z_2))^{-1}\| \cdot \|(\wedge^p T_\omega^{(n)}(z_1) - \wedge^p T_\omega^{(n)}(z_2))\| + 1 \end{aligned}$$

and the same exchanging z_1 and z_2 . This gives the following inequality:

$$\begin{aligned} &\left| \ln \left(\frac{\|\wedge^p T_\omega^{(n)}(z_1)\bar{x}\|}{\|\wedge^p T_\omega^{(n)}(z_2)\bar{x}\|} \right) \right| \\ &\leq \ln \left[\max_{i=1,2} \left\{ \|(\wedge^p T_\omega^{(n)}(z_i))^{-1}\| \right\} \cdot \|\wedge^p T_\omega^{(n)}(z_1) - \wedge^p T_\omega^{(n)}(z_2)\| + 1 \right]. \end{aligned}$$

Since for every $x > 0$, $\ln(x) \leq x - 1$, using (2.11) to be able to apply Lemma 2.6 to the inverse of the transfer matrices, we get:

$$\left| \mathbb{E} \left(\ln \frac{\|\wedge^p T_\omega^{(n)}(z_1)\bar{x}\|}{\|\wedge^p T_\omega^{(n)}(z_2)\bar{x}\|} \right) \right| \leq C_2 \|\wedge^p T_\omega^{(n)}(z_1) - \wedge^p T_\omega^{(n)}(z_2)\|.$$

Lemma 2.8 ensures the existence of a constant $C_4 > 0$ such that: $\|\wedge^p T_\omega^{(n)}(z_1) - \wedge^p T_\omega^{(n)}(z_2)\| \leq C_4 |z_1 - z_2|$. Since C_2 and C_4 do not depend on \bar{x} , we get the second point.

For the third point, it suffices to combine the two previous points. Let $\epsilon \in (0, 1)$ and take (z_1, \bar{x}_1) and (z_2, \bar{x}_2) close enough to have:

$$\begin{aligned} |\Phi(z_1, \bar{x}_1) - \Phi(z_2, \bar{x}_2)| &\leq |\Phi(z_1, \bar{x}_1) - \Phi(z_2, \bar{x}_1)| + |\Phi(z_2, \bar{x}_1) - \Phi(z_2, \bar{x}_2)| \\ &\leq C_2 C_4 |z_1 - z_2| + \epsilon \leq C\epsilon. \end{aligned}$$

This completes the proof. □

Using the inclusion (2.8) and the applications defined at (2.12) and (2.14) one gets, following the strategy of [6], that the group $C^*G_{\mu_z}C$ is p -contracting and L_p -strongly irreducible for every $p \in \{1, \dots, L\}$ and every $z \in \mathbb{C} \setminus \{0\}$ since $C^*U(L, L)C$ is. This implies, for each $p \in \{1, \dots, 2L\}$, (up to the use of the application π in order to apply the classical results in the real setting [24]) the existence of a unique measure $\nu_{p,z}$ on $P(CL_p C^*) \subset P(\mathbb{C}^{2L})$ which is μ_z -invariant and such that :

$$\forall z \in \mathbb{C} \setminus \{0\}, \gamma_1(z) + \dots + \gamma_p(z) = \int_{G_{\mu_z} \times P(\mathbb{C}^{2L})} \ln \frac{\|\wedge^p M\bar{x}\|}{\|\bar{x}\|} d\mu_z(M) d\nu_{p,z}(\bar{x}). \tag{2.21}$$

Following the proof of [26] or [25], one deduce from the integral representation (2.21), from Lemma 2.9 and the use of Banach–Alaoglu’s theorem the continuity of the sums of the Lyapunov exponents hence of the Lyapunov exponents themselves, as functions of the parameter z .

Proposition 2.10 *Let $\epsilon \in (0, 1)$ and $p \in \{1, \dots, 2L\}$. The function $z \mapsto \gamma_p(z)$ is continuous on $\mathbb{C} \setminus \{0\}$.*

We deduce from this continuity result, together with (2.10), a result of positivity of the Lyapunov exponents on some annulus.

Corollary 2.11 *There exists $\epsilon_0 \in (0, 1)$ such that for every $\epsilon \in (0, \epsilon_0]$, the L first Lyapunov exponents are positive on \mathbb{S}_ϵ .*

2.4 Exponential decay of transfer matrices

The positivity of the Lyapunov exponents does not imply directly the exponential decay of the eigenfunctions of the operators \mathbb{U}_ω , as discussed for example in [22], but it already implies the exponential decay of some power of the inverse of products of transfer matrices.

Lemma 2.12 *Let $K \subset \mathbb{C}$ a compact set. There exist constants $C > 0$, $c > 0$ and $s \in (0, 1)$, such that:*

$$\mathbb{E} \left(\left\| \left(T_\omega^{(m)}(z) \dots T_\omega^{(n)}(z) \right)^{-1} v \right\|^s \right) \leq C e^{-c|n-m|} \tag{2.22}$$

for all $z \in K$, for all unit vector v , and for $n, m \in \mathbb{Z}$ such that $|n - m|$ is sufficiently large.

Proof One can follow the proof of [11, Lemma 7.1] or [27, Lemma 3]. It suffices to use (2.11) and the fact that

$$\lim_{|n-m| \rightarrow \infty} \frac{1}{|n - m|} \mathbb{E} \left(\ln \left\| \left(T_\omega^{(m)}(z) \dots T_\omega^{(n)}(z) \right)^{-1} v \right\| \right) = -\gamma_L(z) < 0$$

since for any $A \in GL_L(\mathbb{C})$, the Hilbert–Schmidt norm of its inverse satisfies $\|A^{-1}\|_{HS} = (\sigma_L(A))^{-1}$ where $\sigma_L(A)$ is the smallest singular value of A . \square

3 Exponential decay of the fractional moments of the green’s function

In this Section we provide a proof of the exponential decay of fractional moments of the Green’s matrix, which is the result of Theorem 1.9. We will begin our analysis by reducing the problem to the proof of the exponential decay of some particular coefficients of the Green kernel, following the approaches described in previous works such as [6, 11]. Then, we will adopt a new strategy to establish exponential decay in the reduced case.

The usual scheme to establish exponential decay of fractional moments is to find an explicit expression of the Green’s function in terms of the solutions of the equation $(\mathbb{U}_\omega - z)\phi = 0$ as in [11]. In our case, it is difficult to use this approach due to the

matrix structure of the model which implies a formula (see [12]) whose terms are difficult to control. Therefore, we proceed differently, in two steps: first, we bound the norm of the Green’s function by multiplying it by a non-constant quantity and the norm of the inverses of the transfer matrices which, as we have already established, decay exponentially. Then, we demonstrate that the first quantity is bounded, under a condition on the norm of the matrix α .

3.1 Finite scattering zippers

In this section, we start by explicitly compute the even–odd truncation for the scattering zipper operator. The importance of this step lies in the fact that truncating a unitary operator can lead to a non-unitary operator if the truncation is not done well. Let us first denote explicitly the coefficients of the scattering matrices where we omit the random parameter ω in order to simplify the latter computations.

Let $\omega \in \Omega$. Denote $S_\omega^{(n)} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \in U(2L)_{\text{inv}}$ and take $U, V \in U(L)$. We define the restriction $\mathbb{U}_\omega^{[2n, 2m+1]}$ of \mathbb{U}_ω to an interval $[2n, 2m + 1]$ with $n \leq m$ two elements of \mathbb{Z} .

We construct $\mathbb{V}_\omega^{[2n, 2m+1]}$ and $\mathbb{W}_\omega^{[2n, 2m+1]}$ as follows:

$$\mathbb{V}_\omega^{[2n, 2m+1]} := \bigoplus_{k \in \{n, \dots, m\}} S_\omega^{(2k)} \tag{3.1}$$

$$\mathbb{W}_\omega^{[2n, 2m+1]} := U \oplus \bigoplus_{k \in \{n, \dots, m-1\}} S_\omega^{(2k+1)} \oplus V \tag{3.2}$$

The boundary conditions U and V are placed in \mathbb{W}_ω in such a way that we obtain exactly $2m + 1 - 2n + 1$ L -rows and L -columns starting from $2n$. Thus, $\mathbb{U}_\omega^{[2n, 2m+1]}$ is given by:

$$\begin{matrix} 2n \\ 2n+1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 2m \\ 2m+1 \end{matrix} \begin{pmatrix} S_\omega^{(2n)} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ & 0 & 0 & \cdots & \cdots & 0 & 0 \\ & 0 & 0 & S_\omega^{(2n+2)} & \cdots & \cdots & 0 & 0 \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & S_\omega^{(2m-2)} & 0 & 0 \\ & 0 & 0 & \cdots & \cdots & 0 & 0 & S_\omega^{(2m)} \\ 0 & 0 & \cdots & \cdots & 0 & 0 & S_\omega^{(2m-1)} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & V \end{pmatrix}$$

Since we will mostly deal with the resolvent of $\mathbb{U}_\omega^{[2n, 2m+1]}$, we also give the expression of $\mathbb{U}_\omega^{[2n, 2m+1]} - z\text{Id}$ for $z \in \mathbb{C}$.

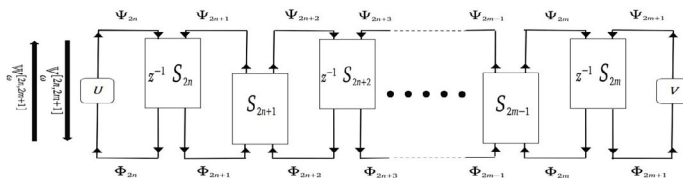
$$\mathbb{U}_\omega^{[2n, 2m+1]} - z\text{Id}$$

$$= \begin{pmatrix} \alpha_{2n}U - z & \beta_{2n}\alpha_{2n+1} & \beta_{2n}\beta_{2n+1} & 0 & \dots & \dots & \dots & 0 \\ \gamma_{2n}U & \delta_{2n}\alpha_{2n+1} - z & \delta_{2n}\beta_{2n+1} & 0 & \dots & \dots & \dots & 0 \\ 0 & \alpha_{2n+2}\gamma_{2n+1} & \alpha_{2n+2}\delta_{2n+1} - z & \beta_{2n+2}\alpha_{2n+3} & \beta_{2n+2}\beta_{2n+3} & 0 & \dots & 0 \\ 0 & \gamma_{2n+2}\gamma_{2n+1} & \gamma_{2n+2}\delta_{2n+1} & \delta_{2n+2}\alpha_{2n+3} - z & \delta_{2n+2}\beta_{2n+3} & 0 & \dots & 0 \\ 0 & 0 & 0 & \alpha_{2n+4}\gamma_{2n+3} & \alpha_{2n+4}\delta_{2n+3} - z & ** & \dots & 0 \\ 0 & 0 & 0 & \gamma_{2n+4}\gamma_{2n+3} & \gamma_{2n+4}\delta_{2n+3} & ** & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \alpha_{2m-2}\gamma_{2m-3} & \alpha_{2m-2}\delta_{2m-3} - z & \beta_{2m-2}\alpha_{2m-1} & \beta_{2m-2}\beta_{2m-1} & 0 \\ 0 & \dots & 0 & \gamma_{2m-2}\gamma_{2m-3} & \gamma_{2m-2}\delta_{2m-3} & \delta_{2m-2}\alpha_{2m-1} - z & \delta_{2m-2}\beta_{2m-1} & 0 \\ 0 & \dots & 0 & 0 & 0 & \alpha_{2m}\gamma_{2m-1} & \alpha_{2m}\delta_{2m-1} - z & \beta_{2m}V \\ 0 & \dots & 0 & 0 & 0 & \gamma_{2m}\gamma_{2m-1} & \gamma_{2m}\delta_{2m-1} & \delta_{2m}V - z \end{pmatrix}$$

Let $z \in \mathbb{C}$ and $\Phi \in \ell^2([2n, 2m+1], \mathbb{C}^L)$ satisfying $\mathbb{U}_\omega^{[2n, 2m+1]} \Phi = z\Phi$. If we define $\Psi = \mathbb{W}_\omega^{[2n, 2m+1]} \Phi$, then $\mathbb{V}_\omega^{[2n, 2m+1]} \Psi = z\Phi$, which leads to $\Phi = z^{-1} \mathbb{V}_\omega^{[2n, 2m+1]} \Psi$. In other words, in the formalism of scattering matrices (or S-matrices), we have:

$$\begin{aligned} \Psi_{2n} &= U \Phi_{2n}, \quad \begin{pmatrix} \Phi_{2n} \\ \Phi_{2n+1} \end{pmatrix} = z^{-1} S_\omega^{(2n)} \begin{pmatrix} \Psi_{2n} \\ \Psi_{2n+1} \end{pmatrix}, \quad \begin{pmatrix} \Psi_{2n+1} \\ \Psi_{2n+2} \end{pmatrix} \\ &= S_\omega^{(2n+1)} \begin{pmatrix} \Phi_{2n+1} \\ \Phi_{2n+2} \end{pmatrix}, \dots \dots \dots, \quad \begin{pmatrix} \Phi_{2m-2} \\ \Phi_{2m-1} \end{pmatrix} \\ &= z^{-1} S_\omega^{(2m-2)} \begin{pmatrix} \Psi_{2m-2} \\ \Psi_{2m-1} \end{pmatrix}, \\ \begin{pmatrix} \Psi_{2m-1} \\ \Psi_{2m} \end{pmatrix} &= S_\omega^{(2m-1)} \begin{pmatrix} \Phi_{2m-1} \\ \Phi_{2m} \end{pmatrix}, \quad V \Phi_{2m+1} = \Psi_{2m+1}. \end{aligned}$$

This yields the following diagram:



Other cases of intervals $[2n, 2m]$, $[2n+1, 2m]$ and $[2n+1, 2m+1]$ are defined in the same manner. The cases where $a = -\infty$ or $b = +\infty$ are similar since we have only to put boundary conditions on one side of the interval.

3.2 Proof of Theorem 1.4

The estimate of the fractional moments of the Green's function has played a prominent role in the proofs of localization for the self-adjoint Anderson model, notably illustrated in the works [28–30]. Our approach to evaluate these moments begins by establishing an upper bound for their expectations. We will draw inspiration from the method presented by Hamza in [26], while adjusting the proofs to adapt the argument with the quasi-one-dimensional character of $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$. This upper bound will subse-

quently serve as a key tool to prove the exponential decay of the fractional moments of the resolvent.

Proof of Theorem 1.4 Let $s \in (0, \frac{1}{4})$, $\omega \in \Omega$ and $z \in \mathbb{C} \setminus \mathbb{S}^1$. Let $\epsilon \in (0, 1)$. For $|z| \leq \epsilon$, the bound (1.17) is clear. Hence, one may assume that $|z| \geq \epsilon$, $|z| \neq 1$. Then, since $(U_\omega - z)^{-1} = \frac{1}{2z} [(U_\omega + z)(U_\omega - z)^{-1} - \text{Id}]$, there exists $\tilde{C}(s) > 0$ such that

$$\forall k, l \in \mathbb{Z}, \mathbb{E} (\|G_\omega(z, k, l)\|^s) \leq \tilde{C}(s) \left(\mathbb{E} \left(\left\| (U_\omega + z)(U_\omega - z)^{-1}(k, l) \right\|^s \right) + \mathbb{I}_L \right)$$

Therefore, to prove (1.17), it suffices to prove the existence of $\widehat{C}(s) > 0$ such that

$$\forall k, l \in \mathbb{Z}, \mathbb{E} \left(\left\| (U_\omega + z)(U_\omega - z)^{-1}(k, l) \right\|^s \right) \leq \widehat{C}(s).$$

We use a finite-rank perturbation method to establish such a bound on the modified resolvent. Let $n \in \mathbb{Z}$. We use the notations introduced in (1.3) and (1.10) which allow us to write:

$$S_\omega^{(2n)} = \begin{pmatrix} \mathbb{I}_L & 0 \\ 0 & \widehat{V}_\omega^{(2n)} e^{i\theta_\omega^{(2n)}} \end{pmatrix} \begin{pmatrix} \alpha & \rho(\alpha)U_\omega^{(2n)} \\ \tilde{\rho}(\alpha) & -\alpha^*U_\omega^{(2n)} \end{pmatrix}.$$

This implies a factorization of U_ω :

$$U_\omega = \mathbb{V}_\omega \mathbb{W}_\omega = \left(\bigoplus_{n \in \mathbb{Z}} \begin{pmatrix} \mathbb{I}_L & 0 \\ 0 & \widehat{V}_\omega^{(2n)} e^{i\theta_\omega^{(2n)}} \end{pmatrix} \begin{pmatrix} \alpha & \rho(\alpha)U_\omega^{(2n)} \\ \tilde{\rho}(\alpha) & -\alpha^*U_\omega^{(2n)} \end{pmatrix} \right) \circ s^L \mathbb{W}_\omega := \mathbb{Y}_\omega \mathbb{D}_\omega \mathbb{V}'_\omega \mathbb{W}_\omega. \tag{3.3}$$

with the unitaries $\mathbb{Y}_\omega = \bigoplus_{n \in \mathbb{Z}} \begin{pmatrix} \mathbb{I}_L & 0 \\ 0 & \widehat{V}_\omega^{(2n)} \end{pmatrix}$, $\mathbb{D}_\omega = \bigoplus_{n \in \mathbb{Z}} \begin{pmatrix} \mathbb{I}_L & 0 \\ 0 & e^{i\theta_\omega^{(2n)}} \end{pmatrix}$ and $\mathbb{V}'_\omega = \bigoplus_{n \in \mathbb{Z}} \begin{pmatrix} \alpha & \rho(\alpha)U_\omega^{(2n)} \\ \tilde{\rho}(\alpha) & -\alpha^*U_\omega^{(2n)} \end{pmatrix} \circ s^L$.

We fix $p \in \{1, \dots, L\}$. For $k \neq l$ even, $A = \{k + 1, l + 1\} \subset \mathbb{Z}$ and $a = \frac{1}{2}(\theta_{\omega,p}^{(k)} + \theta_{\omega,p}^{(l)})$, $b = \frac{1}{2}(\theta_{\omega,p}^{(k)} - \theta_{\omega,p}^{(l)})$, let

$$\eta_{\omega,p}^{(j)} = \begin{cases} a & j \in A \\ 0 & j \notin A \end{cases} \quad \text{and} \quad \xi_{\omega,p}^{(j)} = \begin{cases} b & j = k + 1 \\ -b & j = l + 1 \\ 0 & j \notin A. \end{cases}$$

If one of the k, l is odd, we set $a = b = \frac{1}{2}\theta_{\omega,p}^{(k)}$. Define the diagonal matrices, D_a, D_b and \widehat{D} by:

$$D_a e_{\{j,p\}} = e^{i\eta_{\omega,p}^{(j)}} e_{\{j,p\}}, \quad D_b e_{\{j,p\}} = e^{i\xi_{\omega,p}^{(j)}} e_{\{j,p\}}, \quad \text{and} \\ \widehat{D} e_{\{j,p\}} = \begin{cases} e_{\{j,p\}} & \text{if } j \text{ is even} \\ e^{i\theta_{\omega,p}^{(j)}} e_{\{j,p\}} & \text{if } j \text{ is odd} \end{cases}. \tag{3.4}$$

Using these definitions, we factorize \mathbb{D}_ω as $\mathbb{D}_\omega = D_a D_b \widehat{D}$ and we have:

$$U_\omega = Y_\omega \mathbb{D}_\omega V'_\omega W_\omega = Y_\omega D_a D_b \widehat{D} V'_\omega W_\omega := Y_\omega D_a X_\omega. \tag{3.5}$$

with the unitary operator $X_\omega = D_b \widehat{D} V'_\omega W_\omega$ independent of a . Note that $D_a = Y_\omega^{-1} U_\omega X_\omega^{-1}$ and that in view of the definitions of X_ω and Y_ω , $U_\omega - Y_\omega X_\omega$ is a finite-rank operator.

Let P_A be the orthogonal projection onto the subspace generated by $\{X_\omega^{-1} e_{\{j,p\}} \mid j \in A\}$. Note that $\{X_\omega^{-1} e_{\{j,p\}} \mid j \in \mathbb{Z}, p \in \{1, \dots, L\}\}$ is an orthonormal basis of $l^2(\mathbb{Z}) \otimes \mathbb{C}^L$. One has

$$P_A X_\omega^{-1} Y_\omega^{-1} U_\omega P_A = e^{-ia} I_2. \tag{3.6}$$

Indeed, for $u \in \{X_\omega^{-1} e_{\{j,p\}} : j \in A\}$, there exist real numbers λ and μ such that $u = \lambda X_\omega^{-1} e_{k+1} + \mu X_\omega^{-1} e_{l+1}$. Thus, $X_\omega^{-1} Y_\omega^{-1} U_\omega u = \lambda X_\omega^{-1} D_a e_{k+1} + \mu X_\omega^{-1} D_a e_{l+1} = e^{-ia} (\lambda X_\omega^{-1} e_{k+1} + \mu X_\omega^{-1} e_{l+1}) = e^{-ia} u$. Moreover, $U_\omega - Y_\omega X_\omega = Y_\omega (D_a - \text{Id}) X_\omega = 0$ on $\text{Ran}(\text{Id} - P_A)$ (the vector subspace generated by $\{X_\omega^{-1} e_{\{j,p\}} : j \in \mathbb{Z} \setminus A\}$), since $D_a - \text{Id} = 0$ on $\text{Ran}(\text{Id} - P_A)$. As $(Y_\omega X_\omega)^{-1} U_\omega = e^{-ia} I_2$ on $\text{Ran}(P_A)$,

$$\begin{aligned} U_\omega &= Y_\omega X_\omega (\text{Id} - P_A) + U_\omega P_A = Y_\omega X_\omega (\text{Id} - P_A) + Y_\omega X_\omega (Y_\omega X_\omega)^{-1} U_\omega P_A \\ &= Y_\omega X_\omega (\text{Id} - P_A) + e^{-ia} Y_\omega X_\omega P_A. \end{aligned}$$

For $z \in \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\})$, define

$$F_z = P_A (U_\omega + z) (U_\omega - z)^{-1} P_A, \quad \text{and} \quad \widehat{F}_z = P_A (Y_\omega X_\omega + z) (Y_\omega X_\omega - z)^{-1} P_A. \tag{3.7}$$

One has:

$$\begin{aligned} \widehat{F}_z + \widehat{F}_z^* &= P_A ((Y_\omega X_\omega - z)^{-1})^* \\ &\quad [(Y_\omega X_\omega - z)^* (Y_\omega X_\omega + z) + (Y_\omega X_\omega + z)^* (Y_\omega X_\omega - z)] \\ &\quad (Y_\omega X_\omega - z)^{-1} P_A. \\ &= P_A (2\text{Id} - 2|z|^2) \left((Y_\omega X_\omega - z)^{-1} \right)^* (Y_\omega X_\omega - z)^{-1} P_A. \end{aligned}$$

On the image of P_A , this implies that $\widehat{F}_z + \widehat{F}_z^* < 0$ for $|z| > 1$. Consequently, $-i\widehat{F}_z$ is a dissipative operator. Similarly, $-i\widehat{F}_z^{-1}$ is also a dissipative operator. In the case where $|z| < 1$, we have $i\widehat{F}_z$ and $i\widehat{F}_z^{-1}$ that are dissipative operators.

Next, as in [11, (4.10)], we have

$$F_z - \widehat{F}_z = -2z P_A \left[(Y_\omega X_\omega - z)^{-1} (U_\omega - Y_\omega X_\omega) (U_\omega - z)^{-1} \right] P_A.$$

As $U_\omega - Y_\omega X_\omega$ is zero on $\text{Ran}(\text{Id} - P_A)$ and equals $Y_\omega X_\omega (e^{-ia} - 1) I_2$ on $\text{Ran}(P_A)$, we have:

$$U_\omega - Y_\omega X_\omega = Y_\omega X_\omega (\text{Id} - P_A) + e^{-ia} Y_\omega X_\omega P_A - Y_\omega X_\omega$$

$$\begin{aligned}
 &= \mathbb{Y}_\omega \mathbb{X}_\omega - \mathbb{Y}_\omega \mathbb{X}_\omega P_A + e^{-ia} \mathbb{Y}_\omega \mathbb{X}_\omega P_A - \mathbb{Y}_\omega \mathbb{X}_\omega \\
 &= \mathbb{Y}_\omega \mathbb{X}_\omega \left[e^{-ia} P_A - P_A \right] = \mathbb{Y}_\omega \mathbb{X}_\omega \left[e^{-ia} P_A P_A - P_A P_A \right] \\
 &= \mathbb{Y}_\omega \mathbb{X}_\omega P_A (e^{-ia} - 1) P_A;
 \end{aligned}$$

hence, $F_z - \widehat{F}_z = -2z P_A \left[(\mathbb{Y}_\omega \mathbb{X}_\omega - z)^{-1} (\mathbb{Y}_\omega \mathbb{X}_\omega) P_A (e^{-ia} - 1) P_A (\mathbb{U}_\omega - z)^{-1} \right] P_A$.
 Thus, $F_z - \widehat{F}_z = -P_A (\mathbb{Y}_\omega \mathbb{X}_\omega - z)^{-1} \mathbb{Y}_\omega \mathbb{X}_\omega P_A (e^{-ia} - 1) P_A (2z (\mathbb{U}_\omega - z)^{-1}) P_A$.
 Then, using the equality $2z (\mathbb{U}_\omega - z)^{-1} = (\mathbb{U}_\omega + z) (\mathbb{U}_\omega - z)^{-1} - \text{Id}$, it follows that

$$\begin{aligned}
 F_z - \widehat{F}_z &= P_A (\mathbb{Y}_\omega \mathbb{X}_\omega - z)^{-1} \mathbb{Y}_\omega \mathbb{X}_\omega P_A (e^{-ia} - 1) [F_z - \text{Id}] P_A \\
 &= -P_A [\text{Id} + z (\mathbb{Y}_\omega \mathbb{X}_\omega)] (\mathbb{Y}_\omega \mathbb{X}_\omega - z)^{-1} \mathbb{Y}_\omega \mathbb{X}_\omega P_A (e^{-ia} - 1) [F_z - \text{Id}] P_A \\
 &= P_A \left[\frac{1}{2} (\text{Id} + (\mathbb{Y}_\omega \mathbb{X}_\omega + z) (\mathbb{Y}_\omega \mathbb{X}_\omega - z)^{-1}) P_A (e^{-ia} - 1) [\text{Id} - F_z] \right] P_A \\
 &= P_A \left[\frac{1}{2} (\text{Id} + \widehat{F}_z) (e^{-ia} - 1) [\text{Id} - F_z] \right] P_A.
 \end{aligned}$$

Therefore, $P_A (F_z - \widehat{F}_z) P_A = P_A \left[\frac{1}{2} (\text{Id} + \widehat{F}_z) (e^{-ia} - 1) (\text{Id} - F_z) \right] P_A$, and we can write:

$$F_z - \widehat{F}_z = \frac{1}{2} (\text{Id} + \widehat{F}_z) (e^{-ia} - 1) (\text{Id} - F_z), \tag{3.8}$$

which is equivalent to $F_z \left(\text{Id} + \frac{1}{2} (\text{Id} + \widehat{F}_z) (e^{-ia} - 1) \right) = F_z + \frac{1}{2} (\text{Id} + \widehat{F}_z) (e^{-ia} - 1)$.

$$\text{Hence, } F_z \frac{1}{2} (e^{-ia} - 1) \left(\frac{e^{-ia} + 1}{e^{-ia} - 1} \text{Id} + \widehat{F}_z \right) = \frac{1}{2} (e^{-ia} - 1) \left(\text{Id} + \widehat{F}_z \frac{e^{-ia} + 1}{e^{-ia} - 1} \right).$$

Let us define $m_a := -i \left(\frac{e^{-ia} + 1}{e^{-ia} - 1} \right)$. Then we have:

$$F_z (im_a + \widehat{F}_z) = \text{Id} + im_a \widehat{F}_z. \tag{3.9}$$

Hence, F_z and \widehat{F}_z satisfy the same relation as the one found in the proof of [11, Theorem 3.1] and they have the same dissipative properties. Then, one can follow the proof of [11, Theorem 3.1] for $\tau \equiv \frac{1}{2\pi}$ to finish the proof of (1.17).

Note that if at the beginning of the proof we choose both k and l odd instead of even, we have to write the same proof as in the even case but by writing the factorization (3.3) on the factor \mathbb{W}_ω and replacing $\widehat{V}_\omega^{(2k)}$ by $\widehat{U}_\omega^{(2k)}$ and $\theta_\omega^{(2k)}$ by $\Theta_\omega^{(2k)}$. \square

3.3 Reduction results

In this section, we reduce the problem to structures of odd lengths, and then on those of finite lengths. These reductions imply the need of the invertibility of α . In what follows, we fix $a, b \in \mathbb{Z}$, $|a - b| > 4$.

3.3.1 Reduction to appropriate elements

The goal of this part is to estimate, in expectation, elements of the type $\|G_\omega^{[a,b]}(z, 2n, 2m)\|^s$ and $\|G_\omega^{[a,b]}(z, 2n + 1, 2m)\|^s$ by elements of the type $\|G_\omega^{[a,b]}(z, 2n, 2m + 1)\|^s$ and $\|G_\omega^{[a,b]}(z, 2n + 1, 2m + 1)\|^s$, for which we know explicit expressions. The challenge in proving the reduction to even elements consists of being able to bound the even columns (and rows) of the Green’s matrix by the odd columns (and rows), or more easily, by a sum. We use the notations introduced in Section 3.1. We start with a lemma that allow us to have a useful recurrence relation. We use the representation (1.1) for the scattering matrices $S_\omega^{(n)}$.

Lemma 3.1 *Let $l \in \{a, \dots, b\}$. For every $k \in \{a + 1, \dots, b - 1\}$, $k \neq l$ and every $z \in \mathbb{C} \setminus \mathbb{S}^1$,*

$$[\gamma_k^{-1*} \gamma_{k-1}] G_\omega^{[a,b]}(z, k - 1, l) + \gamma_k^{-1*} [\delta_k + z\alpha_k^*] G_\omega^{[a,b]}(z, k, l) + z G_\omega^{[a,b]}(z, k + 1, l) = 0. \tag{3.10}$$

$$z G_\omega^{[a,b]}(z, k, l) + \beta_k^{-1*} [\alpha_{k+1} + z\delta_k^*] G_\omega^{[a,b]}(z, k + 1, l) + \beta_k^{-1*} \beta_{k+1} G_\omega^{[a,b]}(z, k + 2, l) = 0. \tag{3.11}$$

Similar inequalities are applicable for the columns.

Proof By the definition of the resolvent:

$$[(\mathbb{U}_\omega^{[a,b]} - z\text{Id}) G_\omega^{[a,b]}(z)](k, l) = \begin{cases} \mathbb{I}_L & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$$

Thus, for k even and $l \neq k, k + 1$, we find:

$$[(\mathbb{U}_\omega^{[a,b]} - z\text{Id}) G_\omega^{[a,b]}(z)](k, l) = (\mathbb{U}_\omega^{[a,b]} - z\text{Id})(k, \cdot) \cdot G_\omega^{[a,b]}(z, \cdot, l) = 0$$

This, given the 5-diagonal by blocks structure of the operator, results in:

$$\alpha_k \gamma_{k-1} G_\omega^{[a,b]}(z, k - 1, l) + (\alpha_k \delta_{k-1} - z\mathbb{I}_L) G_\omega^{[a,b]}(z, k, l) + \beta_k \alpha_{k+1} G_\omega^{[a,b]}(z, k + 1, l) + \beta_k \beta_{k+1} G_\omega^{[a,b]}(z, k + 2, l) = 0, \tag{3.12}$$

and similarly for $k + 1$:

$$[(\mathbb{U}_\omega^{[a,b]} - z\text{Id}) \cdot G_\omega^{[a,b]}(z)](k + 1, l) = (\mathbb{U}_\omega^{[a,b]} - z\text{Id})(k + 1, \cdot) \cdot G_\omega^{[a,b]}(z, \cdot, l) = 0.$$

This results in:

$$\gamma_k \gamma_{k-1} G_\omega^{[a,b]}(z, k - 1, l) + \gamma_k \delta_{k-1} G_\omega^{[a,b]}(z, k, l) + (\delta_k \alpha_{k+1} - z\mathbb{I}_L) G_\omega^{[a,b]}(z, k + 1, l) + \delta_k \beta_{k+1} G_\omega^{[a,b]}(z, k + 2, l) = 0 \tag{3.13}$$

Multiplying (3.12) by $\delta_k \beta_k^{-1}$ on the left and taking the difference one gets

$$\begin{aligned} & \left[\delta_k \beta_k^{-1} \alpha_k \gamma_{k-1} - \gamma_k \gamma_{k-1} \right] G_\omega^{[a,b]}(z, k-1, l) + [\delta_k \beta_k^{-1} (\alpha_k \delta - z \mathbb{I}_L) - \gamma_k \delta_{k-1}] G_\omega^{[a,b]} \\ & (z, k, l) + [\delta_k \beta_k^{-1} \beta_k \alpha_{k+1} - (\delta_k \alpha_k - z \mathbb{I}_L)] G_\omega^{[a,b]}(z, k+1, l) \\ & + [\delta_k \beta_k^{-1} \beta_k \beta_{k+1} - \delta_k \beta_{k+1}] G_\omega^{[a,b]}(z, k+2, l) = 0. \end{aligned}$$

After simplification, this yields:

$$\begin{aligned} & \left[(\delta_k \beta_k^{-1} \alpha_k - \gamma_k) \gamma_{k-1} \right] G_\omega^{[a,b]}(z, k-1, l) \\ & + [\delta_k \beta_k^{-1} (\alpha_k \delta_{k-1} - z \mathbb{I}_L) - \gamma_k \delta_{k-1}] G_\omega^{[a,b]}(z, k, l) \\ & + z G_\omega^{[a,b]}(z, k+1, l) = 0. \end{aligned}$$

We have:

$$(\alpha_k \gamma_k^{-1} \delta_k - \beta_k) = (\gamma_k^{-1})^* \text{ and } \beta_k^* \alpha_k = -\delta_k^* \gamma_k \Leftrightarrow \alpha_k \gamma_k^{-1} = -(\beta_k^{-1})^* \delta_k^*.$$

This allows us to write:

$$\left[(\gamma_k^{-1})^* \gamma_{k-1} \right] G_\omega^{[a,b]}(z, k-1, l) + (\gamma_k^{-1})^* [\delta_k + z \alpha_k^*] G_\omega^{[a,b]}(z, k, l) + z G_\omega^{[a,b]}(z, k+1, l) = 0. \tag{3.14}$$

A similar equality can be found by multiplying (3.13) by $\alpha_k \gamma_k^{-1}$ and taking the difference:

$$\begin{aligned} & \left[\alpha_k \gamma_k^{-1} \gamma_k \gamma_{k-1} - \alpha_k \gamma_k \right] G_\omega^{[a,b]}(z, k-1, l) \\ & + [(\alpha_k \gamma_k^{-1} \gamma_k \delta_{k-1}) - (\alpha_k \delta_{k-1} - z \mathbb{I}_L)] G_\omega^{[a,b]}(z, k, l) \\ & + [\alpha_k \gamma_k^{-1} (\delta_k \alpha_{k+1} - z \mathbb{I}_L) - \beta_k \alpha_{k+1}] G_\omega^{[a,b]}(z, k+1, l) \\ & + [\delta_k \gamma_k^{-1} \delta_k \beta_{k+1} - \beta_k \beta_{k+1}] G_\omega^{[a,b]}(z, k+2, l) = 0. \end{aligned}$$

Hence,

$$\begin{aligned} & z G_\omega^{[a,b]}(z, k, l) + [(\alpha_k \gamma_k^{-1} \delta_k - \beta_k) \alpha_{k+1} - z \alpha_k \gamma_k^{-1}] G_\omega^{[a,b]}(z, k+1, l) \\ & + [(\alpha_k \gamma_k^{-1} \delta_k - \beta_k) \beta_{k+1}] G_\omega^{[a,b]}(z, k+2, l) = 0. \end{aligned}$$

For any k , we have:

$$\alpha_k \gamma_k^{-1} \delta_k - \beta_k = (\beta_k^{-1})^* \text{ and } \beta_k^* \alpha_k = -\delta_k^* \gamma_k \Leftrightarrow \alpha_k \gamma_k^{-1} = -(\beta_k^{-1})^* \delta_k^*.$$

This allows us to write:

$$\begin{aligned} & z G_\omega^{[a,b]}(z, k, l) + (\beta_k^{-1})^* [\alpha_{k+1} + z \delta_k^*] G_\omega^{[a,b]}(z, k+1, l) \\ & + (\beta_k^{-1})^* \beta_{k+1} G_\omega^{[a,b]}(z, k+2, l) = 0. \end{aligned} \tag{3.15}$$

To find the equalities for the columns l (while fixing k), it suffices to repeat the same process but start with the multiplication in the reverse direction:

$$\left[G_\omega^{[a,b]}(z)(U_\omega^{[a,b]} - z\text{Id}) \right] (k, l) = \begin{cases} \mathbf{I}_L & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}.$$

□

To exploit these relations, we need to be able to invert $\delta_k + z\alpha_k^*$. This is the subject of the following lemma.

Lemma 3.2 *If $\alpha \in \text{GL}_L(\mathbb{C})$, then for any $\epsilon > 0$, $k \in \mathbb{Z}$ and $z \in \mathbb{S}_\epsilon$, $\delta_k + z\alpha_k^*$ is \mathbb{P} -almost surely invertible.*

Proof Let $z \in \mathbb{S}_\epsilon$ and $k \in \mathbb{Z}$. We define:

$$\begin{aligned} \Omega_{\text{inv}} &= \{ \omega \in \Omega; \delta_k + z\alpha_k^* \in \text{GL}_L(\mathbb{C}) \} \\ &= \{ \omega \in \Omega; \tilde{\rho}^{-1}[zV_\omega^{(k)}\alpha^* - \alpha^*U_\omega^{(k)}] \in \text{GL}_L(\mathbb{C}) \} \\ &= \{ \omega \in \Omega; \det(zV_\omega^{(k)}\alpha^* - \alpha^*U_\omega^{(k)}) \neq 0 \} \end{aligned} \tag{3.16}$$

To show that $\mathbb{P}(\Omega_{\text{inv}}) = 1$, it suffices to prove that there exists $(U_0, V_0) \in \Omega$ such that $\det(zV_0\alpha^* - \alpha^*U_0) \neq 0$, since $U_\omega^{(k)}$ and $V_\omega^{(k)}$ are drawn according to the Haar measure ν_L and since α is invertible.

If $z \neq 1$, we set $(U_0, V_0) = (\mathbf{I}_L, \mathbf{I}_L)$ to get $\det(zV_0\alpha^* - \alpha^*U_0) = (1 - z)^L(-1)^L \det \alpha^* \neq 0$.

If $z = 1$, it suffices to take $(U_0, V_0) = (i\mathbf{I}_L, \mathbf{I}_L)$ to get $\det(zV_0\alpha^* - \alpha^*U_0) = (1 - i)^L \det \alpha^* \neq 0$. □

With Lemma 3.2 one can isolate the term $G_\omega^{[a,b]}(z, k, l)$ in (3.10). We define Ω_{inv} as in (3.16). For $\omega \in \Omega_{\text{inv}}$, the equality (3.10) becomes:

$$\begin{aligned} G_\omega^{[a,b]}(z, k, l) &= \left(\gamma_k^{-1*} [\delta_k + z\alpha_k^*] \right)^{-1} \left[\gamma_k^{-1*} \gamma_{k-1} G_\omega^{[a,b]}(z, k - 1, l) + z G_\omega^{[a,b]}(z, k + 1, l) \right]. \\ &= [\delta_k + z\alpha_k^*]^{-1} \left[\gamma_{k-1} G_\omega^{[a,b]}(z, k - 1, l) + z \gamma_k^{-1*} G_\omega^{[a,b]}(z, k + 1, l) \right]. \end{aligned} \tag{3.17}$$

Hence, to estimate the expectancy $\|G_\omega^{[a,b]}(z, k, l)\|^s$ in terms of $G_\omega^{[a,b]}(z, k - 1, l)$ and $G_\omega^{[a,b]}(z, k + 1, l)$, one needs to estimate the expectancy of $\|(\delta_k + z\alpha_k^*)^{-1}\|^s$. This is exactly Lemma 1.7.

Proof of Lemma 1.7 One has:

$$\begin{aligned} &\mathbb{E} \left[\|(\delta_k + z\alpha_k^*)^{-1}\|^s \right] \\ &= \int_\Omega \|(-zV_\omega^{(k)}\alpha U_\omega^{(k)} + \alpha)^{-1}\|^s d\omega = \int_\Omega \frac{1}{(\sigma_L(\alpha - zV_\omega^{(k)}\alpha U_\omega^{(k)}))^s} d\omega \end{aligned}$$

Here, $\sigma_L(\cdot)$ represents the L^{th} singular value, which is the smallest singular value of the considered matrix. Then,

$$\forall k \in \mathbb{Z}, \sigma_L \left(\alpha - zV_\omega^{(k)}\alpha U_\omega^{(k)} \right) = \sigma_L \left(\alpha - z\widehat{V}_\omega^{(k)} e^{i\theta_\omega^{(k)}} \alpha e^{i\Theta_\omega^{(k)}} \widehat{U}_\omega^{(k)} \right).$$

We can simplify some expressions. Indeed, we know that $\theta_\omega^{(k)}$, $\Theta_\omega^{(k)}$, $\widehat{U}_\omega^{(k)}$ and $\widehat{V}_\omega^{(k)}$ are mutually independent. Furthermore, $\theta_\omega^{(k)} = \text{diag}(\theta_{\omega,1}^{(k)}, \dots, \theta_{\omega,L}^{(k)})$ and the $(\theta_{\omega,j}^{(k)})_{j \in \{1, \dots, L\}}$ are i.i.d. The same goes for $(\Theta_{\omega,j}^{(k)})_{j \in \{1, \dots, L\}}$. Consequently, we can replace $\theta_\omega^{(k)}$ with $\theta_{\omega,1}^{(k)} I_L$ and $\Theta_\omega^{(k)}$ with $\Theta_{\omega,1}^{(k)} I_L$ in the partial expectancy with respect to θ and Θ (and not deterministically!), which gives:

$$\begin{aligned} & \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\alpha - z\widehat{V}_\omega^{(k)} e^{i\theta_\omega^{(k)}} \alpha e^{i\Theta_\omega^{(k)}} \widehat{U}_\omega^{(k)} \right)^{-s} \right) \\ &= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\alpha - z\widehat{V}_\omega^{(k)} e^{i\theta_{\omega,1}^{(k)}} \alpha e^{i\Theta_{\omega,1}^{(k)}} \widehat{U}_\omega^{(k)} \right)^{-s} \right) \\ &= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\alpha - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} \widehat{V}_\omega^{(k)} \alpha \widehat{U}_\omega^{(k)} \right)^{-s} \right). \end{aligned} \tag{3.18}$$

Since $\widehat{U}_\omega^{(k)}$ and $\widehat{V}_\omega^{(k)}$ are in $U(L) \cap H_L(\mathbb{C})$, they have a square root which is also an element of $U(L) \cap H_L(\mathbb{C})$. Hence, one writes $\widehat{U}_\omega^{(k)} = (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} (\widehat{U}_\omega^{(k)})^{\frac{1}{2}}$ and $\widehat{V}_\omega^{(k)} = (\widehat{V}_\omega^{(k)})^{\frac{1}{2}} (\widehat{V}_\omega^{(k)})^{\frac{1}{2}}$ to obtain:

$$\begin{aligned} & \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\alpha - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} \widehat{V}_\omega^{(k)} \alpha \widehat{U}_\omega^{(k)} \right)^{-s} \right) \\ &= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\alpha - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} (\widehat{V}_\omega^{(k)})^{\frac{1}{2}} (\widehat{V}_\omega^{(k)})^{\frac{1}{2}} \alpha (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} \right)^{-s} \right) \\ &= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\left((\widehat{V}_\omega^{(k)})^{\frac{1}{2}} \right)^* \alpha \left((\widehat{U}_\omega^{(k)})^{\frac{1}{2}} \right)^* - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} (\widehat{V}_\omega^{(k)})^{\frac{1}{2}} \alpha (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} \right)^{-s} \right). \end{aligned}$$

Since $(\widehat{U}_\omega^{(k)})^{\frac{1}{2}}$ and $(\widehat{V}_\omega^{(k)})^{\frac{1}{2}}$ belong in particular to $H_L(\mathbb{C})$, we find that

$$\begin{aligned} & \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\alpha - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} \widehat{V}_\omega^{(k)} \alpha \widehat{U}_\omega^{(k)} \right)^{-s} \right) \\ &= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left(\left((\widehat{V}_\omega^{(k)})^{\frac{1}{2}} \alpha (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} (\widehat{V}_\omega^{(k)})^{\frac{1}{2}} \alpha (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} \right)^{-s} \right) \right) \end{aligned} \tag{3.19}$$

$$= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left((\widehat{V}_\omega^{(k)})^{\frac{1}{2}} \left(\alpha - z e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} \alpha \right) (\widehat{U}_\omega^{(k)})^{\frac{1}{2}} \right)^{-s} \right) \tag{3.20}$$

$$= \mathbb{E}_{\theta, \Theta} \left(\sigma_L \left((1 - ze^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} \alpha) \right)^{-s} \right).$$

Furthermore, for every $\omega \in \Omega$, every $z \in \mathbb{C} \setminus \mathbb{S}^1$ and every $k \in \mathbb{Z}$,

$$\sigma_L \left((1 - ze^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})} \alpha) \right) = |1 - ze^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})}| \sigma_L(\alpha).$$

Therefore, we obtain the inequality for the partial expectation:

$$\begin{aligned} \mathbb{E}_{\theta, \Theta} [\|(\delta_k + z\alpha_k^*)^{-1}\|^s] &\leq \int_{[0, 2\pi] \times [0, 2\pi]} \frac{1}{|1 - ze^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})}|^s (\sigma_L(\alpha))^s} d(\theta, \Theta) \\ &= \frac{1}{(\sigma_L(\alpha))^s} \int_0^{2\pi} \frac{1}{|1 - ze^{i\tilde{\theta}}|^s} d\tilde{\theta}, \end{aligned}$$

since $e^{i(\Theta_{\omega,1}^{(k)} + \theta_{\omega,1}^{(k)})}$ follows the uniform law on \mathbb{S}^1 . According to [26, Lemma A.2], for any $z \in \mathbb{C} \setminus \mathbb{S}^1$, $\int_0^{2\pi} \frac{1}{|1 - ze^{i\theta}|^s} d\theta = C(s) < \infty$. Hence, by taking the expectation over $U(L) \times \{-1, 1\}^L \times U(L) \times \{-1, 1\}^L$ of this partial expectation, we obtain the desired result. \square

Remark 3.3 As explained in Remark 1.2, this is for the computations done in (3.19) and (3.20) to be valid that we had to decompose the random unitary phases in a product with a part which is explicitly unitary and Hermitian at the same time. One could imagine that this decomposition could be made for any unitary matrix, and therefore, we could have stick with the definition of the random scattering zipper as in [2]. But if we had done it this way, we would have lost the independancy of the different random terms in the definition of the phases $U_\omega^{(n)}$ and $V_\omega^{(n)}$. Without this independancy, (3.18) could be not true anymore.

Combining the results of Lemma 3.1 and Lemma 1.7, we get the lemma of reduction to the even case.

Lemma 3.4 For $\alpha \in \text{GL}_L(\mathbb{C})$ and $s \in (0, \frac{1}{2})$, there exist $C_1(s), C_2(s) > 0$ such that,

$$\begin{aligned} \mathbb{E} \left(\|G_\omega^{[a,b]}(z, k + 1, l)\|^s \right)^2 &\leq C_1(s) \mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l)\|^{2s} \right) \\ &\quad + C_2(s) \mathbb{E} \left(\|G_\omega^{[a,b]}(z, k + 2, l)\|^{2s} \right). \end{aligned} \tag{3.21}$$

Similarly, for the columns, there exist $\tilde{C}_1(s), \tilde{C}_2(s) > 0$ such that

$$\begin{aligned} \mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l + 1)\|^s \right)^2 &\leq \tilde{C}_1(s) \mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l)\|^{2s} \right) \\ &\quad + \tilde{C}_2(s) \mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l + 2)\|^{2s} \right). \end{aligned} \tag{3.22}$$

Proof For $\omega \in \Omega_{\text{inv}}$, equality (3.10) becomes:

$$\begin{aligned} G_\omega^{[a,b]}(z, k, l) &= \left((\gamma_k^{-1})^* [\delta_k + z\alpha_k^*] \right)^{-1} \\ &\quad \left[(\gamma_k^{-1})^* \gamma_{k-1} G_\omega^{[a,b]}(z, k-1, l) + z G_\omega^{[a,b]}(z, k+1, l) \right]. \\ &= [\delta_k + z\alpha_k^*]^{-1} \left[\gamma_{k-1} G_\omega^{[a,b]}(z, k-1, l) + z (\gamma_k^{-1})^* G_\omega^{[a,b]}(z, k+1, l) \right]. \end{aligned}$$

This gives:

$$\begin{aligned} &\|G_\omega^{[a,b]}(z, k, l)\| \\ &\leq \|(\delta_k + z\alpha_k^*)^{-1}\| \left(\|\gamma_{k-1} G_\omega^{[a,b]}(z, k-1, l)\| + |z| \|\gamma_k^* G_\omega^{[a,b]}(z, k+1, l)\| \right). \end{aligned}$$

Now, we just need to use Hölder’s inequality, Lemma 1.7 and Jensen’s inequality as in [26, Lemma 7.1] to conclude. □

Lemma 3.4 implies the first reduction result.

Proof of Proposition 1.6 The statement of Proposition 1.6 is just a combination of (3.21) and (3.22). □

3.3.2 Reduction to a suitable finite interval

We will now prove that it suffices to bound the norm of the finite Green’s function moments for a restriction to a suitable finite interval to bound the Green’s function on the infinite volume. This reduction is based on comparing the blocks of the infinite matrix with those of the finite matrix. Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$, $a < b$.

Proposition 3.5 For $s \in (0, \frac{1}{2})$ and $\epsilon \in (0, 1)$, there exists $C(s, \epsilon) > 0$ such that

$$\mathbb{E} \left(\|G_\omega^{[a,b]}(z, k, l)\|^s \right)^2 \leq C(s, \epsilon) \mathbb{E} \left(\|G_\omega^{[k,l]}(z, k, l)^{2s} \| \right) \tag{3.23}$$

for every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and all $k, l \in [a + 1, b - 1]$, $|k - l| > 4$.

Proof We only deal with the case $k = 2n, l = 2m$ where $|k - l| \geq 4$ and $m, n \in \mathbb{Z}$, the other cases being treated the same way. Using the definition of $\mathbb{U}_\omega^{[a,b]}$, we have

$$\mathbb{U}_\omega^{[a,b]} = \mathbb{U}_\omega^{[a,2n-1]} \oplus \mathbb{U}_\omega^{[2n,b]} + \Gamma_n^e \tag{3.24}$$

where Γ_n^e is given by

$$\Gamma_n^e(k, l) = \begin{cases} -\beta_{2n-2}\alpha + \beta_{2n-2}V, & k = 2n - 2, l = 2n - 1 \\ -\beta_{2n-2}\beta_{2n-1}, & k = 2n - 2, l = 2n \\ -\delta_{2n-2}\alpha + \delta_{2n-2}V, & k = 2n - 1, l = 2n - 1 \\ -\delta_{2n-2}\beta_{2n-1}, & k = 2n - 1, l = 2n \\ -\alpha_{2n+2}\gamma_{2n+1}, & k = 2n, l = 2n - 1 \\ -\alpha_{2n+2}\delta_{2n+1} + \alpha_{2n+2}U, & k = 2n, l = 2n \\ -\gamma_{2n+2}\gamma_{2n+1}, & k = 2n + 1, l = 2n - 1 \\ -\gamma_{2n+2}\delta_{2n+1} + \gamma_{2n+2}U, & k = 2n + 1, l = 2n \\ 0, & \text{otherwise.} \end{cases}$$

Let $G_\omega^n(z) = G_\omega^{[a, 2n-1]}(z) \oplus G_\omega^{[2n, b]}(z)$. By the first resolvent identity, we have

$$G_\omega^{[a, b]}(z) - G_\omega^n(z) = -G_\omega^{[a, b]}(z)\Gamma_n^e G_\omega^n(z).$$

Consequently, for every $m \geq n + 2$,

$$\begin{aligned} G_\omega^{[a, b]}(z, 2n, 2m) &= \left[\mathbb{I}_L + (\beta_{2n-2}\beta_{2n-1})G_\omega^{[a, b]}(z, 2n, 2n - 2) \right. \\ &\quad + (\alpha_{2n+2}\gamma_{2n+1})G_\omega^{[a, b]}(z, 2n, 2n - 1) \\ &\quad + (\alpha_{2n+2}\delta_{2n+1} + \alpha_{2n+2}U)G_\omega^{[a, b]}(z, 2n, 2n) \\ &\quad \left. + (\gamma_{2n+2}\delta_{2n+1} + \gamma_{2n+2}U)G_\omega^{[a, b]}(z, 2n, 2n + 1) \right] \cdot G_\omega^{[2n, b]}(z, 2n, 2m). \end{aligned} \tag{3.25}$$

Similarly, we have $\mathbb{U}_\omega^{[2n, b]} = \mathbb{U}_\omega^{[2n, 2m]} \oplus \mathbb{U}_\omega^{[2m+1, b]} + \Gamma_m^o$ where Γ_m^o is given by

$$\Gamma_m^o(k, l) = \begin{cases} -\alpha_{2m-2}\gamma_{2m-1} + V\gamma_{2m-1}, & k = 2m, l = 2m - 1 \\ -\alpha_{2m-2}\delta_{2m-1} + V\delta_{2m-1}, & k = 2m, l = 2m \\ -\beta_{2m}\alpha_{2m+1}, & k = 2m, l = 2m + 1 \\ -\beta_{2m}\beta_{2m+2}, & k = 2m, l = 2m + 2 \\ -\gamma_{2m+1}\gamma_{2m-1}, & k = 2m + 1, l = 2m - 1 \\ -\gamma_{2m+1}\delta_{2m} + U\alpha_{2m+1}, & k = 2m + 1, l = 2m \\ -\delta_{2m+1}\alpha_{2m+1} + U\alpha_{2m+1}, & k = 2m + 1, l = 2m + 1 \\ -\delta_{2m+1}\beta_{2m+2} + U\beta_{2m+2}, & k = 2m + 1, l = 2m + 2 \\ 0, & \text{otherwise.} \end{cases}$$

Now, if we define $G_\omega^m(z) = G_\omega^{[2n, 2m]}(z) \oplus G_\omega^{[2m+1, b]}(z)$, once again, we observe that

$$G_\omega^{[2n, b]}(z) - G_\omega^m(z) = -G_\omega^m(z)\Gamma_m^o G_\omega^{[2n, b]}(z).$$

Thus, for any $m \geq n + 2$

$$\begin{aligned} G_\omega^{[2n, b]}(z, 2n, 2m) &= \left[\mathbb{I}_L - [-(\alpha_{2m-2}\gamma_{2m-1} + V\gamma_{2m-1})G_\omega^{[2n, b]}(z, 2m - 1, 2m) \right. \\ &\quad \left. - (\alpha_{2m-2}\delta_{2m-1} + V\delta_{2m-1})G_\omega^{[2n, b]}(z, 2m, 2m) \right] \end{aligned}$$

$$\begin{aligned}
 & - (\gamma_{2m+1}\delta_{2m} + \alpha_{2n+2}U)G_\omega^{[2n,b]}(z, 2m + 1, 2m) \\
 & - (\beta_{2m}\beta_{2m+2})G_\omega^{[2n,b]}(z, 2m + 2, 2m) \Big] G_\omega^{[2n,2m]}(z, 2n, 2m)
 \end{aligned} \tag{3.26}$$

Inserting (3.26) into equation (3.25), then applying Hölder’s inequality and Theorem 1.4, we obtain for all $s \in (0, \frac{1}{2})$ and $\epsilon \in (0, 1)$,

$$\left(\mathbb{E} \left(\left\| G_\omega^{[a,b]}(z, 2n, 2m) \right\|^s \right) \right)^2 \leq C(s, \epsilon) \mathbb{E} \left(\left\| G_z^{[2n,2m]}(z, 2n, 2m + 1) \right\|^{2s} \right).$$

This yields the required result for all n, m such that $m \geq n + 2$. The proof is analogous in the case where $n \geq m + 2$. □

Lemma 1.8 is now a direct application of Proposition 3.5 with $a = -\infty$ and $b = +\infty$. Note that one can take for example $\epsilon = \frac{1}{2}$ to obtain in (3.23) a constant $C(s)$ independent in ϵ as in [11].

3.4 Exponential decay of the reduced case

Now that we have reduce our study to the fractional moments of elements of the form $G_\omega^{[2n,2m+1]}(z, 2n, 2m + 1)$, we are ready to show that their expectations decrease exponentially, which is the statement of Theorem 1.5.

3.4.1 An explicit expression of the Green’s function in terms of transfer matrices

Given the matching between the blocks dimensions in the Green’s matrix and those in the transfer matrices, we formulate an expression of the Green’s function. This expression relies directly on the dimensions of the blocks involved in the product of transfer matrices.

Inspired by the notations in [1], for $m, n \in \mathbb{Z}, n \leq m$, we define:

$$T_{2n}^{2m+1}(z) := T_\omega^{(2m+1)}(z) \cdots T_\omega^{(2n)}(z) = \begin{pmatrix} A_{2n}^{2m+1}(z) & B_{2n}^{2m+1}(z)U \\ C_{2n}^{2m+1}(z) & D_{2n}^{2m+1}(z)U \end{pmatrix} \tag{3.27}$$

where $A_{2n}^{2m+1}(z)$ denote the left upper $L \times L$ block of the product of transfer matrices and the same for the three other blocks.

Note that we remove the ω dependency from the various quantities since the random character plays no role in the following discussion.

Lemma 3.6 *For any $\epsilon \in (0, 1)$, any $z \in \mathbb{S}_\epsilon$, any $U, V \in U(L)$ and any $m, n \in \mathbb{Z}$ such that $|m - n| > 2$:*

1. *the following quantities are invertible:*

$$(a) \ C_{2n}^{2m+1}(z) \pm VA_{2n}^{2m+1}(z), \quad (b) \ B_{2n}^{2m+1}(z)U \pm A_{2n}^{2m+1}(z)$$

$$(c) D_{2n}^{2m+1}(z)U \pm C_{2n}^{2m+1}(z).$$

2. The following quantities are inside the Siegel disk $\mathbb{D}_L = \{Z \in \mathcal{M}_L(\mathbb{C}) \mid ZZ^* < 1\}$:

- (a) $(C_{2n}^{2m+1}(z) - VA_{2n}^{2m+1}(z))^{-1}(VB_{2n}^{2m+1}(z) - D_{2n}^{2m+1}(z)),$
- (b) $(D_{2n}^{2m+1}(z)U - C_{2n}^{2m+1}(z))(B_{2n}^{2m+1}(z)U - A_{2n}^{2m+1}(z))^{-1},$
- (c) $(D_{2n}^{2m+1}(z)U + C_{2n}^{2m+1}(z))(B_{2n}^{2m+1}(z)U + A_{2n}^{2m+1}(z))^{-1}.$

Proof 1(a) and 2(a) follow directly from Theorem 1 in [1]. To prove 1(b) and 2(b), we apply Theorem 1 of [1] to

$$(T_{2n}^{2m+1}(z))^{-1} = \mathcal{L}(T_{2n}^{2m+1}(\frac{1}{z}))^* \mathcal{L} = \begin{pmatrix} (A_{2n}^{2m+1}(\frac{1}{z}))^* & -(C_{2n}^{2m+1}(\frac{1}{z}))^* \\ -(B_{2n}^{2m+1}(\frac{1}{z})U)^* & (D_{2n}^{2m+1}(\frac{1}{z})U)^* \end{pmatrix}.$$

Dropping the $2n$ and $2m + 1$ indices, we find that $-V(A(\frac{1}{z}))^* - (B(\frac{1}{z})U)^*$ is invertible. Therefore, $A(\frac{1}{z})V^* + B(\frac{1}{z})U$ is invertible. Since the mapping $z \mapsto \frac{1}{z}$ is an involution of $\mathbb{C} \setminus \{0\}$, we can deduce that $B(z')U + A(z')V^*$ is invertible for all $z' \in \mathbb{C} \setminus \{0\}$. Since this result holds for any unitary initial conditions U and V , we take $V^* = I_L$ or $V^* = -I_L$ to prove 1(b). Similarly, 2(b) and 2(c) can be shown.

To prove 1(c) we follow the proof of [1, Theorem 1] but with changing the vector $\begin{pmatrix} I_L \\ 0 \end{pmatrix}$ into the vector $\begin{pmatrix} 0 \\ I_L \end{pmatrix}$ in the definition of Φ . Then we apply [1, Lemma 3] with $\Phi^* \mathcal{L} \Phi < 0$ instead of $\Phi^* \mathcal{L} \Phi > 0$ which still implies the invertibility of $DU \pm C$. But since we are in the negative case, $(BU \pm A)(DU \pm C)^{-1}$ is not in \mathbb{D}_L . \square

Lemma 3.7 Let $\epsilon \in (0, 1)$. For all $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and all $m, n \in \mathbb{Z}, |m - n| > 2$,

$$G_\omega^{[2n, 2m+1]}(z, 2n, 2m + 1) = ((C_{2n}^{2m+1} - VA_{2n}^{2m+1}) + (D_{2n}^{2m+1} - VB_{2n}^{2m+1})U)^{-1} \times (L_{2m}^{2m+1} - VK_{2m}^{2m+1}) \tag{3.28}$$

$$G_\omega^{[2n+1, 2m+1]}(z, 2n + 1, 2m + 1) = ((C_{2n+1}^{2m+1} - VA_{2n+1}^{2m+1}) + z(D_{2n+1}^{2m+1} - VB_{2n+1}^{2m+1})U^*)^{-1} (L_{2m}^{2m+1} - VK_{2m}^{2m+1}) \tag{3.29}$$

with

$$K_{2m}^{2m+1} = \frac{1}{z} V_\omega^{(2m+1)} \tilde{\rho}^{-1} V_\omega^{(2m)} \tilde{\rho}^{-1} + V_\omega^{(2m+1)} \tilde{\rho}^{-1} \alpha^* (U_\omega^{(2m)})^* \alpha \tilde{\rho}^{-1} \tag{3.30}$$

$$L_{2m}^{2m+1} = (U_\omega^{(2m+1)})^* \alpha \tilde{\rho}^{-1} V_\omega^{(2m)} \tilde{\rho}^{-1} + z (U_\omega^{(2m+1)})^* \tilde{\rho}^{-1} (U_\omega^{(2m)})^* \alpha \tilde{\rho}^{-1} \tag{3.31}$$

with $U, V \in U(L)$, the two boundary conditions at $2n$ and $2m + 1$ as defined in (3.2).

Proof For (3.28), we know that the quantity $G_\omega^{[2n, 2m+1]}(z, 2n, 2m + 1)$ corresponds to the ϕ_{2n} component of the solution Φ of the equation $(\mathbb{U}_\omega^{[2n, 2m+1]} - z)\Phi = \xi$ for $\xi_k = \delta_{2m+1, k} I_L$. We apply Lemma 4 of [1]:

$$\begin{pmatrix} \phi_{2m+1} \\ \psi_{2m+1} \end{pmatrix} = \begin{pmatrix} \phi_{2m+1} \\ V\phi_{2m+1} \end{pmatrix} = T_{2n}^{2m+1}(z) \begin{pmatrix} I_L & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \phi_{2n} \\ \phi_{2n} \end{pmatrix} + T_{2m}^{2m+1}(z) \begin{pmatrix} -z^{-1} I_L \\ 0 \end{pmatrix}. \tag{3.32}$$

We compute $T_{2m}^{2m+1}(z) \begin{pmatrix} -z^{-1}I_L \\ 0 \end{pmatrix}$ to retrieve the vector $\begin{pmatrix} K_{2m}^{2m+1} \\ L_{2m}^{2m+1} \end{pmatrix}$ given by (3.30) and (3.31) and using the notations introduced at (3.27), (3.32) one gets

$$\begin{pmatrix} \phi_{2m+1} \\ V\phi_{2m+1} \end{pmatrix} = \begin{pmatrix} A_{2n}^{2m+1} & B_{2n}^{2m+1} \\ C_{2n}^{2m+1} & D_{2n}^{2m+1} \end{pmatrix} \begin{pmatrix} I_L & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} \phi_{2n} \\ \phi_{2n} \end{pmatrix} + \begin{pmatrix} K_{2m}^{2m+1} \\ L_{2m}^{2m+1} \end{pmatrix}. \tag{3.33}$$

We multiply the first row by V and calculate the difference of the two lines to find:

$$\phi_{2n} = \left((C_{2n}^{2m+1} - VA_{2n}^{2m+1}) + (D_{2n}^{2m+1} - VB_{2n}^{2m+1})U \right)^{-1} (L_{2m}^{2m+1} - VK_{2m}^{2m+1}). \tag{3.34}$$

For (3.29), we follow the same steps. Note that $U\psi_{2n+1} = z\phi_{2n+1} + \xi_{2n+1} = z\phi_{2n+1}$ implies that $\psi_{2n+1} = zU^*\phi_{2n+1}$. This yields:

$$\begin{pmatrix} \phi_{2m+1} \\ V\phi_{2m+1} \end{pmatrix} = \begin{pmatrix} A_{2n+1}^{2m+1} & B_{2n+1}^{2m+1} \\ C_{2n+1}^{2m+1} & D_{2n+1}^{2m+1} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & zU^* \end{pmatrix} \begin{pmatrix} \phi_{2n+1} \\ \phi_{2n+1} \end{pmatrix} + \begin{pmatrix} K_{2m}^{2m+1} \\ L_{2m}^{2m+1} \end{pmatrix}.$$

We multiply by V and then take the difference of the two lines to find the expression given in the statement. □

Remark 3.8 For the other cases, $G_\omega^{[2n+1,2m]}(z, 2n + 1, 2m)$ and $G_\omega^{[2n,2m]}(z, 2n, 2m)$, the equations $\nabla_\omega\Psi = z\Phi + \xi$ and $\mathbb{W}_\omega\Phi = \Psi$, yield $V\psi_{2m+1} = z\phi_{2m+1} + I_L$ or $\psi_{2m+1} = zV^*\phi_{2m+1} + V^*$. So, we need to multiply the first row by zV^* , add V^* , and take the difference to obtain:

$$G_\omega^{[2n,2m]}(z, 2n, 2m) = [C_{2n}^{2m} - zV^*A_{2n}^{2m} - V^* + (D_{2n}^{2m} - zV^*B_{2n}^{2m} - V^*)U]^{-1} (L_{2m}^{2m} - V^*K_{2m}^{2m} - V^*)$$

with $\begin{pmatrix} K_{2m}^{2m} \\ L_{2m}^{2m} \end{pmatrix} = T_\omega^{(2m)}(z) \begin{pmatrix} -z^{-1}I_L \\ 0 \end{pmatrix}$. As we will see later, the presence of the three extra terms $-V^*$ will prevent to prove directly the exponential decay for these terms which is why we prove before the Lemma 3.4 so that we don't have to estimate directly these terms.

Lemma 3.9 *Let $\epsilon \in (0, 1)$. There exists $\kappa_\epsilon > 0$ such that for every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$,*

$$\|G_\omega^{[2n,2m+1]}(z, 2n, 2m + 1)\| \leq \kappa_\epsilon \left\| \left((C_{2n}^{2m+1} - VA_{2n}^{2m+1}) + (D_{2n}^{2m+1} - VB_{2n}^{2m+1})U \right)^{-1} \right\|. \tag{3.35}$$

Proof It suffices to prove that $\|L_{2m}^{2m+1} - VK_{2m}^{2m+1}\|$ is uniformly bounded in m . One has

$$\begin{aligned} \|L_{2m}^{2m+1} - VK_{2m}^{2m+1}\| &= \left\| (U_\omega^{(2m+1)})^* \alpha \tilde{\rho}^{-1} V_\omega^{(2m)} \tilde{\rho}^{-1} + z(U_\omega^{(2m+1)})^* \tilde{\rho}^{-1} (U_\omega^{(2m)})^* \alpha \tilde{\rho}^{-1} \right. \\ &\quad \left. - V \left(\frac{1}{z} V_\omega^{(2m+1)} \tilde{\rho}^{-1} V_\omega^{(2m)} \tilde{\rho}^{-1} + V_\omega^{(2m+1)} \tilde{\rho}^{-1} \alpha^* (U_\omega^{(2m)})^* \alpha \tilde{\rho}^{-1} \right) \right\| \end{aligned}$$

$$\leq \|\tilde{\rho}^{-1}\|^2 \left[\|\alpha\| \cdot \|\rho^{-1}\| + \frac{1}{|z|} + \|\alpha\| \|\tilde{\rho}^{-1}\|^2 (|z| + \|\alpha\|) \right]$$

For $\|\alpha\| < 1$ and $\epsilon \in (0, 1)$, by Lemma 2.4 and since $z \in \mathbb{S}_\epsilon$:

$$\|L_{2m}^{2m+1} - VK_{2m}^{2m+1}\| \leq \frac{1}{1 - \|\alpha\|^2} \left[\frac{\|\alpha\|}{\sqrt{1 - \|\alpha\|^2}} + \frac{1}{1 - \epsilon} + (1 - \epsilon)\|\alpha\| + \|\alpha\|^2 \right] := \kappa_\epsilon.$$

□

Remark 3.10 Note that since in Theorem 1.5 $\epsilon \in (0, \epsilon_0]$ and $\epsilon_0 \in (0, 1)$ is fixed by Corollary 2.11, the term $\frac{1}{1 - \epsilon}$ in κ_ϵ is in the interval $(1, \frac{1}{1 - \epsilon_0})$ and thus is bounded.

To estimate the Green matrix using transfer matrices presents notable complexity: in the scalar context, this process is made possible through the property $|a \cdot b| = |a| \cdot |b|$. However, in the matrix context, this direct relationship does not hold, as we only have sub-multiplicativity of norms. Thus, we opt for a different method.

3.4.2 Estimate of the Green kernel by products of transfer matrices

Our first step is to establish a lower bound on the norms of the inverses of transfer matrices, which corresponds to the norms of the transfer matrices themselves. Hence, we have to prove that the norm of a new term needs to be uniformly bounded by a constant.

Lemma 3.11 *Let $\epsilon \in (0, 1)$. There exists $\kappa_\epsilon > 0$ such that for all $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and all $m, n \in \mathbb{Z}$ such that $|m - n| > 2$:*

$$\|(T_{2n}^{2m+1}(z))^{-1}\| \geq \kappa_\epsilon \frac{\|G_\omega^{[2n, 2m+1]}(z, 2n, 2m + 1)\|}{\|H_{2n}^{2m+1}(F_{2n}^{2m+1})^{-1} - G_{2n}^{2m+1}(E_{2n}^{2m+1})^{-1}\|} \tag{3.36}$$

where

$$E_{2n}^{2m+1} = (C_{2n}^{2m+1} - VA_{2n}^{2m+1}) + (D_{2n}^{2m+1} - VB_{2n}^{2m+1})U. \tag{3.37}$$

$$F_{2n}^{2m+1} = (D_{2n}^{2m+1} - VB_{2n}^{2m+1})U - (C_{2n}^{2m+1} - VA_{2n}^{2m+1}). \tag{3.38}$$

$$G_{2n}^{2m+1} = A_{2n}^{2m+1} + V^*C_{2n}^{2m+1} + (B_{2n}^{2m+1} + V^*D_{2n}^{2m+1})U. \tag{3.39}$$

$$H_{2n}^{2m+1} = (B_{2n}^{2m+1} + V^*D_{2n}^{2m+1})U - (A_{2n}^{2m+1} + V^*C_{2n}^{2m+1}). \tag{3.40}$$

Remark 3.12 Since in Theorem 1.5 $\epsilon \in (0, \epsilon_0]$, once $\epsilon_0 \in (0, 1)$ is fixed, κ_ϵ can be taken uniform in ϵ . Thus, κ can be taken such that it only depends on the norm of the matrix α .

Proof Throughout this proof, we will omit the indices $2n$ and $2m + 1$ for brevity. With a direct computation, we get with the expressions given in (3.37), (3.38), (3.39) and (3.40),

$$(T_{2n}^{2m+1}(z))^{-1} = \begin{pmatrix} A & BU \\ C & DU \end{pmatrix}^{-1} = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} -V & I_L \\ I_L & V^* \end{pmatrix} \right)^{-1}$$

$$\begin{aligned} & \left(\begin{array}{cc} (C-VA)+(D-VB)U & (D-VB)U-(C-VA) \\ A+V^*C+(B+V^*D)U & (B+V^*D)U-(A+V^*C) \end{array} \right)^{-1} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} I_L & -I_L \\ I_L & I_L \end{pmatrix} \right)^{-1} \\ &= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} -V & I_L \\ I_L & V^* \end{pmatrix} \right)^{-1} \begin{pmatrix} E & F \\ G & H \end{pmatrix}^{-1} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} I_L & -I_L \\ I_L & I_L \end{pmatrix} \right)^{-1}. \end{aligned}$$

Using Lemma 3.6 1(a) and 2(a) and the Neumann power serie lemma, $E = (C - VA) + (D - VB)U$ is invertible, so we can compute the inverse of $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$ using the Schur complement :

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix}^{-1} = \begin{pmatrix} E^{-1} + E^{-1}F \left(\frac{M}{E}\right)^{-1} GE^{-1} & E^{-1}F \left(\frac{M}{E}\right)^{-1} \\ \left(\frac{M}{E}\right)^{-1} GE^{-1} & \left(\frac{M}{E}\right)^{-1} \end{pmatrix},$$

where $\left(\frac{M}{E}\right) = H - GE^{-1}F$ is the Schur complement of E . Thus, we obtain:

$$\begin{aligned} (T_{2n}^{2m+1}(z))^{-1} &= \left(\frac{1}{\sqrt{2}} \begin{pmatrix} -V & I_L \\ I_L & V^* \end{pmatrix} \right)^{-1} \begin{pmatrix} E^{-1} + E^{-1}F \left(\frac{M}{E}\right)^{-1} GE^{-1} & E^{-1}F \left(\frac{M}{E}\right)^{-1} \\ \left(\frac{M}{E}\right)^{-1} GE^{-1} & \left(\frac{M}{E}\right)^{-1} \end{pmatrix} \\ &\quad \left(\frac{1}{\sqrt{2}} \begin{pmatrix} I_L & -I_L \\ I_L & I_L \end{pmatrix} \right)^{-1}. \end{aligned}$$

By taking the Frobenius norm, we obtain:

$$\|(T_{2n}^{2m+1}(z))^{-1}\| = \left\| \begin{pmatrix} E^{-1} + E^{-1}F \left(\frac{M}{E}\right)^{-1} GE^{-1} & E^{-1}F \left(\frac{M}{E}\right)^{-1} \\ \left(\frac{M}{E}\right)^{-1} GE^{-1} & \left(\frac{M}{E}\right)^{-1} \end{pmatrix} \right\| \tag{3.41}$$

since the matrices $\frac{1}{\sqrt{2}} \begin{pmatrix} -V & I_L \\ I_L & V^* \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} I_L & -I_L \\ I_L & I_L \end{pmatrix}$ are unitary. We know that the Frobenius norm of a square matrix is always greater than or equal to the sum of the norms of its blocks. Therefore,

$$\left\| \begin{pmatrix} E^{-1} + E^{-1}F \left(\frac{M}{E}\right)^{-1} GE^{-1} & E^{-1}F \left(\frac{M}{E}\right)^{-1} \\ \left(\frac{M}{E}\right)^{-1} GE^{-1} & \left(\frac{M}{E}\right)^{-1} \end{pmatrix} \right\| \geq \|E^{-1}F \left(\frac{M}{E}\right)^{-1}\|.$$

Furthermore, by the sub-multiplicativity of the Frobenius norm,

$$\|E^{-1}F \left(\frac{M}{E}\right)^{-1}\| \geq \frac{\|E^{-1}F \left(\frac{M}{E}\right)^{-1} \left(F \left(\frac{M}{E}\right)^{-1}\right)^{-1}\|}{\left\| \left(F \left(\frac{M}{E}\right)^{-1}\right)^{-1} \right\|} = \frac{\|E^{-1}\|}{\left\| \left(\frac{M}{E}\right) F^{-1} \right\|}. \tag{3.42}$$

Combining (3.41) and (3.42), we have the following inequality:

$$\|(T_{2n}^{2m+1}(z))^{-1}\| \geq \frac{\|E^{-1}\|}{\left\| \left(\frac{M}{E}\right) F^{-1} \right\|}.$$

To conclude, it suffices to observe that $(\frac{M}{E}) F^{-1} = HF^{-1} - GE^{-1}$ and, using (3.37) and Lemma 3.9, $\kappa \|G_{\omega}^{[2n, 2m+1]}(z, 2n, 2m + 1)\| \leq \|E^{-1}\|$, where κ is the constant obtained in Lemma 3.9 for $\epsilon = \epsilon_0$. □

With Lemma 3.11, to control the norm of the Green kernel by the norm of $(T_{2n}^{2m+1}(z))^{-1}$, it suffices to prove that there exists a constant $C_{\alpha} > 0$ such that:

$$\forall m, n \in \mathbb{Z}, |m - n| \geq 2, \|H_{2n}^{2m+1}(F_{2n}^{2m+1})^{-1} - G_{2n}^{2m+1}(E_{2n}^{2m+1})^{-1}\| \leq C_{\alpha}. \tag{3.43}$$

To achieve this, we propose a strategy which consists in constructing a sequence of matrices which will follow a sub-arithmetic-geometric progression. This particular progression eventually reaches a maximum threshold, meaning it does not exceed a certain limit, provided that the value of the norm of α meet a specific condition. We set $\|\cdot\|$ as the Frobenius norm on $\mathcal{M}_L(\mathbb{C})$.

Proposition 3.13 *There exists $r_0 \in (0, 1)$, $p_0 > 2$ and $C_{r_0, p_0} > 0$ such that for every $\|\alpha\| \leq r_0$, every $\epsilon \in (0, 1)$, every $z \in \mathbb{S}_{\epsilon}$ and every $m, n \in \mathbb{Z}$, $|m - n| > p_0$,*

$$\|H_{2n}^{2m+1}(F_{2n}^{2m+1})^{-1} - G_{2n}^{2m+1}(E_{2n}^{2m+1})^{-1}\| \leq C_{r_0, p_0}. \tag{3.44}$$

Proof Step 1 During the next computations, we drop the indices $2n, 2m + 1$. Using (3.37), (3.38), (3.39) and (3.40),

$$\begin{aligned} HF^{-1} - GE^{-1} &= ((B + V^*D)U - (A + V^*C))((D - VB)U - (C - VA))^{-1} \\ &\quad - (A + V^*C + (B + V^*D)U)((D - VB)U + (C - VA))^{-1} \\ &= (BU - A + V^*(DU - C))(DU - C - V(BU - A))^{-1} \\ &\quad - (BU + A + V^*(DU + C))(DU + C - V(A + BU))^{-1}. \end{aligned}$$

According to Lemma 3.6, $BU - A, BU + A, DU - C$ and $DU + C$ are invertible and we find:

$$\begin{aligned} HF^{-1} - GE^{-1} &= -V^* \left(I_L - (DU - C)(BU - A)^{-1}V^* \right)^{-1} \\ &\quad + V^* \left(I_L - V(BU - A)(DU - C)^{-1} \right)^{-1} \\ &\quad + V^* \left(I_L - (DU + C)(BU + A)^{-1}V^* \right)^{-1} \\ &\quad - V^* \left(I_L - V(BU + A)(DU + C)^{-1} \right)^{-1}. \end{aligned} \tag{3.45}$$

If one sets $M = (DU - C)(BU - A)^{-1}V^*$ and $N = (DU + C)(BU + A)^{-1}V^*$, by Lemma 3.6 they are both invertible and they both lie in \mathbb{D}_L . Therefore, $I_L - M$ and $I_L - N$ are invertible and the same for $I_L - M^{-1}$ and $I_L - N^{-1}$. Moreover, one has $(I_L - M^{-1})^{-1} = -M(I_L - M)^{-1}$ and $(I_L - N^{-1})^{-1} = -N(I_L - N)^{-1}$. Rewriting

(3.45), one gets

$$HF^{-1} - GE^{-1} = V^* \left(-(\mathbb{I}_L + M)(\mathbb{I}_L - M)^{-1} + (\mathbb{I}_L + N)(\mathbb{I}_L - N)^{-1} \right). \tag{3.46}$$

Applying the Frobenius norm and since V^* is unitary and $M, N \in \mathbb{D}_L$,

$$\|HF^{-1} - GE^{-1}\| \leq 2 \left(\|(\mathbb{I}_L - M)^{-1}\| + \|(\mathbb{I}_L - N)^{-1}\| \right). \tag{3.47}$$

Step 2 We will begin by bounding $\|(\mathbb{I}_L - M_{2n}^{2m+1})^{-1}\|$. Since by Lemma 3.6, $M_{2n}^{2m+1} \in \mathbb{D}_L$,

$$\|(\mathbb{I}_L - M_{2n}^{2m+1})^{-1}\| \leq \frac{1}{1 - \|M_{2n}^{2m+1}\|}.$$

To show that the upper bound $\frac{1}{1 - \|M_{2n}^{2m+1}\|}$ is bounded, it suffices to prove the existence of a constant $C \in (0, 1)$ independent of n and m , such that for every $m, n \in \mathbb{Z}$, $|m - n|$ large enough,

$$\|M_{2n}^{2m+1}\| = \|(D_{2n}^{2m+1}U - C_{2n}^{2m+1})(B_{2n}^{2m+1}U - A_{2n}^{2m+1})^{-1}V^*\| \leq C. \tag{3.48}$$

We start by finding a recurrence relationship between the ranks $2m$ and $2m + 1$ for n fixed. Writing

$$\begin{aligned} \begin{pmatrix} A_{2n}^{2m+1}(z) & B_{2n}^{2m+1}(z)U \\ C_{2n}^{2m+1}(z) & D_{2n}^{2m+1}(z)U \end{pmatrix} &= T_{2n}^{2m+1}(z) = T_{2m}^{2m+1}(z)T_{2n}^{2m-1}(z) \\ &= \begin{pmatrix} W_{2m}^{2m+1}(z) & X_{2m}^{2m+1}(z) \\ Y_{2m}^{2m+1}(z) & Z_{2m}^{2m+1}(z) \end{pmatrix} \cdot \begin{pmatrix} A_{2n}^{2m-1}(z) & B_{2n}^{2m-1}(z)U \\ C_{2n}^{2m-1}(z) & D_{2n}^{2m-1}(z)U \end{pmatrix} \end{aligned}$$

with

$$\begin{aligned} W_{2m}^{2m+1}(z) &= \frac{1}{z}V_{\omega}^{(2m+1)}(\tilde{\rho}(\alpha))^{-1}V_{\omega}^{(2m)}(\tilde{\rho}(\alpha))^{-1} + V_{\omega}^{(2m+1)}(\tilde{\rho}(\alpha))^{-1}\alpha^*(U_{\omega}^{(2m)})^*(\rho(\alpha))^{-1}, \\ X_{2m}^{2m+1}(z) &= -\frac{1}{z}V_{\omega}^{(2m+1)}(\tilde{\rho}(\alpha))^{-1}V_{\omega}^{(2m)}(\tilde{\rho}(\alpha))^{-1}\alpha^* - V_{\omega}^{(2m+1)}(\tilde{\rho}(\alpha))^{-1}\alpha^*(U_{\omega}^{(2m)})^*(\rho(\alpha))^{-1}, \\ Y_{2m}^{2m+1}(z) &= -(U_{\omega}^{(2m+1)})^*\alpha(\tilde{\rho}(\alpha))^{-1}V_{\omega}^{(2m)}(\tilde{\rho}(\alpha))^{-1} \\ &\quad - z(U_{\omega}^{(2m+1)})^*(\tilde{\rho}(\alpha))^{-1}(U_{\omega}^{(2m)})^*\alpha(\tilde{\rho}(\alpha))^{-1}, \\ Z_{2m}^{2m+1}(z) &= (U_{\omega}^{(2m+1)})^*\alpha(\tilde{\rho}(\alpha))^{-1}V_{\omega}^{(2m)}(\tilde{\rho}(\alpha))^{-1}\alpha^* + z(U_{\omega}^{(2m+1)})^*(\tilde{\rho}(\alpha))^{-1}(U_{\omega}^{(2m)})^*\alpha(\rho(\alpha))^{-1}, \end{aligned}$$

yields the following relationships (dropping the z dependence):

$$\begin{aligned} A_{2n}^{2m+1} &= W_{2m}^{2m+1}A_{2n}^{2m-1} + X_{2m}^{2m+1}C_{2n}^{2m-1}, \quad B_{2n}^{2m+1}U = W_{2m}^{2m+1}B_{2n}^{2m-1}U + X_{2m}^{2m+1}D_{2n}^{2m-1}U \\ C_{2n}^{2m+1} &= Y_{2m}^{2m+1}A_{2n}^{2m-1} + Z_{2m}^{2m+1}C_{2n}^{2m-1}, \quad D_{2n}^{2m+1}U = Y_{2m}^{2m+1}B_{2n}^{2m-1}U + Z_{2m}^{2m+1}D_{2n}^{2m-1}U. \end{aligned}$$

Summing and subtracting these, we find:

$$\begin{aligned}
 A_{2n}^{2m+1} \pm B_{2n}^{2m+1}U &= W_{2m}^{2m+1}(A_{2n}^{2m-1} \pm B_{2n}^{2m-1}U) + X_{2m}^{2m+1}(C_{2n}^{2m-1} \pm D_{2n}^{2m-1}U). \\
 C_{2n}^{2m+1} \pm D_{2n}^{2m+1}U &= Y_{2m}^{2m+1}(A_{2n}^{2m-1} \pm B_{2n}^{2m-1}U) + Z_{2m}^{2m+1}(C_{2n}^{2m-1} \pm D_{2n}^{2m-1}U).
 \end{aligned}$$

This establishes the relationship:

$$\begin{aligned}
 &(D_{2n}^{2m+1}U - C_{2n}^{2m+1})(B_{2n}^{2m+1}U - A_{2n}^{2m+1})^{-1} \\
 &= \left[Y_{2m}^{2m+1}(A_{2n}^{2m-1} - B_{2n}^{2m-1}U) + Z_{2m}^{2m+1}(C_{2n}^{2m-1} - D_{2n}^{2m-1}U) \right] \\
 &\quad \times \left[W_{2m}^{2m+1}(A_{2n}^{2m-1} - B_{2n}^{2m-1}U) + X_{2m}^{2m+1}(C_{2n}^{2m-1} - D_{2n}^{2m-1}U) \right]^{-1}
 \end{aligned}$$

Under the condition of the invertibility of W_{2m}^{2m+1} , we factorize $W_{2m}^{2m+1}(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)$ on the right in the inverse to get:

$$\begin{aligned}
 &(D_{2n}^{2m+1}U - C_{2n}^{2m+1})(B_{2n}^{2m+1}U - A_{2n}^{2m+1})^{-1} \\
 &= \left[Y_{2m}^{2m+1} + Z_{2m}^{2m+1}(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1} \right] (W_{2m}^{2m+1})^{-1} \\
 &\quad \times \left[I_L + X_{2m}^{2m+1}(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}(W_{2m}^{2m+1})^{-1} \right]^{-1}
 \end{aligned}$$

Under the condition of the invertibility of W_{2m}^{2m+1} and assuming that

$$\|X_{2m}^{2m+1}\| \cdot \|(W_{2m}^{2m+1})^{-1}\| < 1, \tag{3.49}$$

using Neumann power serie, we can write:

$$\begin{aligned}
 &\|(D_{2n}^{2m+1}U - C_{2n}^{2m+1})(B_{2n}^{2m+1}U - A_{2n}^{2m+1})^{-1}\| \\
 &\leq \frac{(\|Y_{2m}^{2m+1}\| + \|Z_{2m}^{2m+1}\| \cdot \|(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}\|) \cdot \|(W_{2m}^{2m+1})^{-1}\|}{1 - \|X_{2m}^{2m+1}\| \cdot \|(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}\| \cdot \|(W_{2m}^{2m+1})^{-1}\|}.
 \end{aligned} \tag{3.50}$$

Therefore, we find:

$$\begin{aligned}
 \|(I_L - M_{2n}^{2m+1})^{-1}\| &\leq \frac{1}{1 - \|(D_{2n}^{2m+1}U - C_{2n}^{2m+1})(B_{2n}^{2m+1}U - A_{2n}^{2m+1})^{-1}\|} \\
 &\leq \frac{1}{1 - \frac{(\|Y_{2m}^{2m+1}\| + \|Z_{2m}^{2m+1}\|) \cdot \|(W_{2m}^{2m+1})^{-1}\|}{1 - \|X_{2m}^{2m+1}\| \cdot \|(W_{2m}^{2m+1})^{-1}\| \cdot \|(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}\|}}
 \end{aligned}$$

under the condition

$$\frac{(\|Y_{2m}^{2m+1}\| + \|Z_{2m}^{2m+1}\| \cdot \|(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}\|) \cdot \|(W_{2m}^{2m+1})^{-1}\|}{1 - \|X_{2m}^{2m+1}\| \cdot \|(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}\| \cdot \|(W_{2m}^{2m+1})^{-1}\|} < 1. \tag{3.51}$$

which is implied by the condition

$$(\|Y_{2m}^{2m+1}\| + \|Z_{2m}^{2m+1}\| + \|X_{2m}^{2m+1}\|) \cdot \|(W_{2m}^{2m+1})^{-1}\| \leq 1. \tag{3.52}$$

since $\|(C_{2n}^{2m-1} - D_{2n}^{2m-1}U)(A_{2n}^{2m-1} - B_{2n}^{2m-1}U)^{-1}\| < 1$. We set, with n still fixed,

$$x_m := \|X_{2m}^{2m+1}\|, \quad y_m := \|Y_{2m}^{2m+1}\|, \quad z_m := \|Z_{2m}^{2m+1}\|, \quad w_m := \|(W_{2m}^{2m+1})^{-1}\| \tag{3.53}$$

and for all $m \geq n + 1$,

$$f_m^{(n)} = \|(D_{2n}^{2m+1}U - C_{2n}^{2m+1})(B_{2n}^{2m+1}U - A_{2n}^{2m+1})^{-1}\|. \tag{3.54}$$

With these notations, we write:

$$\begin{aligned} \|(I_L - M_{2n}^{2m+1})^{-1}\| &\leq \frac{1}{1 - f_m^{(n)}} \leq \frac{1}{1 - \frac{y_m w_m + z_m w_m}{1 - x_m w_m f_{m-1}^{(n)}}} \\ &\leq \frac{1 - x_m w_m f_{m-1}^{(n)}}{1 - y_m w_m - z_m w_m - x_m w_m f_{m-1}^{(n)}} \\ &\leq \frac{1}{1 - y_m w_m - z_m w_m - x_m w_m f_{m-1}^{(n)}} \end{aligned}$$

Thus, we obtain a recurrence relationship for all $m \geq n + 1$:

$$f_m^{(n)} \leq y_m w_m + z_m w_m + x_m w_m f_{m-1}^{(n)}. \tag{3.55}$$

Step 3 Our goal now is to show that the sequence $(f_m^{(n)})_{m \geq n+1}$ is uniformly bounded in both m and n , by a constant strictly smaller than 1, at least for $|m - n|$ large enough. Iterating the sub-arithmetic-geometric relationship (3.55) and still under the assumption $x_m w_m < 1$, one finds

$$\begin{aligned} \forall m \geq n + 1, \quad f_m^{(n)} &\leq (y_m w_m + z_m w_m)(1 + x_m w_m + \dots + (x_m w_m)^{m-n-1}) \\ &\quad + (x_m w_m)^{m-n} f_{m-n}^{(n)} \end{aligned} \tag{3.56}$$

$$= \frac{y_m w_m + z_m w_m}{1 - x_m w_m} + \left(f_{m-n}^{(n)} - \frac{y_m w_m + z_m w_m}{1 - x_m w_m} \right) (x_m w_m)^{m-n} \tag{3.57}$$

$$\leq \frac{y_m w_m + z_m w_m}{1 - x_m w_m} + \left(1 - \frac{y_m w_m + z_m w_m}{1 - x_m w_m} \right) (x_m w_m)^{m-n} \tag{3.58}$$

since we know that for every $m, n, f_m^{(n)} < 1$, using Lemma 3.6. It remains to estimate the term $\frac{y_m w_m + z_m w_m}{1 - x_m w_m}$. Using Lemma 2.4, one gets

$$\forall m \in \mathbb{Z}, y_m \leq (1 + |z|) \frac{\|\alpha\|}{1 - \|\alpha\|^2}, \quad x_m \leq \left(1 + \frac{1}{|z|}\right) \frac{\|\alpha\|}{1 - \|\alpha\|^2} \quad \text{and}$$

$$z_m \leq \|\alpha\| \frac{|z| + \|\alpha\|}{1 - \|\alpha\|^2}. \tag{3.59}$$

Then, to estimate w_m , we write :

$$\begin{aligned} (W_{2m}^{2m+1}(z))^{-1} &= \left(\frac{1}{z} V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} V_\omega^{(2m)}(\tilde{\rho}(\alpha))^{-1} + V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} \alpha^* (U_\omega^{(2m)})^* (\rho(\alpha))^{-1}\right)^{-1}, \\ &= z \left(V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} V_\omega^{(2m)}(\tilde{\rho}(\alpha))^{-1}\right)^{-1} \\ &\quad \left(I_L + z V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} \alpha^* (U_\omega^{(2m)})^* (\rho(\alpha))^{-1} \left(V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} V_\omega^{(2m)}(\tilde{\rho}(\alpha))^{-1}\right)^{-1}\right)^{-1}. \end{aligned}$$

Using Lemma 2.4,

$$\begin{aligned} \|V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} \alpha^* (U_\omega^{(2m)})^* (\rho(\alpha))^{-1} \left(V_\omega^{(2m+1)}(\tilde{\rho}(\alpha))^{-1} V_\omega^{(2m)}(\tilde{\rho}(\alpha))^{-1}\right)^{-1}\| \\ \leq |z| \|\alpha\| \frac{(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2}. \end{aligned} \tag{3.60}$$

For any $\epsilon > 0$ and any $z \in \mathbb{S}_\epsilon$, the upper bound in (3.60) tends to 0 when $\|\alpha\|$ tends to 0. Hence, there exists $r_1 \in (0, 1)$ independent of ϵ such that for any α such that $\|\alpha\| \leq r_1$ and any $z \in \mathbb{S}_\epsilon$, $|z| \|\alpha\| \frac{(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2} < 1$. For such α , $W_{2m}^{2m+1}(z)$ is invertible and using again Lemma 2.4,

$$\begin{aligned} \forall m \in \mathbb{Z}, w_m = \|(W_{2m}^{2m+1}(z))^{-1}\| &\leq \frac{|z|(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - |z| \|\alpha\| \frac{(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2}} \\ &= \frac{|z|(1 - \|\alpha\|^2)(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2 - |z| \|\alpha\| (2 - \sqrt{1 - \|\alpha\|^2})^2}. \end{aligned} \tag{3.61}$$

Combining (3.59) and (3.61), for any α such that $\|\alpha\| \leq r_1$, any $\epsilon > 0$ and any $z \in \mathbb{S}_\epsilon$,

$$\forall m \in \mathbb{Z}, x_m w_m \leq \frac{(1 + |z|) \|\alpha\| (2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2 - |z| \|\alpha\| (2 - \sqrt{1 - \|\alpha\|^2})^2} := \mu_{z, \alpha}. \tag{3.62}$$

and

$$\forall m \in \mathbb{Z}, \frac{y_m w_m + z_m w_m}{1 - x_m w_m} \leq \frac{\left((1 + |z|) \frac{\|\alpha\|}{1 - \|\alpha\|^2} + \|\alpha\| \frac{|z| + \|\alpha\|}{1 - \|\alpha\|^2} \right) \frac{|z|(1 - \|\alpha\|^2)(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2 - |z| \|\alpha\| (2 - \sqrt{1 - \|\alpha\|^2})^2}}{1 - \left(1 + \frac{1}{|z|}\right) \frac{\|\alpha\|}{1 - \|\alpha\|^2} \frac{|z|(1 - \|\alpha\|^2)(2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2 - |z| \|\alpha\| (2 - \sqrt{1 - \|\alpha\|^2})^2}}$$

$$= \frac{\|\alpha\| (1 + 2|z| + \|\alpha\|) |z| (2 - \sqrt{1 - \|\alpha\|^2})^2}{1 - \|\alpha\|^2 - (2|z| + 1)\|\alpha\| (2 - \sqrt{1 - \|\alpha\|^2})^2} := \lambda_{z,\alpha} \tag{3.63}$$

For any $\epsilon > 0$ and any $z \in \mathbb{S}_\epsilon$, both $\lambda_{z,\alpha}$ and $\mu_{z,\alpha}$ tends to 0 when $\|\alpha\|$ tends to 0. Hence, one can find $r_2 \in (0, 1)$ independent of ϵ such that for any α such that $\|\alpha\| \leq r_2$ and any $z \in \mathbb{S}_\epsilon$, $\mu_{z,\alpha} \in (0, 1)$. Then, for such α , $\mu_{z,\alpha}^{|m-n|}$ tends to 0 when $|m - n|$ tends to infinity. Moreover, one can also find $r_3 \in (0, 1)$ independent of ϵ such that for any α such that $\|\alpha\| \leq r_3$ and any $z \in \mathbb{S}_\epsilon$, $\lambda_{z,\alpha} \in (0, \frac{1}{2}]$.

From these two facts and inequality (3.58), one deduces that for every $\epsilon \in (0, 1)$, for any α such that $\|\alpha\| \leq r_0 := \min(r_1, r_2, r_3)$, there exists $p_0 > 2$, and $\tilde{C}_{r_0,p_0} \in (0, 1)$ such that for any $m, n \in \mathbb{Z}$, $|m - n| \geq p_0$ and any $z \in \mathbb{S}_\epsilon$,

$$0 < f_m^{(n)} \leq \lambda_{z,\alpha} + (1 - \lambda_{z,\alpha})\mu_{z,\alpha}^{|m-n|} \leq \tilde{C}_{r_0,p_0}. \tag{3.64}$$

Following the exact same procedure, we obtain the same bound \tilde{C}_{r_0,p_0} for the second term $\|(I_L - N)^{-1}\|$, since the change of sign in front of A and C does not change any of the estimates in norm. We finally deduce (3.44) with $C_{r_0,p_0} = \frac{2}{1 - \tilde{C}_{r_0,p_0}}$.

Step 4 To finish the proof it remains to discuss the assumptions made during the previous steps. In particular we have to give conditions on α to satisfy the conditions of invertibility of W_{2m}^{2m+1} and to satisfy conditions (3.49) and (3.52). Actually, invertibility of W_{2m}^{2m+1} was discussed at (3.60) and is implied by the condition $\|\alpha\| \leq r_1$. Condition (3.49) is implied by $\lambda_{z,\alpha} < 1$ and in particular is satisfied for $\|\alpha\| \leq r_2$. Condition (3.52) is also implied by $\lambda_{z,\alpha} < 1$ and leads to the same condition on α . □

Proof of Theorem 1.5 Using Proposition 3.13, Lemma 3.11 implies that

$$\|G_\omega^{[2n, 2m+1]}(z, 2n, 2m + 1)\| \leq \frac{1}{\kappa} C_{r_0,p_0} \|(T_{2n}^{2m+1}(z))^{-1}\|$$

with κ uniform in $\epsilon \in (0, \epsilon_0]$. Taking the power s and the expectancy and applying Lemma 2.12, one gets the statement of Theorem 1.5. □

To prove the decay estimate for the general Green’s function, it suffices to combine all the previous results. Actually, to prove Theorem 1.9, one directly follow the proof of [26, Theorem 7.1] since we already proved Proposition 1.6 and Proposition 3.5.

4 Dynamical localization

In this section we prove dynamical localization for $\{\mathbb{U}_\omega\}_{\omega \in \Omega}$ and thus demonstrate the main result of the article, Theorem 1.3. To do this, we will rely on two important aspects: firstly, the use of the exponential decay of the moments of the resolvent to obtain an estimate of the second order moments of the resolvent ; secondly, we connect the powers of the operator to its resolvent through a general lemma about unitary operators.

4.1 Proof of Theorem 1.10

Proof Let $\omega \in \Omega$. We use here the factorization (3.3) of $U_\omega = Y_\omega D_\omega V'_\omega W_\omega$, with the unitaries $Y_\omega = \bigoplus_{j \in \mathbb{Z}} \begin{pmatrix} I_L & 0 \\ 0 & \widehat{V}_\omega^{(2j)} \end{pmatrix}$, $D_\omega = \bigoplus_{j \in \mathbb{Z}} \begin{pmatrix} I_L & 0 \\ 0 & e^{i\theta_\omega^{(2j)}} \end{pmatrix}$ and $V'_\omega = \bigoplus_{j \in \mathbb{Z}} \begin{pmatrix} \alpha & \rho(\alpha)U_\omega^{(2j)} \\ \widehat{\rho}(\alpha) & -\alpha^*U_\omega^{(2j)} \end{pmatrix} \circ s^L$.

We set $X_\omega = V'_\omega W_\omega$. Let $n \in \mathbb{Z}$ be fixed and define, for $\delta = (\delta_1, \dots, \delta_L) \in [0, 2\pi]^L$, the diagonal by blocks operator

$$Y_\omega^\delta := \left(\bigoplus_{j < n} \begin{pmatrix} I_L & 0 \\ 0 & \widehat{V}_\omega^{(2j)} e^{i\theta_\omega^{(2j)}} \end{pmatrix} \right) \oplus \begin{pmatrix} I_L & 0 \\ 0 & \widehat{V}_\omega^{(2n)} e^{i(\theta_\omega^{(2n)} - \delta)} \end{pmatrix} \oplus \left(\bigoplus_{j > n} \begin{pmatrix} I_L & 0 \\ 0 & \widehat{V}_\omega^{(2j)} e^{i\theta_\omega^{(2j)}} \end{pmatrix} \right)$$

and $U_\omega^\delta := Y_\omega^\delta X_\omega$. (4.1)

Recall that we denote by $(e_{\{n,k\}})_{n \in \mathbb{Z}, k \in \{1, \dots, L\}}$ the canonical basis of $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^L$ and let P_{2n+1} be the projection onto $\text{Span}(e_{\{2n+1,1\}}, \dots, e_{\{2n+1,L\}})$. Identifying P_{2n+1} with its matrix in the canonical basis, one has $P_{2n+1} = \text{diag}(0, \dots, 0, I_L, 0, \dots, 0)$ with I_L in the $(2n + 1)^{\text{th}}$ position. The projection P_{2n+1} corresponds to the projection on the $(2n + 1)^{\text{th}}$ block. We define $\eta_{2n} = e^{i(\theta_\omega^{(2n)} - \delta)} - e^{i\theta_\omega^{(2n)}}$ to get:

$$U_\omega^\delta = Y_\omega(D_\omega + \eta_{2n} P_{2n+1})X_\omega = U_\omega + Y_\omega \eta_{2n} P_{2n+1} X_\omega. \tag{4.2}$$

Indeed, the $2L \times 2L$ block $\begin{pmatrix} I_L & 0 \\ 0 & \widehat{V}_\omega^{(2n)} e^{i(\theta_\omega^{(2n)} - \delta)} \end{pmatrix}$ corresponds to the $(2n)^{\text{th}}$ and the $(2n + 1)^{\text{th}}$ blocks and in view of the position of the $L \times L$ block $\widehat{V}_\omega^{(2n)} e^{i(\theta_\omega^{(2n)} - \delta)}$, it is the P_{2n+1} projector which appears in (4.2) and not the P_{2n} projector.

By the geometric resolvent identity,

$$\forall z \in \mathbb{C} \setminus \mathbb{S}^1, (U_\omega^\delta - z) - (U_\omega - z)^{-1} = -(U_\omega^\delta - z)^{-1} [Y_\omega \eta_{2n} P_{2n+1} X_\omega] (U_\omega - z)^{-1}.$$

We multiply on the left by X_ω and on the right by Y_ω to get:

$$X_\omega (U_\omega^\delta - z) Y_\omega - X_\omega (U_\omega - z)^{-1} Y_\omega = -X_\omega (U_\omega^\delta - z)^{-1} Y_\omega [\eta_{2n} P_{2n+1}] X_\omega (U_\omega - z)^{-1} Y_\omega.$$

We define $F_\delta(z) := X_\omega (U_\omega^\delta - z)^{-1} Y_\omega$ and $F(z) = X_\omega (U_\omega - z)^{-1} Y_\omega$. Then,

$$F_\delta(z) - F(z) = -F_\delta(z) (\eta_{2n} P_{2n+1}) F(z).$$

Let $k \in \{1, \dots, L\}$. For $\delta = (0, \dots, 0, \delta_k, 0, \dots, 0)$, $\eta_{2n} = \text{diag}(0, \dots, e^{i(\theta_{\omega,k}^{(2n)} - \delta_k)} - e^{i\theta_{\omega,k}^{(2n)}}, \dots, 0)$ and we set $\eta_{2n,k} = e^{i(\theta_{\omega,k}^{(2n)} - \delta_k)} - e^{i\theta_{\omega,k}^{(2n)}}$. Furthermore, for $(m, k) \in \mathbb{Z} \times \{1, \dots, L\}$ and $k \in \{1, \dots, L\}$,

$$\langle e_{\{m,l\}} | (F_\delta(z) - F(z)) e_{\{2n+1,k\}} \rangle = \langle e_{\{m,l\}} | (-F_\delta(z) (\eta_{2n} P_{2n+1}) F(z)) e_{\{2n+1,k\}} \rangle.$$

We define $F_\delta(z, l, k) := \langle e_{\{m,l\}} | (F_\delta(z) - F(z)) e_{\{2n+1,k\}} \rangle$. For all δ :

$$|F_\delta(z, k, k)| = |\langle e_{\{m,k\}} | F_\delta(z) e_{\{2n+1,k\}} \rangle| = \left| \langle e_{\{m,k\}} | \mathbb{X}_\omega(\mathbb{U}_\omega^\delta - z)^{-1} \mathbb{Y}_\omega e_{\{2n+1,k\}} \rangle \right|$$

$$\leq \|e_{\{m,k\}}\| \|\mathbb{X}_\omega(\mathbb{U}_\omega^\delta - z)^{-1} \mathbb{Y}_\omega\| \|e_{\{2n+1,k\}}\| = \|(\mathbb{U}_\omega^\delta - z)^{-1}\| \leq \frac{1}{1 - |z|}.$$

Then :

$$F_\delta(z, l, k) = \frac{F(z, l, k)}{1 + \eta_{2n,k} F(z, k, k)} = \frac{1}{\eta_{2n,k} + (F(z, k, k))^{-1}} \cdot \frac{F(z, l, k)}{F(z, k, k)}. \tag{4.3}$$

But: $F(z, k, k) = \frac{1}{\eta_{2n,k} + (F(z, k, k))^{-1}} \times 1$ and $1 - |z| \leq |\eta_{2n,k} + (F(z, k, k))^{-1}|$. Thus, we also have $1 - |\eta_{2n,k} + (F(z, k, k))^{-1}| \leq |z|$.

Recall that $\eta_{2n,k} = e^{i\theta_{\omega,k}^{(2n)} - \delta_k} - e^{i\theta_{\omega,k}^{(2n)}}$ and introduce $\delta_k \in [0, 2\pi]$ such that:

$$e^{-i\delta_k} = \frac{1 - e^{-i\theta_{\omega,k}^{(2n)}} (F(z, k, k))^{-1}}{|1 - e^{i\theta_{\omega,k}^{(2n)}} (F(z, k, k))^{-1}|}.$$

With this choice of δ_k , following the proof of [11, Proposition 5.1], from equation (5.17) to equation (5.20), we get:

$$(1 - |z|^2) |F_\delta(z, l, k)|^2 \leq \frac{(1 - |y|^2) |1 - y|^s}{|e^{-i\delta_k} - y|^2} |F(z, l, k)|^s.$$

By taking the expectation and integrating over δ_k ,

$$\mathbb{E} \left((1 - |z|^2) |F(z, l, k)|^2 \right) = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{E} \left((1 - |z|^2) |F(z, l, k)|^2 \right) d\delta_k.$$

Moreover, since the Haar measure on the unit circle is invariant by rotations, the random variables $e^{i\theta_{\omega,k}^{(2n)} - \delta_k}$ and $e^{i\theta_{\omega,k}^{(2n)}}$ have the same distribution. Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} \mathbb{E} \left((1 - |z|^2) |F(z, l, k)|^2 \right) d\delta_k = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{E} \left((1 - |z|^2) |F_\delta(z, l, k)|^2 \right) d\delta_k.$$

By Fubini’s theorem, we have:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \mathbb{E} \left((1 - |z|^2) |F_\delta(z, l, k)|^2 \right) d\delta_k &= \mathbb{E} \left((1 - |z|^2) \frac{1}{2\pi} \int_0^{2\pi} |F_\delta(z, l, k)|^2 d\delta_k \right) \\ &\leq \mathbb{E} \left(\frac{1}{2\pi} \sup_{|y| < 1} \int_0^{2\pi} \frac{(1 - |y|^2) |1 - y|^s}{|e^{-i\delta_k} - y|^2} |F(z, l, k)|^s d\delta_k \right) \\ &\leq 2^s \mathbb{E} \left(|F(z, l, k)|^s \times \sup_{|y| < 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |y|^2)}{|e^{-i\delta_k} - y|^2} d\delta_k \right) \end{aligned}$$

since $|1 - y|^s \leq 2^s$ for $y \in \mathbb{C}$ such that $|y| < 1$. Using [11, Eq.(5.24)],

$$\mathbb{E} \left((1 - |z|^2) |F(z, k, l)|^2 \right) \leq 2^s \mathbb{E} \left(|F(z, k, l)|^s \right) \text{ for all } s \in (0, 1).$$

We have:

$$\begin{aligned} \mathbb{E} \left(|F(z, l, k)|^s \right) &= \mathbb{E} \left(\left| \langle e_{\{m,l\}} \mid \mathbb{X}_\omega (\mathbb{U}_\omega - z)^{-1} \mathbb{Y}_\omega e_{\{2n+1,k\}} \rangle \right|^s \right) \\ &= \mathbb{E} \left(\left| \langle \mathbb{X}_\omega^* e_{\{m,l\}} \mid (\mathbb{U}_\omega - z)^{-1} \mathbb{Y}_\omega e_{\{2n+1,k\}} \rangle \right|^s \right). \end{aligned}$$

We use the band structure of \mathbb{X}_ω^* , apply Minkowski’s inequality and use the fact that the coefficients of \mathbb{X}_ω are bounded to write:

$$\mathbb{E} \left(|F(z, l, k)|^s \right) \leq C_1(s) \sum_{|j-l| \leq 4L} \mathbb{E} \left(\left| \langle \mathbb{X}_\omega^* e_{\{m,j\}} \mid (\mathbb{U}_\omega - z)^{-1} \mathbb{Y}_\omega e_{\{2n+1,k\}} \rangle \right|^s \right).$$

Since \mathbb{Y}_ω is block diagonal with unitary blocks in $U(L)$,

$$\begin{aligned} \forall m, n \in \mathbb{Z}, \forall j, k \in \{1, \dots, L\}, \quad & \left| \langle e_{\{m,j\}} \mid (\mathbb{U}_\omega - z)^{-1} \mathbb{Y}_\omega e_{\{2n+1,k\}} \rangle \right| \\ & \leq \|G_\omega(z, \tilde{m}, 2n + 1)\|^s \end{aligned}$$

where \tilde{m} is the index of the block containing $mL + j$ and equals $\left\lceil \frac{mL+j}{L} \right\rceil$. Thus,

$$\mathbb{E} \left(|F(z, l, k)|^s \right) \leq C_1(s) \sum_{|j-l| \leq 4L} \mathbb{E} \left(\|G_\omega(z, \left\lceil \frac{mL+j}{L} \right\rceil, 2n + 1)\|^s \right).$$

Now, we need to connect $\mathbb{E} \left(|F(z, l, k)|^s \right)$ and $\mathbb{E} \left((1 - |z|^2) \left| \langle e_{\{m,l\}} \mid (\mathbb{U}_\omega - z)^{-1} e_{\{2n+1,k\}} \rangle \right|^2 \right)$.

Since \mathbb{Y}_ω is a direct sum of matrices of the form $\begin{pmatrix} 1_L & 0 \\ 0 & \widehat{V}_\omega^{(2j)} e^{i\theta_\omega^{(2j)}} \end{pmatrix}$, we have $\mathbb{Y}_\omega e_{\{2n+1,k\}} = \widehat{V}_\omega^{(2n)} e^{i\theta_\omega^{(2n)}} e_{\{2n+1,k\}}$, and:

$$\begin{aligned} \left| \langle e_{\{m,l\}} \mid (\mathbb{U}_\omega - z)^{-1} e_{\{2n+1,k\}} \rangle \right|^2 &= \left| \langle \mathbb{X}_\omega e_{\{m,l\}} \mid \mathbb{X}_\omega (\mathbb{U}_\omega - z)^{-1} \mathbb{Y}_\omega \mathbb{Y}_\omega^* e_{\{2n+1,k\}} \rangle \right|^2 \\ &= \left| \langle \mathbb{X}_\omega e_{\{m,l\}} \mid F(z) \mathbb{Y}_\omega^* e_{\{2n+1,k\}} \rangle \right|^2 \\ &= \left| \sum_{j=1}^L \langle \mathbb{X}_\omega e_{\{m,l\}} \mid F(z) ((\widehat{V}_\omega^{(2n)} e^{i\theta_\omega^{(2n)}})^*)_{j,k} e_{\{2n+1,j\}} \rangle \right|^2 \\ &= \left| \sum_{j=1}^L ((\widehat{V}_\omega^{(2n)} e^{i\theta_\omega^{(2n)}})^*)_{j,k} \langle \mathbb{X}_\omega e_{\{m,l\}} \mid F(z) e_{\{2n+1,j\}} \rangle \right|^2 \end{aligned}$$

By the Cauchy–Schwarz inequality, we obtain:

$$\begin{aligned} & \left| \langle e_{\{m,l\}} | (\mathbb{U}_\omega - z)^{-1} e_{\{2n+1,k\}} \rangle \right|^2 \\ & \leq \left(\sum_{j=1}^L \left| \langle (\widehat{V}_\omega^{(2n)})^* \rangle_{j,k} \right|^2 \right) \left(\sum_{j=1}^L \left| \langle \mathbb{X}_\omega e_{\{m,l\}} | F(z) e_{\{2n+1,j\}} \rangle \right|^2 \right) \end{aligned}$$

Since $\widehat{V}_\omega^{(2n)}$ is a unitary matrix, its k^{th} column is of norm 1 and $\sum_{j=1}^L \left| \langle (\widehat{V}_\omega^{(2n)})^* \rangle_{j,k} \right|^2 = 1$. Hence,

$$\left| \langle e_{\{m,l\}} | (\mathbb{U}_\omega - z)^{-1} e_{\{2n+1,k\}} \rangle \right|^2 \leq \sum_{j=1}^L \left| \langle \mathbb{X}_\omega e_l | F(z) e_{\{2n+1,j\}} \rangle \right|^2.$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left((1 - |z|^2) \left| \langle e_{\{m,l\}} | (\mathbb{U}_\omega - z)^{-1} e_{\{2n+1,k\}} \rangle \right|^2 \right) \\ & \leq \sum_{j=1}^L \mathbb{E} \left((1 - |z|^2) \left| \langle \mathbb{X}_\omega e_{\{m,l\}} | F(z) e_{\{2n+1,j\}} \rangle \right|^2 \right) \end{aligned}$$

Due to the band structure of the operator \mathbb{X}_ω , there exists a constant $C_2(s)$ such that:

$$\begin{aligned} & \mathbb{E} \left((1 - |z|^2) \left| \langle e_{\{m,l\}} | (\mathbb{U}_\omega - z)^{-1} e_{\{2n+1,k\}} \rangle \right|^2 \right) \\ & \leq C_2(s) \sum_{|p-l| \leq 4L} \sum_{j=1}^L \mathbb{E} \left((1 - |z|^2) |F(z, p, j)|^2 \right) \\ & \leq C_2(s) 2^s \sum_{|p-l| \leq 4L} \sum_{j=1}^L \mathbb{E} (|F(z, p, j)|^s) \\ & \leq C_1(s) C_2(s) 2^s \sum_{|p-l| \leq 4L} \sum_{j=1}^L \sum_{|q-p| \leq 4L} \\ & \mathbb{E} \left(\left\| G_\omega \left(z, \left[\frac{p-l+q}{L} \right], \left[\frac{(2n+1)L+j}{L} \right] \right) \right\|^s \right). \end{aligned}$$

We conclude the proof with the exponential decay of $\mathbb{E}(\|G_\omega(z, p-l+q, (2n+1)L+j)\|^s)$ given by Theorem 1.9 and the fact that each of the three sums has a finite number of terms. □

4.2 Proof of Theorem 1.3

First of all, recall that for U a unitary operator,

$$\forall n \in \mathbb{Z}, U^n = \lim_{r \rightarrow 1^-} \frac{1-r^2}{2\pi} \int_0^{2\pi} (U - re^{i\theta})^{-1} (U^{-1} - re^{-i\theta})^{-1} e^{in\theta} d\theta. \quad (4.4)$$

Using this formula, of which we find an elementary proof in [26], we can prove Theorem 1.3 following exactly the proof of [26, Theorem 8.2], since we already have obtained the main input of this proof, estimate (1.22).

Data Availability this manuscript does not contain any result based upon any data.

Declarations

Conflict of interest all authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

References

1. Marin, L., Schulz-Baldes, H.: Scattering zippers and their spectral theory. *Journal of Spectral Theory* **3**(1), 47–82 (2013)
2. Boumaza, H., Marin, L.: Absence of absolutely continuous spectrum for random scattering zippers. *Journal of Mathematical Physics* **56**(2) (2015)
3. Carmona, R., Lacroix, J.: *Spectral Theory of Random Schrödinger Operators*. Springer, ??? (2012)
4. Kirsch, W.: An invitation to random Schrödinger operators, 1–119 (2008)
5. Blatter, G., Browne, D.A.: Zener tunneling and localization in small conducting rings. *Physical Review B* **37**(8), 3856 (1988)
6. Asch, J., Bourget, O., Joye, A.: Localization properties of the chalker–coddington model. In: *Annales Henri Poincaré*, vol. 11, pp. 1341–1373 (2010). Springer
7. Bourget, O., Howland, J.S., Joye, A.: Spectral analysis of unitary band matrices. *Communications in mathematical physics* **234**, 191–227 (2003)
8. Simon, B.: Orthogonal Polynomials on the Unit Circle. American Mathematical Soc., ??? (2005)
9. Schulz-Baldes, H.: Rotation numbers for Jacobi matrices with matrix entries. *Math. Phys. Electron. J.* **13**, 40 (2007). Id/No 5
10. Cantero, M.J., Moral, L., Velázquez, L.: Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle. *Linear Algebra Appl.* **362**, 29–56 (2003). [https://doi.org/10.1016/S0024-3795\(02\)00457-3](https://doi.org/10.1016/S0024-3795(02)00457-3)
11. Hamza, E., Joye, A., Stolz, G.: Dynamical localization for unitary anderson models. *Mathematical Physics, Analysis and Geometry* **12**, 381–444 (2009)
12. Damanik, D., Pushnitski, A., Simon, B.: The analytic theory of matrix orthogonal polynomials. *Surveys in Approximation Theory* **4**, 1–85 (2008)
13. Hamza, E., Joye, A., Stolz, G.: Localization for random unitary operators. *Letters in Mathematical Physics* **75**(3), 255–272 (2006)
14. Joye, A.: Density of states and Thouless formula for random unitary band matrices. *Ann. Henri Poincaré* **5**(2), 347–379 (2004)
15. Joye, A.: Fractional moment estimates for random unitary operators. *Lett. Math. Phys.* **72**(1), 51–64 (2005). <https://doi.org/10.1007/s11005-005-3256-8>
16. Joye, A.: Random unitary models and their localization properties. In: *Entropy and the Quantum II*. 2nd Arizona School of Analysis with Applications, University of Arizona, Tucson, AZ, USA, March 15–19, 2010, pp. 117–134. Providence, RI: American Mathematical Society (AMS), ??? (2011)
17. Aizenman, M., Warzel, S.: *Random Operators* vol. 168. American Mathematical Soc., ??? (2015)

18. Asch, J., Bourget, O., Joye, A.: Dynamical localization of the chalker-coddington model far from transition. *Journal of Statistical Physics* **147**, 194–205 (2012)
19. Cedzich, C., Werner, A.H.: Anderson localization for electric quantum walks and skew-shift CMV matrices. *Commun. Math. Phys.* **387**(3), 1257–1279 (2021). <https://doi.org/10.1007/s00220-021-04204-w>
20. Zhu, X.: Localization for random CMV matrices. *J. Approx. Theory* 298, 20 (2024) <https://doi.org/10.1016/j.jat.2023.106008> . Id/No 106008
21. Bucaj, V., Damanik, D., Fillman, J., Gerbuz, V., VandenBoom, T., Wang, F., Zhang, Z.: Localization for the one-dimensional anderson model via positivity and large deviations for the lyapunov exponent. *Transactions of the American Mathematical Society* **372**(5), 3619–3667 (2019)
22. Boumaza, H.: Localization for random quasi-one-dimensional models. *Journal of Mathematical Physics* **64**(9) (2023)
23. Kotani, S., Simon, B.: Stochastic schrödinger operators and jacobi matrices on the strip. *Communications in mathematical physics* **119**, 403–429 (1988)
24. Bougerol, P., Lacroix, J.: Products of random matrices with applications to Schrödinger operators. Birkhäuser, Boston, MA (1985)
25. Boumaza, H.: Exposants de lyapunov et densité d'états intégrée pour des opérateurs de schrödinger continus à valeurs matricielles. PhD thesis, Université Paris-Diderot-Paris VII (2007)
26. Hamza, E.F.: Localization properties for the unitary Anderson model. The University of Alabama at Birmingham (PhD Thesis 2007)
27. Boumaza, H.: Localization for a matrix-valued anderson model. *Mathematical Physics, Analysis and Geometry* **12**(3), 255–286 (2009)
28. Graf, G.M.: Anderson localization and the space-time characteristic of continuum states. *Journal of Statistical Physics* **75**, 337–346 (1994)
29. Aizenman, M., Molchanov, S.: Localization at large disorder and at extreme energies: An elementary derivations. *Communications in Mathematical Physics* **157**, 245–278 (1993)
30. Aizenman, M., Graf, G.M.: Localization bounds for an electron gas. *Journal of Physics A: Mathematical and General* **31**(32), 6783 (1998)

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