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The band spectrum of the periodic Airy–Schrödinger operator on the real line

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Abstract

We introduce the periodic Airy–Schrödinger operator and we describe its band spectrum. This is an example of solvable model with a periodic potential which is not differentiable at its extrema. We prove that there exists a sequence of explicit constants giving upper bounds of the semiclassical parameter for which explicit estimates are valid. We completely determine the behaviour of the edges of the first spectral band with respect to the semiclassical parameter. Then, we investigate the spectral bands and gaps situated in the range of the potential. We prove precise estimates on the widths of these spectral bands and these spectral gaps and we determine an upper bound on the integrated spectral density in this range. Finally, we get estimates of the edges of spectral bands and thus of the widths of spectral bands and spectral gaps which are stated for values of the semiclassical parameter in fixed intervals.

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1. Introduction and outline of the paper

For periodic Schrödinger operators, the semiclassical behaviour of the bottom of the spectrum and of the widths of the spectral bands and gaps is well known when the potential is analytic

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or C^∞. We introduce a periodic Schrödinger operator whose potential is not differentiable at its minima and maxima points and for which we still determine the semiclassical behaviour of the band edges and of the widths of the spectral bands and gaps.

The spectrum of the Schrödinger operator for a linear potential well is known. The eigenvalues are given by the absolute values of the zeroes of the first Airy function and its derivative. Our periodic Schrödinger operator with singular potential is a periodization of the truncated linear potential well. We expect to find a spectrum made of spectral bands, each containing one of the eigenvalues of the Schrödinger operator with a linear potential well. This is proven in the present paper in Theorem 2.5.

1.1. From the model to the semiclassical problem

Let $2L_0 \in \mathbb{R}_+^*$ be a characteristic length modelling the distance between two ions in a one dimensional periodic lattice of ions. The behaviour of electrons of mass m in this lattice is modelled by the following $2L_0$-periodic Schrödinger operator acting on the Sobolev space $\mathsf{H}^2(\mathbb{R})$,

$$H = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V, \tag{1.1}$$

where $\hbar$ is the reduced Planck constant and V is the $2L_0$-periodic function on $\mathbb{R}$ defined by

$$\forall z \in [-L_0, L_0],\ V(z) = V_0\left(\tfrac{|z|}{L_0} - 1\right),$$

$V_0 \in \mathbb{R}_+^*$ being a reference potential. The ions, in this model, are located at points $2nL_0$ for $n \in \mathbb{Z}$, this points corresponding to the minima of the potential V.

We call H the *periodic Airy–Schrödinger operator* on $\mathbb{R}$.

Since V is periodic and locally integrable, the theory of periodic operators ([12]) asserts that the operator H has purely absolutely continuous spectrum and that this spectrum is the union of spectral bands:

$$\sigma(H) = \bigcup_{p\geq 0} [E^p_{\min}, E^p_{\max}],$$

where $E^p_{\min}$ and $E^p_{\max}$ are the spectral band edges. For $p \geq 0$, we shall call $[E^p_{\min}, E^p_{\max}]$ the p-th spectral band and $(E^p_{\max}, E^{p+1}_{\min})$ the p-th spectral gap. We will state precisely these notations and characterize these spectral band edges in Section 3.2.

Traditional results describe the spectral bands near the minimum of a C^2 potential ([5]). We generalize this analysis to the case of the potential V which is not differentiable on its minima points. We are able to count the number of spectral bands in $[-V_0, 0]$, the range of the potential V, for any value of a dimensionless parameter defined in (1.3), and to describe precisely these spectral bands.

In order to describe the spectrum of H, one considers the equation on $\psi \in \mathsf{H}^2(\mathbb{R})$,

$$-\frac{\hbar^2}{2m}\psi'' + V(z)\psi = E\psi\ . \tag{1.2}$$

As V is affine on every interval $[nL_0, (n+1)L_0]$ for every $n \in \mathbb{Z}$, one recognizes in (1.2), after rescaling and translating the variable, the Airy equation: $u'' = zu$. The canonical solutions u and v (with Cauchy data $(1,0)$ and $(0,1)$) will be used throughout our study, as well as the classical pair of solutions (Ai, Bi).

We introduce the semiclassical parameter h

$$\mathsf{h} := \frac{\hbar}{L_0(2mV_0)^{\frac{1}{2}}}. \tag{1.3}$$

Notation. For any real number E, we set $\mathbf{E} = \frac{E}{V_0}$. In particular, we set, for any $p \geq 0$, $\mathbf{E}^p_{\min} = \frac{E^p_{\min}}{V_0}$ and $\mathbf{E}^p_{\max} = \frac{E^p_{\max}}{V_0}$.

We use the semiclassical parameter h to rewrite the periodic Airy–Schrödinger operator in a form analog to the operator studied in [5].

Definition 1.1. Let $\mathbf{H}$ be the periodic Schrödinger operator acting on the Sobolev space $\mathsf{H}^2(\mathbb{R})$ and defined by

$$\mathbf{H} = -\mathsf{h}^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi + \mathbf{V}(x) \tag{1.4}$$

with $\mathbf{V}$ the 2-periodic function equal to $\mathbf{V}(x) = |x| - 1$ on the interval $[-1, 1]$.

Then, the equation (1.2) is equivalent to:

$$\mathbf{H}\phi = \mathbf{E}\phi, \quad \phi \in \mathsf{H}^2(\mathbb{R}). \tag{1.5}$$

One of the purposes of semiclassical analysis is to prove estimates, when h tends to 0, of quantities depending on h. If these estimates are valid for any value of the semiclassical parameter in a fixed interval $(0, \mathsf{h}_0]$ with $\mathsf{h}_0 > 0$, we want to say that the considered estimate is universal in $(0, \mathsf{h}_0]$. This notion of universal estimate in the context of semiclassical analysis follows the discussion of [6] about the domain of validity of the semiclassical analysis for their model.

In the literature, many results of semiclassical analysis are stated in this setting, providing estimates uniform on $\mathsf{h} \in (0, \mathsf{h}_0]$ (see [13,11]). An explicit constant h_0 is found in [7], though, in general, the constant is not explicit. For general references about semiclassical analysis we refer to the textbooks [3,9].

We give explicit values of h_0, for estimates of the spectral band edges and thus of the widths of the spectral bands and gaps. These estimates take the following form. Let $\mathbf{E}_p$ be an edge of the p-th spectral band $[\mathbf{E}^p_{\min}, \mathbf{E}^p_{\max}]$. If $\mathbf{E}_p$ is in the range of $\mathbf{V}$, we prove that there exist an explicit $\mathsf{h}_0(p) > 0$ and universal constants $C_p > 0$ and $\lambda_p \neq 0$ such that for every $\mathsf{h} \in (0, \mathsf{h}_0(p))$,

$$\left|\mathbf{E}_p - \left(-1 + \mathfrak{a}_p \mathsf{h}^{\frac{2}{3}} + \lambda_p \mathsf{h}^{\frac{2}{3}} e^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \mathfrak{a}_p)^{\frac{3}{2}}}\right)\right| \leq C_p \mathsf{h}^{\frac{5}{3}} (1 - \mathfrak{a}_p \mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}} e^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \mathfrak{a}_p)^{\frac{3}{2}}}, \tag{1.6}$$

where $\mathfrak{a}_p$ is defined in Section 2.4. Moreover, (1.6) extends to $\mathsf{h} = 0$ by continuity in h of $\mathbf{E}_p$ (setting $\mathbf{E}_p(0) = -1$) and the fact that the other quantities admit finite limits when h tends to 0.

1.2. Outline of the paper

In Section 3.1, we describe the localization of the zeroes of the canonical solutions of the Airy equation and their derivatives. These intervals of localization imply an important result of separation and ordering of the zeroes of the canonical solutions and their derivatives. In Section 3.2, the equations on the edges of the spectral bands are derived from the classical theory of periodic operators. The result on ordering the roots of these equations is the key result which allows to distinguish the upper edge of a spectral band from the lower edge of the next one among the solutions of the equations obtained in Section 3.2. This numbering of the spectral edges is performed in Sections 4 and 5.

Section 3 is devoted to the study of families of strictly monotonous and continuous functions which allows, in Sections 4 and 5, to describe the solutions to the equations which define the spectral edges. Section 3 gives also the graphical interpretation of these equations, in terms of the functions $\frac{v}{u}$ and $\frac{v'}{u'}$, which has guided our analysis throughout this paper.

Section 4.2 is devoted to the proof of Theorem 2.2. This proof contains most of the results used later in Section 6 to get all the uniform estimates of the widths of the spectral bands and of the spectral gaps. We also investigate in Section 4.3 the behaviour of the upper edge of the first spectral band in the semiclassical limit $\mathsf{h} \to 0$ as well as for h tends to infinity. In Section 5 we characterize the spectral edges of all the spectral bands in the range of $\mathbf{V}$, we count these bands and we prove Theorem 2.4. We also prove a result on the integrated spectral density in the range of $\mathbf{V}$.

In Appendix B, a variant of the Sturm–Picone lemma showing interlacing results on zeroes of solutions of general ordinary differential equations leads, in Appendix C, to monotonicity of auxiliary functions used in the sequel. This is a key point of the arguments developed in this paper.

2. Notations and main results

2.1. The canonical solutions of the Airy equation

Recall that u and v are the canonical solutions of the Airy equation, satisfying

$$u(0) = 1,\ u'(0) = 0 \quad \text{and} \quad v(0) = 0,\ v'(0) = 1,$$

which Wronskian is 1. One has the expression of u and v in terms of the classical Airy functions Ai and Bi:

$$\forall x \in \mathbb{R},\ u(x) = \pi(Bi'(0)Ai(x) - Ai'(0)Bi(x)) \tag{2.1}$$

and

$$\forall x \in \mathbb{R},\ v(x) = \pi(Ai(0)Bi(x) - Bi(0)Ai(x)). \tag{2.2}$$

Both u and v are analytic functions on $\mathbb{R}$. Moreover, u and v are strictly increasing and positive on $(0, +\infty)$. Thus, the zeroes of u, v and their derivatives are all non-positive real numbers.

Notation. We denote by

1. $\{-\tilde{c}_{2j}\}_{j\geq 0}$ and $\{-\tilde{c}_{2j+1}\}_{j\geq 0}\cup\{0\}$ the set of the zeroes of respectively u and u',
2. $\{-c_{2j+1}\}_{j\geq 0}\cup\{0\}$ and $\{-c_{2j}\}_{j\geq 0}$ the set of the zeroes of respectively v and v'.

A direct consequence of Proposition 3.1 is: $\forall p\geq 0,\ -\tilde{c}_p < -c_p$.

2.2. *The first spectral band*

Our first result gives the initialization of the counting of the spectral bands which are included in the range of $\mathbf{V}$, $[-1,0]$.

Theorem 2.1. *For* $\mathsf{h}\geq c_0^{-\frac{3}{2}}$, *there is no spectral gap of* $\mathbf{H}$ *in* $[-1,0]$. *The first spectral gap intersects* $[-1,0)$ *as soon as* $0<\mathsf{h}<c_0^{-\frac{3}{2}}$.

We get precise estimates of the ground state $\mathbf{E}^0_{\min}$ for values of h in a fixed interval of the form $(0,\mathsf{h}_0)$ and when h tends to infinity. Before stating them, we need to introduce notations for the zeroes of the Airy function Ai and its derivative.

Notation. We denote by $\{-a_j\}_{j\geq 1}$ the set of the zeroes of Ai and by $\{-\tilde{a}_j\}_{j\geq 1}$ the set of the zeroes of Ai' where the real numbers $-a_j$ and $-\tilde{a}_j$ are arranged in decreasing order. These sets are both subsets of $(-\infty,0]$. Moreover, for every $j\geq 1$, $-a_j\in(-\tilde{a}_{j+1},-\tilde{a}_j)$. We set $\alpha=-\frac{Ai(0)}{Ai'(0)}>0$. An approximate value of α is: $\alpha\simeq 1{,}372$.

All the quantities in (2.4) below are continuous in h at 0 if one replaces $\mathbf{E}^0_{\min}$ by -1 and all the other quantities by their limit.

Theorem 2.2. *We have the following estimates on* $\mathbf{E}^0_{\min}$*:*

1. *For every* $\mathsf{h}>0$

$$-1<\mathbf{E}^0_{\min}<\min\left(-\tfrac{1}{2},\ -1+\tilde{a}_1\mathsf{h}^{\frac{2}{3}}\right). \tag{2.3}$$

2. *There exists a universal constant* $M_0>0$ *such that, for every* $\mathsf{h}\in(0,\tilde{a}_1^{-\frac{3}{2}})$,

$$\left|\mathbf{E}^0_{\min}-\left(-1+\tilde{a}_1\mathsf{h}^{\frac{2}{3}}+\alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}\right)\right|\leq M_0\mathsf{h}^{\frac{5}{3}}(1-\tilde{a}_1\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}} \tag{2.4}$$

Remark 2.1. In the proof of (2.4), we get a family of explicit constants $M_{0,\tau}$ depending on a technical parameter $\tau>0$ for which (2.4) is valid. Optimizing in τ, we find an approximate value of the minimum of $\tau\mapsto M_{0,\tau}$ equal to 7271. We can take this value for the constant M_0. Note that this value is not proven to be optimal for the universal estimate (2.4).

When h tends to infinity, the first spectral band of $\mathbf{H}$ satisfies:

Theorem 2.3.

1. *One has,*

$$\lim_{\mathsf{h}\to+\infty} \mathbf{E}^0_{\min} = -\frac{1}{2} \quad \textit{and} \quad \lim_{\mathsf{h}\to+\infty} \mathbf{E}^0_{\max} = +\infty.$$

2. *More precisely, when* h *tends to infinity,*

$$\mathbf{E}^0_{\min} = -\frac{1}{2} - \frac{1}{120}\frac{1}{\mathsf{h}^2} + \mathcal{O}\left(\frac{1}{\mathsf{h}^4}\right). \tag{2.5}$$

2.3. Counting and estimates of the spectral bands in the range of V

We have estimates on the widths of the spectral bands and the spectral gaps which are located in the range of $\mathbf{V}$.

Two h-dependent integers are of interest in this paper:

1. the unique integer p_0 such that

$$\tilde{c}_{p_0+1} - \tilde{c}_{p_0} < \mathsf{h}^{-\frac{2}{3}} \leq \tilde{c}_{p_0} - \tilde{c}_{p_0-1} \quad \text{when} \quad \mathsf{h} \geq c_0^{-\frac{3}{2}}; \tag{2.6}$$

2. the unique integer k_0 such that

$$c_{k_0} < \mathsf{h}^{-\frac{2}{3}} < \tilde{c}_{k_0} \quad \text{or} \quad \tilde{c}_{k_0} \leq \mathsf{h}^{-\frac{2}{3}} < c_{k_0+1} \quad \text{when} \quad \mathsf{h} < c_0^{-\frac{3}{2}}. \tag{2.7}$$

Denote the integer part of a real number x by $[x]$. One has $k_0 = \left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right]$ or $k_0 = \left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right] - 1$.

Let $p \geq 0$ an integer and denote by $\delta_p = \mathbf{E}^p_{\max} - \mathbf{E}^p_{\min}$ the width of the p-th spectral band of $\mathbf{H}$ and by $\gamma_p = \mathbf{E}^{p+1}_{\min} - \mathbf{E}^p_{\max}$ the width of the p-th spectral gap.

Theorem 2.4. *Let* $\mathsf{h} < c_0^{-\frac{3}{2}}$.

1. *The* $k_0 + 1$ *first spectral bands are included in the range of* $\mathbf{V}$.
2. *One has, for every* $p \in \{2, \dots, k_0\}$,

$$0 < \delta_p \leq \left(\frac{\pi}{3} + \frac{7}{3\pi}\frac{p+\frac{1}{3}}{p(p+\frac{2}{3})}\right)\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \frac{h^{\frac{2}{3}}}{p^{\frac{1}{3}}}, \tag{2.8}$$

and for every $p \in \{2, \dots, k_0 - 1\}$,

$$0 < 2\left(\left(\tfrac{7}{6}\right)^{\frac{2}{3}} - 1\right)\left(\tfrac{\pi}{3}\right)^{\frac{2}{3}} \frac{h^{\frac{2}{3}}}{(p+1)^{\frac{1}{3}}} < \gamma_p \leq \left(\pi + \frac{7}{3\pi}\frac{p}{p^2-1}\right)\left(\frac{3}{\pi}\right)^{\frac{1}{3}} \frac{h^{\frac{2}{3}}}{(p-1)^{\frac{1}{3}}}. \tag{2.9}$$

In particular, none of the gaps in $\sigma(\mathbf{H}) \cap [-1, 0]$ *is empty.*

Thanks to $k_0^{-\frac{1}{3}}$ is of order $\mathsf{h}^{\frac{1}{3}}$, the highest spectral band in the range of $\mathbf{V}$ is of size of order h.

We do not prove in this paper a lower bound of δ_p. A still open conjecture is whether or not δ_p has an exponential lower bound.

For any $\beta > 1$, the inequality in (2.9) shows that $(p^\beta \gamma_p)_{p\geq 2}$ is not bounded when h tends to 0. It was expected since, by results of Hochstadt, it would imply that V is a smooth function (see [8]). Moreover, an exponentially small upper bound of γ_p is characteristic from the analyticity of V (see [14]).

The inequality (2.8) implies an upper bound for the spectral density in the range of the potential $\mathbf{V}$ in the semiclassical limit. Let $k_0(\mathsf{h})$ be the integer defined in (2.7). For any $\mathsf{h} < c_0^{-\frac{3}{2}}$, we denote by $D(\mathsf{h})$ the sum of the lengths of the $k_0 + 1$ first spectral bands (which are all included in the range of $\mathbf{V}$) divided by the length of the range of $\mathbf{V}$ (which is equal to 1):

$$\forall \mathsf{h} < c_0^{-\frac{3}{2}},\ D(\mathsf{h}) = \sum_{p=0}^{k_0(\mathsf{h})} \delta_p.$$

Corollary 2.1. *When* h *tends to* 0, $D(\mathsf{h})$ *admits a limit denoted by* D_V. *Moreover,*

$$0 < D_V \leq \left(\frac{2}{3}\right)^{\frac{1}{3}}. \tag{2.10}$$

The limit D_V can be interpreted as the integrated spectral density in the range of the potential $\mathbf{V}$ in the semiclassical limit.

Note that the number of gaps intersecting the range of $\mathbf{V}$ increases by one each time the semiclassical parameter is equal to one of the numbers $c_p^{-\frac{3}{2}}$, $p \geq 0$. To complete the first point of Theorem 2.4 we observe that the roots of the canonical solutions of the Airy equation and their derivatives characterize the values of h for which a spectral band either enters in the range of the potential or completes its entrance:

Proposition 2.1. *There exists a unique spectral band for which either the upper or the lower edge is equal to* 0 *if and only if* $\mathsf{h} \in \{c_p^{-\frac{3}{2}}, \tilde{c}_p^{-\frac{3}{2}}\}_{p\geq 0}$.

The counting of energy levels in the range of the potential can be found in [4], in the simple case of the rectangular potential hole, which is not periodic (and thus [4] counts eigenvalues and not spectral bands). In [4], explicit values of the “size parameter” for which an eigenvalue enters the range of the potential are not given.

2.4. Universal estimates for the spectral bands and gaps

Thanks to the explicit form of the bands, universal estimates are proven.

Notation. Let $j \geq 0$ and define the real numbers $\mathfrak{a}_{2j} = \tilde{a}_{j+1}$ and $\mathfrak{a}_{2j+1} = a_{j+1}$.

Theorem 2.5.

1. *Let* $p \geq 0$*. The shifted and rescaled* p*-th spectral band,*
$$[\mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^p_{\min}), \mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^p_{\max})]$$
tends to the singleton $\{\mathfrak{a}_p\}$ *when* h *tends to* 0*.*
2. *There exists a universal constant* $K_{2j} > 0$ *such that for every* $\mathsf{h} \in (0, c_{2j}^{-\frac{3}{2}}]$*,*
$$\left|\delta_{2j} - 2\alpha\sqrt{3}\frac{(u'(-\mathfrak{a}_{2j}))^2}{\mathfrak{a}_{2j}}\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\mathfrak{a}_{2j})^{\frac{3}{2}}}\right| \leq K_{2j}\mathsf{h}^{\frac{5}{3}}(1-\mathfrak{a}_{2j}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\mathfrak{a}_{2j})^{\frac{3}{2}}} \tag{2.11}$$
when $p = 2j$ *and there exists a universal constant* $K_{2j+1} > 0$ *such that for every* $\mathsf{h} \in (0, c_{2j+1}^{-\frac{3}{2}}]$*,*
$$\left|\delta_{2j+1} - 2\alpha\sqrt{3}(u(-\mathfrak{a}_{2j+1}))^2\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\mathfrak{a}_{2j+1})^{\frac{3}{2}}}\right| \leq K_{2j+1}\mathsf{h}^{\frac{5}{3}}(1-\mathfrak{a}_{2j+1}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\mathfrak{a}_{2j+1})^{\frac{3}{2}}} \tag{2.12}$$
when $p = 2j+1$*.*
3. *Let* $\mathsf{h} \in (0, c_0^{-\frac{3}{2}}]$*. For every* $p \in \{0, \ldots, \left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right]\}$*, (2.11) holds true if* $p = 2j$ *is even and (2.12) holds true if* $p = 2j+1$ *is odd.*

Remark 2.2. Formulas (2.11) and (2.12) hold true when one replaces, for $\mathsf{h} = 0$, δ_p by 0 and all the quantities by their limit.

Remark 2.3. Explicit formulas for the universal constants K_{2j} and K_{2j+1} are deduced from those given for M_{2j}, $\tilde{M}_{2j}$, M_{2j+1} and $\tilde{M}_{2j+1}$ in the proof of Proposition 6.1. Note that we did not prove that these universal constants are optimal.

Remark 2.4. Using Taylor formula for the exponential function, estimate (2.11) implies that there exists a universal constant $K'_{2j} > 0$ such that, for every $\mathsf{h} \in (0, c_{2j}^{-\frac{3}{2}}]$,
$$\left|\delta_{2j} - 2\alpha\sqrt{3}\frac{(u'(-\mathfrak{a}_{2j}))^2}{\mathfrak{a}_{2j}}\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}\mathsf{h}^{-1}+2\mathfrak{a}_{2j}\mathsf{h}^{-\frac{1}{3}}}\right| \leq K'_{2j}\mathsf{h}\mathsf{e}^{-\frac{4}{3}\mathsf{h}^{-1}+2\mathfrak{a}_{2j}\mathsf{h}^{-\frac{1}{3}}} \tag{2.13}$$
and we have a similar estimate for p odd. In (2.13), the factor $\frac{4}{3}$ in front of the h^{-1} is the Agmon distance between two consecutive minima of the potential $\mathbf{V}$. Hence, (2.13) shows an extra term $2\mathfrak{a}_{2j}\mathsf{h}^{-\frac{1}{3}}$ for the tunnelling effect compared, for example, to the quadratic double well.

The first statement of Theorem 2.5 shows the convergence of the band spectrum of the periodic Airy–Schrödinger operator to the pure point spectrum of the Schrödinger operator for a linear potential well. Note that the eigenspace associated with $\mathfrak{a}_{2j}$ is spanned by $x \mapsto Ai(|x| - \mathfrak{a}_{2j})$ and the eigenspace associated with $\mathfrak{a}_{2j+1}$ is spanned by $x \mapsto \mathrm{sign}(x) \cdot Ai'(|x| - \mathfrak{a}_{2j+1})$.

The spectral bands obtained for this non-C^1 potential are larger than the ones obtained in the case of a C^2 potential. Indeed, Theorem 1.1 of [5] shows spectral bands of widths proportional to an exponential term equal to

$$\mathsf{h}\mathsf{e}^{-\frac{4}{3}\mathsf{h}^{-1}(1-(2p+1)\mathsf{h})^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\mathsf{h}^{\frac{1}{4}}\right)\right). \tag{2.14}$$

This difference of behaviour is expected since the size of the spectral bands depends strongly on the regularity of the potential. In [5], the potential is assumed to be at least two times differentiable at its minima and maxima points while in our case, it is not even differentiable at these points.

The width of a spectral gap is close to the difference of two consecutive eigenvalues of the Airy–Schrödinger operator multiplied by the $\mathsf{h}^{\frac{2}{3}}$ factor.

Theorem 2.6.

1. *For every $p \geq 0$, there exists a universal constant $\tilde{K}_p > 0$ such that, for every $\mathsf{h} \in (0, \mathfrak{a}_{p+1}^{-\frac{3}{2}})$,*

$$\left|\gamma_p - (\mathfrak{a}_{p+1} - \mathfrak{a}_p)\mathsf{h}^{\frac{2}{3}}\right| \leq \tilde{K}_p \mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \mathfrak{a}_{p+1})^{\frac{3}{2}}}. \tag{2.15}$$

2. *Let $\mathsf{h} \in (0, c_1^{-\frac{3}{2}}]$. For every $p \in \{0, \dots, \left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right] - 1\}$, (2.15) holds true.*

Remark 2.5. Notice that, for every $\mathsf{h} > 0$, $\mathsf{h} \leq \mathfrak{a}_{\left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right]}^{-\frac{3}{2}}$, which justifies the second statement of Theorem 2.6.

Sharper estimates for the widths of the gaps are found in Proposition 6.2.

We can also prove, using the bootstrap technique proof of Theorem 2.2 and after reversing the numbering of the bands, a result similar to Theorem 8.1 of [10] (it is the case $\mu \in [-Ch, 0]$ in its notations) for the widths of the gaps and the bands, but with a different order of magnitude of these widths. These differences are due to the fact that in [10, Theorem 8.1], the potential is supposed to be analytic and in our case it is only piecewise analytic.

3. Preliminaries to the computation of the band edges

3.1. Localization of the zeroes of the canonical solutions

The aim of this Section is to obtain precise intervals of localization of the roots of u, v, u' and v'. This will be a consequence of the expressions of these canonical solutions in terms of fractional Bessel functions.

For this purpose, one defines functions $P(\nu,\cdot)$ and $Q(\nu,\cdot)$ for any real number ν through the Bessel functions J_ν (see [11])

$$P(\nu,\xi)=\sqrt{\tfrac{\pi\xi}{2\sin(\nu\pi)}}\left(J_\nu(\xi)\sin\left(\xi+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)+J_{-\nu}(\xi)\sin\left(\xi-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right)$$

and

$$Q(\nu,\xi)=\sqrt{\tfrac{\pi\xi}{2\sin(\nu\pi)}}\left(J_\nu(\xi)\cos\left(\xi+\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)-J_{-\nu}(\xi)\cos\left(\xi-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right).$$

The functions $P(\nu,\cdot)$ and $Q(\nu,\cdot)$ have known expansions. In particular one has, for every $\xi>\frac{1}{\sqrt{26}}$, $P(\frac{1}{3},\xi)>0$ and

$$\forall\xi>\frac{1}{\sqrt{13}},\ \left|\frac{Q(\frac{1}{3},\xi)}{P(\frac{1}{3},\xi)}\right|<\frac{5}{36\xi}. \tag{3.1}$$

Moreover, for every $\xi>\frac{1}{\sqrt{22}}$, $P(\frac{2}{3},\xi)>0$ and

$$\forall\xi>\frac{1}{\sqrt{11}},\ \left|\frac{Q(\frac{2}{3},\xi)}{P(\frac{2}{3},\xi)}\right|<\frac{7}{12\xi}. \tag{3.2}$$

Notation. For every $x>0$, we set $\xi=\frac{2}{3}x^{\frac{3}{2}}$.

We have, for every $x>0$,

$$u(-x)=2\pi^{\frac{1}{2}}x^{-\frac{1}{4}}Ai'(0)\left(\sin\left(\xi-\tfrac{7\pi}{12}\right)P\left(\tfrac{1}{3},\xi\right)+\cos\left(\xi-\tfrac{7\pi}{12}\right)Q\left(\tfrac{1}{3},\xi\right)\right), \tag{3.3}$$

$$u'(-x)=-2\pi^{\frac{1}{2}}x^{\frac{1}{4}}Ai'(0)\left(\cos\left(\xi-\tfrac{7\pi}{12}\right)P\left(\tfrac{2}{3},\xi\right)-\sin\left(\xi-\tfrac{7\pi}{12}\right)Q\left(\tfrac{2}{3},\xi\right)\right), \tag{3.4}$$

$$v(-x)=-2\pi^{\frac{1}{2}}x^{-\frac{1}{4}}Ai(0)\left(\sin\left(\xi+\tfrac{\pi}{12}\right)P\left(\tfrac{1}{3},\xi\right)+\cos\left(\xi+\tfrac{\pi}{12}\right)Q\left(\tfrac{1}{3},\xi\right)\right), \tag{3.5}$$

$$v'(-x)=-2\pi^{\frac{1}{2}}x^{\frac{1}{4}}Ai(0)\left(-\cos\left(\xi+\tfrac{\pi}{12}\right)P\left(\tfrac{2}{3},\xi\right)+\sin\left(\xi+\tfrac{\pi}{12}\right)Q\left(\tfrac{2}{3},\xi\right)\right) \tag{3.6}$$

Heuristically, from these expressions and using (3.1) and (3.2), approximations of the zeroes of $u(-x)$, $u'(-x)$, $v(-x)$ and $v'(-x)$ for $x>0$ are given by:

$$\xi\pm\frac{\pi}{12}=n\pi+\delta\frac{\pi}{2},\quad n\in\mathbb{N},\ \delta\in\{0,1\},$$

the zeroes of the cosinus and sinus in (3.3), (3.4), (3.5) and (3.6).

Note that the roots of u, v, u' and v' are interlaced with the zeroes of Ai and Ai' thanks to (2.1) and (2.2). Thus, for every $j\geq 0$,

$$-\tilde{c}_{2j}\in(-a_{j+1},-\tilde{a}_{j+1}),\ -\tilde{c}_{2j+1}\in(-\tilde{a}_{j+2},-a_{j+1}), \tag{3.7}$$

$$-c_{2j}\in(-a_{j+1},-\tilde{a}_{j+1})\text{ and }-c_{2j+1}\in(-\tilde{a}_{j+2},-a_{j+1}). \tag{3.8}$$

In order to give more precise intervals of localization of the zeroes of the canonical solutions and their derivatives we use their expressions in terms of the fractional Bessel functions J_ν.

We have, for every $x > 0$,

$$u(-x) = -\tfrac{2\pi}{\sqrt{3}} Ai'(0)\sqrt{x} \cdot J_{-\frac{1}{3}}(\xi), \tag{3.9}$$

$$u'(-x) = -\tfrac{2\pi}{\sqrt{3}} Ai'(0) x \cdot J_{\frac{2}{3}}(\xi), \tag{3.10}$$

$$v(-x) = -\tfrac{2\pi}{\sqrt{3}} Ai(0)\sqrt{x} \cdot J_{\frac{1}{3}}(\xi), \tag{3.11}$$

$$v'(-x) = \tfrac{2\pi}{\sqrt{3}} Ai(0) x \cdot J_{-\frac{2}{3}}(\xi). \tag{3.12}$$

These expressions are obtained from the expressions of u and v in terms of the classical Airy functions, using the expressions valid for any $x > 0$,

$$Ai(-x) = \frac{1}{3}\sqrt{x}\left(J_{\frac{1}{3}}(\xi) + J_{-\frac{1}{3}}(\xi)\right), \quad Bi(-x) = \frac{1}{\sqrt{3}}\sqrt{x}\left(J_{-\frac{1}{3}}(\xi) - J_{\frac{1}{3}}(\xi)\right),$$

$$Ai'(-x) = -\frac{1}{3}x\left(J_{-\frac{2}{3}}(\xi) - J_{\frac{2}{3}}(\xi)\right) \quad \text{and} \quad Bi'(-x) = \frac{1}{\sqrt{3}}x\left(J_{-\frac{2}{3}}(\xi) + J_{\frac{2}{3}}(\xi)\right)$$

and the equalities $Bi(0) = \sqrt{3}Ai(0)$ and $Bi'(0) = -\sqrt{3}Ai'(0)$.

From the expressions (3.9), (3.10), (3.11) and (3.12) and the variations of the fractional Bessel functions (see [1, 9.1]), we also obtain the variations of u and v.

1. u is positive on $(-\tilde{c}_0, +\infty)$ and for every $j \geq 0$, $(-1)^j u$ is negative on $[-\tilde{c}_{2j+2}, -\tilde{c}_{2j}]$. It is strictly increasing on $(-\tilde{c}_1, +\infty)$, and for every $j \geq 0$, $(-1)^j u$ is strictly decreasing on $[-\tilde{c}_{2j+3}, -\tilde{c}_{2j+1}]$.
2. v is positive on $[0, +\infty)$, negative on $[-c_1, 0]$ and for every $j \geq 0$, $(-1)^j v$ is positive on $[-c_{2j+3}, -c_{2j+1}]$. It is strictly increasing on $(-c_0, +\infty)$, and for every $j \geq 0$, $(-1)^j v$ is strictly decreasing on $[-c_{2j+2}, -c_{2j}]$.

Notation. We introduce, for every $p \geq 0$,

$$\xi_p = \frac{2}{3} c_p^{\frac{3}{2}} \quad \text{and} \quad \tilde{\xi}_p = \frac{2}{3} \tilde{c}_p^{\frac{3}{2}}.$$

Proposition 3.1. *Let $j \geq 0$. One has*

$$\xi_{2j} \in \left[\frac{5\pi}{12} + j\pi - \frac{7}{12(j\pi + \frac{\pi}{3})}, \frac{5\pi}{12} + j\pi + \frac{7}{12(j\pi + \frac{\pi}{3})}\right], \tag{3.13}$$

$$\xi_{2j+1} \in \left[\frac{11\pi}{12} + j\pi - \frac{5}{36(j\pi + \frac{5\pi}{6})}, \frac{11\pi}{12} + j\pi + \frac{5}{36(j\pi + \frac{5\pi}{6})}\right], \tag{3.14}$$

$$\tilde{\xi}_{2j} \in \left[\frac{7\pi}{12} + j\pi - \frac{5}{36(j\pi + \frac{\pi}{2})}, \frac{7\pi}{12} + j\pi + \frac{5}{36(j\pi + \frac{\pi}{2})}\right], \tag{3.15}$$

$$\tilde{\xi}_{2j+1} \in \left[\frac{13\pi}{12} + j\pi - \frac{7}{12(j+1)\pi}, \frac{13\pi}{12} + j\pi + \frac{7}{12(j+1)\pi}\right]. \tag{3.16}$$

Proof. It can be deduced directly from the expressions (3.9), (3.10), (3.11) and (3.12) and the expansions of the zeroes of the fractional Bessel functions given in [1, 9.5.12]. Actually, these specific intervals of localization of ξ_p and $\tilde{\xi}_p$ are found using the expressions (3.3), (3.4), (3.5) and (3.6) and thanks to the inequalities (3.1) and (3.2). For example, for $\tilde{\xi}_{2j}$, one determines the sign of $u(-x)$ at each x corresponding to an edge of the interval given in (3.15) by using (3.3) and (3.1). Since the two signs are opposite, it is just a consequence of the intermediate value theorem. □

3.2. *Characterization of the spectral band edges*

The band edges are characterized through the theory of Bloch decomposition for periodic Schrödinger operators ([12, XIII]).

Let $\omega \in [-1,1]$. We start by considering the restriction $\mathbf{H}(\omega)$ of $\mathbf{H}$ to $\mathsf{H}^2([-1,1])$, the Sobolev space of functions $\psi \in \mathsf{H}^2(\mathbb{R})$ which satisfy

$$\psi(1) = \mathrm{e}^{\mathrm{i}(\pi\omega+\pi)}\psi(-1) \quad \text{and} \quad \psi'(1) = \mathrm{e}^{\mathrm{i}(\pi\omega+\pi)}\psi'(-1). \tag{3.17}$$

According to [12], $\mathbf{H}$ is the direct integral of the operators $\mathbf{H}(\omega)$:

$$\mathbf{H} = \int_{[-1,1]}^{\oplus} \mathbf{H}(\omega)\mathrm{d}\omega.$$

This decomposition in direct integral allows to recover the spectrum of $\mathbf{H}$ from the spectra of the $\mathbf{H}(\omega)$'s.

To determine the spectral band edges of $\mathbf{H}$, we solve the equation (1.5) on $[-1,1]$ with the boundary conditions (3.17). A pair of fundamental solutions is $x \mapsto u(-\mathsf{h}^{-\frac{2}{3}}\mathbf{E} + \mathsf{h}^{-\frac{2}{3}}|x|)$ and $x \mapsto \mathrm{sign}(x) \cdot v(-\mathsf{h}^{-\frac{2}{3}}\mathbf{E} + \mathsf{h}^{-\frac{2}{3}}|x|)$. It leads to solve the four equations in $\mathbf{X} = \mathsf{h}^{-\frac{2}{3}}\mathbf{E}$:

$$v'(-\mathbf{X})u'(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) - v'(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X})u'(-\mathbf{X}) = 0, \tag{3.18}$$

$$u(-\mathbf{X})v(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) - v(-\mathbf{X})u(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = 0, \tag{3.19}$$

$$v(-\mathbf{X})u'(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) - u(-\mathbf{X})v'(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = 0, \tag{3.20}$$

$$v'(-\mathbf{X})u(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) - u'(-\mathbf{X})v(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = 0. \tag{3.21}$$

The general theory of periodic operators asserts that the set of solutions of (3.18) and (3.19) after multiplication by $\mathsf{h}^{\frac{2}{3}}$ is exactly $\{\mathbf{E}^0_{\min}, \mathbf{E}^1_{\max}, \mathbf{E}^2_{\min}, \mathbf{E}^3_{\max}, \dots\}$ and the set of solutions of (3.20) and (3.21) (again after multiplication by $\mathsf{h}^{\frac{2}{3}}$) is exactly $\{\mathbf{E}^0_{\max}, \mathbf{E}^1_{\min}, \mathbf{E}^2_{\max}, \mathbf{E}^3_{\min}, \dots\}$.

Then, using [12, Theorem XIII.90], the spectrum of $\mathbf{H}$ is the band spectrum:

$$\sigma(\mathbf{H}) = \bigcup_{p \geq 0} \left[\mathbf{E}^p_{\min}, \mathbf{E}^p_{\max}\right].$$

Moreover $\sigma(\mathbf{H})$ is purely absolutely continuous and $\mathbf{H}$ has no eigenvalues.

Remark 3.1. The spectral band edges are the zeroes of the two solutions of (1.5) and their derivatives, associated with the Cauchy data $(1, 0)$ and $(0, 1)$. They are respectively the even and odd canonical solutions of (1.5).

One must exert caution for solving the equations (3.18), (3.19), (3.20) and (3.21) with the particular values of h such that

$$\mathsf{h}^{-\frac{2}{3}} \in \{\tilde{c}_q - \tilde{c}_r, \mid q > r \geq 0\} := \mathsf{Z}.$$

Then, for $\mathsf{h}^{-\frac{2}{3}} = \tilde{c}_q - \tilde{c}_r \in \mathsf{Z}$, in the set $\{c_p, \tilde{c}_p\}_{p\geq 0}$, $\mathbf{X} = \mathsf{h}^{-\frac{2}{3}}\tilde{c}_r$ is the unique solution of the equation: (3.18) if q and r are odd, (3.19) if q and r are even, (3.20) if q is odd and r is even and (3.21) if q is even and r is odd.

Conversely, if $\mathsf{h}^{-\frac{2}{3}} \notin \mathsf{Z}$, then none of the $\mathsf{h}^{-\frac{2}{3}}\tilde{c}_p$ is a solution in $-\mathbf{X}$ of any of the equations (3.18), (3.19), (3.20) and (3.21).

Assumption. From now on, we assume that $\mathsf{h}^{-\frac{2}{3}} \notin \mathsf{Z}$.

Note that, however, all our results hold true when $\mathsf{h}^{-\frac{2}{3}} \in \mathsf{Z}$, this assumption is only made for convenience's sake.

With the assumption above, the band edges of the spectral bands of $\mathbf{H}$ are solutions of the four following equations:

$$\frac{v'}{u'}(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = \frac{v'}{u'}(-\mathbf{X}), \quad \text{for } \mathbf{X} \notin \{\tilde{c}_{2j+1} - \mathsf{h}^{-\frac{2}{3}}\}_{j\geq 0} \cup \{\tilde{c}_{2j+1}\}_{j\geq 0}, \tag{3.22}$$

$$\frac{v}{u}(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = \frac{v}{u}(-\mathbf{X}), \quad \text{for } \mathbf{X} \notin \{\tilde{c}_{2j} - \mathsf{h}^{-\frac{2}{3}}\}_{j\geq 0} \cup \{\tilde{c}_{2j}\}_{j\geq 0}, \tag{3.23}$$

$$\frac{v'}{u'}(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = \frac{v}{u}(-\mathbf{X}), \quad \text{for } \mathbf{X} \notin \{\tilde{c}_{2j+1} - \mathsf{h}^{-\frac{2}{3}}\}_{j\geq 0} \cup \{\tilde{c}_{2j}\}_{j\geq 0}, \tag{3.24}$$

$$\frac{v}{u}(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = \frac{v'}{u'}(-\mathbf{X}), \quad \text{for } \mathbf{X} \notin \{\tilde{c}_{2j} - \mathsf{h}^{-\frac{2}{3}}\}_{j\geq 0} \cup \{\tilde{c}_{2j+1}\}_{j\geq 0}. \tag{3.25}$$

3.3. Variations of $\frac{v}{u}$ and $\frac{v'}{u'}$

Using the value of the Wronskian of u and v one has:

$$\forall x \in [0, +\infty), \ \left(\frac{v}{u}\right)'(x) = \frac{1}{u^2(x)} \text{ and } \forall x \in (0, +\infty), \ \left(\frac{v'}{u'}\right)'(x) = -\frac{x}{(u'(x))^2}.$$

Similarly,

$$\forall x \in (0, +\infty), \ \left(\frac{u}{v}\right)'(x) = -\frac{1}{(v(x))^2} \text{ and } \forall x \in [0, +\infty), \ \left(\frac{u'}{v'}\right)'(x) = \frac{x}{(v'(x))^2}.$$

Thus, the functions $\frac{v}{u}$ and $\frac{v'}{u'}$ have the following behaviour (Fig. 1).

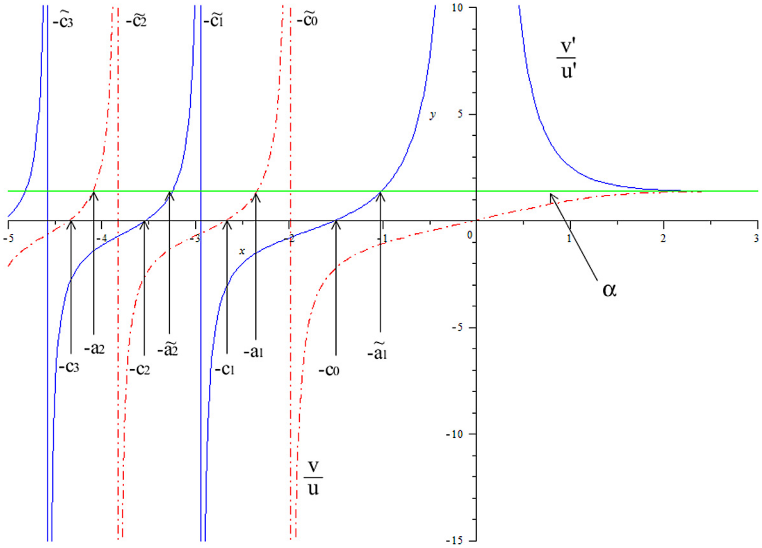

Fig. 1. Graphs of $\frac{v}{u}$ and $\frac{v'}{u'}$.

1. On every interval $(-\tilde{c}_{2j+2}, -\tilde{c}_{2j})$, the function $\frac{v}{u}$ is continuous, strictly increasing and is a bijection from $(-\tilde{c}_{2j+2}, -\tilde{c}_{2j})$ to $\mathbb{R}$. The function $\frac{v}{u}$ is continuous, strictly increasing and is a bijection from $(-\tilde{c}_0, +\infty)$ to $(-\infty, \alpha)$.
2. On every interval $(-\tilde{c}_{2j+3}, -\tilde{c}_{2j+1})$, $\frac{v'}{u'}$ is continuous, strictly increasing and is a bijection from $(-\tilde{c}_{2j+3}, -\tilde{c}_{2j+1})$ to $\mathbb{R}$. The function $\frac{v'}{u'}$ is continuous and strictly increasing on $(-\tilde{c}_1, 0)$ from $-\infty$ to $+\infty$. It is also continuous, strictly decreasing and a bijection from $(0, +\infty)$ to $(\alpha, +\infty)$.

We remark that α is the common limit at infinity of the two functions $\frac{v}{u}$ and $\frac{v'}{u'}$, thanks to the limits $\frac{Bi(x)}{Ai(x)} \xrightarrow[x\to+\infty]{} +\infty$ and $\frac{Bi'(x)}{Ai'(x)} \xrightarrow[x\to+\infty]{} -\infty$.

3.4. *Some auxiliary functions*

One sets, for $x \geq 0$ and $z \in \mathbb{R}$,

$$f_x(z) = v'(x-z)u(x) - u'(x-z)v(x) = \pi \left(Bi'(x-z)Ai(x) - Ai'(x-z)Bi(x)\right) \tag{3.26}$$

and

$$g_x(z) = v(x-z)u(x) - u(x-z)v(x) = \pi \left(Bi(x-z)Ai(x) - Ai(x-z)Bi(x)\right). \tag{3.27}$$

The expressions in terms of the Airy functions allow us to use classical properties of the Ai and Bi functions instead of the properties of u and v when it makes proofs easier.

The functions f_x and g_x are non-zero solutions of differential equations of the form $u'' = p(x)u$ which satisfy the assumptions of Sturm's theorem (see [2]), thus their zeroes are isolated on the real line. We denote by

$$z_0(x) < z_2(x) < \cdots < z_{2j}(x) < \ldots$$

the zeroes of f_x arranged in increasing order. Then, since 0 is the first zero of g_x for every x, we denote by

$$0 < z_1(x) < z_3(x) < \cdots < z_{2j+1}(x) < \ldots$$

the zeroes of g_x arranged in increasing order.

We characterize these zeroes and prove that none of them is negative.

Let $j \geq 0$ an integer. Let $x \geq 0$ and denote by $\psi_{2j}(x)$ the unique solution of the equation

$$\frac{v'}{u'}(z) = \frac{v}{u}(x),\ z \in [-c_{2j}, -\tilde{a}_{j+1}). \tag{3.28}$$

We also denote by $\psi_{2j+1}(x)$ the unique solution of the equation

$$\frac{v}{u}(z) = \frac{v}{u}(x),\ z \in [-c_{2j+1}, -a_{j+1}). \tag{3.29}$$

Lemma 3.1. *For every $k \geq 0$, the function ψ_k is well defined, continuous and strictly increasing.*

Proof. Let $j \geq 0$. Recall that α is the common limit of $\frac{v}{u}$ and $\frac{v'}{u'}$ at infinity. The function $\frac{v'}{u'}$ is a bijection from $[-c_{2j}, -\tilde{a}_{j+1})$ to $[0, \alpha)$. Since for every $x \geq 0$, $\frac{v}{u}(x) \in [0, \alpha)$, we have:

$$\forall x \geq 0,\ \psi_{2j}(x) = \left(\frac{v'}{u'}\Big|_{[-c_{2j}, -\tilde{a}_{j+1})} \right)^{-1} \left(\frac{v}{u}(x) \right).$$

Thus, the function ψ_{2j} is well defined and it is continuous by continuity of $\frac{v}{u}$ on $[0, +\infty)$ and of the inverse of $\frac{v'}{u'}$ on $[0, \alpha)$. Since $\frac{v'}{u'}$ is strictly increasing on $[-c_{2j}, -\tilde{a}_{j+1})$, its reciprocal function is strictly increasing on $[0, \alpha)$ and since $\frac{v}{u}$ is strictly increasing on $[0, +\infty)$, we deduce that ψ_{2j} is strictly increasing on $[0, +\infty)$.

The function $\frac{v}{u}$ is a bijection from $(-c_{2j+1}, -a_{j+1}]$ to $[0, \alpha)$. With the same arguments as before, we have that

$$\forall x \geq 0,\ \psi_{2j+1}(x) = \left(\frac{v}{u}\Big|_{(-c_{2j+1}, -a_{j+1}]} \right)^{-1} \left(\frac{v}{u}(x) \right)$$

and thus ψ_{2j+1} is well defined, continuous and strictly increasing. □

Lemma 3.2. *Let $k \geq 0$. Then, for every $x \geq 0$,*

$$z_k(x) \geq 0 \quad \text{and} \quad z_k(x) = x - \psi_k(x).$$

Therefore, z_k is continuous on $[0, +\infty)$. Moreover, for every $k \geq 0$, the function z_k is strictly increasing from $[0, +\infty)$ to $[c_k, +\infty)$.

This result is proven in Appendix C.

Let $j \geq 0$. For $x \geq 0$, denote by $\psi^{2j}(x) \in (-a_{j+1}, -\tilde{c}_{2j}]$ the unique solution of the equation

$$\frac{v}{u}(z) = \frac{v'}{u'}(x), \; z \in (-a_{j+1}, -\tilde{c}_{2j}]. \tag{3.30}$$

We also denote by $\psi^{2j+1}(x) \in (-\tilde{a}_{j+2}, -\tilde{c}_{2j+1})$ the unique solution of the equation

$$\frac{v'}{u'}(z) = \frac{v'}{u'}(x), \; z \in (-\tilde{a}_{j+2}, -\tilde{c}_{2j+1}). \tag{3.31}$$

Lemma 3.3. *For every $k \geq 0$, the function ψ^k is well defined, continuous and strictly decreasing on* $[0, +\infty)$.

Proof. We assume that k is even, $k = 2j$ for one $j \geq 0$. The function $\frac{v}{u}$ is a continuous bijection from $(-a_{j+1}, -\tilde{c}_{2j}]$ to $[\alpha, +\infty)$. Since for every $x \geq 0$, $\frac{v'}{u'}(x) \in [\alpha, +\infty)$, we have:

$$\forall x \geq 0, \; \psi^{2j}(x) = \left(\frac{v}{u}\bigg|_{(-a_{j+1}, -\tilde{c}_{2j}]} \right)^{-1} \left(\frac{v'}{u'}(x) \right).$$

Then ψ^{2j} is well defined, continuous and it is the unique solution of (3.30). Moreover, $\left(\frac{v}{u}\right)_{2j}^{-1}$ is a strictly increasing function from $[\alpha, +\infty)$ to the interval $(-a_{j+1}, -\tilde{c}_{2j}]$ and $\frac{v'}{u'}$ is strictly decreasing on $[0, +\infty)$. Thus, ψ^{2j} is strictly decreasing on $[0, +\infty)$.

We assume that k is odd, $k = 2j + 1$ for one $j \geq 0$. The function $\frac{v'}{u'}$ is a continuous bijection from $(-\tilde{a}_{j+2}, -\tilde{c}_{2j+1})$ to $[\alpha, +\infty)$. Since for every $x \geq 0$, $\frac{v'}{u'}(x) \in [\alpha, +\infty)$, we write:

$$\forall x \geq 0, \; \psi^{2j+1}(x) = \left(\frac{v'}{u'}\bigg|_{(-\tilde{a}_{j+2}, -\tilde{c}_{2j+1})} \right)^{-1} \left(\frac{v'}{u'}(x) \right).$$

Then ψ^{2j+1} is well defined, continuous and it is the unique solution of (3.31). Moreover, $\left(\frac{v'}{u'}\Big|_{(-\tilde{a}_{j+2}, -\tilde{c}_{2j+1})} \right)^{-1}$ is a strictly increasing function from $[\alpha, +\infty)$ to the interval $(-\tilde{a}_{j+2}, -\tilde{c}_{2j+1})$ and $\frac{v'}{u'}$ is strictly decreasing on $[0, +\infty)$. Thus, ψ^{2j+1} is strictly decreasing on $[0, +\infty)$. $\square$

4. The first spectral band

4.1. Lower bound of the continuous spectrum

For $\mathbf{E} < -1$, we show that there are no solutions to equations (3.22), (3.23), (3.25) and (3.24). This writes:

Proposition 4.1. *For every* $\mathsf{h} > 0$, $\sigma(\mathbf{H}) \subset [-1, +\infty)$.

Proof. We remark that on the interval $[0,+\infty)$, $\frac{v}{u}<\alpha$ and $\frac{v'}{u'}>\alpha$. Thus, these two functions cannot have a common value on this interval and equations (3.25) and (3.24) do not have any solution with $0<-\mathsf{h}^{-\frac{2}{3}}-\mathbf{X}<-\mathbf{X}$.

Moreover, we already know that $\frac{v}{u}$ is strictly increasing on $[0,+\infty)$ and $\frac{v'}{u'}$ is strictly decreasing on $(0,+\infty)$. Since $-\mathsf{h}^{-\frac{2}{3}}-\mathbf{X}\neq-\mathbf{X}$ the equations (3.22), (3.23) do not have any solution when $0<-\mathsf{h}^{-\frac{2}{3}}-\mathbf{X}<-\mathbf{X}$. □

Remark 4.1. This result holds true for every h strictly positive. In particular we do not need to assume the semiclassical parameter h to be small.

4.2. The bottom of the spectrum

The bottom of the spectrum is a solution of either the equation (3.22) or the equation (3.23) with $-\mathbf{X}\geq 0$ and thus $-\mathsf{h}^{-\frac{2}{3}}-\mathbf{X}<0$.

We start by proving that for every $\mathsf{h}>0$, the equation (3.22) has a unique solution with $-\mathbf{X}>0$ and $-\mathsf{h}^{-\frac{2}{3}}-\mathbf{X}\in[-\tilde{a}_1,0)$.

The function $\frac{v'}{u'}$ is an increasing continuous bijection from $(-\tilde{a}_1,0)$ to $[\alpha,+\infty)$ and thus, since for every $x>0$, $\frac{v'}{u'}(x)\in[\alpha,+\infty)$, one defines:

$$\forall x>0,\ \psi(x)=\left(\frac{v'}{u'}\Big|_{(-\tilde{a}_1,0)}\right)^{-1}\left(\frac{v'}{u'}(x)\right). \tag{4.1}$$

The function ψ does not belong to the family of the functions ψ_{2k+1}, due to the difference of behaviour of $\frac{v'}{u'}$ on $(-\tilde{a}_1,+\infty)$ compared to $(-\infty,-\tilde{a}_1)$.

The function ψ is a continuous decreasing bijection from $(0,+\infty)$ to $(-\tilde{a}_1,0)$ and $x\mapsto x-\psi(x)$ is a continuous increasing bijection from $(0,+\infty)$ to $(0,+\infty)$. Thus,

$$\forall\mathsf{h}>0,\ \exists!\tilde{x}>0,\ \tilde{x}-\psi(\tilde{x})=\mathsf{h}^{-\frac{2}{3}}.$$

One sets $\tilde{\mathbf{X}}=-\tilde{x}$ and since $\tilde{x}=\mathsf{h}^{-\frac{2}{3}}+\psi(\tilde{x})\in(-\tilde{a}_1+\mathsf{h}^{-\frac{2}{3}},\mathsf{h}^{-\frac{2}{3}})$ one has

$$-\mathsf{h}^{-\frac{2}{3}}<\tilde{\mathbf{X}}<-\mathsf{h}^{-\frac{2}{3}}+\tilde{a}_1. \tag{4.2}$$

Observe that $\tilde{\mathbf{X}}$ is the smallest solution of (3.22) and that it is a continuous function on $\mathsf{h}\in(0,+\infty)$.

Now we turn to the smallest solution of (3.23). Since the function z_1 is an increasing continuous bijection from $[0,+\infty)$ to $[c_1,+\infty)$,

$$\forall\mathsf{h}\leq c_1^{-\frac{3}{2}},\exists!x_1\geq 0,\ z_1(x_1)=x_1-\psi_1(x_1)=\mathsf{h}^{-\frac{2}{3}}.$$

One sets $\check{\mathbf{X}}_1=-x_1$ and since $x_1=\mathsf{h}^{-\frac{2}{3}}+\psi_1(x_1)\in(-c_1+\mathsf{h}^{-\frac{2}{3}},-a_1+\mathsf{h}^{-\frac{2}{3}})$ one has

$$-\mathsf{h}^{-\frac{2}{3}}+a_1<\check{\mathbf{X}}_1<-\mathsf{h}^{-\frac{2}{3}}+c_1.$$

Since $\tilde{a}_1 < a_1$ we have $\tilde{\mathbf{X}} < \check{\mathbf{X}}_1$ and thus, if $\mathbf{X}^0_{\mathrm{min}} = \mathsf{h}^{-\frac{2}{3}}\mathbf{E}^0_{\mathrm{min}}$, $\mathbf{X}^0_{\mathrm{min}} = \tilde{\mathbf{X}}$. In particular, $\mathbf{X}^0_{\mathrm{min}}$ is the smallest solution of (3.22). Moreover, it is the smallest solution among those of (3.22) and (3.23).

Now that we have identified the bottom of the spectrum of $\mathbf{H}$, we prove the estimates stated in Theorem 2.2.

Proof of Theorem 2.2. **(1)** Since $\mathbf{X}^0_{\mathrm{min}} = \tilde{\mathbf{X}}$, by (4.2) we have, for every $\mathsf{h} > 0$, $-1 < \mathbf{E}^0_{\mathrm{min}} < -1 + \tilde{a}_1\mathsf{h}^{\frac{2}{3}}$ and $\mathbf{E}^0_{\mathrm{min}}$ tends to -1 when h tends to 0.

Let $\mathsf{h} > 0$. By the previous inequality, $-\mathbf{E}^0_{\mathrm{min}} \in (-\tilde{a}_1\mathsf{h}^{\frac{2}{3}} + 1, 1)$. We recall that, since u and v are linear combinations of Ai and Bi and since $\mathbf{X}^0_{\mathrm{min}}$ is a solution of (3.22), it also satisfies:

$$\tfrac{Bi'}{Ai'}(-\mathbf{X}^0_{\mathrm{min}} - \mathsf{h}^{-\frac{2}{3}}) = \tfrac{Bi'}{Ai'}(-\mathbf{X}^0_{\mathrm{min}}), \quad -\mathbf{X}^0_{\mathrm{min}} - \mathsf{h}^{-\frac{2}{3}} \in (-\tilde{a}_1, 0).$$

We introduce the functions

$$F_{\pm}(\cdot,\mathsf{h}) \ : \ \begin{array}{rcl} I_{\pm} & \to & \mathbb{R} \\ x & \mapsto & \frac{Bi'}{Ai'}\left(x - \frac{\mathsf{h}^{-\frac{2}{3}}}{2}\right) - \frac{Bi'}{Ai'}\left(x + \frac{\mathsf{h}^{-\frac{2}{3}}}{2}\right) \end{array}$$

with $I_- = \left(-\frac{1}{2}\mathsf{h}^{-\frac{2}{3}}, \frac{1}{2}\mathsf{h}^{-\frac{2}{3}}\right)$ and $I_+ = \left(-\tilde{a}_1 + \frac{1}{2}\mathsf{h}^{-\frac{2}{3}}, \frac{1}{2}\mathsf{h}^{-\frac{2}{3}}\right)$. We remark that, for every $\mathsf{h} > 0$, the unique zero of $F_{\pm}(\cdot,\mathsf{h})$ on $I_{\pm}$ is $-\mathbf{X}^0_{\mathrm{min}} - \frac{1}{2}\mathsf{h}^{-\frac{2}{3}}$.

Thanks to the fact that 0 is the point of maximum of $\frac{Bi'}{Ai'}$ on $(-\tilde{a}_1, +\infty)$, one has:

$$F_{\pm}\left(\tfrac{1}{2}\mathsf{h}^{-\frac{2}{3}}, \mathsf{h}\right) = \tfrac{Bi'}{Ai'}(0) - \tfrac{Bi'}{Ai'}(\mathsf{h}^{-\frac{2}{3}}) > 0.$$

First case: $\mathsf{h} \leq (2\tilde{a}_1)^{-\frac{3}{2}}$. In this case, $-\tilde{a}_1 + \frac{1}{2}\mathsf{h}^{-\frac{2}{3}} > 0$ hence the unique zero of $F_+(\cdot,\mathsf{h})$ on I_+ is strictly positive, hence $-\mathbf{X}^0_{\mathrm{min}} - \frac{1}{2}\mathsf{h}^{-\frac{2}{3}} > 0$ which implies that $\mathbf{E}^0_{\mathrm{min}} < -\frac{1}{2}$. Thus, point (1) is proven for every $\mathsf{h} \leq (2\tilde{a}_1)^{-\frac{3}{2}}$.

Second case: $\mathsf{h} > (2\tilde{a}_1)^{-\frac{3}{2}}$. We prove that the unique zero of $F_{\pm}(\cdot,\mathsf{h})$ on $I_{\pm}$ is strictly positive. Since $F_{\pm}(\frac{1}{2}\mathsf{h}^{-\frac{2}{3}},\mathsf{h}) > 0$, it is sufficient to prove that $F_{\pm}(0,\mathsf{h}) < 0$. Indeed, if $\mathsf{h} > \tilde{a}_1^{-\frac{3}{2}}$, then $0 \in I_-$ and we study the unique zero of $F_-(\cdot,\mathsf{h})$ in I_-. If $(2\tilde{a}_1)^{-\frac{3}{2}} < \mathsf{h} \leq \tilde{a}_1^{-\frac{3}{2}}$, then $-\tilde{a}_1 + \frac{1}{2}\mathsf{h}^{-\frac{2}{3}} < 0$ and $0 \in I_+$. In this case, we study the unique zero of $F_+(\cdot,\mathsf{h})$ in I_+.

We have:

$$F_{\pm}(0,\mathsf{h}) = \tfrac{Bi'}{Ai'}\left(-\tfrac{1}{2}\mathsf{h}^{-\frac{2}{3}}\right) - \tfrac{Bi'}{Ai'}\left(\tfrac{1}{2}\mathsf{h}^{-\frac{2}{3}}\right).$$

Let $y = \frac{1}{2}\mathsf{h}^{-\frac{2}{3}}$ so that $y \in (0, \tilde{a}_1)$ and set:

$$\forall y \in (0, \tilde{a}_1), \ G(y) = \tfrac{Bi'}{Ai'}(-y) - \tfrac{Bi'}{Ai'}(y).$$

One has, for every $y \in (0, \tilde{a}_1)$,

$$G'(y) = \frac{y}{\pi(Ai'(-y))^2(Ai'(y))^2}\left((Ai'(-y) - Ai'(y))(Ai'(-y) + Ai'(y))\right).$$

On $(0, \tilde{a}_1)$, $\frac{y}{\pi(Ai'(-y))^2(Ai'(y))^2} > 0$ and as Ai' is negative on $(-\tilde{a}_1, +\infty)$, $Ai'(-y) + Ai'(y) < 0$ for every $y \in (0, \tilde{a}_1)$.

Let

$$K : \begin{array}{ccc} (0, \tilde{a}_1) & \to & \mathbb{R} \\ y & \mapsto & Ai'(-y) - Ai'(y) \end{array}.$$

Then $K(0) = 0$ and

$$\forall y \in (0, \tilde{a}_1),\ K'(y) = -Ai''(-y) - Ai''(y) = y(Ai(-y) - Ai(y)).$$

But, the Airy function Ai is decreasing on $(-\tilde{a}_1, +\infty)$ hence, for $y \in (0, \tilde{a}_1)$, $Ai(-y) - Ai(y) > 0$ and $K'(y) > 0$. Thus, K is strictly increasing on $(0, \tilde{a}_1)$ and

$$\forall y \in (0, \tilde{a}_1),\ K(y) > K(0) = 0.$$

Thus, for every $y \in (0, \tilde{a}_1)$, $G'(y) < 0$. Since $G(0) = 0$, for every $y \in (0, \tilde{a}_1)$, $G(y) < 0$, which rewrites

$$\forall \mathsf{h} > (2\tilde{a}_1)^{-\frac{3}{2}},\ F_\pm(0, \mathsf{h}) < 0.$$

Thus, the unique zero of $F_\pm(\cdot, \mathsf{h})$ in $I_\pm$ is strictly positive. Thus, $-\mathbf{X}^0_{\min} - \frac{1}{2}\mathsf{h}^{-\frac{2}{3}} \in \left(0, \frac{1}{2}\mathsf{h}^{-\frac{2}{3}}\right)$ and $\mathbf{E}^0_{\min} < -\frac{1}{2}$.

Taking in account the result in both cases, point (1) is proven.

(2) Since $\mathbf{X}^0_{\min}$ is the smallest solution of (3.22), if one sets $X = -\mathbf{X}^0_{\min} - \mathsf{h}^{-\frac{2}{3}} + \tilde{a}_1$, then X satisfies

$$\frac{u'}{v'}(X - \tilde{a}_1) = \frac{u'}{v'}(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1), \quad X \in [0, \tilde{a}_1]. \tag{4.3}$$

Let $\tau > 0$. We assume that $\mathsf{h} \le (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$ and thus $X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 \ge \tau$.

By Lemma A.5, equality (4.3) implies that

$$\left|\frac{u'}{v'}(X - \tilde{a}_1) - \frac{1}{\alpha} + \frac{\sqrt{3}}{\alpha}\mathrm{e}^{-\frac{4}{3}(X+\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}\right| \le 2.83\left(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1\right)^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}(X+\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}. \tag{4.4}$$

Let $\epsilon = \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}$, which amounts to $\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 = \left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{2}{3}}$. Note that ϵ is well defined if and only if $\mathsf{h}^{-\frac{2}{3}} > \tilde{a}_1$ hence our choice of restricting ourselves to $\mathsf{h} \le (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$ for a given $\tau > 0$.

We use the following identity valid for strictly positive real numbers a and b:

$$a^{\frac{3}{2}} - b^{\frac{3}{2}} - \tfrac{3}{2} b^{\frac{1}{2}}(a-b) = (a-b)^2 \frac{2\sqrt{\frac{a}{b}}+1}{2\left(\sqrt{\frac{a}{b}}+1\right)^2} b^{-\frac{1}{2}}. \tag{4.5}$$

Define Y through $X = \epsilon Y$ and consider $a = X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 = \epsilon Y + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1$ and $b = \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 = \left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{2}{3}}$. One has,

$$\begin{aligned} -\tfrac{4}{3}\left(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1\right)^{\frac{3}{2}} + \tfrac{4}{3}\left(\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} + 2\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}} \epsilon Y \\ = \left(-\tfrac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}} (\epsilon Y)^2 Q(\epsilon, Y) \end{aligned}$$

where

$$Q(\epsilon, Y) = \frac{4}{3} \frac{2\frac{(\epsilon Y + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{1}{2}}}{\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}} + 1}{2\left(\frac{(\epsilon Y + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{1}{2}}}{\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}} + 1\right)^2}.$$

The condition $\mathsf{h} \leq (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$ implies $0 < \epsilon \leq \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}} < 1$.

Since $\epsilon Y + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 = -\mathbf{X}^0_{\min} > 0$,

$$\frac{(\epsilon Y + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{1}{2}}}{\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}} \geq 0.$$

If $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is defined, for every $x \in \mathbb{R}_+$, by $\varphi(x) = \frac{2x+1}{2(x+1)^2}$, then φ is decreasing on $\mathbb{R}_+$, $\varphi(0) = \frac{1}{2}$ and φ tends to 0 when x tends to $+\infty$. Thus, for every $\epsilon \in [0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$ and every Y,

$$0 < Q(\epsilon, Y) \leq \frac{2}{3}. \tag{4.6}$$

We have,

$$\mathrm{e}^{-\frac{4}{3}\left(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1\right)^{\frac{3}{2}}} = \epsilon \cdot \mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}} \epsilon Y + \left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}} (\epsilon Y)^2 Q(\epsilon, Y)}. \tag{4.7}$$

The condition $X \in [0, \tilde{a}_1)$ implies that

$$\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{2}{3}} = -\tilde{a}_1 + \mathsf{h}^{-\frac{2}{3}} \leq X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 \leq \tilde{a}_1 + \left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{2}{3}}.$$

Moreover,

$$\left(\left(\frac{3}{4}\right)^{\frac{2}{3}} + \frac{\tilde{a}_1}{|\ln(\epsilon)|^{\frac{2}{3}}}\right)^{-\frac{3}{2}} \frac{1}{|\ln(\epsilon)|} \leq \left(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1\right)^{-\frac{3}{2}} \leq \frac{4}{3}\frac{1}{|\ln(\epsilon)|}. \tag{4.8}$$

Using (4.4) and the identity $\frac{1}{\alpha} = \frac{u'}{v'}(-\tilde{a}_1)$, which comes directly from the expressions of u and v in terms of Ai and Bi, for every $\epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$,

$$\begin{aligned}\left|\frac{1}{\epsilon}\left(\frac{u'}{v'}(\epsilon Y - \tilde{a}_1) - \frac{u'}{v'}(-\tilde{a}_1)\right) + \frac{\sqrt{3}}{\alpha}\mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y}\mathrm{e}^{\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2 Q(\epsilon,Y)}\right| \leq \\ \frac{4}{3}\frac{\sqrt{3}}{\alpha}\frac{2.83}{|\ln(\epsilon)|}\mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y}\mathrm{e}^{\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2 Q(\epsilon,Y)}.\end{aligned} \tag{4.9}$$

To estimate the left member of (4.9) one uses:

$$\frac{u'}{v'}(\epsilon Y - \tilde{a}_1) - \frac{u'}{v'}(-\tilde{a}_1) = \int_0^1 \left(\frac{u'}{v'}\right)'(-\tilde{a}_1 + t\epsilon Y)\epsilon Y \mathrm{d}t = \int_0^1 \frac{-\tilde{a}_1 + t\epsilon Y}{(v'(-\tilde{a}_1 + t\epsilon Y))^2}\epsilon Y \mathrm{d}t.$$

Using the upper bound for $\frac{x}{(v'(x))^2}$ and $x \leq 0$ given in (A.1), and since $\tilde{a}_1 - \frac{1}{2}\epsilon Y \in (\frac{\tilde{a}_1}{2}, \tilde{a}_1]$ and $\frac{1}{\alpha} > \frac{\tilde{a}_1}{2}$,

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ \left|\frac{u'}{v'}(\epsilon Y - \tilde{a}_1) - \frac{u'}{v'}(-\tilde{a}_1)\right| \geq \frac{1}{\alpha}\epsilon Y. \tag{4.10}$$

Using (4.9), (4.10) and (4.6),

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ Y \leq \sqrt{3}\mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y}\mathrm{e}^{\frac{2}{3}\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2}\left(1 + 2.83\frac{4}{\sqrt{3}}\left(\frac{1}{|\ln(\epsilon)|}\right)\right),$$

which rewrites

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ Y\mathrm{e}^{2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y} \leq \sqrt{3}\mathrm{e}^{\frac{2}{3}\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2}\left(1 + 2.83\frac{4}{\sqrt{3}}\left(\frac{1}{|\ln(\epsilon)|}\right)\right).$$

But, $\epsilon Y = X \in [0, \tilde{a}_1)$ is bounded and thus (4.11) where $B_\tau > 0$ is independent on h and Y,

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ Y\mathrm{e}^{2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y} \leq \sqrt{3}\mathrm{e}^{\frac{2}{3}\tau^{-\frac{1}{2}}\tilde{a}_1^2}\left(1 + 2.83\sqrt{3}\tau^{-\frac{3}{2}}\right) =: B_\tau \tag{4.11}$$

and

$$\forall\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ \epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}Y\mathrm{e}^{2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y}\le\epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}B_\tau.$$

The function $x\mapsto x\mathrm{e}^{2x}$ is C^1 and strictly increasing on $\mathbb{R}$, let us denote by k its reciprocal function which is also C^1. From (4.11) one gets

$$\forall\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ \epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}Y\le k\left(\epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}B_\tau\right).$$

Since k is of class C^1, $k(0)=0$, $k'(0)=1$ and since for every $\epsilon\in(0,1)$, $\epsilon\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\in[0,\frac{1}{2}]$, for every $\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$,

$$k\left(\epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}B_\tau\right)\le B_\tau\epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}+\frac{1}{2}B_\tau^2\sup_{s\in[0,\frac{1}{2}B_\tau]}|k''(s)|\cdot\epsilon^2\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{2}{3}}.$$

Thus,

$$\forall\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ Y\le B_\tau+\frac{1}{2}B_\tau^2\sup_{s\in[0,\frac{1}{2}B_\tau]}|k''(s)|\cdot\epsilon\left(-\tfrac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}},\tag{4.12}$$

from which we deduce that Y is bounded by $C_\tau:=B_\tau+B_\tau^2\left(1+2\ln(1+\frac{1}{2}B_\tau)\right)$, since for every $s\ge0$, $k''(s)\le4+8\ln(1+\frac{s}{2})$. Thus, $|X|\le C_\tau\epsilon$, namely

$$|X|\le C_\tau\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}$$

and since $X=-\mathbf{X}^0_{\min}-\mathsf{h}^{-\frac{2}{3}}+\tilde{a}_1$, we already have

$$\left|\mathbf{E}^0_{\min}+1-\tilde{a}_1\mathsf{h}^{\frac{2}{3}}\right|\le C_\tau\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}.$$

We refine the estimate. Using Taylor formula and since for every $\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$, $\epsilon Y\in[0,C_\tau\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]\subset[0,C_\tau]$,

$$\forall\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ \left|\frac{u'}{v'}(\epsilon Y-\tilde{a}_1)-\frac{u'}{v'}(-\tilde{a}_1)-\left(\frac{u'}{v'}\right)'(-\tilde{a}_1)(\epsilon Y)\right|\le3C_\tau^2\epsilon^2\tag{4.13}$$

since

$$\sup_{s\in[-\tilde{a}_1,-\tilde{a}_1+C_\tau]}\left|\left(\frac{u'}{v'}\right)''(s)\right|\le\sup_{s\in[-\tilde{a}_1,+\infty)}\left|\left(\frac{u'}{v'}\right)''(s)\right|\le3.\tag{4.14}$$

Using (4.6) and (4.12) and since, for every $\epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$, $\epsilon\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}} \in [0, \frac{1}{2}]$ and $\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}} \in (0, \tau^{-\frac{1}{2}}]$,

$$\begin{aligned}\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ &\left|-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y + \left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2 Q(\epsilon, Y)\right| \\ &\leq \left(\frac{3}{4}\right)^{\frac{1}{3}}\left(2C_\tau + \frac{2}{3}C_\tau^2\tau^{-1}\right)\epsilon|\ln(\epsilon)|^{\frac{1}{3}}.\end{aligned} \tag{4.15}$$

Let $D_\tau = \left(\frac{3}{4}\right)^{\frac{1}{3}}\left(2C_\tau + \frac{2}{3}C_\tau^2\tau^{-1}\right)$. Using Taylor formula for the exponential function,

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ \left|\mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y + \left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2 Q(\epsilon, Y)} - 1\right| \leq D_\tau \epsilon|\ln(\epsilon)|^{\frac{1}{3}}. \tag{4.16}$$

One deduces from (4.9) and (4.16) that, for every $\epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$,

$$\left|\left(\frac{u'}{v'}\right)'(-\tilde{a}_1)\cdot Y + \frac{\sqrt{3}}{\alpha}\right| \leq 3C_\tau^2\epsilon + \frac{\sqrt{3}}{\alpha}D_\tau\epsilon|\ln(\epsilon)|^{\frac{1}{3}} + \frac{4}{3}\frac{\sqrt{3}}{\alpha}\frac{2.83}{|\ln(\epsilon)|}D_\tau\epsilon|\ln(\epsilon)|^{\frac{1}{3}}.$$

Since for $\epsilon \in (0, 1)$, $\epsilon|\ln(\epsilon)| \in (0, \frac{1}{2}]$, $\epsilon|\ln(\epsilon)|^{\frac{1}{3}} \in (0, \frac{1}{2}]$ and $\epsilon|\ln(\epsilon)|^{\frac{4}{3}} \in (0, \frac{1}{2}]$, if

$$L_\tau := \frac{3}{2}C_\tau^2 + \frac{\sqrt{3}}{2\alpha}D_\tau + 2.83\cdot\frac{2}{\sqrt{3}\alpha}D_\tau,$$

$L_\tau > 0$ is independent on h and

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}],\ \left|\left(\frac{u'}{v'}\right)'(-\tilde{a}_1)\cdot Y + \frac{\sqrt{3}}{\alpha}\right| \leq \frac{L_\tau}{|\ln(\epsilon)|}.$$

Since $\left(\frac{u'}{v'}\right)'(-\tilde{a}_1) = \frac{-\tilde{a}_1}{(v'(-\tilde{a}_1))^2}$, for every $\mathsf{h} \leq (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$,

$$\left|Y - \frac{\sqrt{3}}{\alpha}\frac{(v'(-\tilde{a}_1))^2}{\tilde{a}_1}\right| \leq L_\tau\frac{(v'(-\tilde{a}_1))^2}{\tilde{a}_1}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{-\frac{3}{2}}. \tag{4.17}$$

But, $\frac{1}{\alpha}\frac{(v'(-\tilde{a}_1))^2}{\tilde{a}_1} = \alpha\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}$. Thus, for every $\mathsf{h} \leq (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$,

$$\left|X - \alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}}\right| \leq L_\tau\alpha^2\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}}$$

and finally, using $\mathbf{X}^0_{\min} = -X - \mathsf{h}^{-\frac{2}{3}} + \tilde{a}_1$ and multiplying by $\mathsf{h}^{\frac{2}{3}}$, for every $\mathsf{h} \le (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$,

$$\begin{aligned}\left|\mathbf{E}^0_{\min} + 1 - \tilde{a}_1 \mathsf{h}^{\frac{2}{3}} + \alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}\right| \le \\ L_\tau \alpha^2 \frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{-\frac{3}{2}}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}.\end{aligned} \tag{4.18}$$

Since $-1 < \mathbf{E}^0_{\min} < -1 + \tilde{a}_1 \mathsf{h}^{\frac{2}{3}}$, for every $\mathsf{h} \in [(\tilde{a}_1 + \tau)^{-\frac{3}{2}}, \tilde{a}_1^{-\frac{3}{2}})$,

$$\begin{aligned}\left|\mathbf{E}^0_{\min} + 1 - \tilde{a}_1 \mathsf{h}^{\frac{2}{3}} + \alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}\right| \le \\ \tilde{a}_1 \tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{-\frac{3}{2}}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}.\end{aligned} \tag{4.19}$$

Let $M_{0,\tau} := \max\left(\alpha^2 \frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1} L_\tau, \tilde{a}_1 \tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}\right)$. Then, (2.4) is valid for $M_{0,\tau}$ for every $\tau > 0$. Remark that the smallest value of $M_{0,\tau}$ is obtained for τ such that $\alpha^2 \frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1} L_\tau = \tilde{a}_1 \tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}$ whose approximate values are $\tau \simeq 3.075$ and $M_{0,3.075} \simeq 7271$. □

Remark 4.2. We had to consider equation (4.3) involving $\frac{u'}{v'}$ instead of $\frac{v'}{u'}$ because $\left(\frac{v'}{u'}\right)''(x)$ tends to $+\infty$ when x tends to 0^+.

4.3. *The upper edge of the first spectral band*

The upper edge of the first spectral band is given by the smallest value of $\mathbf{X}$ among the solutions of equations (3.24) and (3.25).

We start by assuming that $\mathsf{h} \in (\tilde{c}_0^{-\frac{3}{2}}, c_0^{-\frac{3}{2}}]$. In this case (3.25) has no solution with $-\mathbf{X} > 0$ and we prove that (3.24) has a unique solution such that $-\mathbf{X} > 0$ and $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X} \in [-c_0, -\tilde{a}_1)$. Indeed, the function $z_0 : [0, +\infty) \to [c_0, +\infty)$ is a continuous bijection and

$$\forall \mathsf{h} \le c_0^{-\frac{3}{2}},\ \exists! x_0 \ge 0,\ z_0(x_0) = x_0 - \psi_0(x_0) = \mathsf{h}^{-\frac{2}{3}}.$$

One sets $\check{\mathbf{X}}_0 = -x_0$ and since $x_0 = \mathsf{h}^{-\frac{2}{3}} + \psi_0(x_0) \in [-c_0 + \mathsf{h}^{-\frac{2}{3}}, -\tilde{a}_1 + \mathsf{h}^{-\frac{2}{3}})$ one has

$$-\mathsf{h}^{-\frac{2}{3}} + \tilde{a}_1 < \check{\mathbf{X}}_0 \le -\mathsf{h}^{-\frac{2}{3}} + c_0. \tag{4.20}$$

Thus, for $\mathsf{h} \in (\tilde{c}_0^{-\frac{3}{2}}, c_0^{-\frac{3}{2}}]$, $\mathbf{X}^0_{\max} = \check{\mathbf{X}}_0$.

We then assume that $\mathsf{h} \le \tilde{c}_0^{-\frac{3}{2}}$. In this case, (3.24) has still a unique solution such that $-\mathbf{X} > 0$ and $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X} \in [-c_0, -\tilde{a}_1)$, namely $\check{\mathbf{X}}_0$, but one can also find a solution of (3.25) with $-\mathbf{X} > 0$. Indeed, the function from $[0, +\infty) \to [\tilde{c}_0, +\infty)$ which maps $x \ge 0$ to $x - \psi^0(x)$ is a continuous strictly increasing bijection. Thus,

$$\forall \mathsf{h} \le \tilde{c}_0^{-\frac{3}{2}},\ \exists! \tilde{x}_0 \ge 0,\ \tilde{x}_0 - \psi^0(\tilde{x}_0) = \mathsf{h}^{-\frac{2}{3}}.$$

One sets $\tilde{\mathbf{X}}_0 = -\tilde{x}_0$ and since $\tilde{x}_0 = \mathsf{h}^{-\frac{2}{3}} + \psi^0(\tilde{x}_0) \in (-a_1 + \mathsf{h}^{-\frac{2}{3}}, -\tilde{c}_0 + \mathsf{h}^{-\frac{2}{3}}]$ one has

$$-\mathsf{h}^{-\frac{2}{3}} + \tilde{c}_0 \le \tilde{\mathbf{X}}_0 < -\mathsf{h}^{-\frac{2}{3}} + a_1.$$

Since $c_0 < \tilde{c}_0$, one has $\check{\mathbf{X}}_0 < \tilde{\mathbf{X}}_0$ which implies $\mathbf{X}^0_{\max} = \check{\mathbf{X}}_0$ and $\mathbf{X}^1_{\min} = \tilde{\mathbf{X}}_0$.

As we identified $\mathbf{X}^0_{\max}$ among all the solutions of (3.24) and (3.25), we give more precise estimates. Notice that (4.21) can be extended for $\mathsf{h} = 0$ by continuity.

Proposition 4.2. *We have the following estimates on* $\mathbf{E}^0_{\max}$*:*

1. *For every* $\mathsf{h} < c_0^{-\frac{3}{2}}$*,*

$$-1 + \tilde{a}_1 \mathsf{h}^{\frac{2}{3}} < \mathbf{E}^0_{\max} < -1 + c_0 \mathsf{h}^{\frac{2}{3}}.$$

2. *There exists a universal constant* $\tilde{M}_0 > 0$ *such that, for every* $\mathsf{h} \in [0, c_0^{-\frac{3}{2}}]$*,*

$$\left| \mathbf{E}^0_{\max} - \left(-1 + \tilde{a}_1 \mathsf{h}^{\frac{2}{3}} + \alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1} \mathsf{h}^{\frac{2}{3}} \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}} \right) \right| \le \tilde{M}_0 \mathsf{h}^{\frac{5}{3}} (1 - \tilde{a}_1 \mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}} \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}}. \tag{4.21}$$

Proof. **(1)** The first point is (4.20) multiplied by $\mathsf{h}^{\frac{2}{3}}$ and the inequality is strict since z_0 is strictly increasing.

(2) For the second point, we follow the proof of point (2) of Theorem 2.2. We assume that $\mathsf{h} \le c_0^{-\frac{3}{2}}$ (for the identification of $\mathbf{E}^0_{\max}$) and thus $\mathsf{h} \le (\tilde{a}_1 + \tau)^{-\frac{3}{2}}$ with $\tau = c_0 - \tilde{a}_1 > 0$. One sets $X = -\mathbf{X}^0_{\max} - \mathsf{h}^{-\frac{2}{3}} + \tilde{a}_1$. Then X satisfies

$$\frac{u'}{v'}(X - \tilde{a}_1) = \frac{u}{v}(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1), \quad X \in [-c_0 + \tilde{a}_1, 0). \tag{4.22}$$

Using Lemma A.6 and (4.22),

$$\left| \frac{u'}{v'}(X - \tilde{a}_1) - \frac{1}{\alpha} + \frac{\sqrt{3}}{\alpha} \mathrm{e}^{-\frac{4}{3}(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}} \right| \le 2.61 \cdot \left(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1 \right)^{-\frac{3}{2}} \mathrm{e}^{-\frac{4}{3}(X + \mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}}. \tag{4.23}$$

Again, we set $\epsilon = \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - \tilde{a}_1)^{\frac{3}{2}}} \in (0, \mathrm{e}^{-\frac{4}{3}(c_0 - \tilde{a}_1)^{\frac{3}{2}}}]$. We also define Y as in the proof of point (2) of Theorem 2.2. Then, equality (4.7) holds and the condition $X \in [-c_0 + \tilde{a}_1, 0)$ implies that

$$-c_0+\tilde{a}_1+\Big(-\tfrac{3}{4}\ln(\epsilon)\Big)^{\frac{2}{3}}=-c_0+\mathsf{h}^{-\frac{2}{3}}\leq X+\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1\leq\Big(-\tfrac{3}{4}\ln(\epsilon)\Big)^{\frac{2}{3}}.$$

Moreover, for every $\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$,

$$\frac{4}{3}\frac{1}{|\ln(\epsilon)|}\leq\Big(X+\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1\Big)^{-\frac{3}{2}}\leq\left(\Big(\frac{3}{4}\Big)^{\frac{2}{3}}+\frac{\tilde{a}_1-c_0}{|\ln(\epsilon)|^{\frac{2}{3}}}\right)^{-\frac{3}{2}}\frac{1}{|\ln(\epsilon)|}.$$

Using (4.23) and the relation $\frac{1}{\alpha}=\frac{u'}{v'}(-\tilde{a}_1)$, one gets, for every $\epsilon\in(0,\mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$,

$$\begin{aligned}&\left|\frac{1}{\epsilon}\left(\frac{u'}{v'}(\epsilon Y-\tilde{a}_1)-\frac{u'}{v'}(-\tilde{a}_1)\right)+\frac{\sqrt{3}}{\alpha}\mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y}\mathrm{e}^{\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2Q(\epsilon Y)}\right|\leq\\&\qquad\frac{4}{3}\frac{\sqrt{3}}{\alpha}\frac{2.61}{|\ln(\epsilon)|}\mathrm{e}^{-2\left(-\frac{3}{4}\ln(\epsilon)\right)^{\frac{1}{3}}\epsilon Y}\mathrm{e}^{\left(-\frac{3}{4}\ln(\epsilon)\right)^{-\frac{1}{3}}(\epsilon Y)^2Q(\epsilon Y)}.\end{aligned}\tag{4.24}$$

Inequality (4.10) holds and, using the fact that $\epsilon Y=X\in[-c_0-\tilde{a}_1,0)$, one gets that Y is bounded and there exists an explicit constant $C>0$ (similar to $C_{c_0-\tilde{a}_1}$ with 2.83 changed by 2.61) such that

$$|X|\leq C\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}.$$

Then, the bootstrap argument gives the existence of $L>0$ (similar to $L_{c_0-\tilde{a}_1}$ with 2.83 changed by 2.61), such that

$$\forall h<c_0^{-\frac{3}{2}},\ \left|Y+\alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\right|\leq L\alpha^2\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{-\frac{3}{2}}$$

and for every $h<c_0^{-\frac{3}{2}}$,

$$\begin{aligned}&\left|\mathbf{E}^0_{\max}+1-\tilde{a}_1\mathsf{h}^{\frac{2}{3}}-\alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}\right|\leq\\&\qquad\alpha^2L\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}},\end{aligned}\tag{4.25}$$

which proves the second point for $\tilde{M}_0=\alpha^2L\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}$. Note that an approximate value for $\tilde{M}_0$ is 2.32×10^9. □

We deduce from the estimates in h of $\mathbf{E}^0_{\min}$ and $\mathbf{E}^0_{\max}$ an estimate of the width of the first spectral band.

Proposition 4.3. *There exists a universal constant* $K_0 > 0$ *such that, for every* $\mathsf{h} \in (0, c_0^{-\frac{3}{2}}]$,

$$\left| (\mathbf{E}^0_{\max} - \mathbf{E}^0_{\min}) - 2\alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}} \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}} \right| \leq K_0 \mathsf{h}^{\frac{5}{3}} (1-\tilde{a}_1\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}} \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}. \tag{4.26}$$

Proof. Indeed one sets $K_0 = \max(M_0, \tilde{M}_0) = \tilde{M}_0 \simeq 2.32 \times 10^9$ and (4.26) is a consequence of (2.4) and (4.21). □

For $\mathsf{h} \geq c_0^{-\frac{3}{2}}$, the situation changes. The range of the periodic potential $\mathbf{V}$ is included in the first spectral band.

Proposition 4.4. *If* $\mathsf{h} \geq c_0^{-\frac{3}{2}}$, $\mathbf{E}^0_{\max} \geq 0$ *and we have*

$$\left[\min\left(-\frac{1}{2}, -1+\tilde{a}_1\mathsf{h}^{\frac{2}{3}}\right), 0\right] \subset \left[\mathbf{E}^0_{\min}, \mathbf{E}^0_{\max}\right].$$

Proof. If $\mathsf{h} \geq c_0^{-\frac{3}{2}}$, no solution of (3.24) or (3.25) satisfies $-\mathbf{X} > 0$. Thus, $\mathbf{E}^0_{\max} \geq 0$. Using the upper bound on $\mathbf{E}^0_{\min}$ given in Theorem 2.2, we have $-1+\tilde{a}_1\mathsf{h}^{\frac{2}{3}} \in [\mathbf{E}^0_{\min}, \mathbf{E}^0_{\max}]$. Using point (1) of Theorem 2.2, we also have $-\frac{1}{2} \in [\mathbf{E}^0_{\min}, \mathbf{E}^0_{\max}]$, which proves the proposition. □

Proposition 4.4 along with Proposition 4.2 imply Theorem 2.1. The following proposition states more precisely the behaviour of $\mathbf{E}^0_{\max}$.

Proposition 4.5. *Let* $\mathsf{h} \geq c_0^{-\frac{3}{2}}$.

1. *If* $\mathsf{h} \in [c_0^{-\frac{3}{2}}, (\tilde{c}_1 - \tilde{c}_0)^{-\frac{3}{2}})$, *then* $\mathbf{E}^0_{\max} \in (-1+\tilde{c}_0\mathsf{h}^{\frac{2}{3}}, 0]$.
2. *If* $\mathsf{h} > (\tilde{c}_1 - \tilde{c}_0)^{-\frac{3}{2}}$, *let* p_0 *defined in* (2.6). *Then,* $\mathbf{E}^0_{\max} \in [-1+\tilde{c}_{p_0}\mathsf{h}^{\frac{2}{3}}, \tilde{c}_{p_0+1}\mathsf{h}^{\frac{2}{3}}]$ *or* $\mathbf{E}^0_{\max} \in [-1+\tilde{c}_{p_0-1}\mathsf{h}^{\frac{2}{3}}, \tilde{c}_{p_0}\mathsf{h}^{\frac{2}{3}}]$.
3.

$$\lim_{\mathsf{h}\to+\infty} \mathbf{E}^0_{\max} = +\infty. \tag{4.27}$$

Proof. **(2)** Since $(\tilde{c}_{p+1} - \tilde{c}_p)_{p\geq 0}$ is strictly decreasing and converges to 0, for any $\mathsf{h} \in ((\tilde{c}_1 - \tilde{c}_0)^{-\frac{3}{2}}, +\infty)$, the integer $p_0 \geq 1$ defined in (2.6) is well defined and unique.

$\mathbf{X}^0_{\max}$ is a solution of either (3.25) or (3.24). Let $k \geq 1$. The restriction of the function $\frac{v}{u}$ to $(-\tilde{c}_{2k}, -\tilde{c}_{2k-2})$ is a strictly increasing and continuous bijection from $(-\tilde{c}_{2k}, -\tilde{c}_{2k-2})$ to $\mathbb{R}$ and $\frac{v'}{u'}$ induce a strictly increasing and continuous bijection from $(-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$ to $\mathbb{R}$. Then, studying the sign of f_x for $x \in (-\tilde{c}_{2k}, -\tilde{c}_{2k-2})$ and using the Sturm–Picone's lemma as in the proof of Lemma C.2, we prove that the function

$$x \mapsto x - \left(\frac{v'}{u'}\Big|_{(-\tilde{c}_{2k+1},-\tilde{c}_{2k-1})} \right)^{-1} \left(\frac{v}{u}\Big|_{(-\tilde{c}_{2k},-\tilde{c}_{2k-2})} \right)(x)$$

is strictly increasing and continuous from $(-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$ to $(\tilde{c}_{2k+1} - \tilde{c}_{2k}, \tilde{c}_{2k-1} - \tilde{c}_{2k-2})$.

Thus, (3.24) admits a unique solution $\underline{\mathbf{X}}_k$ with $-\underline{\mathbf{X}}_k \in (-\tilde{c}_{2k}, -\tilde{c}_{2k-2})$ and $-\mathsf{h}^{-\frac{2}{3}} - \underline{\mathbf{X}}_k \in (-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$.

The sign of f_x is deduced from the signs of u, v, u' and v' on the interval $(-\tilde{c}_{2k+1}, -\tilde{c}_{2k-2})$, since $x \in (-\tilde{c}_{2k}, -\tilde{c}_{2k-2})$ and $x - z \in (-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$. For example, we have on $(-\tilde{c}_{2k+1}, -c_{2k+1})$,

$$(-1)^k u < 0, \quad (-1)^k u' > 0, \quad (-1)^k v > 0, \quad (-1)^k v' < 0$$

and on $(-c_{2k+1}, -\tilde{c}_{2k})$,

$$(-1)^k u < 0, \quad (-1)^k u' > 0, \quad (-1)^k v < 0, \quad (-1)^k v' < 0$$

and the signs alternate on the intervals $(-\tilde{c}_{2k}, -c_{2k})$, $(-c_{2k}, -\tilde{c}_{2k-1})$, $(-\tilde{c}_{2k-1}, -c_{2k-1})$ and $(-c_{2k-1}, -\tilde{c}_{2k-2})$.

The restriction of the function $\frac{v}{u}$ to $(-\tilde{c}_{2k+2}, -\tilde{c}_{2k})$ is a strictly increasing and continuous bijection from $(-\tilde{c}_{2k+2}, -\tilde{c}_{2k})$ to $\mathbb{R}$. We set for every $x \in (-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$ and every $z \in (-\tilde{c}_{2k+1} + \tilde{c}_{2k+2}, -\tilde{c}_{2k-1} + \tilde{c}_{2k})$, $\bar{g}_x(z) = v(x-z)u'(x) - u(x-z)v'(x)$. Then $\bar{g}_x$ satisfies the Airy equation and using the signs of u, v, u' and v' given above and a Sturm–Picone's argument as in Lemma C.2, we prove that $x \mapsto x - \left(\frac{v}{u}\Big|_{(-\tilde{c}_{2k+2},-\tilde{c}_{2k})} \right)^{-1}_{2k} \left(\frac{v'}{u'}\Big|_{(-\tilde{c}_{2k+1},-\tilde{c}_{2k-1})} \right)(x)$ is strictly increasing and continuous from $(-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$ to $(-\tilde{c}_{2k+1} + \tilde{c}_{2k+2}, -\tilde{c}_{2k-1} + \tilde{c}_{2k})$.

Thus, (3.25) admits a unique solution $\bar{\mathbf{X}}_k$ with $-\bar{\mathbf{X}}_k \in (-\tilde{c}_{2k+1}, -\tilde{c}_{2k-1})$ and $-\mathsf{h}^{-\frac{2}{3}} - \bar{\mathbf{X}}_k \in (-\tilde{c}_{2k+2}, -\tilde{c}_{2k})$.

Since $\tilde{c}_{p_0+1} - \tilde{c}_{p_0} < \mathsf{h}^{-\frac{2}{3}} < \tilde{c}_{p_0} - \tilde{c}_{p_0-1}$, we have either $\mathbf{X}^0_{\max} = \underline{\mathbf{X}}_k$ or $\mathbf{X}^0_{\max} = \bar{\mathbf{X}}_k$ for k equal to the integer part of $\frac{p_0}{2}$.

We deduce that $-\mathbf{X}^0_{\max} \in [-\tilde{c}_{p_0+1}, -\tilde{c}_{p_0-1}]$ and $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^0_{\max} \in [-\tilde{c}_{p_0+2}, -\tilde{c}_{p_0}]$, or $-\mathbf{X}^0_{\max} \in [-\tilde{c}_{p_0}, -\tilde{c}_{p_0-2}]$ and $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^0_{\max} \in [-\tilde{c}_{p_0+1}, -\tilde{c}_{p_0-1}]$. After multiplication by $\mathsf{h}^{\frac{2}{3}}$, this proves the second point of Proposition 4.5.

(1) If $\mathsf{h} \in [c_0^{-\frac{3}{2}}, (\tilde{c}_1 - \tilde{c}_0)^{-\frac{3}{2}})$, then $\mathsf{h}^{-\frac{2}{3}} \in (\tilde{c}_2 - \tilde{c}_1, \tilde{c}_0)$. The function $\frac{v'}{u'}$ induces a strictly increasing and continuous bijection from $(-\tilde{c}_1, 0)$ to $\mathbb{R}$. Then, the function $x \mapsto x - \left(\frac{v}{u}\Big|_{(-\tilde{c}_2,-\tilde{c}_0)} \right)^{-1} \left(\frac{v'}{u'}\Big|_{(-\tilde{c}_1,0)} \right)(x)$ is strictly increasing and continuous from $(-\tilde{c}_1, 0)$ to $(\tilde{c}_2 - \tilde{c}_1, \tilde{c}_0)$, using again Sturm–Picone's Lemma with $\bar{g}_x$ for $x \in (-\tilde{c}_1, 0)$. Thus, (3.25) admits a unique solution $\bar{\mathbf{X}}_0$ with $-\bar{\mathbf{X}}_0 \in (-\tilde{c}_1, 0)$ and $-\mathsf{h}^{-\frac{2}{3}} - \bar{\mathbf{X}}_0 \in (-\tilde{c}_2, -\tilde{c}_0)$. Since $\mathbf{X}^0_{\max} = \bar{\mathbf{X}}_0$, we proved the first point after multiplication by $\mathsf{h}^{\frac{2}{3}}$.

(3) The integer p_0 tends to $+\infty$ when h tends to $+\infty$. Indeed, using the asymptotics of the $\tilde{c}_p$ deduced from those of the $\tilde{\xi}_p$ as in (3.15) and (3.16), one has that there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1 \mathsf{h}^2 \leq p_0 \leq C_2 \mathsf{h}^2$ for h large enough. Since $\tilde{c}_p \xrightarrow[p \to +\infty]{} +\infty$, we get (4.27) and prove the third point. $\square$

Proof of Theorem 2.3. **(2)** We start with the proof of the second point since it will imply the first limit in the first point of Theorem 2.3. We look at the behaviour of $\mathbf{E}^0_{\min}$ when h tends to infinity and thus $\mathsf{h}^{-\frac{2}{3}}$ tends to 0. Since $\mathbf{X}^0_{\min}$ is a solution of (3.22) with $-\mathbf{X}^0_{\min} > 0$ and $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^0_{\min} \in (-\tilde{a}_1, 0)$, when $\mathsf{h}^{-\frac{2}{3}}$ tends to 0, both $-\mathbf{X}^0_{\min}$ and $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^0_{\min}$ tends to 0. In order to avoid the technical difficulty induced by the fact that $\frac{v'}{u'}$ tends to $+\infty$ at 0, we notice that $\mathbf{X}^0_{\min}$ is also the unique solution in $(-\mathsf{h}^{-\frac{2}{3}}, 0)$ of the equation

$$\frac{u'}{v'}(-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}) = \frac{u'}{v'}(-\mathbf{X}). \tag{4.28}$$

Note that u and v stand for f and g in [1, 10.4.3]. Thus, for x in a neighbourhood of 0 where v' does not vanish,

$$\left(\frac{u'}{v'}\right)'(x) = \left(\frac{u'}{v'}\right)'(0) + \frac{1}{2}x^2 - \frac{2}{15}x^5 + \mathcal{O}(x^8). \tag{4.29}$$

Let $y = -\mathbf{X}^0_{\min} - \frac{\mathsf{h}^{-\frac{2}{3}}}{2}$. Then, (4.28) rewrites

$$\frac{u'}{v'}\left(y - \frac{\mathsf{h}^{-\frac{2}{3}}}{2}\right) = \frac{u'}{v'}\left(y + \frac{\mathsf{h}^{-\frac{2}{3}}}{2}\right). \tag{4.30}$$

Since $-\mathsf{h}^{-\frac{2}{3}} < \mathbf{X}^0_{\min} < 0$, $|y| \le \frac{\mathsf{h}^{-\frac{2}{3}}}{2}$ and $y = \mathcal{O}(\mathsf{h}^{-\frac{2}{3}})$. Thus equation (4.30) and equality (4.29) imply

$$y - \frac{2}{15}y^4 - \frac{4}{3}\left(\frac{\mathsf{h}^{-\frac{2}{3}}}{2}\right)^3 y^2 = \frac{2}{15}\left(\frac{\mathsf{h}^{-\frac{2}{3}}}{2}\right)^4 + \mathcal{O}(\mathsf{h}^{-\frac{14}{3}}). \tag{4.31}$$

Then, $y = \mathcal{O}(\mathsf{h}^{-\frac{8}{3}})$. Hence, $y = \frac{1}{120}\mathsf{h}^{-\frac{8}{3}} + \mathcal{O}(\mathsf{h}^{-\frac{14}{3}})$. After multiplication by $\mathsf{h}^{\frac{2}{3}}$, this proves (2.5).

(1) The first limit is a direct consequence of point (2). The second limit follows from point (3) of Proposition 4.5. □

4.4. The first spectral gap

Similarly to the estimates for the edges of the first spectral band, we prove the following estimate for $\mathbf{E}^1_{\min}$.

Proposition 4.6. *There exists a universal constant $M_1 > 0$ such that, for every $\mathsf{h} \in (0, a_1^{-\frac{3}{2}})$,*

$$\left|\mathbf{E}^1_{\min} - \left(-1 + a_1\mathsf{h}^{\frac{2}{3}} - \alpha\sqrt{3}(u(-a_1))^2\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - a_1)^{\frac{3}{2}}}\right)\right| \le M_1\mathsf{h}^{\frac{5}{3}}(1 - a_1\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - a_1)^{\frac{3}{2}}}. \tag{4.32}$$

Proof. We follow the proof of point (2) of Theorem 2.2. Let $\tau > 0$. We assume that $\mathsf{h} \leq (a_1+\tau)^{-\frac{3}{2}}$. One sets $X = -\mathbf{X}^1_{\min} - \mathsf{h}^{-\frac{2}{3}} + a_1$ which satisfies

$$\frac{v}{u}(X-a_1) = \frac{v'}{u'}(X+\mathsf{h}^{-\frac{2}{3}}-a_1), \quad X \in [0, -\tilde{c}_0 + a_1). \tag{4.33}$$

Using Lemma A.3, (4.33) implies that

$$\begin{aligned}\left|\frac{v}{u}(X-a_1) - \alpha - \alpha\sqrt{3}\mathrm{e}^{-\frac{4}{3}(X+\mathsf{h}^{-\frac{2}{3}}-a_1)^{\frac{3}{2}}}\right| \leq \\ A_\tau\left(X+\mathsf{h}^{-\frac{2}{3}}-a_1\right)^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}(X+\mathsf{h}^{-\frac{2}{3}}-a_1)^{\frac{3}{2}}}.\end{aligned} \tag{4.34}$$

We set $\epsilon = \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_1)^{\frac{3}{2}}} \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}]$ and $Y = \frac{1}{\epsilon}X$. Since $\alpha = \frac{v}{u}(-a_1)$ and using the fact that $\left(\frac{v}{u}\right)' = \frac{1}{u^2}$ is bounded from below by $\frac{\alpha}{2}$ by (A.7), one shows with a similar proof as (4.10) that:

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}], \ \left|\frac{v}{u}(X-a_1) - \frac{v}{u}(-a_1)\right| \geq \frac{\alpha}{2}\epsilon Y.$$

Then, using that $\epsilon Y = X \in [0, -\tilde{c}_0 + a_1]$ is bounded, one gets that Y is bounded and there exists $C_\tau > 0$ (similar to the constant C_τ in the proof of the estimate of $\mathbf{E}^0_{\min}$ with A_τ instead of 2.83) such that

$$|X| \leq C_\tau \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_1)^{\frac{3}{2}}}.$$

Using Taylor formula as in (4.13) and since $\left(\frac{v}{u}\right)''$ is bounded by 1 on $[0, +\infty)$,

$$\forall \epsilon \in (0, \mathrm{e}^{-\frac{4}{3}\tau^{\frac{3}{2}}}], \ \left|\frac{v}{u}(\epsilon Y - a_1) - \frac{v}{u}(-a_1) - \left(\frac{v}{u}\right)'(-a_1)(\epsilon Y)\right| \leq C_\tau^2\epsilon^2.$$

But, $\left(\frac{v}{u}\right)'(-a_1) = \frac{1}{(u(-a_1))^2}$ and we get, similarly to (4.17), the existence of $L_\tau > 0$ such that

$$\forall \mathsf{h} \leq (a_1+\tau)^{-\frac{3}{2}}, \ \left|Y - \alpha\sqrt{3}(u(-a_1))^2\right| \leq (u(-a_1))^2 L_\tau (\mathsf{h}^{-\frac{2}{3}} - a_1)^{-\frac{3}{2}}$$

from which we obtain (4.32) on $(0, (a_1+\tau)^{-\frac{3}{2}}]$ for the constant $(u(-a_1))^2 L_\tau$.

Since $-1 + \tilde{c}_0\mathsf{h}^{\frac{2}{3}} < \mathbf{E}^1_{\min} < -1 + \tilde{a}_1\mathsf{h}^{\frac{2}{3}}$, for every $\mathsf{h} \in [(a_1+\tau)^{-\frac{3}{2}}, a_1^{-\frac{3}{2}})$,

$$\begin{aligned}\left|\mathbf{E}^1_{\min} + 1 - \tilde{a}_1\mathsf{h}^{\frac{2}{3}} + \alpha\sqrt{3}\frac{(u'(-\tilde{a}_1))^2}{\tilde{a}_1}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_1)^{\frac{3}{2}}}\right| \leq \\ (a_1 - \tilde{c}_0)\tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}(\mathsf{h}^{-\frac{2}{3}} - a_1)^{-\frac{3}{2}}\mathsf{h}^{\frac{2}{3}}\mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_1)^{\frac{3}{2}}}.\end{aligned} \tag{4.35}$$

Let $M_{1,\tau} := \max\left((u(-a_1))^2 L_\tau, (a_1-\tilde{c}_0)\tau^{\frac{3}{2}} \mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}\right)$. Then, (4.32) is valid for $M_{1,\tau}$ for every $\tau > 0$. Remark that the smallest value of $M_{1,\tau}$ is obtained for τ such that $(u(-a_1))^2 L_\tau = (a_1 - \tilde{c}_0)\tau^{\frac{3}{2}} \mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}$ whose approximate values are $\tau \simeq 3.01$ and $M_{1,3.01} \simeq 1942$. □

Remark 4.3. In this proof we had to consider equation (4.33) in this form and not with $\frac{u}{v}$ and $\frac{u'}{v'}$ as in the proof of Theorem 2.2, since $\left(\frac{v}{u}\right)''$ is bounded on $[0,+\infty)$ and $\left(\frac{u}{v}\right)''$ is not.

Combining the estimates of $\mathbf{E}^0_{\max}$ and $\mathbf{E}^1_{\min}$ we deduce an estimate of the first spectral gap of $\mathbf{H}$.

Proposition 4.7. *There exists a universal constant $\tilde{K}_0 > 0$ such that, for every $\mathsf{h} \in (0, a_1^{-\frac{3}{2}})$,*

$$\left|(\mathbf{E}^1_{\min} - \mathbf{E}^0_{\max}) - \left((a_1 - \tilde{a}_1)\mathsf{h}^{\frac{2}{3}} - \alpha\sqrt{3}(u(-a_1))^2 \mathsf{h}^{\frac{2}{3}} \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - a_1)^{\frac{3}{2}}}\right)\right| \le \tilde{K}_0 \mathsf{h}^{\frac{5}{3}}(1 - a_1\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}} \mathrm{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}} - a_1)^{\frac{3}{2}}}. \tag{4.36}$$

Proof. We combine (4.32) and point (2) of Proposition 4.2, use $a_1 > \tilde{a}_1$ and we get (4.36) with $\tilde{K}_0 = \max(\tilde{M}_0, M_1)$. □

5. Counting the spectral bands in the range of V

In this section, we prove Theorem 2.4 by determining the band edges which are contained in the interval $[-1, 0]$ for a fixed h.

Proposition 5.1. *Let $p \ge 0$ and assume that $\mathsf{h} \le \tilde{c}_p^{-\frac{3}{2}}$. Then, for every $k \in \{0, \ldots, p\}$,*

1. *If $k = 2j$ is even, (3.25) has a unique solution $\hat{\mathbf{X}}_{2j}$ with $-\hat{\mathbf{X}}_{2j} \in [0,+\infty)$, $-\mathsf{h}^{-\frac{2}{3}} - \hat{\mathbf{X}}_{2j} \in (-a_{j+1}, -\tilde{c}_{2j}]$ and satisfying:*

$$-\mathsf{h}^{-\frac{2}{3}} + \tilde{c}_{2j} \le \hat{\mathbf{X}}_{2j} < -\mathsf{h}^{-\frac{2}{3}} + a_{j+1}. \tag{5.1}$$

2. *If $k = 2j+1$ is odd, (3.22) has a unique solution $\hat{\mathbf{X}}_{2j+1}$ with $-\hat{\mathbf{X}}_{2j+1} \in [0,+\infty)$, $-\mathsf{h}^{-\frac{2}{3}} - \hat{\mathbf{X}}_{2j+1} \in (-\tilde{a}_{j+2}, -\tilde{c}_{2j+1}]$ and satisfying:*

$$-\mathsf{h}^{-\frac{2}{3}} + \tilde{c}_{2j+1} \le \hat{\mathbf{X}}_{2j+1} < -\mathsf{h}^{-\frac{2}{3}} + \tilde{a}_{j+2}. \tag{5.2}$$

Proof. By Lemma 3.3, for every $k \ge 0$, the function $x \mapsto x - \psi^k(x)$ is a strictly increasing and continuous bijection from $[0,+\infty)$ to $[\tilde{c}_k, +\infty)$. Thus, if $\mathsf{h}^{-\frac{2}{3}} \ge \tilde{c}_p \ge \tilde{c}_k$, there exists a unique $x^k \ge 0$ such that $\mathsf{h}^{-\frac{2}{3}} = x^k - \psi^k(x^k)$. Let $\hat{\mathbf{X}}_k$ be such that $\hat{\mathbf{X}}_k = -x^k$. Then, if $k = 2j$, $\hat{\mathbf{X}}_{2j}$ is the unique solution of (3.25) with $-\hat{\mathbf{X}}_{2j} \in [0,+\infty)$ and $-\mathsf{h}^{-\frac{2}{3}} - \hat{\mathbf{X}}_{2j} \in (-a_{j+1}, -\tilde{c}_{2j}]$. Moreover,

$$-a_{j+1} < -\hat{\mathbf{X}}_k - \mathsf{h}^{-\frac{2}{3}} \leq -\tilde{c}_k < 0 \leq -\hat{\mathbf{X}}_k,$$

and we get (5.1). If $k = 2j+1$, $\hat{\mathbf{X}}_{2j+1}$ is the unique solution of (3.22) with $-\hat{\mathbf{X}}_{2j+1} \in [0, +\infty)$ and $-\mathsf{h}^{-\frac{2}{3}} - \hat{\mathbf{X}}_{2j+1} \in (-\tilde{a}_{j+2}, -\tilde{c}_{2j+1}]$. Moreover,

$$-\tilde{a}_{j+2} < -\hat{\mathbf{X}}_k - \mathsf{h}^{-\frac{2}{3}} \leq -\tilde{c}_k < 0 \leq -\hat{\mathbf{X}}_k,$$

and we get (5.2). □

Proposition 5.2. *Assume that* $\mathsf{h} \leq c_p^{-\frac{3}{2}}$. *Then, for every* $k \in \{0, \ldots, p\}$,

1. *If* $k = 2j$ *is even,* (3.24) *has a unique solution* $\check{\mathbf{X}}_{2j}$ *with* $-\check{\mathbf{X}}_{2j} \in [0, +\infty)$, $-\mathsf{h}^{-\frac{2}{3}} - \check{\mathbf{X}}_{2j} \in [c_{2j}, -\tilde{a}_{j+1})$ *and satisfying:*

$$-\mathsf{h}^{-\frac{2}{3}} + \tilde{a}_{j+1} < \check{\mathbf{X}}_{2j} \leq -\mathsf{h}^{-\frac{2}{3}} + c_{2j}. \tag{5.3}$$

2. *If* $k = 2j+1$ *is odd,* (3.23) *has a unique solution* $\check{\mathbf{X}}_{2j+1}$ *with* $-\check{\mathbf{X}}_{2j+1} \in [0, +\infty)$, $-\mathsf{h}^{-\frac{2}{3}} - \check{\mathbf{X}}_{2j+1} \in [c_{2j+1}, -a_{j+1})$ *and satisfying:*

$$-\mathsf{h}^{-\frac{2}{3}} + a_{j+1} < \check{\mathbf{X}}_{2j+1} \leq -\mathsf{h}^{-\frac{2}{3}} + c_{2j+1}. \tag{5.4}$$

Proof. Let $k \in \{0, \ldots, p\}$. Since $\mathsf{h} \leq c_p^{-\frac{3}{2}}$, we have $\mathsf{h}^{-\frac{2}{3}} \in [c_k, +\infty)$. Thanks to Lemma C.2, z_k is continuous and strictly increasing and there exists a unique real number $x_k \geq 0$ such that $\mathsf{h}^{-\frac{2}{3}} = z_k(x_k)$. Let $\check{\mathbf{X}}_k$ be such that $\check{\mathbf{X}}_k = -x_k$. Then, $\check{\mathbf{X}}_{2j}$ is the unique solution of (3.24) such that $-\check{\mathbf{X}}_{2j} \in [0, +\infty)$ and $-\mathsf{h}^{-\frac{2}{3}} - \check{\mathbf{X}}_{2j} \in [c_{2j}, -\tilde{a}_{j+1})$. Moreover,

$$-c_k \leq -\check{\mathbf{X}}_k - \mathsf{h}^{-\frac{2}{3}} < -\tilde{a}_{j+1}$$

and we get (5.3).

Similarly, $\check{\mathbf{X}}_{2j+1}$ is the unique solution of (3.23) such that $-\check{\mathbf{X}}_{2j+1} \in [0, +\infty)$ and $-\mathsf{h}^{-\frac{2}{3}} - \check{\mathbf{X}}_{2j+1} \in [c_{2j+1}, -a_{j+1})$. Moreover,

$$-c_k \leq -\check{\mathbf{X}}_k - \mathsf{h}^{-\frac{2}{3}} < -a_{j+1}$$

and we get (5.4). □

We deduce from Proposition 5.1 and Proposition 5.2 the following proposition on the p first spectral bands and the $p-1$ first spectral gaps of the operator H.

Proposition 5.3. *Let* $p \geq 0$. *Assume that* $\mathsf{h} \leq \tilde{c}_p^{-\frac{3}{2}}$.

1. *For every* $k \in \{0, \ldots, p\}$, $\mathbf{E}_{\min}^{k+1} = \mathsf{h}^{\frac{2}{3}} \hat{\mathbf{X}}_k$ *and* $\mathbf{E}_{\max}^{k} = \mathsf{h}^{\frac{2}{3}} \check{\mathbf{X}}_k$.

2. *We have the estimates on the spectral gaps:*

$$\forall j \geq 1,\ 0 < (\tilde{c}_{2j-1} - c_{2j-1})\mathsf{h}^{\frac{2}{3}} \leq \mathbf{E}^{2j}_{\min} - \mathbf{E}^{2j-1}_{\max} \leq (\tilde{a}_{j+1} - a_j)\mathsf{h}^{\frac{2}{3}}$$

and

$$\forall j \geq 0,\ 0 < (\tilde{c}_{2j} - c_{2j})\mathsf{h}^{\frac{2}{3}} \leq \mathbf{E}^{2j+1}_{\min} - \mathbf{E}^{2j}_{\max} \leq (a_{j+1} - \tilde{a}_{j+1})\mathsf{h}^{\frac{2}{3}}.$$

In particular, all the spectral gaps in $\sigma(\mathbf{H})$ are not empty.

Proof. **(1)** Using the estimates obtained on $\hat{\mathbf{X}}_k$ and $\check{\mathbf{X}}_k$ and using the fact that $c_k < \tilde{c}_k$, we have $\check{\mathbf{X}}_k < \hat{\mathbf{X}}_k$. Since

$$-1 < \mathbf{E}^0_{\min} < -1 + \tilde{a}_1\mathsf{h}^{\frac{2}{3}} < \mathsf{h}^{\frac{2}{3}}\check{\mathbf{X}}_0 < \mathsf{h}^{\frac{2}{3}}\hat{\mathbf{X}}_0,$$

we have $\mathbf{E}^0_{\max} = \mathsf{h}^{\frac{2}{3}}\check{\mathbf{X}}_0$ and $\mathbf{E}^1_{\min} = \mathsf{h}^{\frac{2}{3}}\hat{\mathbf{X}}_0$. Then using $\check{\mathbf{X}}_k < \hat{\mathbf{X}}_k$, we deduce the first point.

(2) These two estimates are deduced directly from the estimates proven on $\hat{\mathbf{X}}_k$ and $\check{\mathbf{X}}_k$ in Proposition 5.1 and Proposition 5.2. We just have to be careful with the fact that $\mathbf{E}^{2j}_{\min} = \mathsf{h}^{\frac{2}{3}}\hat{\mathbf{X}}_{2j-1}$ and $\mathbf{E}^{2j+1}_{\min} = \mathsf{h}^{\frac{2}{3}}\hat{\mathbf{X}}_{2j}$ and to use the right estimate in Proposition 5.1 depending on the parity of k. □

Propositions 5.1, 5.2 and 5.3 imply the proof of Proposition 2.1.

Proof of Proposition 2.1. For every $p \geq 0$, $-\mathbf{E}^p_{\max}(\mathsf{h}) = (z_p)^{-1}(\mathsf{h}^{-\frac{2}{3}})$ and since z_p is strictly increasing and continuous on $[0, +\infty)$, $\mathsf{h} \mapsto \mathbf{E}^p_{\max}(\mathsf{h})$ is strictly increasing and continuous on $[0, +\infty)$. Since for every $p \geq 0$, $\mathbf{E}^p_{\max}(c_p^{-\frac{3}{2}}) = 0$, $c_p^{-\frac{3}{2}}$ is the unique zero in $[0, +\infty)$ of the function $\mathsf{h} \mapsto \mathbf{E}^p_{\max}(\mathsf{h})$.

Since for every $p \geq 0$, $-\mathbf{E}^{p+1}_{\min}(\mathsf{h}) = (z^p)^{-1}(\mathsf{h}^{-\frac{2}{3}})$ (where $z^p : x \mapsto x - \psi^p(x)$ is strictly increasing and continuous on $[0, +\infty)$), $\mathsf{h} \mapsto \mathbf{E}^{p+1}_{\min}(\mathsf{h})$ is also strictly increasing and continuous on $[0, +\infty)$, and since for every $p \geq 0$, $\mathbf{E}^{p+1}_{\min}(\tilde{c}_p^{-\frac{3}{2}}) = 0$, $\tilde{c}_p^{-\frac{3}{2}}$ is the unique zero in $[0, +\infty)$ of the function $\mathsf{h} \mapsto \mathbf{E}^{p+1}_{\min}(\mathsf{h})$. □

The estimates in Propositions 5.1, 5.2 and 5.3 combined with the intervals given in Lemma 3.1 lead to the proof of Theorem 2.4. Before that, we prove a technical lemma.

Lemma 5.1. *For every $y \in \mathbb{R}^*_+$, let $I(y) = (\frac{3}{2})^{\frac{1}{3}} \frac{y^{\frac{3}{2}}+1}{y^2+y+1}$. Then, for every $\eta > 0$ and every real number $0 < b < a$ such that $\frac{a-b}{b} \in [0, \eta]$,*

$$(a-b)b^{-\frac{1}{3}} I((1+\eta)^{\frac{2}{3}}) \leq \left(\tfrac{3}{2}a\right)^{\frac{2}{3}} - \left(\tfrac{3}{2}b\right)^{\frac{2}{3}} \leq (a-b)b^{-\frac{1}{3}} I(1). \tag{5.5}$$

Proof. One checks that $I(1)=(\frac{3}{2})^{\frac{1}{3}}$, $I'(1)=-\frac{1}{6}$ and that $I'<0$ on $[1,+\infty[$, hence

$$I((1+\eta)^{\frac{2}{3}})\le I\left((\tfrac{a}{b})^{\frac{2}{3}}\right)\le I(1).$$

Since

$$I\left((\tfrac{a}{b})^{\frac{2}{3}}\right)=(a-b)^{-1}\left(\left(\tfrac{3}{2}a\right)^{\frac{2}{3}}-\left(\tfrac{3}{2}b\right)^{\frac{2}{3}}\right)b^{\frac{1}{3}},$$

we get (5.5). □

Proof of Theorem 2.4. Let $\mathsf{h}<c_0^{-\frac{3}{2}}$. The first point in Theorem 2.4 is a direct consequence of point (1) of Proposition 5.3 and Propositions 5.1 and 5.2 which ensure that for every $k\in\{0,\dots,k_0\}$, $\hat{\mathbf{E}}_k$ and $\check{\mathbf{E}}_k$ are in $[-1,0]$.

For the second point, using Propositions 5.1, 5.2 and 5.3 one deduces that

$$\forall p\in\{2,\dots,k_0\},\ (\tilde{c}_p-c_p)\mathsf{h}^{\frac{2}{3}}\le\mathbf{E}_{\min}^{p+1}-\mathbf{E}_{\max}^{p}\le(\tilde{c}_p-\tilde{c}_{p-2})\mathsf{h}^{\frac{2}{3}}$$

and

$$\forall p\in\{2,\dots,k_0\},\ 0<\mathbf{E}_{\max}^{p}-\mathbf{E}_{\min}^{p}\le(c_p-\tilde{c}_{p-1})\mathsf{h}^{\frac{2}{3}}.$$

Let $p\in\{2,\dots,k_0\}$. Assume that p is even, that is $p=2j$ for $j\ge 1$. Then,

$$\tilde{c}_{2j}-\tilde{c}_{2j-2}=\left(\tfrac{3}{2}\tilde{\xi}_{2j}\right)^{\frac{2}{3}}-\left(\tfrac{3}{2}\tilde{\xi}_{2j-2}\right)^{\frac{2}{3}}.$$

Using (3.15) in Lemma 3.1, we have $\tilde{\xi}_{2j}-\tilde{\xi}_{2j-2}\in\left[\pi-\frac{5}{9\pi}\frac{2j}{(2j)^2-1},\pi+\frac{5}{9\pi}\frac{2j}{(2j)^2-1}\right]$ and

$$\tilde{\xi}_{2j-2}^{-\frac{1}{3}}\le\left(-\frac{5\pi}{12}+j\pi-\frac{5}{18\pi}\right)^{-\frac{1}{3}}\le\left(\frac{(2j-1)\pi}{2}\right)^{-\frac{1}{3}}.$$

Thus, by Lemma 5.1,

$$\tilde{c}_{2j}-\tilde{c}_{2j-2}\le\left(\pi+\frac{5}{9\pi}\frac{2j}{(2j)^2-1}\right)\left(\frac{3}{2}\right)^{\frac{1}{3}}\left(\frac{(2j-1)\pi}{2}\right)^{-\frac{1}{3}}.$$

If p is odd, that is $p=2j+1$ for $j\ge 1$, then, using (3.16) in Lemma 3.1, we have $\tilde{\xi}_{2j+1}-\tilde{\xi}_{2j-1}\in\left[\pi-\frac{7}{3\pi}\frac{2j+1}{(2j+1)^2-1},\pi+\frac{7}{3\pi}\frac{2j+1}{(2j+1)^2-1}\right]$ and $\tilde{\xi}_{2j-1}^{-\frac{1}{3}}\le\left(\frac{2j\pi}{2}\right)^{-\frac{1}{3}}$. Thus, by Lemma 5.1,

$$\tilde{c}_{2j+1}-\tilde{c}_{2j-1}\le\left(\pi+\frac{7}{3\pi}\frac{2j+1}{(2j+1)^2-1}\right)\left(\frac{3}{2}\right)^{\frac{1}{3}}\left(\frac{2j\pi}{2}\right)^{-\frac{1}{3}}.$$

Since $\frac{7}{3\pi} > \frac{5}{9\pi}$, we have

$$\forall p \in \{2, \dots, k_0\},\ \tilde{c}_p - \tilde{c}_{p-2} \le \left(\pi + \frac{7}{3\pi}\frac{p}{p^2-1}\right)\left(\frac{3}{\pi}\right)^{\frac{1}{3}}(p-1)^{-\frac{1}{3}}$$

which proves (2.9). The proof of the upper bound in (2.8) is similar. We estimate both $c_{2j} - \tilde{c}_{2j-1}$ and $c_{2j+1} - \tilde{c}_{2j}$ for $j \ge 1$ by using (3.13), (3.16), (3.15) and (3.14) to obtain that $\xi_{2j} - \tilde{\xi}_{2j-1} \in \left[\frac{\pi}{3} - \frac{7}{3\pi}\frac{2j+\frac{1}{3}}{2j(2j+\frac{2}{3})}, \frac{\pi}{3} + \frac{7}{3\pi}\frac{2j+\frac{1}{3}}{2j(2j+\frac{2}{3})}\right]$ and $\xi_{2j+1} - \tilde{\xi}_{2j} \in \left[\frac{\pi}{3} - \frac{5}{9\pi}\frac{2j+1+\frac{1}{3}}{(2j+1)(2j+1+\frac{2}{3})}, \frac{\pi}{3} + \frac{5}{9\pi}\frac{2j+1+\frac{1}{3}}{(2j+1)(2j+1+\frac{2}{3})}\right]$. We also have that, for every $p \in \{2, \dots, k_0\}$, $\tilde{\xi}_p^{-\frac{1}{3}} \le \left(\frac{(p+1)\pi}{2}\right)^{-\frac{1}{3}}$. Since $\frac{7}{3\pi} > \frac{5}{9\pi}$, we get the upper bound of (2.8) by using Lemma 5.1.

It remains to prove the lower bound in (2.8). We have to find a lower bound of $\tilde{c}_p - c_p$ for every $p \in \{2, \dots, k_0\}$. Using (3.15) and (3.13) we get for every $j \ge 1$,

$$\tilde{\xi}_{2j} - \xi_{2j} \in \left[\frac{\pi}{6} - \frac{229}{432\pi}, \frac{\pi}{6} + \frac{229}{432\pi}\right] \subset \left[\frac{\pi}{9}, \frac{2\pi}{9}\right].$$

We have

$$\frac{\tilde{\xi}_{2j} - \xi_{2j}}{\xi_{2j}} \le \frac{\frac{2\pi}{9}}{\frac{5\pi}{12} + \pi - \frac{7}{16\pi}} \le \frac{\frac{2\pi}{9}}{\frac{17\pi}{12}} \le \frac{8}{51} \le \frac{1}{6}.$$

Moreover, since $\frac{5}{6} + \frac{7}{8\pi^2} < 1$,

$$(\xi_{2j})^{-\frac{1}{3}} \ge \left(\frac{5\pi}{12} + \frac{2j\pi}{2} + \frac{7}{16\pi}\right)^{-\frac{1}{3}} \ge \left(\frac{\pi}{2}\right)^{-\frac{1}{3}}(2j+1)^{-\frac{1}{3}}.$$

Thus, we take $\eta = \frac{1}{6}$ and use the lower bound in (5.5) to get

$$I\left(\left(\frac{7}{6}\right)^{\frac{2}{3}}\right)\frac{2^{\frac{1}{3}}\pi^{\frac{2}{3}}}{9}(2j+1)^{-\frac{1}{3}} \le \tilde{c}_{2j} - c_{2j}.$$

For $p = 2j+1$, since $\frac{\pi}{9} < \frac{\pi}{6} - \frac{97}{264\pi}$ and $\frac{5}{6} + \frac{5}{33\pi^2} < 1$, and taking $\eta = \frac{1}{2}$, we get a larger lower bound which is

$$I\left(\left(\frac{3}{2}\right)^{\frac{2}{3}}\right)\frac{2^{\frac{1}{3}}\pi^{\frac{2}{3}}}{9}(2j+2)^{-\frac{1}{3}} \le \tilde{c}_{2j+1} - c_{2j+1}.$$

It allows to conclude that the lower bound valid for every $p \in \{2, \dots, k_0\}$ is the one obtained for p even. This proves the lower bound in (2.8). □

Proof of Corollary 2.1. Recall that the integer $k_0(\mathsf{h})$ defined in (2.7) is equal to $\left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right]$ or to $\left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right] - 1$. Using (2.8), one has:

$$\forall \mathsf{h} < c_0^{-\frac{3}{2}},\ 0 < \sum_{p=2}^{k_0(\mathsf{h})} \delta_p \leq \sum_{p=2}^{k_0(\mathsf{h})} \left(\frac{\pi}{3} + \frac{7}{3\pi} \frac{p+\frac{1}{3}}{p(p+\frac{2}{3})} \right) \left(\frac{3}{\pi} \right)^{\frac{1}{3}} \frac{\mathsf{h}^{\frac{2}{3}}}{p^{\frac{1}{3}}}. \tag{5.6}$$

Since $x \mapsto \frac{1}{x^{\frac{1}{3}}}$ and $x \mapsto \frac{1}{x^{\frac{1}{3}}} \frac{x+\frac{1}{3}}{x(x+\frac{2}{3})}$ are decreasing functions on $[1, +\infty)$, by comparison between sums and integrals,

$$\frac{3}{2}\mathsf{h}^{\frac{2}{3}} \left((k_0(\mathsf{h})+1)^{\frac{2}{3}} - 2^{\frac{2}{3}} \right) \leq \sum_{p=2}^{k_0(\mathsf{h})} \frac{\mathsf{h}^{\frac{2}{3}}}{p^{\frac{1}{3}}} \leq \frac{3}{2}\mathsf{h}^{\frac{2}{3}} \left((k_0(\mathsf{h}))^{\frac{2}{3}} - 1 \right)$$

and

$$\mathsf{h}^{\frac{2}{3}} \int_2^{k_0(\mathsf{h})+1} \frac{1}{x^{\frac{1}{3}}} \frac{x+\frac{1}{3}}{x(x+\frac{2}{3})} \mathrm{d}x \leq \sum_{p=2}^{k_0(\mathsf{h})} \frac{p+\frac{1}{3}}{p(p+\frac{2}{3})} \frac{\mathsf{h}^{\frac{2}{3}}}{p^{\frac{1}{3}}} \leq \mathsf{h}^{\frac{2}{3}} \int_1^{k_0(\mathsf{h})} \frac{1}{x^{\frac{1}{3}}} \frac{x+\frac{1}{3}}{x(x+\frac{2}{3})} \mathrm{d}x.$$

Since

$$\int \frac{1}{x^{\frac{1}{3}}} \frac{x+\frac{1}{3}}{x(x+\frac{2}{3})} \mathrm{d}x = -\frac{3}{2} \frac{1}{x^{\frac{1}{3}}} - \left(\frac{3}{16}\right)^{\frac{1}{3}} \ln\left(x^{\frac{1}{3}} + \left(\frac{2}{3}\right)^{\frac{1}{3}} \right) +$$
$$\left(\frac{3}{2^7}\right)^{\frac{1}{3}} \ln\left(x^{\frac{2}{3}} - \left(\frac{2}{3}x\right)^{\frac{1}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}} \right) + \frac{3^{\frac{5}{6}}}{2^{\frac{4}{3}}} \arctan\left(\frac{2^{\frac{2}{3}}}{3^{\frac{1}{6}}} (x^{\frac{1}{3}} - \frac{1}{3^{\frac{1}{2}}}) \right),$$

and since $k_0(\mathsf{h})$ is equal to $\left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right]$ or $\left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right] - 1$, one gets that

$$\sum_{p=2}^{k_0(\mathsf{h})} \frac{\mathsf{h}^{\frac{2}{3}}}{p^{\frac{1}{3}}} \xrightarrow[\mathsf{h}\to 0]{} \frac{3}{2}\left(\frac{4}{3\pi}\right)^{\frac{2}{3}} \quad \text{and} \quad \sum_{p=2}^{k_0(\mathsf{h})} \frac{p+\frac{1}{3}}{p(p+\frac{2}{3})} \frac{\mathsf{h}^{\frac{2}{3}}}{p^{\frac{1}{3}}} \xrightarrow[\mathsf{h}\to 0]{} 0.$$

Thus,

$$\sum_{p=2}^{k_0(\mathsf{h})} \left(\frac{\pi}{3} + \frac{7}{3\pi} \frac{p+\frac{1}{3}}{p(p+\frac{2}{3})} \right) \left(\frac{3}{\pi} \right)^{\frac{1}{3}} \frac{\mathsf{h}^{\frac{2}{3}}}{p^{\frac{1}{3}}} \xrightarrow[\mathsf{h}\to 0]{} \left(\frac{2}{3}\right)^{\frac{1}{3}}. \tag{5.7}$$

The function $\mathsf{h} \mapsto \sum_{p=0}^{k_0(\mathsf{h})} \delta_p$ is increasing and for every $\mathsf{h} > 0$, $\sum_{p=0}^{k_0(\mathsf{h})} \delta_p \leq (0 - \mathbf{E}^0_{\min}) \leq 1$, since the spectrum of $\mathbf{H}$ is included in $[-1, +\infty]$ by Proposition 4.1. Thus $\mathsf{h} \mapsto D(\mathsf{h})$ admits a limit in $\mathbb{R}$ at $+\infty$. Then, (5.6) and (5.7) imply (2.10). □

6. Uniform estimates of the widths of the spectral bands and gaps

Proposition 5.3 allows us to identify the spectral band edges among the solutions of (3.25), (3.24), (3.22) and (3.23). Using proofs similar to those of the estimates and asymptotics of $\mathbf{E}^0_{\text{min}}$, $\mathbf{E}^0_{\text{max}}$ and $\mathbf{E}^1_{\text{min}}$, one gets the following estimates for the spectral band edges.

Proposition 6.1. *Let $j \geq 0$.*

1. *There exists a universal constant $M_{2j} > 0$ such that for every $\mathsf{h} \in (0, \tilde{a}_{j+1}^{-\frac{3}{2}})$,*

$$\left|\mathbf{E}^{2j}_{\text{min}} - \left(-1 + \tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}} - \alpha\sqrt{3}\frac{(u'(-\tilde{a}_{j+1}))^2}{\tilde{a}_{j+1}}\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_{j+1})^{\frac{3}{2}}}\right)\right| \leq M_{2j}\mathsf{h}^{\frac{5}{3}}(1-\tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_{j+1})^{\frac{3}{2}}} \tag{6.1}$$

and there exists a universal constant $\tilde{M}_{2j} > 0$ such that for every $\mathsf{h} \in (0, c_{2j}^{-\frac{3}{2}}]$,

$$\left|\mathbf{E}^{2j}_{\text{max}} - \left(-1 + \tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}} + \alpha\sqrt{3}\frac{(u'(-\tilde{a}_{j+1}))^2}{\tilde{a}_{j+1}}\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_{j+1})^{\frac{3}{2}}}\right)\right| \leq \tilde{M}_{2j}\mathsf{h}^{\frac{5}{3}}(1-\tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_{j+1})^{\frac{3}{2}}}. \tag{6.2}$$

2. *There exists a universal constant $M_{2j+1} > 0$ such that for every $\mathsf{h} \in (0, a_{j+1}^{-\frac{3}{2}})$,*

$$\left|\mathbf{E}^{2j+1}_{\text{min}} - \left(-1 + a_{j+1}\mathsf{h}^{\frac{2}{3}} - \alpha\sqrt{3}(u(-a_{j+1}))^2\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_{j+1})^{\frac{3}{2}}}\right)\right| \leq M_{2j+1}\mathsf{h}^{\frac{5}{3}}(1-a_{j+1}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_{j+1})^{\frac{3}{2}}} \tag{6.3}$$

and there exists a universal constant $\tilde{M}_{2j+1} > 0$ such that for every $\mathsf{h} \in (0, c_{2j+1}^{-\frac{3}{2}}]$,

$$\left|\mathbf{E}^{2j+1}_{\text{max}} - \left(-1 + a_{j+1}\mathsf{h}^{\frac{2}{3}} + \alpha\sqrt{3}(u(-a_{j+1}))^2\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_{j+1})^{\frac{3}{2}}}\right)\right| \leq \tilde{M}_{2j+1}\mathsf{h}^{\frac{5}{3}}(1-a_{j+1}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_{j+1})^{\frac{3}{2}}}. \tag{6.4}$$

3. *Let $h \in (0, c_1^{-\frac{3}{2}}]$. For every $p \in \{0, \dots, \left[\frac{4}{3\pi}\frac{1}{\mathsf{h}}\right]\}$, (6.1) and (6.2) hold true if $p = 2j$ is even and (6.3) and (6.4) hold true if $p = 2j+1$ is odd.*

Remark 6.1. The estimates (6.1), (6.2), (6.3) and (6.4) can be extended to $\mathsf{h} = 0$ by continuity on $[0, +\infty)$ of the spectral band edges and since all the quantities have finite limits when h tends to 0.

Proof. For $j = 0$ we already obtained the estimate of $\mathbf{E}^0_{\min}$. For every $j \geq 1$, $\mathbf{X}^{2j}_{\min}$ is the unique solution of (3.22) with $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^{2j}_{\min} \in [-\tilde{a}_{j+1}, -\tilde{c}_{2j-1})$. Using Lemma A.1, the function $x \mapsto \frac{x}{(v'(x))^2}$ is smaller than $-\frac{1}{\alpha}$ on the interval $[-\tilde{a}_{j+1}, -\tilde{c}_{2j-1})$. Thus, the scheme of the proof of the estimate of $\mathbf{E}^0_{\min}$ can be followed and leads to (6.1), using the inequality $-1 + \tilde{c}_{2j-1}\mathsf{h}^{\frac{2}{3}} < \mathbf{E}^{2j}_{\min} < -1 + \tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}}$. Setting $B_{2j,\tau} := \sqrt{3}\mathrm{e}^{\frac{2}{3}\tau^{-\frac{1}{2}}\tilde{a}^2_{j+1}}(1 + 2.83\sqrt{3}\tau^{-\frac{3}{2}})$ and defining $C_{2j,\tau}$, $D_{2j,\tau}$ and $L_{2j,\tau}$ as in the proof of the estimate of $\mathbf{E}^0_{\min}$ but with $B_{2j,\tau}$ replacing B_τ, one sets $M_{2j,\tau} := \max(\alpha^2\frac{(u'(-\tilde{a}_{j+1}))^2}{\tilde{a}_{j+1}}L_{2j,\tau}, \tilde{a}_{j+1}\tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}})$. Any value of $\tau > 0$ provides a universal constant M_{2j}. Thus, we set: $M_{2j} := \inf_{\tau>0} M_{2j,\tau}$, which is obtained for τ such that $\alpha^2\frac{(u'(-\tilde{a}_{j+1}))^2}{\tilde{a}_{j+1}}L_{2j,\tau} = \tilde{a}_{j+1}\tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}}$.

For every $j \geq 0$, $\mathbf{X}^{2j}_{\max}$ is the unique solution of (3.24) with $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^{2j}_{\max} \in [-c_{2j}, -\tilde{a}_{j+1})$. Since the function $x \mapsto \frac{x}{(v'(x))^2}$ is smaller than $-\frac{1}{\alpha}$ on the interval $[-c_{2j}, -\tilde{a}_{j+1}]$, the scheme of the proof of the estimate of $\mathbf{E}^0_{\max}$ can be followed and leads to (6.2). Defining L_{2j} as in the proof of the estimate on $\mathbf{E}^0_{\max}$ but for $\tau = c_{2j} - \tilde{a}_{j+1}$, one sets $\tilde{M}_{2j} := \alpha^2\frac{(u'(-\tilde{a}_{j+1}))^2}{\tilde{a}_{j+1}}L_{2j}$.

For every $j \geq 0$, $\mathbf{X}^{2j+1}_{\min}$ is the unique solution of (3.25) with $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^{2j+1}_{\min} \in [-a_{j+1}, -\tilde{c}_{2j})$. Since the function $\frac{1}{u^2}$ is greater than $\frac{\alpha}{2}$ on this interval, the scheme of the proof of the estimate of $\mathbf{E}^1_{\min}$ can be followed and leads to (6.3), using the inequality $-1 + \tilde{c}_{2j}\mathsf{h}^{\frac{2}{3}} < \mathbf{E}^{2j+1}_{\min} < -1 + a_{j+1}\mathsf{h}^{\frac{2}{3}}$. As for $\mathbf{E}^{2j}_{\min}$, we define $L_{2j+1,\tau}$ with A_τ given in Lemma A.3 instead of 2.83 and a_{j+1} instead of $\tilde{a}_{j+1}$. Then we set $M_{2j+1,\tau} := \max((u(-a_{j+1}))^2L_{2j+1,\tau}, (a_{j+1} - \tilde{c}_{2j})\tau^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}\tau^{\frac{3}{2}}})$ and we set: $M_{2j+1} := \inf_{\tau>0} M_{2j+1,\tau}$.

For every $j \geq 0$, $\mathbf{X}^{2j+1}_{\max}$ is the unique solution of (3.23) with $-\mathsf{h}^{-\frac{2}{3}} - \mathbf{X}^{2j+1}_{\max} \in [-c_{2j+1}, -a_{j+1})$. Since the function $\frac{1}{u^2}$ is greater than $\frac{\alpha}{2}$ on this interval and using the estimate (A.19), combining the proofs of the estimates of $\mathbf{E}^0_{\max}$ and $\mathbf{E}^1_{\min}$ leads to (6.2). Defining L_{2j+1} for $\tau = c_{2j+1} - a_{j+1}$, one sets $\tilde{M}_{2j+1} := (u(-a_{j+1}))^2L_{2j+1}$.

The last statement is a direct consequence of the counting of the number of spectral bands in the range of $\mathbf{V}$ done in Theorem 2.4 and the fact that both sequences $(c_p^{-\frac{3}{2}})_{p\geq 0}$ and $(\mathfrak{a}_p^{-\frac{3}{2}})_{p\geq 0}$ are strictly decreasing and interlaced. □

Proposition 6.1 implies Theorem 2.5.

Proof of Theorem 2.5. The first statement of Theorem 2.5 is about the convergence of the spectral bands to the zeroes of the Airy function Ai and its derivative, and is a consequence of the first two terms of (6.1), (6.2), (6.3) and (6.4). More precisely, for every $j \geq 0$ and every $\mathsf{h} \in (0, c_{2j}^{-\frac{3}{2}})$,

$$\mathsf{h}^{-\frac{2}{3}}(1 + \mathbf{E}^{2j}_{\min}) = \tilde{a}_{j+1} + \mathcal{O}\left(\mathrm{e}^{-\frac{4}{3}\frac{1}{\mathsf{h}}(1-\tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}})^{\frac{3}{2}}}\right)$$

and

$$\mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^{2j}_{\max}) = \tilde{a}_{j+1} + \mathcal{O}\left(\mathsf{e}^{-\frac{4}{3}\frac{1}{\mathsf{h}}(1-\tilde{a}_{j+1}\mathsf{h}^{\frac{2}{3}})^{\frac{3}{2}}}\right)$$

and both $\mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^{2j}_{\min})$ and $\mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^{2j}_{\max})$ tends to $\tilde{a}_{j+1}$ when h tends to 0.

Using (6.3) and (6.4) we prove similarly that both $\mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^{2j+1}_{\min})$ and $\mathsf{h}^{-\frac{2}{3}}(1+\mathbf{E}^{2j+1}_{\max})$ tends to a_{j+1} when h tends to 0.

The estimate (2.11) is obtained from (6.1) and (6.2) for $K_{2j} = \max(\tilde{M}_{2j}, M_{2j})$.

The estimate (2.12) is obtained from (6.4) and (6.3) for

$$K_{2j+1} = \max(\tilde{M}_{2j+1}, M_{2j+1}).$$

The last statement is a direct consequence of the counting of the number of spectral bands in the range of $\mathbf{V}$ done in Theorem 2.4 and the fact that the sequence $(c_p^{-\frac{3}{2}})_{p\geq 0}$ is strictly decreasing. □

Proposition 6.2. *Let $j \geq 0$. There exists a universal constant $\tilde{K}_{2j} > 0$ such that, for every $\mathsf{h} \in (0, a_{j+1}^{-\frac{3}{2}})$,*

$$\begin{aligned} &\left|\gamma_{2j} - \left((a_{j+1}-\tilde{a}_{j+1})\mathsf{h}^{\frac{2}{3}} - \alpha\sqrt{3}(u(-a_{j+1}))^2\mathsf{h}^{\frac{2}{3}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_{j+1})^{\frac{3}{2}}}\right)\right| \\ &\qquad \leq \tilde{K}_{2j}\mathsf{h}^{\frac{5}{3}}(1-a_{j+1}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-a_{j+1})^{\frac{3}{2}}}, \end{aligned} \tag{6.5}$$

and a universal constant $\tilde{K}_{2j+1} > 0$ such that, for every $\mathsf{h} \in (0, \tilde{a}_{j+2}^{-\frac{3}{2}})$,

$$\begin{aligned} &\left|\gamma_{2j+1} - \left((\tilde{a}_{j+2}-a_{j+1})\mathsf{h}^{\frac{2}{3}} - \alpha\sqrt{3}\frac{(u'(-\tilde{a}_{j+2}))^2}{\tilde{a}_{j+2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_{j+2})^{\frac{3}{2}}}\right)\right| \\ &\qquad \leq \tilde{K}_{2j+1}\mathsf{h}^{\frac{5}{3}}(1-\tilde{a}_{j+2}\mathsf{h}^{\frac{2}{3}})^{-\frac{3}{2}}\mathsf{e}^{-\frac{4}{3}(\mathsf{h}^{-\frac{2}{3}}-\tilde{a}_{j+2})^{\frac{3}{2}}}. \end{aligned} \tag{6.6}$$

Proof. The estimate (6.5) is obtained for $\tilde{K}_{2j} = \max(\tilde{M}_{2j}, M_{2j+1})$ and is a consequence of (6.3), (6.2) and the fact that $a_{j+1} > c_{2j} > \tilde{a}_{j+1}$.

The estimate (6.6) is obtained for $\tilde{K}_{2j+1} = \max(\tilde{M}_{2j+1}, M_{2j+2})$ and is a consequence of (6.1), (6.4) and the fact that $\tilde{a}_{j+2} > c_{2j+1} > a_{j+1}$. □

This proposition implies directly Theorem 2.6 of the introduction. The last statement of Theorem 2.6 is a direct consequence of the counting of the number of spectral gaps in the range of $\mathbf{V}$ done in Theorem 2.4 and the fact that the sequence $(\mathfrak{a}_p^{-\frac{3}{2}})_{p\geq 0}$ is strictly decreasing.

7. Conclusion

Let us summarize some of the results obtained in this article.

1. We obtained estimates of the widths of the spectral bands and the spectral gaps, which are universal in precise intervals of the semiclassical parameter, for a periodic and symmetric potential which is not differentiable at its extrema. Our estimates rely on a thorough analysis of the special functions involved here.
2. As expected, the spectral bands are exponentially thin in the semiclassical limit, but the width is somewhat larger than in the regular case.
3. Thanks to the rather explicit estimates, all our results are obtained for $\mathsf{h} \in [0, \mathsf{h}_0)$ where h_0 is explicit and depends only on the number of the spectral band.
4. Counting results of the spectral bands and gaps situated in the range of the potential **V** are obtained.
5. We prove an upper bound on the integrated spectral density in the range of the potential.

Appendix A. Uniform estimates for $\frac{u}{v}$, $\frac{u'}{v'}$, $\frac{v}{u}$, $\frac{v'}{u'}$ and their derivatives

The real number α provides lower and upper bounds for the derivatives of $\frac{u}{v}$, $\frac{u'}{v'}$, $\frac{v}{u}$ and $\frac{v'}{u'}$ on the negative half-line.

Lemma A.1. *For every* $x \in (-\infty, 0) \setminus \{-c_{2j+1}\}_{j\geq 0}$,

$$-\frac{1}{v^2(x)} \leq -\frac{1}{\alpha} \tag{A.1}$$

and for every $x \in (-\infty, 0] \setminus \{-c_{2j}\}_{j\geq 0}$,

$$\frac{x}{(v'(x))^2} \leq \max\left(-\frac{1}{\alpha}, x\right). \tag{A.2}$$

Proof. Using (2.1), the equality $Bi(0) = \sqrt{3}Ai(0)$, one has, for every $x \in (-\infty, 0) \setminus \{-c_{2j+1}\}_{j\geq 0}$,

$$-\frac{1}{v^2(x)} = -\frac{1}{\alpha} \cdot \frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{1}{(-\sqrt{3}Ai(x) + Bi(x))^2}. \tag{A.3}$$

Let $M = \sqrt{Ai^2 + Bi^2}$ be the Airy modulus. The function M is strictly increasing on $(-\infty, 0]$ (see [11]) and we have, for every $x \in (-\infty, 0) \setminus \{-c_{2j+1}\}_{j\geq 0}$,

$$\frac{1}{(-\sqrt{3}Ai(x) + Bi(x))^2} = \frac{(\sqrt{3}Ai(x) + Bi(x))^2}{(M(x))^4} \geq \frac{1}{(M(x))^2}.$$

With (A.3), it leads to

$$-\frac{1}{v^2(x)} \leq -\frac{1}{\alpha} \cdot \frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{1}{(M(x))^2} \leq -\frac{1}{\alpha} \cdot \frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{1}{(M(0))^2},$$

for every $x \in (-\infty, 0) \setminus \{-c_{2j+1}\}_{j\geq 0}$. Since $M(0)^2 = 4(Ai(0))^2$, we get, for every $x \in (-\infty, 0) \setminus \{-c_{2j+1}\}_{j\geq 0}$,

$$-\frac{1}{v^2(x)} \leq -\frac{1}{\alpha} \cdot \frac{\alpha}{4\pi^2(Ai(0))^4} \leq -\frac{1}{\alpha}$$

since $\frac{\alpha}{4\pi^2(Ai(0))^4} \simeq 2.19 > 1$, and (A.1) is proven.

Derivating (2.1) and using the equality $Bi(0) = \sqrt{3}Ai(0)$, one has, for every $x \in (-\infty, 0] \setminus \{-c_{2j}\}_{j\geq 0}$,

$$\frac{x}{(v'(x))^2} = \frac{1}{\alpha} \cdot \frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{x}{(-\sqrt{3}Ai'(x) + Bi'(x))^2}. \tag{A.4}$$

Let $N = \sqrt{(Ai')^2 + (Bi')^2}$. Then, $x \mapsto (N(x))^{-2}$ and $x \mapsto x(N(x))^{-2}$ are strictly increasing on $(-\infty, 0]$ and we have, for every $x \in (-\infty, 0] \setminus \{-c_{2j}\}_{j\geq 0}$,

$$\frac{x}{(-\sqrt{3}Ai'(x) + Bi'(x))^2} \leq \frac{4x}{(N(x))^2} \leq \frac{4x}{(N(0))^2}.$$

Since $N(0)^2 = 4(Ai'(0))^2$, (A.4) implies that for every $x \in (-\infty, -\frac{1}{12\alpha}] \setminus \{-c_{2j}\}_{j\geq 0}$,

$$\frac{x}{(v'(x))^2} \leq \frac{1}{\alpha} \cdot \frac{\alpha}{\pi^2(Ai(0)Ai'(0))^2} \cdot \left(-\frac{1}{12\alpha}\right) = -\frac{1}{\alpha}, \tag{A.5}$$

since $\pi^2(Ai(0)Ai'(0))^2 = 12$. Moreover, on $(-c_0, 0]$, $x \mapsto \frac{x}{(v'(x))^2}$ is strictly concave and its graph is under its tangent at 0. Since $-c_0 < \frac{1}{12\alpha}$, one has:

$$\forall x \in [-\tfrac{1}{12\alpha}, 0],\ \frac{x}{(v'(x))^2} \leq x. \tag{A.6}$$

Finally, (A.5) and (A.6) implies (A.2). □

Lemma A.2. *For every* $x \in (-\infty, 0] \setminus \{\tilde{c}_{2j}\}_{j\geq 0}$,

$$\frac{1}{u^2(x)} > \frac{\alpha}{2} \tag{A.7}$$

and for every $x \in (-\infty, 0) \setminus \{\tilde{c}_{2j+1}\}_{j\geq 0}$,

$$-\frac{x}{(u'(x))^2} > \alpha. \tag{A.8}$$

Proof. For every $x \in (-\infty, 0] \setminus \{\tilde{c}_{2j}\}_{j\geq 0}$,

$$\frac{1}{u^2(x)} = \frac{\alpha}{2} \cdot \frac{2\alpha}{\pi^2(Ai(0))^2} \cdot \frac{1}{(\sqrt{3}Ai(x) + Bi(x))^2}. \tag{A.9}$$

If M is the Airy modulus,

$$\forall x \leq 0,\ (\sqrt{3}Ai(x) + Bi(x))^2 \leq 4(M(x))^2 \leq 4(M(0))^2.$$

With (A.9), for every $x \in (-\infty, 0] \setminus \{\tilde{c}_{2j}\}_{j\geq 0}$,

$$\frac{1}{u^2(x)} \geq \frac{\alpha}{2} \cdot \frac{2\alpha}{\pi^2(Ai(0))^2} \cdot \frac{1}{4((Ai(0))^2+(Bi(0))^2)}.$$

Since $\frac{2\alpha}{\pi^2(Ai(0))^2} \cdot \frac{1}{4((Ai(0))^2+(Bi(0))^2)} \simeq 1.09$, one deduces (A.7).

One also has, for every $x \in (-\infty, 0) \setminus \{\tilde{c}_{2j+1}\}_{j\geq 0}$,

$$-\frac{x}{(u'(x))^2} = \alpha \cdot \frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{-x}{(\sqrt{3}Ai'(x)+Bi'(x))^2}. \tag{A.10}$$

The function $x \mapsto |x|^{-\frac{1}{4}}N(x)$ is strictly increasing on $(-\infty, 0)$ (see [11]) and:

$$\forall x < 0,\ \frac{-x}{(\sqrt{3}Ai'(x)+Bi'(x))^2} \geq \frac{-x \cdot |x|^{-\frac{1}{2}}}{4(N(x))^2|x|^{-\frac{1}{2}}}.$$

Since $x \mapsto \frac{|x|^{\frac{1}{2}}}{(N(x))^2|x|^{-\frac{1}{2}}}$ is strictly decreasing on $(-\infty, 0)$, one has:

$$\forall x < -\tilde{a}_1,\ \frac{-x}{(\sqrt{3}Ai'(x)+Bi'(x))^2} \geq \frac{\tilde{a}_1}{4(N(-\tilde{a}_1))^2}.$$

As $(N(-\tilde{a}_1))^2 = |Bi'(-\tilde{a}_1)|$, with (A.10) one has, for every $x \in (-\infty, -\tilde{a}_1] \setminus \{\tilde{c}_{2j}\}_{j\geq 0}$,

$$-\frac{x}{(u'(x))^2} \geq \alpha \cdot \frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{\tilde{a}_1}{4|Bi'(-\tilde{a}_1)|}.$$

With $\frac{\alpha}{\pi^2(Ai(0))^2} \cdot \frac{\tilde{a}_1}{4|Bi'(-\tilde{a}_1)|} \simeq 2.74$, we deduce that

$$\forall x < -\tilde{a}_1,\ -\frac{x}{(u'(x))^2} > \alpha.$$

Moreover, the function $x \mapsto -\frac{x}{(u'(x))^2}$ is convex on the interval $(-\tilde{c}_1, 0)$ and its minimum on this interval has an approximate value equal to $2.03 > \alpha$. Since $-\tilde{c}_1 < -\tilde{a}_1$, we deduce (A.8). □

We also get estimates on the positive half-line.

Lemma A.3. *Let $\tau > 0$. There exists $A_\tau > 0$ such that, for every $x \geq \tau$,*

$$\left|\frac{v'}{u'}(x) - \alpha - \alpha\sqrt{3}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}}\right| \leq A_\tau x^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}}. \tag{A.11}$$

Proof. Let $\tau > 0$ and $x \geq \tau$. One has:

$$\begin{aligned}\frac{v'}{u'}(x) &= \frac{Ai(0)Bi'(x) - Bi(0)Ai'(x)}{-Ai'(0)Bi'(x) + Bi'(0)Ai'(x)} = \alpha \frac{1-\sqrt{3}\frac{Ai'(x)}{Bi'(x)}}{1+\sqrt{3}\frac{Ai'(x)}{Bi'(x)}} \\ &= \alpha - 2\alpha\sqrt{3}\frac{Ai'(x)}{Bi'(x)} + 3\alpha\left(\frac{Ai'(x)}{Bi'(x)}\right)^2\left[1 + \frac{1-\sqrt{3}\frac{Ai'(x)}{Bi'(x)}}{1+\sqrt{3}\frac{Ai'(x)}{Bi'(x)}}\right]. \end{aligned} \tag{A.12}$$

For $x \geq \tau$,

$$0 \leq 1 + \frac{1-\sqrt{3}\frac{Ai'(x)}{Bi'(x)}}{1+\sqrt{3}\frac{Ai'(x)}{Bi'(x)}} \leq \frac{2}{1+\sqrt{3}\frac{Ai'(\tau)}{Bi'(\tau)}}. \tag{A.13}$$

Using [1, 10.4.61, 10.4.66] and since each error term in these expansions is bounded in absolute value by the first neglected term of the expansion ([11, Section 4.4.1]), one has for every $x \geq \tau$,

$$Ai'(x) = -\frac{1}{2}\pi^{-\frac{1}{2}}x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}} + R_1 \text{ with } |R_1| \leq \frac{7}{96}\pi^{-\frac{1}{2}}x^{\frac{1}{4}}x^{-\frac{3}{2}}e^{-\frac{2}{3}x^{\frac{3}{2}}} \tag{A.14}$$

and

$$Bi'(x) = \pi^{-\frac{1}{2}}x^{\frac{1}{4}}e^{\frac{2}{3}x^{\frac{3}{2}}} + R_2 \text{ with } |R_2| \leq \frac{7}{48}\pi^{-\frac{1}{2}}x^{\frac{1}{4}}x^{-\frac{3}{2}}e^{\frac{2}{3}x^{\frac{3}{2}}}. \tag{A.15}$$

Thus, for every $x \geq \tau$,

$$\begin{aligned}&\frac{Ai'(x)}{Bi'(x)} = -\frac{1}{2}e^{-\frac{4}{3}x^{\frac{3}{2}}} + R_3 \text{ with} \\ &\qquad |R_3| \leq x^{-\frac{3}{2}}e^{-\frac{4}{3}x^{\frac{3}{2}}}\left(\frac{7}{48} + \frac{3}{2}\left(\frac{7}{48}\right)^2\tau^{-\frac{3}{2}} + \frac{1}{2}\left(\frac{7}{48}\right)^3\tau^{-3}\right).\end{aligned} \tag{A.16}$$

We also remark that for every $x \geq \tau$,

$$\left(\frac{Ai'(x)}{Bi'(x)}\right)^2 = \frac{1}{4}e^{-\frac{8}{3}x^{\frac{3}{2}}} - e^{-\frac{4}{3}x^{\frac{3}{2}}}R_3 + R_3^2 \text{ and } e^{-\frac{8}{3}x^{\frac{3}{2}}} \leq x^{-\frac{3}{2}}e^{-\frac{4}{3}x^{\frac{3}{2}}}. \tag{A.17}$$

Let $A_{\tau,0} = \frac{7}{48} + \frac{3}{2}\left(\frac{7}{48}\right)^2\tau^{-\frac{3}{2}} + \frac{1}{2}\left(\frac{7}{48}\right)^3\tau^{-3}$. Combining (A.12), (A.13), (A.16) and (A.17), we get (A.11) for

$$A_\tau = A_{\tau,0} + \frac{3}{2}\alpha\frac{1}{1+\sqrt{3}\frac{Ai'(\tau)}{Bi'(\tau)}}\left[1 + A_{\tau,0} + \tau^{-\frac{3}{2}}A_{\tau,0}^2\right]. \quad \square \tag{A.18}$$

Remark A.1. From (A.18), we get that A_τ tends to $+\infty$ when τ tends to 0.

Lemma A.4. *For every* $x \geq 0$*,*

$$\left| \frac{v}{u}(x) - \alpha + \alpha\sqrt{3}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}} \right| \leq 2.46 \cdot x^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}}. \tag{A.19}$$

Proof. A similar proof as the one of Lemma A.3 leads to (A.19) for $x \geq 1$ since $\tilde{A}_0 + \frac{3}{2}\alpha\left(1 + \tilde{A}_0 + \tau^{-\frac{3}{2}}\tilde{A}_0^2\right) \simeq 2.458$, where $\tilde{A}_0 = \frac{5}{48} + \frac{3}{2}\left(\frac{5}{48}\right)^2 + \frac{1}{2}\left(\frac{5}{48}\right)^3$ and since, for every $x \geq 0$, $\frac{2}{1+\sqrt{3}\frac{Ai(x)}{Bi(x)}} \in [0, 2]$. For $x \in [0, 1]$, $x \mapsto \frac{v}{u}(x)x^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}x^{\frac{3}{2}}} - \alpha x^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}x^{\frac{3}{2}}} + \alpha\sqrt{3}x^{\frac{3}{2}}$ is a continuous increasing function whose approximate value at 1 is 0.684. Since $0.684 < 2.46$, we deduce (A.19). □

Lemma A.5. *For every* $x \geq 0$*,*

$$\left| \frac{u'}{v'}(x) - \frac{1}{\alpha} + \frac{\sqrt{3}}{\alpha}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}} \right| \leq 2.83 \cdot x^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}}. \tag{A.20}$$

Proof. Let $x \geq 0$. One has:

$$\frac{u'}{v'}(x) = \frac{1}{\alpha} - \frac{2\sqrt{3}}{\alpha}\frac{Ai'(x)}{Bi'(x)} + \frac{3}{\alpha}\left(\frac{Ai'(x)}{Bi'(x)}\right)^2 \left[1 + \frac{1 + \sqrt{3}\frac{Ai'(x)}{Bi'(x)}}{1 - \sqrt{3}\frac{Ai'(x)}{Bi'(x)}}\right]. \tag{A.21}$$

For $x \geq 0$,

$$0 \leq 1 + \frac{1 + \sqrt{3}\frac{Ai'(x)}{Bi'(x)}}{1 - \sqrt{3}\frac{Ai'(x)}{Bi'(x)}} \leq 2. \tag{A.22}$$

Combining (A.21), (A.22), (A.16) and (A.17), we get (A.20) for $x \geq 1$, since $A_{1,0} + \frac{3}{\alpha}\left(1 + A_{1,0} + A_{1,0}^2\right) \simeq 2.828 < 2.83$, where $A_{1,0}$ is given after (A.17) for $\tau = 1$. For $x \in [0, 1]$, $x \mapsto \frac{u'}{v'}(x)x^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}x^{\frac{3}{2}}} - \frac{1}{\alpha}x^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}x^{\frac{3}{2}}} + \frac{\sqrt{3}}{\alpha}x^{\frac{3}{2}}$ is a continuous positive function which admits a unique maximum smaller than 1. Since $1 < 2.83$, we deduce (A.20). □

Lemma A.6. *For every* $x \geq 0$*,*

$$\left| \frac{u}{v}(x) - \frac{1}{\alpha} + \frac{\sqrt{3}}{\alpha}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}} \right| \leq 2.61 \cdot x^{-\frac{3}{2}}\mathrm{e}^{-\frac{4}{3}x^{\frac{3}{2}}}. \tag{A.23}$$

Proof. A similar proof as the one of Lemma A.5 leads to (A.23) for $x \geq 1$, since $\tilde{A} + \frac{3}{\alpha}\left(1 + \tilde{A} + \tilde{A}\right) \simeq 2.604 < 2.61$ with $\tilde{A} = \frac{5}{48} + \frac{3}{2}\left(\frac{5}{48}\right)^2 + \frac{1}{2}\left(\frac{5}{48}\right)^3$. Indeed, for every $x \geq 0$, $\frac{2}{1+\sqrt{3}\frac{Ai(x)}{Bi(x)}} \in [0, 2]$. Moreover, for $x \in [0, 1]$, $x \mapsto \frac{u}{v}(x)x^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}x^{\frac{3}{2}}} - \frac{1}{\alpha}x^{\frac{3}{2}}\mathrm{e}^{\frac{4}{3}x^{\frac{3}{2}}} + \frac{\sqrt{3}}{\alpha}x^{\frac{3}{2}}$ is a continuous function, increasing from 0 to its value at $x = 1$ which is approximately equal to 2.594. Since $2.594 < 2.61$, we deduce (A.23). □

Appendix B. Sturm–Picone's lemmas

In this Appendix we prove a Sturm's formula and a version of the Sturm–Picone's formula adapted to the setting of the proof of Lemma C.2.

Lemma B.1. *Let $a < b$ be two real numbers and let $g_1, g_2 \in C^0([a,b])$. Let z be a solution of $-z'' + g_1 z = 0$ and let y be a solution of $-y'' + g_2 y = 0$. Then:*

$$\big(yz' - zy'\big)' = (g_1 - g_2)yz. \tag{B.1}$$

Proof. We have:

$$y(-z'' + g_1 z) - z(-y'' + g_1 y) = -yz'' + zy'' = -(yz' - zy')'$$

We also have:

$$(-y'' + g_1 y) - (-y'' + g_2 y) = (g_1 - g_2)y.$$

Then,

$$y(-z'' + g_1 z) - z\big(-z'' + g_2 z + (g_1 - g_2)y\big) = -(yz' - zy')'.$$

Since $-z'' + g_1 z = 0$ and $-y'' + g_2 y = 0$, we finally have:

$$-(yz' - zy')' = -(g_1 - g_2)yz,$$

which proves (B.1). □

Lemma B.2. *Let $a < b$ be two real numbers and let $q_1, q_2, g \in C^0([a,b]) \cap C^1((a,b))$, $q_1 > q_2 > 0$. Let z be a solution of $-(q_1 z')' + gz = 0$ and let y be a solution of $-(q_2 y')' + gy = 0$ with $y > 0$ on $(a,b]$. Then:*

$$\left(\frac{z}{y}(q_1 yz' - q_2 y'z)\right)' = (q_1 - q_2)(z')^2 + q_2\left(z' - \frac{y'z}{y}\right)^2. \tag{B.2}$$

Moreover, if there exists $\eta > 0$ such that $q_1 - q_2 > \eta$, then there exists $A > 0$ such that

$$\left[\frac{z}{y}(q_1 yz' - q_2 y'z)\right]_a^b \geq A\int_a^b z^2(x)\mathrm{d}x. \tag{B.3}$$

Proof. We have, on the interval (a,b),

$$\begin{aligned}\left(\frac{z}{y}(q_1 yz' - q_2 y'z)\right)' &= \left(q_1 zz' - q_2\frac{y'}{y}z^2\right)' = (q_1 z')'z + q_1(z')^2 - (q_2 y')'\frac{z^2}{y} - q_2 y'\left(\frac{z^2}{y}\right)' \\ &= gz^2 + q_1(z')^2 - gy\frac{z^2}{y} - q_2 y'\left(\frac{z^2}{y}\right)' = q_1(z')^2 - q_2 y'\left(\frac{z^2}{y}\right)'\end{aligned}$$

$$
\begin{aligned}
&= (q_1 - q_2)(z')^2 + q_2\left((z')^2 - y'\left(\frac{z^2}{y}\right)'\right)\\
&= (q_1 - q_2)(z')^2 + q_2\left((z')^2 - 2y'\frac{zz'}{y} + \frac{(y')^2}{y^2}z^2\right)\\
&= (q_1 - q_2)(z')^2 + q_2\left(z' - \frac{y'z}{y}\right)^2.
\end{aligned}
$$

This proves (B.2). Then, integrating (B.2) between a and b and using Poincaré inequality in the last inequality, there exists $A > 0$ (depending on η, a and b) such that

$$
\begin{aligned}
\left[\frac{z}{y}(q_1 y z' - q_2 y' z)\right]_a^b &= \int_a^b (q_1(x) - q_2(x))(z'(x))^2 + q_2(x)\left(z'(x) - \frac{y'(x)z(x)}{y(x)}\right)^2 \mathrm{d}x\\
&\geq \int_a^b (q_1(x) - q_2(x))(z'(x))^2 \mathrm{d}x\\
&\geq \eta \int_a^b (z'(x))^2 \mathrm{d}x \geq A \int_a^b (z(x))^2 \mathrm{d}x.
\end{aligned}
$$

This proves (B.3). □

Appendix C. The monotonicity arguments

We have defined the functions f_x, g_x and the functions z_k for $k \geq 0$ in Section 3.4.

Lemma C.1. *Let $k \geq 0$. Then, for every $x \geq 0$,*

$$z_k(x) \geq 0 \quad \text{and} \quad z_k(x) = x - \psi_k(x).$$

Therefore, z_k is continuous on $[0, +\infty)$. Moreover, for every $j \geq 0$ and every $x \geq 0$,

$$
\begin{aligned}
0 < x + \tilde{a}_1 < z_0(x) \leq x + c_0 < \cdots < x + \tilde{a}_{j+1} < z_{2j}(x) \leq x + c_{2j}\\
< x + a_{j+1} < z_{2j+1}(x) \leq x + c_{2j+1} \ldots
\end{aligned}
\tag{C.1}
$$

Proof. In this proof it will be easier to use the expressions in terms of Airy functions for f_x and g_x since we will use classical properties of the Ai and Bi functions and in particular the fact that Ai' is strictly negative on the positive real half-line, which is not the case for u'.

For $x \geq 0$, $Ai(x) > 0$, $Bi(x) > 0$, Bi' is strictly positive on $[0, +\infty)$ and Ai' is strictly negative on $[0, +\infty)$. If $z \leq 0$, $x - z \geq 0$ and $f_x(z) > 0$. Thus, $z_0(x) > 0$. Then, 0 is a zero of g_x and since $\frac{Bi}{Ai}$ is strictly increasing on $[0, +\infty)$, and for $z < 0$, $x - z > x$ and $g_x(z) > 0$. So, 0 is the first zero of g_x. In particular, for every $k \geq 0$, $z_k(x) \geq 0$.

Let $j \geq 0$. We remark that, by definition of ψ_{2j}, we have $f_x(x - \psi_{2j}(x)) = 0$, and by definition of ψ_{2j+1}, we have $g_x(x - \psi_{2j+1}(x)) = 0$. Moreover, for $x - z \notin \{-\tilde{c}_{2j+1}\}_{j\geq 0}$, by unicity of $\psi_{2j}(x)$ in $[-c_{2j}, -\tilde{a}_{j+1})$, $x - \psi_{2j}(x)$ is the unique zero of f_x in $(x + \tilde{a}_{j+1}, x + c_{2j}]$. Since we have $f_x(x + \tilde{a}_{j+1}) = Bi'(-\tilde{a}_{j+1})Ai(x) \neq 0$, the set of the zeroes of f_x is exactly $\{x - \psi_{2j}(x),\ j \geq 0\}$. Thus, for every $j \geq 0$, $z_{2j}(x) = x - \psi_{2j}(x)$.

For $x - z \notin \{-\tilde{c}_{2j}\}_{j\geq 0}$, by unicity of $\psi_{2j+1}(x)$ in $[-c_{2j+1}, -a_{j+1})$, $x - \psi_{2j+1}(x)$ is the unique zero of g_x in $(x + a_{j+1}, x + c_{2j+1}]$. Since we have $g_x(x + a_{j+1}) = Bi(-a_{j+1})Ai(x) \neq 0$, the set of the zeroes of g_x is exactly $\{0\} \cup \{x - \psi_{2j+1}(x),\ j \geq 0\}$. Thus, for every $j \geq 0$, $z_{2j+1}(x) = x - \psi_{2j+1}(x)$.

By Lemma 3.1, we deduce that z_k is continuous on $[0, +\infty)$. Since for every $j \geq 0$, $\psi_{2j}(x) \in [-c_{2j}, -\tilde{a}_{j+1})$ and $\psi_{2j+1}(x) \in [-c_{2j+1}, -a_{j+1})$, we deduce (C.1). □

We now prove further properties of the functions f_x and g_x and in particular their signs and their variations.

Proposition C.1. *For every $x \geq 0$, the functions f_x and g_x from $\mathbb{R}$ to $\mathbb{R}$ have the following properties:*

1. *$\forall z \in \mathbb{R}$, $g_x'(z) = -f_x(z)$ and $f_x'(z) = -(x - z)g_x(z)$.*
2. *g_x satisfies the Airy equation: $g_x'' = (x - z)g_x$ and f_x satisfies the ordinary differential equation on $\mathbb{R} \setminus \{x\}$:*

$$\left(\frac{f_x'}{x - z}\right)' = f_x. \tag{C.2}$$

Proof. This results from direct computations. □

Proposition C.2. *For every $x \geq 0$, the functions f_x and g_x from $\mathbb{R}$ to $\mathbb{R}$ have the following properties:*

1. *The function f_x' vanishes exactly on 0, x, and $z_{2j+1}(x)$ for every $j \geq 0$. It is strictly negative on $(-\infty, 0)$, strictly positive on $(0, x)$, strictly negative on $(x, z_1(x))$ and, for every $j \geq 1$, $(-1)^{j+1} f_x'$ is strictly positive on $(z_{2j-1}(x), z_{2j+1}(x))$.*
2. *The function f_x is strictly positive on $(-\infty, z_0(x))$ and, for every $j \geq 1$, $(-1)^{j+1} f_x$ is strictly positive on $(z_{2j-2}(x), z_{2j}(x))$.*
3. *The function g_x' vanishes exactly on $z_{2j}(x)$ for every $j \geq 0$. It is strictly negative on $(-\infty, z_0(x))$ and, for every $j \geq 1$, $(-1)^{j+1} g_x'$ is strictly positive on $(z_{2j-2}(x), z_{2j}(x))$.*
4. *The function g_x is strictly positive on $(-\infty, 0)$, strictly negative on $(0, z_1(x))$ and, for every $j \geq 1$, $(-1)^{j+1} g_x$ is strictly positive on the interval (z_{2j-1}, z_{2j+1}).*

Proof. We will again use the expressions in terms of Airy functions for f_x and g_x.

(1) From the expression of f_x', it is clear that $f_x'(0) = f_x'(x) = 0$. We have already proven in Lemma 3.2 that for $z \in (-\infty, 0)$, $f_x'(z) < 0$. Then, for $z \in (0, x)$, $x - z > 0$, $x - z < x$, $\frac{Bi}{Ai}(x - z) < \frac{Bi}{Ai}(x - z)$ and $f_x'(z) > 0$. From $f_x'(z) = -(x - z)g_x(z)$ and Lemma 3.2, we know that the remaining zeroes of f_x' are exactly the $z_{2j+1}(x)$ for $j \geq 0$. We also have $f_x'(x + a_1) = a_1 Bi(-a_1)Ai(x)$ with $a_1 > 0$, $Bi(-a_1) < 0$ and $Ai(x) > 0$, thus $f_x'(x + a_1) < 0$. Since f_x' is of constant sign in $(x, z_1(x))$, one deduce that f_x' is strictly negative on $(x, z_1(x))$. To finish the proof of point (1), it is sufficient to remark that f_x' is of constant sign on every interval $(z_{2j-1}(x), z_{2j+1}(x))$ for $j \geq 1$. But, $x + a_{2j} \in$

$(z_{4j-3}(x), z_{4j-1}(x))$ and $f'_x(x + a_{2j}) = a_{2j} Bi(-a_{2j}) Ai(x) > 0$, since $Bi(-a_{2j}) > 0$. Thus, f'_x is strictly positive on $(z_{4j-3}(x), z_{4j-1}(x))$. Similarly, $x + a_{2j+1} \in (z_{4j-1}(x), z_{4j+1}(x))$ and $f'_x(x + a_{2j+1}) = a_{2j+1} Bi(-a_{2j+1}) Ai(x) < 0$, since $Bi(-a_{2j+1}) < 0$. Thus, f'_x is strictly negative on $(z_{4j-1}(x), z_{4j+1}(x))$.

(2) We have already proven in Lemma 3.2 that for $z \in (-\infty, 0)$, $f_x(z) > 0$. We also have $f_x(0) = \frac{1}{\pi} > 0$ since it is the value of the Wronskian of Ai and Bi and thus, for every $z \in (-\infty, x)$, $f_x(z) \geq \frac{1}{\pi}$. Since $z_0(x)$ is the first zero of f_x, this function is strictly positive on $(-\infty, z_0(x))$. We remark that f_x is of constant sign on every interval $(z_{2j-2}(x), z_{2j}(x))$ for $j \geq 1$. But, $x + \tilde{a}_{2j+1} \in (z_{4j-2}(x), z_{4j}(x))$ and $f_x(x + \tilde{a}_{2j+1}) = Bi'(-\tilde{a}_{2j+1}) Ai(x) > 0$, since $Bi'(-\tilde{a}_{2j+1}) > 0$. Thus, f_x is strictly positive on $(z_{4j-2}(x), z_{4j}(x))$. Similarly, $x + \tilde{a}_{2j+2} \in (z_{4j}(x), z_{4j+2}(x))$ and $f_x(x + \tilde{a}_{2j+2}) = Bi'(-\tilde{a}_{2j+2}) Ai(x) < 0$, since $Bi'(-\tilde{a}_{2j+2}) < 0$. Thus, f_x is strictly negative on $(z_{4j}(x), z_{4j+2}(x))$.

(3) It is deduced directly from point (1) of Proposition C.1 and point (2).

(4) It comes from point (1) of Proposition C.1, point (1) and the fact that for $z \geq z_1(x)$, $z > x$ and $x - z < 0$. □

We have now all the ingredients needed to prove that z_k is a strictly increasing function.

Lemma C.2. *For every $k \geq 0$, the function z_k is strictly increasing from* $[0, +\infty)$ *to* $[c_k, +\infty)$.

Proof. We will separate the proof in two cases, depending on the parity of k.

Case 1: $k = 2j$ for $j \geq 0$. Let $0 < x_1 < x_2$. We want to prove that $z_{2j}(x_1) \leq z_{2j}(x_2)$. Assume that $z_{2j}(x_2) < z_{2j}(x_1)$. Let $\delta > 0$ be such that $z_{2j}(x_2) + \delta < z_{2j}(x_1)$. We use (C.1) to get

$$\begin{aligned} z_{2j-1}(x_1) < x_1 + \tilde{a}_{j+1} < x_2 + \tilde{a}_{j+1} < z_{2j}(x_2) < z_{2j}(x_2) + \delta < \\ z_{2j}(x_1) < x_1 + a_{j+1} < x_2 + a_{j+1} < z_{2j+1}(x_2). \end{aligned}$$

In particular, $x_1 - (z_{2j}(x_2) + \delta) < 0$, $x_2 - (z_{2j}(x_2) + \delta) < 0$ and

$$(z_{2j}(x_2) + \delta) \in (z_{2j-1}(x_1), z_{2j}(x_1)) \cap (z_{2j}(x_2), z_{2j+1}(x_2)).$$

Thus, using Proposition C.2,

$$(-1)^j f_{x_1}(z_{2j}(x_2) + \delta) > 0, \quad (-1)^j f'_{x_1}(z_{2j}(x_2) + \delta) < 0, \tag{C.3}$$

$$\text{and} \quad (-1)^j f_{x_2}(z_{2j}(x_2) + \delta) < 0, \quad (-1)^j f'_{x_2}(z_{2j}(x_2) + \delta) < 0. \tag{C.4}$$

There exists $\eta > 0$ such that, for every $z \in [z_{2j}(x_2) + \delta, z_{2j}(x_1)]$, $\frac{1}{x_1 - z} - \frac{1}{x_2 - z} \geq \eta$. Moreover, since $(z_{2j}(x_2) + \delta, z_{2j}(x_1)) \subset (z_{2j-1}(x_1), z_{2j}(x_1))$, for every $z \in (z_{2j}(x_2) + \delta, z_{2j}(x_1))$, $(-1)^{j+1} f_{x_1}(z) > 0$. Similarly, since we have the inclusion $(z_{2j}(x_2) + \delta, z_{2j}(x_1)) \subset (z_{2j}(x_2), z_{2j+1}(x_2))$, for every $z \in (z_{2j}(x_2) + \delta, z_{2j}(x_1))$, $(-1)^{j+1} f_{x_2}(z) < 0$. Then, applying Lemma B.2,

$$\int_{z_{2j}(x_2)+\delta}^{z_{2j}(x_1)} \left(\frac{f_{x_1}(z)}{f_{x_2}(z)} \left(\frac{f'_{x_1}(z) f_{x_2}(z)}{x_1 - z} - \frac{f_{x_1}(z) f'_{x_2}(z)}{x_2 - z} \right) \right)' \mathrm{d}z > 0. \tag{C.5}$$

Since $f_{x_1}(z_{2j}(x_1)) = 0$, the integral in the left side of equality (C.5) is equal to

$$-\frac{f_{x_1}(z_{2j}(x_2)+\delta) f'_{x_1}(z_{2j}(x_2)+\delta)}{x_1 - z_{2j}(x_2) - \delta} + \frac{f^2_{x_1}(z_{2j}(x_2)+\delta)}{x_2 - z_{2j}(x_2) - \delta} \cdot \frac{f'_{x_2}(z_{2j}(x_2)+\delta)}{f_{x_2}(z_{2j}(x_2)+\delta)} < 0 \tag{C.6}$$

by the use of (C.3) and (C.4). But (C.6) contradicts (C.5) and thus we must have $z_{2j}(x_1) \leq z_{2j}(x_2)$. The function z_{2j} is an increasing function from $[0, +\infty)$ to $[c_k, +\infty)$.

It remains to prove that z_{2j} is strictly increasing. If z_{2j} is not strictly increasing, since it is increasing and continuous, there exists an interval in $[0, +\infty)$ on which z_{2j} is constant. But, z_{2j} is also analytic on $[0, +\infty)$ since one can prove that actually the functions ψ_{2j} are analytic. Thus, if it is constant on an interval, it should be constant everywhere which is not the case, so z_{2j} is actually strictly increasing.

Case 2: $k = 2j + 1$ for $j \geq 0$. Let $x_1 < x_2$. We will show by induction on $j \geq 0$ that $z_{2j+1}(x_1) < z_{2j+1}(x_2)$.

For $j = 0$, we directly apply the classical interlacing zeroes theorem of Sturm with potentials $q(z) = -(x_2 - z) < p(z) = -(x_1 - z)$, since g_{x_1} satisfies $-g''_{x_1} + p g_{x_1} = 0$ and g_{x_2} satisfies $-g''_{x_2} + q g_{x_2} = 0$. Applying this theorem between 0 which is a common zero to g_{x_1} and g_{x_2} and $z_1(x_2)$ which is the first strictly positive zero of g_{x_2} one gets that g_{x_1} admits a zero in the interval $(0, z_1(x_2))$. Since $z_1(x_1)$ is the smallest strictly positive zero of g_{x_1}, we necessarily have $z_1(x_1) \in (0, z_1(x_2))$ and $z_1(x_1) < z_1(x_2)$. Thus, z_1 is strictly increasing.

Let $j \geq 1$. We assume by induction that $z_{2j-1}(x_1) < z_{2j-1}(x_2)$ and we want to prove that $z_{2j+1}(x_1) < z_{2j+1}(x_2)$. We assume the contrary: $z_{2j+1}(x_2) \leq z_{2j+1}(x_1)$. Then we have

$$z_{2j-1}(x_1) < z_{2j-1}(x_2) < z_{2j}(x_2) < z_{2j+1}(x_2) \leq z_{2j+1}(x_1). \tag{C.7}$$

We apply Lemma B.1 to g_{x_1} and g_{x_2} between $z_{2j-1}(x_2)$ and $z_{2j+1}(x_2)$ to get

$$\int_{z_{2j-1}(x_2)}^{z_{2j+1}(x_2)} \left(g_{x_2}(z) g'_{x_1}(z) - g_{x_1} g'_{x_2}(z)\right)' \mathrm{d}z = \int_{z_{2j-1}(x_2)}^{z_{2j+1}(x_2)} (x_1 - x_2) g_{x_1}(z) g_{x_2}(z) \mathrm{d}z. \tag{C.8}$$

But, using (C.7), we have $(z_{2j-1}(x_2), z_{2j+1}(x_2)) \subset (z_{2j-1}(x_1), z_{2j+1}(x_1))$. Using Proposition C.2,

$$\forall z \in (z_{2j-1}(x_2), z_{2j+1}(x_2)),\ (-1)^j g_{x_1}(z) < 0 \text{ and } (-1)^j g_{x_2}(z) < 0.$$

Since $x_1 - x_2 < 0$,

$$\int_{z_{2j-1}(x_2)}^{z_{2j+1}(x_2)} (x_1 - x_2) g_{x_1}(z) g_{x_2}(z) \mathrm{d}z < 0. \tag{C.9}$$

We have,

$$\int_{z_{2j-1}(x_2)}^{z_{2j+1}(x_2)} \big(g_{x_2}(z)g'_{x_1}(z) - g_{x_1}g'_{x_2}(z)\big)' \,\mathrm{d}z =$$
$$- g_{x_1}(z_{2j+1}(x_2))g'_{x_2}(z_{2j+1}(x_2)) + g_{x_1}(z_{2j-1}(x_2))g'_{x_2}(z_{2j-1}(x_2)).$$

But, using again Proposition C.2,

$$\forall z \in [z_{2j-1}(x_2), z_j(x_2)),\ (-1)^j g'_{x_2}(z) < 0$$

and

$$\forall z \in (z_{2j}(x_2), z_{2j+1}(x_2)],\ (-1)^j g'_{x_2}(z) > 0.$$

In particular,

$$(-1)^j g_{x_1}(z_{2j+1}(x_2)) < 0, \quad (-1)^j g'_{x_2}(z_{2j+1}(x_2)) > 0,$$

and

$$(-1)^j g_{x_1}(z_{2j-1}(x_2)) < 0, \quad (-1)^j g'_{x_2}(z_{2j-1}(x_2)) < 0.$$

Thus,

$$\int_{z_{2j-1}(x_2)}^{z_{2j+1}(x_2)} \big(g_{x_2}(z)g'_{x_1}(z) - g_{x_1}g'_{x_2}(z)\big)' \,\mathrm{d}z > 0$$

which contradicts (C.9). So we have $z_{2j+1}(x_1) < z_{2j+1}(x_2)$ and z_{2j+1} is strictly increasing.

We have thus proven by induction that for every $j \geq 0$, z_{2j+1} is strictly increasing from $[0, +\infty)$ to $[c_{2j+1}, +\infty)$. This finishes the proof of Lemma C.2. □

References

[1] M.A. Abramovitz, I.A. Stegun, Handbook for Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th edition, Applied Mathematics Series, vol. 55, 1990.

[2] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Company, Inc., New York–Toronto–London, 1955.

[3] M. Dimassi, J. Sjöstrand, Spectral Asymptotics in the Semi-Classical Limit, London Mathematical Society Lecture Note Series, vol. 268, Cambridge University Press, Cambridge, 1999.

[4] S. Flügge, Practical Quantum Mechanics, Classics in Mathematics, Springer-Verlag, Berlin, 1999.

[5] E.M. Harrell, The band-structure of a one-dimensional, periodic system in a scaling limit, Ann. Physics 119 (2) (1979) 351–369.

[6] B. Helffer, O. Lafitte, The semiclassical regime for ablation front models, Arch. Ration. Mech. Anal. 183 (3) (2007) 371–409.

[7] B. Helffer, A. Martinez, D. Robert, Ergodicité et limite semi-classique, Comm. Math. Phys. 109 (2) (1987) 313–326.

[8] H. Hochstadt, Asymptotic estimates for the Sturm–Liouville spectrum, Comm. Pure Appl. Math. 14 (1961) 749–764.

[9] A. Martinez, An Introduction to Semiclassical and Microlocal Analysis, Universitext, Springer-Verlag, New York, 2002.

[10] C. März, Spectral asymptotics near the potential maximum for Hill's equation, Asymptot. Anal. 5 (1992) 221–267.

[11] F.W.J. Olver, Asymptotics and Special Functions, Computer Science and Scientific Computing, Academic Press, San Diego, 1974.

[12] M. Reed, B. Simon, Methods of Modern Mathematical Physics IV: Analysis of Operators, Academic Press, New York, 1978.

[13] D. Robert, Analyse semi-classique de l'effet tunnel, in: Séminaire Bourbaki, vol. 1985/1986, Astérisque 145–146 (5) (1987) 257–281.

[14] E. Trubowitz, The inverse problem for periodic potentials, Comm. Pure Appl. Math. 30 (1977) 321–342.