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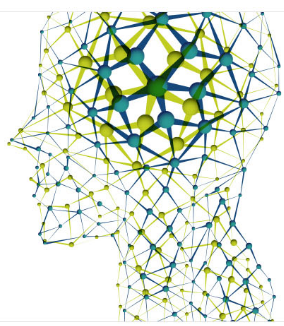

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Absence of absolutely continuous spectrum for random scattering zippers

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A scattering zipper is a system obtained by concatenation of scattering events with equal even number of incoming and outgoing channels. The associated scattering zipper operator is the unitary analog of Jacobi matrices with matrix entries. For infinite identical events and independent and identically distributed random phases, Lyapunov exponents positivity is proved and yields absence of absolutely continuous spectrum by Kotani's theory. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4906809]

I. RANDOM SCATTERING ZIPPERS

A. Deterministic scattering zippers

Let L be an integer larger than 1. An infinite scattering zipper describes consecutive scattering events with a fixed number $2L$ of incoming and outgoing channels each. It is described by a sequence $(S_n)_{n\in\mathbb{Z}}$ of unitary scattering matrices in the unitary group $\mathrm{U}(2L)$. More precisely, the matrices S_n are chosen to belong to the following subset:

$$\mathrm{U}(2L)_{\mathrm{inv}} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{U}(2L) \;\middle|\; \alpha,\gamma,\delta \in \mathcal{M}_L(\mathbb{C}) \text{ and } \beta \in \mathrm{GL}_L(\mathbb{C}) \right\}. \tag{1}$$

The scattering zipper operator $\mathbb{U}$ associated to the sequence $(S_n)_{n\in\mathbb{Z}}$ is the operator acting on $\ell^2(\mathbb{Z},\mathbb{C}^L)$ and defined as

$$\mathbb{U} = \mathbb{V}\,\mathbb{W}\,, \tag{2}$$

for the two unitary operators $\mathbb{V}$ and $\mathbb{W}$ acting on $\ell^2(\mathbb{Z},\mathbb{C}^L)$ and given by

$$\mathbb{V} = \begin{pmatrix} \ddots & & & \\ & S_0 & & \\ & & S_2 & \\ & & & \ddots \end{pmatrix}, \qquad \mathbb{W} = \begin{pmatrix} \ddots & & & \\ & S_{-1} & & \\ & & S_1 & \\ & & & \ddots \end{pmatrix}, \tag{3}$$

where the $2L\times 2L$ blocks in $\mathbb{V}$ are shifted to the left by L with respect to those of $\mathbb{W}$ along the diagonal.

The action of $\mathbb{U}$ on its wave functions is as follows. Let $z\in\mathbb{C}$ and let $\phi=(\phi_n)_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z},\mathbb{C}^L)$ be a solution of $\mathbb{U}\phi = z\phi$. Let $\psi=\mathbb{W}\phi$. Then, $\mathbb{V}\psi = z\phi$ and $\phi = z^{-1}\mathbb{V}\psi$. Rewriting these relations in terms of scattering matrices, one has

$$\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = S_1\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}, \quad \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = z^{-1}S_2\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \quad \begin{pmatrix}\psi_2\\ \psi_3\end{pmatrix} = S_3\begin{pmatrix}\phi_2\\ \phi_3\end{pmatrix}, \quad \begin{pmatrix}\phi_3\\ \phi_4\end{pmatrix} = z^{-1}S_4\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} \quad \ldots. \tag{4}$$

The shift of the blocks in $\mathbb{V}$ allows to consider the model of scattering zipper as a generalization of both the Blatter-Browne model, introduced to understand the Zener tunneling effect[3] and the

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Chalker-Coddington model on a strip.[2] Indeed, the operator $\mathbb{U}$ has the same unitary band matrix structure as the Blatter-Browne model and the Chalker-Coddington model on a strip, but with an arbitrary large width for the band. Note that since the operator $\mathbb{U}$ is a special case of unitary band matrix, general results on unitary band matrices apply.[7] Some of these results will be used in Sec. IV. However, the scattering zipper model is different from the Chalker-Coddington model on the whole lattice $\mathbb{Z}^2$ since it is one-dimensional and not two-dimensional.

The scattering zipper model can also be considered as a matrix-valued generalization of the CMV matrices. The CMV matrices are the unitary analog of Jacobi matrices and they were introduced in the first place to study orthonormal polynomials on the circle (see Refs. 26 and 27 for a large review on that topic). On the other side, Jacobi matrices were generalized to Jacobi matrices with matrix coefficients.[24,25] Scattering zippers are introduced in order to have a unitary analog of Jacobi matrices with matrix coefficients which shares with CMV matrices spectral properties and a way of representing their coefficients. Thus, the factorization in two diagonal by block operators for CMV matrices[8] is used in (2) as a definition for scattering zippers. In the following diagram, horizontal arrows represent the change from scalar-valued to matrix-valued operators and vertical arrows represent the "unitarization" of the models.

$$\begin{array}{ccc} \text{Jacobi matrices} & \longrightarrow & \text{Jacobi matrices with matrix coefficients} \\ \downarrow & & \downarrow \\ \text{CMV matrices} & \longrightarrow & \text{Scattering zippers} \end{array}$$

From the spectral point on view, Verblunsky's theorem states that any sequence in $\mathbb{S}^1$ is the sequence of Verblunsky coefficients for a unique probability measure on the unit circle.[26] Thus, to each CMV matrix, one can associate a unique probability measure on the unit circle which is its spectral measure. A matrix-valued version of CMV matrices was also considered in Ref. 11 and corresponds to a particular case of scattering zippers with all the phases set to I_{L}, where I_{L} is the identity matrix of order L. But, this matrix-valued version of CMV matrices does not satisfy an analog of Verblunsky's theorem: (see also Ref. 10). One of the motivations to introduce scattering zippers was to fill this gap. It was proven in Ref. 23 that, for every scattering zipper, there exists a unique matrix-valued probability measure on $\mathbb{S}^1$ which is the spectral measure of the scattering zipper. Thus, scattering zippers satisfy an analog of Verblunsky's theorem.

The structure of the elements in $\mathrm{U}(2L)_{\mathrm{inv}}$ is now investigated. Invertibility of the upper right entries of size $L \times L$ of the S_n's is the only hypothesis done on these unitary matrices. This hypothesis ensures that the scattering is effective so that $\mathbb{U}$ does not decouple into a direct sum of two or more operators. It suffices to assume invertibility of the upper right block because of the shift by L of the blocks of $\mathbb{V}$ with respect to those of $\mathbb{W}$. Equivalent to the condition that β is invertible is the condition $\alpha^*\alpha < I_{\mathrm{L}}$. Furthermore, one has the representation[23]

$$\mathrm{U}(2L)_{\mathrm{inv}} = \{ S(\alpha,U,V) \in \mathrm{U}(2L) \mid \alpha^*\alpha < I_{\mathrm{L}} \text{ and } U,V \in \mathrm{U}(L)\} , \tag{5}$$

where

$$S(\alpha,U,V) = \begin{pmatrix} \alpha & \rho(\alpha)U \\ V\widetilde{\rho}(\alpha) & -V\alpha^*U \end{pmatrix} \text{ and } \rho(\alpha) = (I_{\mathrm{L}} - \alpha\alpha^*)^{\frac{1}{2}}, \ \ \widetilde{\rho}(\alpha) = (I_{\mathrm{L}} - \alpha^*\alpha)^{\frac{1}{2}} . \tag{6}$$

In what follows, when there is no ambiguity, ρ and $\widetilde{\rho}$ denote, respectively, $\rho(\alpha)$ and $\widetilde{\rho}(\alpha)$. The sequence $(\alpha_n)_{n\in\mathbb{Z}}$ corresponding to the sequence $(S_n)_{n\in\mathbb{Z}}$ is called the Verblunsky sequence associated to $\mathbb{U}$.

B. Random scattering zippers

The aim of this section is to introduce the random setting which allows to define a random version of the scattering zipper $\mathbb{U}$. It will be achieved by defining an ergodic family of scattering zippers whose phases are randomly picked in the unitary group.

Let $\Omega_0 = \mathrm{U}(L) \times \mathrm{U}(L)$, $\mathcal{B}_0$ the Borelian σ-algebra over $\mathrm{U}(L) \times \mathrm{U}(L)$ for the usual Lie group topology, and $\mathbb{P}_0 = \nu_L \otimes \nu_L$, where ν_L is the Haar measure on $\mathrm{U}(L)$. Then, $\mathbb{P}_0$ is a probability measure on Ω_0.

We define the product probability space

$$(\Omega, \mathcal{B}, \mathbb{P}) = \left((\Omega_0)^{\mathbb{Z}},\ \otimes_{n\in\mathbb{Z}} \mathcal{B}_0,\ \otimes_{n\in\mathbb{Z}} \mathbb{P}_0 \right).$$

Notation. For $\omega \in \Omega$ and $n \in \mathbb{Z}$, we denote $\omega_n \in \Omega_0$ more explicitly by: $\omega_n = (U_n(\omega), V_n(\omega))$.

Let $\omega \in \Omega$ and let $(\alpha_n)_{n\in\mathbb{Z}}$ be a sequence of matrices in $\mathcal{M}_L(\mathbb{C})$ such that for every $n \in \mathbb{Z}$, $\alpha_n^* \alpha_n < I_{\mathrm{L}}$. Then, for every $n \in \mathbb{Z}$, one considers the random unitary matrix $S_n(\omega) \in \mathrm{U}(2L)_{\mathrm{inv}}$ defined by

$$S_n(\omega) = S(\alpha_n, U_n(\omega), V_n(\omega)).$$

The sequence $((U_n(\omega), V_n(\omega)))_{n\in\mathbb{Z}}$ is a sequence of independent and identically distributed (*i.i.d.* for short) random variables defined on the probability space $(\Omega_0,\ \mathcal{B}_0,\ \mathbb{P}_0)$. Therefore, $(S_n(\omega))_{n\in\mathbb{Z}}$ is a sequence of independent random matrices in $\mathrm{U}(2L)_{\mathrm{inv}}$, but not necessarily identically distributed since it depends on the sequence $(\alpha_n)_{n\in\mathbb{Z}}$.

Associated to this sequence $(S_n(\omega))_{n\in\mathbb{Z}}$, one defines the operators $\mathbb{V}_\omega$, $\mathbb{W}_\omega$, and $\mathbb{U}_\omega = \mathbb{V}_\omega\, \mathbb{W}_\omega$ as in (2) and (3).

Definition 1. We call ***random scattering zipper*** *associated to $(S_n(\omega))_{n\in\mathbb{Z}}$, the family of random operators $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$.*

In order to study spectral properties of random scattering zippers, the spectrum of the operator $\mathbb{U}_\omega$ must not depend on the random parameter ω, at least $\mathbb{P}$-almost surely. This property will be a consequence of the ergodicity of the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$. To have this ergodicity property, it suffices to make the following assumption.

Assumption A. The Verblunsky sequence is constant, equal to some $\alpha \neq 0$.

Remark 1. Assumption A also implies that the sequence $(S_n(\omega))_{n\in\mathbb{Z}}$ is a sequence of i.i.d. random matrices in $\mathrm{U}(2L)_{\mathrm{inv}}$.

Notation. Denote the shift $\tau : \Omega \to \Omega$ defined by

$$\forall \omega \in \Omega,\ \forall n \in \mathbb{Z},\ (\tau(\omega))_n \ = \ \omega_{n+2}.$$

The shift τ is ergodic on $(\Omega, \mathcal{B}, \mathbb{P})$. Moreover, using τ and the Assumption A, one shows that the family of random operators $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ is $2\mathbb{Z}$-ergodic. Indeed, for $p \in \mathbb{Z}$, let U_p be the unitary operator on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ defined by

$$\forall \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L),\ \forall n \in \mathbb{Z},\ (U_p \phi)_n = \phi_{n-p}.$$

Then, for every $\omega \in \Omega$ and every $m \in \mathbb{Z}$, $\mathbb{U}_{\tau^m(\omega)} = U_{2m} \mathbb{U}_\omega U_{2m}^*$. The $2\mathbb{Z}$-ergodicity instead of $\mathbb{Z}$-ergodicity as in Anderson models is due to the fact that the scattering matrices are picked in $\mathrm{U}(2L)$ and that the blocks of $\mathbb{V}$ are shifted only by L and not $2L$. The induced overlapping of the blocks of $\mathbb{V}$ and $\mathbb{W}$ is an obstruction to the $\mathbb{Z}$-ergodicity.

Let $\mathbb{S}^1$ denote the unit circle in $\mathbb{C}$: $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Since for every $\omega \in \Omega$, $\mathbb{U}_\omega$ is unitary, its spectrum $\sigma(\mathbb{U}_\omega)$ is a subset of $\mathbb{S}^1$. Moreover, using the $2\mathbb{Z}$-ergodicity of the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$, there exists $\Sigma \subset \mathbb{S}^1$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, $\Sigma = \sigma(\mathbb{U}_\omega)$. The set Σ is called the almost-sure spectrum of $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$.

There also exist Σ_{pp}, Σ_{ac}, and Σ_{sc}, subsets of $\mathbb{S}^1$, such that for $\mathbb{P}$-almost every $\omega \in \Omega$, $\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(\mathbb{U}_\omega)$, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(\mathbb{U}_\omega)$, and $\Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(\mathbb{U}_\omega)$, respectively, the pure point, absolutely continuous, and singular continuous spectrum of $\mathbb{U}_\omega$. This is a general property of ergodic unitary operators. For a proof of these results, one can follow Chap. V in Ref. 9. Although the results of Ref. 9 are stated for self-adjoint operators, the proofs only rely on the existence of a spectral measure for the operator and properties of functional calculus, which in our setting is a consequence of the spectral theorem for unitary operators (see Ref. 22, Chapter 31).

The introduction of randomness in deterministic scattering zipper model (2) is necessary to be able to consider this model as a generalization of the Chalker-Coddington model on a strip where the phases are random.[2] In this last model, the phases appear as diagonal coefficients of two by two matrices, those being the diagonal blocks of the operator defining the Chalker-Coddington model on a strip. In the random scattering zipper, the phases $(U_n(\omega))_{n\in\mathbb{Z}}$ and $(V_n(\omega))_{n\in\mathbb{Z}}$ are no longer scalar-valued random variables, but they are matrix-valued in $\mathrm{U}(L)$. Thus, assuming that these random variables are distributed according to the Haar measure on $\mathrm{U}(L)$ is a direct generalization of taking phases distributed according to the Haar measure on the unit circle.[2]

The one-dimensional Anderson model has been thoroughly investigated[12] and generalization to quasi-one dimensional Anderson model (Anderson model with a matrix-valued potential) has been considered.[5] The random scattering zipper is the same kind of generalization with the random CMV model playing the role of the Anderson model. Instead of considering scalar-valued coefficients in our infinite matrices, we consider matrix-valued coefficients. In this comparison with the Anderson model, Assumption A has to be understood as the analog of the single-site potential assumption made in the Anderson model: a compactly supported potential is shifted throughout $\mathbb{Z}$ and only the random variables differ at each point of the lattice $\mathbb{Z}$. This assumption is necessary to have ergodicity of the considered family of random operators and thus to be able to apply the well-known results of the theory of ergodic families of operators.

One can also see the random scattering zipper as a unitary version of the quasi-one dimensional Anderson model. There already exists a unitary version of the scalar-valued Anderson model which has been studied by Hamza, Joye, and Stolz in Ref. 15, but until the introduction of the random scattering zipper, there was no unitary version of the matrix-valued Anderson model.

C. Main results

After having precisely defined the studied model, we state the three main results of this paper. For the definition of the Lyapunov exponents associated to the random scattering zipper $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$, see Sec. II.

Theorem 1. *Under Assumption A, for every* $z\in\mathbb{S}^1$, $\gamma_1(z)>\gamma_2(z)>\cdots>\gamma_L(z)>0$.

Using Theorem 1 and adapting some results of Kotani's theory, one deduces the absence of absolutely continuous spectrum for $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$.

Theorem 2. *Under Assumption A, the random scattering zipper* $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ *has no absolutely continuous spectrum:* $\Sigma_{\mathrm{ac}}=\emptyset$.

A more general result giving a relation between the multiplicity of the absolutely continuous spectrum of $\mathbb{U}_\omega$ and the number of vanishing Lyapunov exponents is shown in Sec. IV. This result is a version of the Ishii-Pastur theorem valid for general ergodic scattering zippers which do not necessarily satisfy Assumption A. For $j\in\{1,\ldots,L\}$, we set

$$Z_j=\left\{z\in\mathbb{S}^1\mid \exists\, l_1,\ldots,l_{2j}\in\{1,\ldots,2L\},\ \gamma_{l_1}(z)=\cdots=\gamma_{l_{2j}}(z)=0\right\}.$$

Then, we have

Theorem 3. *Let* $(\alpha_n)_{n\in\mathbb{Z}}$ *be such that the associated scattering zipper* $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ *is ergodic. For* $\mathbb{P}$*-almost every* $\omega\in\Omega$*, the multiplicity of the absolutely continuous spectrum of* $\mathbb{U}_\omega$ *on* Z_j *is at most* $2j$.

Theorem 2 is deduced directly from Theorem 3.

Let us finish this introduction by giving the outline of the article. In Sec. II, we present the formalism of transfer matrices for the random scattering zipper, and we define the Lyapunov exponents associated to these transfer matrices. We also recall some of the first properties of these exponents. In Sec. III, we prove Theorem 1 by reducing it to an algebraic result on the Lie group generated by the transfer matrices of $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$, the Fürstenberg group. Finally, in Sec. IV, we prove Theorem 2 by adapting many ideas of Kotani and Simon[21] to the setting of random scattering zippers. In particular, we characterize the multiplicity of the absolutely continuous spectrum in terms of the number of vanishing Lyapunov exponents on a subset of full measure of the unit circle and thus prove Theorem 3.

Results on the positivity of the Lyapunov exponents are already known for some models of unitary band matrices (see Refs. 7 and 16). The article of Bourget, Howland, and Joye[7] was a pioneering work on the study of one-channel unitary models and it was followed by additional work by Joye.[17,18] For a review of these results and for a presentation of the associated physical models, one can read Joye.[19] These unitary models deal with scalar coefficients compared to our model which deals with matrix coefficients.

The positivity of the Lyapunov exponents for matrix-valued models is also known for matrix-valued Anderson models, both in the discrete and continuous settings (see Ref. 13 for the discrete case and Ref. 6 for the continuous case). The Lie algebraic techniques used to prove positivity of the Lyapunov exponents for these models, and on which Theorem 1 is based upon, were developed in the 1980s and 1990s in several papers and in particular the work of Goldsheid and Margulis[14] and the work of Klein, Lacroix, and Speis.[20] For all these models, the positivity of the Lyapunov exponents implies absence of absolutely continuous spectrum through the results of Kotani's theory.[7,21] The positivity of the Lyapunov exponents is a strong sign of the Anderson localization and one can expect that random scattering zippers exhibit such localization and also exhibit strong dynamical localization.[5]

II. TRANSFER MATRICES AND LYAPUNOV EXPONENTS

A. Transfer matrices

In this section, we present the formalism of the transfer matrices which allows to reduce the understanding of the asymptotic behaviour of the solutions of the equation

$$\mathbb{U}_\omega \phi = z\phi, \quad \text{for } z \in \mathbb{S}^1, \tag{7}$$

to the understanding of the asymptotic behaviour of a product of random matrices in the Lorentz group.

The dynamics of the wave functions of $\mathbb{U}_\omega$ are difficult to understand by using only relations (4), since the action of S_n on ϕ and ψ depends on the parity of n. Therefore, one needs to rewrite (4) in a more suitable form. Instead of looking at the "input-output" relations through S_n, one needs to understand how to pass from $\binom{\phi_n}{\psi_n}$ to $\binom{\phi_{n+1}}{\psi_{n+1}}$ for ϕ a solution of (7) and $\psi = \mathbb{W}\phi$. This is done by transforming the scattering matrices which lie in $\mathrm{U}(2L)_{\mathrm{inv}}$ into elements of the Lorentz group.

The Lorentz group $\mathrm{U}(L,L)$ of signature (L,L) is defined to be the set of $2L \times 2L$ matrices preserving the form

$$\mathcal{L} = \begin{pmatrix} I_{\mathrm{L}} & 0 \\ 0 & -I_{\mathrm{L}} \end{pmatrix}. \tag{8}$$

Then, the application from $\mathrm{U}(2L)_{\mathrm{inv}}$ to $\mathrm{U}(L,L)$,

$$\varphi : \begin{array}{ccc} \mathrm{U}(2L)_{\mathrm{inv}} & \to & \mathrm{U}(L,L) \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} & \mapsto & \begin{pmatrix} \gamma - \delta\beta^{-1}\alpha & \delta\beta^{-1} \\ -\beta^{-1}\alpha & \beta^{-1} \end{pmatrix} \end{array}$$

is a bijection. Let $z \in \mathbb{S}^1$. With ϕ and ψ as above, the following relations hold:[23]

$$\forall n \in \mathbb{Z},\ \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix} = \varphi(z^{-1} S_{2n}(\omega)) \begin{pmatrix} \psi_{2n-1} \\ \phi_{2n-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \psi_{2n+1} \\ \phi_{2n+1} \end{pmatrix} = \varphi(S_{2n+1}(\omega)) \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix}. \tag{9}$$

Thus,

$$\forall n \in \mathbb{Z},\ \begin{pmatrix} \phi_{2n+2} \\ \psi_{2n+2} \end{pmatrix} = \varphi(z^{-1} S_{2n+2}(\omega)) \cdot \varphi(S_{2n+1}(\omega)) \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix} \tag{10}$$

and

$$\forall n \in \mathbb{Z},\ \begin{pmatrix} \phi_{2n+1} \\ \psi_{2n+1} \end{pmatrix} = \varphi(z^{-1} S_{2n+1}(\omega)) \cdot \varphi(S_{2n}(\omega)) \begin{pmatrix} \phi_{2n-1} \\ \psi_{2n-1} \end{pmatrix}. \tag{11}$$

Moreover, one has

$$\forall n \in \mathbb{Z},\ \begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix} = \varphi(z^{-1}S_{2n}(\omega)) \cdot \varphi(S_{2n-1}(\omega)) \cdots \varphi(z^{-1}S_2(\omega)) \cdot \varphi(S_1(\omega)) \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} \tag{12}$$

and

$$\forall n \in \mathbb{Z},\ \begin{pmatrix} \phi_{2n+1} \\ \psi_{2n+1} \end{pmatrix} = \varphi(z^{-1}S_{2n+1}(\omega)) \cdot \varphi(S_{2n}(\omega)) \cdots \varphi(z^{-1}S_3(\omega)) \cdot \varphi(S_2(\omega)) \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}. \tag{13}$$

Therefore, because of Assumption A and because of the *i.i.d.* character of the sequence $((U_n(\omega), V_n(\omega)))_{n\in\mathbb{Z}}$ in Ω, the asymptotic behaviour of both sequences $\left(\begin{pmatrix} \phi_{2n} \\ \psi_{2n} \end{pmatrix}\right)_{n\in\mathbb{Z}}$ and $\left(\begin{pmatrix} \phi_{2n+1} \\ \psi_{2n+1} \end{pmatrix}\right)_{n\in\mathbb{Z}}$ is the same and is given by the asymptotic behaviour of the product in (12) or (13). It leads to consider the application $T(z,\cdot) : \Omega \to \mathrm{U}(L,L)$,

$$\forall \omega \in \Omega,\ T(z,\omega) = \begin{pmatrix} V_0(\omega) & 0 \\ 0 & (U_0(\omega))^* \end{pmatrix} \hat{T}_0(z) \begin{pmatrix} V_1(\omega) & 0 \\ 0 & (U_1(\omega))^* \end{pmatrix} \hat{T}_1 \tag{14}$$

with

$$\hat{T}_0(z) = \begin{pmatrix} z^{-1}\widetilde{\rho}^{-1} & \widetilde{\rho}^{-1}\alpha^* \\ \alpha\widetilde{\rho}^{-1} & z\rho^{-1} \end{pmatrix} \quad \text{and} \quad \hat{T}_1 = \begin{pmatrix} \widetilde{\rho}^{-1} & \widetilde{\rho}^{-1}\alpha^* \\ \alpha\widetilde{\rho}^{-1} & \rho^{-1} \end{pmatrix},$$

since the following relation holds:[23]

$$\forall \omega \in \Omega,\ \forall z \in \mathbb{S}^1,\ \forall n \in \mathbb{Z},\ T(z,\tau^n(\omega)) = \varphi(z^{-1}S_{2n}(\omega)) \cdot \varphi(S_{2n+1}(\omega)). \tag{15}$$

Definition 2. Let $z \in \mathbb{S}^1$, $n \in \mathbb{Z}$, and $\omega \in \Omega$. The nth transfer matrix associated to the operator $\mathbb{U}_\omega$ is the matrix $T(z,\tau^n(\omega))$.

The sequence $(T(z,\tau^n(\omega)))_{n\in\mathbb{Z}}$ is an *i.i.d.* sequence of random matrices in $\mathrm{U}(L,L)$ because of the *i.i.d.* character of the sequence $((U_n(\omega),V_n(\omega)))_{n\in\mathbb{Z}}$ in Ω and because of Assumption A. It was necessary to consider as transfer matrices a product of two matrices $\varphi(z^{-1}S_{2n}(\omega)) \cdot \varphi(S_{2n+1}(\omega))$ and not just one of these matrices to have the property that they are identically distributed.

The transfer matrices $T(z,\cdot)$ generate the cocycle Φ on the ergodic dynamical system $(\Omega,\mathcal{B},\mathbb{P},(\tau^n)_{n\in\mathbb{Z}})$ with $\Phi(z,\cdot,\cdot) : \Omega\times\mathbb{Z} \to \mathrm{U}(L,L)$,

$$\Phi(z,\omega,n) = \begin{cases} T(z,\tau^{n-1}(\omega))\dots T(z,\omega) & \text{if} \quad n > 0 \\ I_{2L} & \text{if} \quad n = 0. \\ (T(z,\tau^n(\omega))^{-1}\dots(T(z,\tau^{-1}(\omega))^{-1} & \text{if} \quad n < 0 \end{cases}$$

Due to (12) and (13), for fixed $z \in \mathbb{S}^1$ and $\omega \in \Omega$, the asymptotic behaviour of the solutions of (7) is the same as the asymptotic behaviour of the sequence $(||\Phi(z,\omega,n)||)_{n\in\mathbb{Z}}$, where $||\ ||$ is any norm on $\mathrm{U}(L,L)$.

B. Lyapunov exponents

The Lyapunov exponents associated to the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ are defined by using the cocycle Φ.

Proposition 1. Let $z \in \mathbb{S}^1$. For $\mathbb{P}$-almost every $\omega \in \Omega$, the following limits exist and are equal:

$$\Psi(z,\omega) := \lim_{n\to+\infty} ((\Phi(z,\omega,n))^*\Phi(z,\omega,n))^{1/2n} = \lim_{n\to-\infty} ((\Phi(z,\omega,n))^*\Phi(z,\omega,n))^{1/2|n|}. \tag{16}$$

For every $k \in \{1,\dots,2L\}$, denoted by $\lambda_k(z,\omega)$, the eigenvalues of $\Psi(z,\omega)$ arranged in decreasing order. Then, there exist real numbers $\lambda_k(z) \geq 0$ such that for $\mathbb{P}$-almost every $\omega \in \Omega$, $\lambda_k(z,\omega) = \lambda_k(z)$.

Proof. This is a direct consequence of Oseledets theorem applied to the cocycle $\Phi(z,\cdot,\cdot)$. Indeed, according to Ref. 1 (Remark 3.4.10), one can apply Ref. 1 (Theorem 3.4.11) on $\mathbb{C}^{2L}$ instead of $\mathbb{R}^{2L}$. □

Notation. Denote by Ω_{Lyap} a subset of Ω such that

- $\mathbb{P}(\Omega_{\text{Lyap}}) = 1$,
- for every $\omega \in \Omega_{\text{Lyap}}$, limits (16) exist,
- for every $\omega \in \Omega_{\text{Lyap}}$, for every $k \in \{1,\dots,2L\}$, $\lambda_k(z,\omega) = \lambda_k(z)$.

Proposition 1 leads to the definition of the Lyapunov exponents associated to the ergodic family of operators $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$.

Definition 3. The $2L$ Lyapunov exponents associated to the ergodic family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ are defined by

$$\forall\, k \in \{1,\dots,2L\},\ \gamma_k(z) := \log(\lambda_k(z)).$$

Since the transfer matrices lie in the group $\mathrm{U}(L,L)$ and thus, for every $n \in \mathbb{Z}$, the matrix $\Phi(z,\omega,n)$ lies in this group too, we get a symmetry relation for the Lyapunov exponents

$$\forall k\ \in \{0,\dots L\},\ \gamma_{2L-k+1}(z) = -\gamma_k(z).$$

It implies that the Lyapunov exponents arrange by pairs of opposite real numbers

$$\gamma_1(z) \geq \gamma_2(z) \geq \cdots \geq \gamma_L(z) \geq 0 \geq \gamma_{L+1}(z) = -\gamma_L(z) \geq \cdots \geq \gamma_{2L}(z) = -\gamma_1(z). \tag{17}$$

Therefore, only the L first Lyapunov exponents are relevant to prove the non-vanishing of all of them.

In Sec. III, the Lyapunov exponents associated to the cocycle Φ are proven to be all distincts. Thus, using inequality (17), the L first Lyapunov exponents are all positive.

III. POSITIVITY OF THE LYAPUNOV EXPONENTS

A. The Fürstenberg group

Let $z \in \mathbb{S}^1$. To prove that the Lyapunov exponents are all distinct and strictly positive, one considers the Fürstenberg group associated to $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$, which is the closed subgroup of $\mathrm{U}(L,L)$,

$$G(z)\ =\ \overline{\langle \mathrm{supp}\mu_z\rangle} \subset \mathrm{U}(L,L),$$

generated by μ_z, the common law of all the transfer matrices $T(z,\tau^n(\omega))$, for every $n \in \mathbb{Z}$. The closure is taken for the topology on $\mathrm{U}(L,L)$ which is induced by the usual topology on $\mathcal{M}_{2L}(\mathbb{C})$.

Since $(T(z,\tau^n(\omega)))_{n\in\mathbb{Z}}$ is an *i.i.d.* sequence of random matrices in $\mathrm{U}(L,L)$, using Assumption A and the fact that the Haar measure on $\mathrm{U}(L)$ is supported by the whole unitary group, we get the following internal description of the Fürstenberg group:

$$G(z)\ =\ \overline{\left\langle \left\{ \begin{pmatrix} V_0 & 0 \\ 0 & (U_0)^* \end{pmatrix} \hat{T}_0(z) \begin{pmatrix} V_1 & 0 \\ 0 & (U_1)^* \end{pmatrix} \hat{T}_1 \middle| (U_0,V_0,U_1,V_1) \in \mathrm{U}(L)^4 \right\} \right\rangle}. \tag{18}$$

The Fürstenberg group is therefore the closure of the group generated by the first transfer matrix when the random phases take all possibles values in the unitary group. This description of the elements of $G(z)$ is used to prove that the subgroup $G(z)$ is actually equal to the whole group $\mathrm{U}(L,L)$.

Proposition 2. For any $z \in \mathbb{S}^1$, $G(z)\ =\ \mathrm{U}(L,L)$.

Proof. By connexity of $\mathrm{U}(L,L)$, it suffices to show that the Lie algebras of $G(z)$ and $\mathrm{U}(L,L)$ are equal. The Lie algebra associated to $\mathrm{U}(L,L)$ is given by

$$\mathfrak{u}(L,L) = \mathrm{Lie}(\mathrm{U}(L,L)) = \{T \in \mathcal{M}_{2L}(\mathbb{C}) \mid T^*\mathcal{L} + \mathcal{L}T = 0\}.$$

Thus,

$$\mathfrak{u}(L,L) = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in \mathcal{M}_{2L}(\mathbb{C}) \;\middle|\; A^* = -A, D^* = -D, (A,B,D) \in \mathcal{M}_L(\mathbb{C}) \right\}.$$

Denote $\mathrm{Lie}(G(z)) = \mathfrak{g}(z)$. The proof divides in numerous steps.

Step 1. Setting $U_0 = V_0 = U_1 = V_1 = I_{\mathrm{L}}$, one gets $\hat{T}_0(z)\hat{T}_1 \in G(z)$. As $G(z)$ is a multiplicative group, $(\hat{T}_0(z)\hat{T}_1)^{-1}$ also belongs to $G(z)$. Thus, setting $U_1 = V_1 = I_{\mathrm{L}}$, letting U_0, V_0 free and multiplying at the right by $(\hat{T}_0(z)\hat{T}_1)^{-1}$ shows that

$$\forall (U_0, V_0) \in \mathrm{U}(L)^2,\ \begin{pmatrix} V_0 & 0 \\ 0 & U_0^* \end{pmatrix} \in G(z).$$

Step 2. By Step 1, one has $\mathrm{U}(L) \oplus \mathrm{U}(L) \subset G(z)$. This implies that

$$\mathrm{Lie}(\mathrm{U}(L)) \oplus \mathrm{Lie}(\mathrm{U}(L)) \subset \mathfrak{g}(z).$$

Since $\mathrm{Lie}(\mathrm{U}(L)) = \{A \in \mathcal{M}_L(\mathbb{C}) | A^* = -A\}$,

$$\mathfrak{a}_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| A^* = -A, D^* = -D, (A, D) \in \mathcal{M}_L(\mathbb{C})^2 \right\} \subset \mathfrak{g}(z).$$

Step 3. For $V_0 = U_0 = I_{\mathrm{L}}$ and for any U_1, V_1, $\hat{T}_0(z) \left(\begin{smallmatrix} V_1 & 0 \\ 0 & U_1^* \end{smallmatrix}\right) \hat{T}_1 \in G(z)$. Therefore,

$$\hat{T}_1^{-1} \begin{pmatrix} V_1 & 0 \\ 0 & U_1^* \end{pmatrix} \hat{T}_1 \ = \ (\hat{T}_0(z)\hat{T}_1)^{-1} \hat{T}_0(z) \begin{pmatrix} V_1 & 0 \\ 0 & U_1^* \end{pmatrix} \hat{T}_1 \in G(z).$$

For $j \in \{1, \dots, L\}$ and $t \in \mathbb{R}$, picking $V_1 = \mathrm{diag}(1, \dots, 1, e^{\mathrm{i}t}, 1, \dots, 1)$ with $e^{\mathrm{i}t}$ at the jth place, $U_1 = I_{\mathrm{L}}$ and derivating at $t = 0$ yields

$$\forall j \in \{1, \dots, L\},\ \mathrm{i}\hat{T}_1^{-1} \begin{pmatrix} E_{jj} & 0 \\ 0 & 0 \end{pmatrix} \hat{T}_1 \in \mathfrak{g}(z),$$

where $E_{kl} \in \mathcal{M}_L(\mathbb{C})$ is the matrix with all entries zero except the (k, l) entry which is one. Then, summing over j gives

$$\mathrm{i}\hat{T}_1^{-1} \begin{pmatrix} I_{\mathrm{L}} & 0 \\ 0 & 0 \end{pmatrix} \hat{T}_1 \in \mathfrak{g}(z).$$

Writing $\hat{T}_1$ as a block matrix, by direct computation,

$$\mathrm{i} \begin{pmatrix} 0 & \widetilde{\rho}^{-2}\alpha^* \\ -\alpha\widetilde{\rho}^{-2} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

Step 4. Since $\alpha \neq 0$ and $\widetilde{\rho}^{-2}$ is invertible, $\widetilde{\rho}^{-2}\alpha^* \neq 0$. So, there exists a couple of indices $(j_0, k_0) \in \{1, \dots, L\}^2$ such that $(\widetilde{\rho}^{-2}\alpha^*)_{(j_0, k_0)} \neq 0$. Let $c = (\widetilde{\rho}^{-2}\alpha^*)_{(j_0, k_0)}$. Since the Lie algebra $\mathfrak{g}(z)$ is stable by brackets,

$$\left[\mathrm{i} \begin{pmatrix} 0 & 0 \\ 0 & E_{j_0 j_0} \end{pmatrix}, \left[\mathrm{i} \begin{pmatrix} E_{k_0 k_0} & 0 \\ 0 & 0 \end{pmatrix}, \mathrm{i} \begin{pmatrix} 0 & \widetilde{\rho}^{-2}\alpha^* \\ -\alpha\widetilde{\rho}^{-2} & 0 \end{pmatrix} \right] \right] \ = \ \mathrm{i} \begin{pmatrix} 0 & cE_{k_0 j_0} \\ -\overline{c}E_{j_0 k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z).$$

Thus, we have shown that there exists an index $(j_0, k_0) \in \{1, \dots, L\}^2$ and $c \in \mathbb{C}$, with $c \neq 0$ such that

$$\mathrm{i} \begin{pmatrix} 0 & cE_{k_0 j_0} \\ -\overline{c}E_{j_0 k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{19}$$

Step 5. Let $(j, k) \in \{1, \dots, L\}^2$ and $y \in \mathbb{C}$. By Step 2, the matrix $\left(\begin{smallmatrix} yE_{jk} - \overline{y}E_{kj} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ belongs to $\mathfrak{g}(z)$ as the block $yE_{jk} - \overline{y}E_{kj}$ is anti-hermitian.

Using Step 4, for any $k \in \{1, \dots, L\} \setminus \{k_0\}$ and any $y \in \mathbb{C}$, one has

$$\left[\begin{pmatrix} yE_{kk_0} - \overline{y}E_{k_0 k} & 0 \\ 0 & 0 \end{pmatrix}, \mathrm{i} \begin{pmatrix} 0 & cE_{k_0 j_0} \\ -\overline{c}E_{j_0 k_0} & 0 \end{pmatrix} \right] = \mathrm{i} \begin{pmatrix} 0 & ycE_{kj_0} \\ -\overline{yc}E_{j_0 k} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{20}$$

By Step 2,

$$\mathrm{i} \begin{pmatrix} 0 & 0 \\ 0 & E_{jj_0} + E_{j_0 j} \end{pmatrix} \in \mathfrak{g}(z).$$

Then, for any $j \in \{1,\dots,L\}\backslash\{j_0\}$,

$$\left[\mathrm{i}\begin{pmatrix} 0 & ycE_{kj_0} \\ -\overline{yc}E_{j_0k} & 0 \end{pmatrix}, \mathrm{i}\begin{pmatrix} 0 & 0 \\ 0 & E_{jj_0}+E_{j_0j} \end{pmatrix}\right] = \begin{pmatrix} 0 & -ycE_{kj} \\ -\overline{yc}E_{jk} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{21}$$

Setting $y = -c^{-1}$ or $y = -ic^{-1}$ in (21), one gets:

$$\forall (j,k) \in \{1,\dots,L\}^2\backslash\{(j_0,k_0)\}, \ \begin{pmatrix} 0 & E_{kj} \\ E_{jk} & 0 \end{pmatrix} \in \mathfrak{g}(z) \quad \text{and} \quad \begin{pmatrix} 0 & \mathrm{i}E_{kj} \\ -\mathrm{i}E_{jk} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{22}$$

Step 6. In the case $(j,k) = (j_0,k_0)$, by the same computation as in (20), one obtains

$$\mathrm{i}\begin{pmatrix} 0 & (y-\overline{y})cE_{k_0j_0} \\ -(\overline{y}-y)\overline{c}E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{23}$$

For $y = -\frac{\mathrm{i}}{2c}$,

$$\begin{pmatrix} 0 & E_{k_0j_0} \\ E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{24}$$

If $\mathrm{Re}(c) \neq 0$, then $\mathrm{Im}(ic) \neq 0$ and, using (19) and (24),

$$\frac{1}{\mathrm{Im}(ic)}\left(\mathrm{i}\begin{pmatrix} 0 & cE_{k_0j_0} \\ -\overline{c}E_{j_0k_0} & 0 \end{pmatrix} - \mathrm{Re}(ic)\begin{pmatrix} 0 & E_{k_0j_0} \\ E_{j_0k_0} & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & \mathrm{i}E_{k_0j_0} \\ -\mathrm{i}E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{25}$$

If $\mathrm{Re}(c) = 0$, then $\mathrm{Im}(c) \neq 0$ since $c \neq 0$. Taking $y = -\frac{\mathrm{i}}{2\mathrm{Im}(c)}$ in (23), one has again

$$\begin{pmatrix} 0 & \mathrm{i}E_{k_0j_0} \\ -\mathrm{i}E_{j_0k_0} & 0 \end{pmatrix} \in \mathfrak{g}(z). \tag{26}$$

Let

$$\mathfrak{a}_2 = \left\{ \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \middle| B \in \mathcal{M}_L(\mathbb{C}) \right\}.$$

Since the elements of $\mathfrak{g}(z)$ constructed in (22), (24), and (26) form a basis of $\mathfrak{a}_2$ (as a real vector space), one has

$$\mathfrak{a}_2 \subset \mathfrak{g}(z).$$

Step 7. By Step 2 and Step 6, one gets that $\mathfrak{u}(L,L) = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \subset \mathfrak{g}(z)$. Therefore, $\mathfrak{u}(L,L) = \mathfrak{g}(z)$, which ends the proof. □

Remark 2. In the case $\alpha = 0$, one can show that $\mathfrak{g}(z) = \mathfrak{a}_1 \subsetneq \mathfrak{u}(L,L)$. The hypothesis $\alpha \neq 0$ is therefore needed to prove Proposition 2.

B. Proof of Theorem 1

This section is devoted to the proof of the positivity of the Lyapunov exponents of the ergodic family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$. We explain how to reduce the proof of Theorem 1 to the result of Proposition 2.

Proof of Theorem 1. We can follow the strategy of Ref. 2 (Theorem 6.1). Let $z \in \mathbb{S}^1$. By Proposition 2, $G(z) = \mathrm{U}(L,L)$. By Cayley transform, the group $G(z)$ is unitarily equivalent to the complex symplectic group. Indeed, if $C \in \mathcal{M}_{2L}(\mathbb{C})$ is the matrix

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} I_{\mathrm{L}} & -iI_{\mathrm{L}} \\ I_{\mathrm{L}} & iI_{\mathrm{L}} \end{pmatrix}$$

and J is the matrix

$$J = \begin{pmatrix} 0 & -I_{\rm L} \\ I_{\rm L} & 0 \end{pmatrix},$$

one has

$$\mathrm{U}(L,L) = C\mathrm{Sp}_{\rm L}(\mathbb{C})C^*,$$

where

$$\mathrm{Sp}_{\rm L}(\mathbb{C}) = \{M \in \mathcal{M}_{2L}(\mathbb{C}) | M^* J M = J\}.$$

Since the results of Ref. 4 are stated for the real symplectic group, one can introduce, as in Ref. 2, the following application which splits the real and imaginary parts of the matrices in $\mathcal{M}_{2L}(\mathbb{C})$,

$$\pi : \begin{array}{ccc} \mathcal{M}_{2L}(\mathbb{C}) & \to & \mathcal{M}_{4L}(\mathbb{R}) \\ A + iB & \mapsto & \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \end{array}.$$

Then, $\pi(C^*\mathrm{U}(L,L)C) \subset \mathrm{Sp}_{2\rm L}(\mathbb{R})$. With this setting, one deduces immediately from Ref. 2 (Lemma 6.3) that $\pi(C^* \cdot \mathrm{U}(L,L) \cdot C)$ is L_{2p}-strongly irreducible for every $p \in \{1,\ldots,L\}$ (for a definition of the notion L_p-strong irreducibility, see Ref. 4 (Definition A.IV.3.3)). To adapt the proof of Ref. 2 (Lemma 6.4) to scattering zippers, one only needs to perform a permutation of lines and columns of the matrix with $2L$ distinct singular values which is defined in Ref. 2. More precisely, let P denote the permutation matrix which send, for $j \in \{1,\ldots,L\}$, the $(L+j)-$th line of the identity matrix of order $2L$ to the line $2j$ and, for $k \in \{0,\ldots,L-1\}$, the kth line to the line $2k+1$. Then, multiplying on left and right by P the diagonal by blocks matrix used in Ref. 2 (Lemma 6.4), one gets an element of $\mathrm{U}(L,L)$ with $2L$ distinct singular values. Thus, $\pi(C^* \cdot \mathrm{U}(L,L) \cdot C)$ is $2p$-contracting for every $p \in \{1,\ldots,L\}$ (for a definition of p-contractivity, see Ref. 4 (Definition A.IV.1.1)).

By applying Ref. 4 (Proposition A.IV.3.4) to the group $\pi(C^* \cdot \mathrm{U}(L,L) \cdot C)$, all Lyapunov exponents are distincts. Moreover, using inequalities (17), the L first Lyapunov exponents are positive. □

IV. ABSENCE OF ABSOLUTELY CONTINUOUS SPECTRUM

In this section, we prove that Theorem 1 implies absence of absolutely continuous spectrum for the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$. For this purpose, we prove an analog of the theorem of Ishii and Pastur on the characterization of the absolutely continuous spectrum in term of zeros of the Lyapunov exponents. More precisely, we obtain a unitary version of the matrix-valued analog of Ishii-Pastur's theorem which was proven in Ref. 21 (Theorem 5.4).

Except at the end of this section, when we use Theorem 1 to prove Theorem 2, we will not use Assumption A. We only need to assume that the sequence $(\alpha_n)_{n\in\mathbb{Z}}$ which defines the operators $\mathbb{U}_\omega$ is such that the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ is ergodic. Of course, Assumption A ensures ergodicity of this family but although it was needed to prove positivity of the Lyapunov exponents in Theorem 1, it is not needed to prove Theorem 4.

Recall that for $\omega \in \Omega_{\rm Lyap}$, the operator $\mathbb{U}_\omega$ has $2L$ Lyapunov exponents which can be regrouped by pairs of opposite real numbers

$$\gamma_1(z) \geq \cdots \geq \gamma_L(z) \geq 0 \geq \gamma_{L+1}(z) = -\gamma_L(z) \geq \cdots \geq \gamma_{2L}(z) = -\gamma_1(z).$$

Also recall the definition of the sets Z_j given in the introduction. For $j \in \{1,\ldots,L\}$, we set

$$Z_j = \{z \in \mathbb{S}^1 \,|\, \exists\, l_1,\ldots,l_{2j} \in \{1,\ldots,2L\},\ \gamma_{l_1}(z) = \cdots = \gamma_{l_{2j}}(z) = 0\}.$$

A sequence $\varphi \in (\mathbb{C}^L)^{\mathbb{Z}}$ is said to be *polynomially bounded* if there exist $C > 0$ and $p \geq 1$ such that

$$\forall n \in \mathbb{Z},\ ||\varphi_n||_{\mathbb{C}^L} \leq C(1+|n|)^p,$$

where $||\cdot||_{\mathbb{C}^L}$ is any norm on $\mathbb{C}^L$. With these definitions, we can state the first result of this section.

Proposition 3. Let $j \in \{1,\ldots,L\}$ and let $z \in Z_j$ be fixed. Let $\omega \in \Omega_{\rm Lyap}$. Then, every subspace of the space

$$\{\varphi \in (\mathbb{C}^L)^{\mathbb{Z}} \mid \mathbb{U}_\omega \varphi = z\varphi,\ \varphi \notin \ell^2(\mathbb{Z}) \otimes \mathbb{C}^L \text{ and } \varphi \text{ is polynomially bounded } \}$$

has dimension at most $2j$.

Proof. We set

$$V_{\mathrm{sol}}(z) = \{\varphi \in (\mathbb{C}^L)^{\mathbb{Z}} \mid \mathbb{U}_\omega \varphi = z\varphi\},$$
$$V_{\mathrm{P}}(z) = \{\varphi \in V_{\mathrm{sol}}(z) \mid \varphi \text{ is polynomially bounded}\},$$

and

$$V_{\ell^2}(z) = \{\varphi \in V_{\mathrm{sol}}(z) \mid \varphi \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^L\}.$$

To prove the proposition, one has to show that

$$\dim V_{\mathrm{P}}(z) \leq 2j + \dim V_{\ell^2}(z).$$

For $\varphi = (\varphi_n)_{n\in\mathbb{Z}}$ and $\psi = (\psi_n)_{n\in\mathbb{Z}}$ in $V_{\mathrm{sol}}(z)$, we set

$$W(\varphi,\psi) = \begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}.$$

Then, W is an antisymmetric form on $V_{\mathrm{sol}}(z)$. Moreover, since ψ is in $V_{\mathrm{sol}}(z)$, it is uniquely determined by $\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}$ and since $\mathcal{L}$ is non-singular, W is non-degenerate. Indeed, let $\psi \in V_{\mathrm{sol}}(z)$ be such that for every $\varphi \in V_{\mathrm{sol}}(z)$, $W(\varphi,\psi) = 0$. Then, we apply this equality with $2L$ different φ's, those being such that $\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}$ are the $2L$ vectors of the canonical basis of $\mathbb{C}^{2L}$. Writing these $2L$ relations, we obtain a Cramer system with unique solution $\mathcal{L}\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = 0$. Since $\mathcal{L}$ is non-singular, we get $\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = 0$ and since ψ is uniquely determined by $\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}$, we get $\psi = 0$.

Now, since W is a non-degenerate antisymmetric form on $V_{\mathrm{sol}}(z)$, if V_1 and V_2 are two subspaces of $V_{\mathrm{sol}}(z)$ such that

$$\forall \varphi \in V_1,\ \forall \psi \in V_2,\ W(\varphi,\psi) = 0, \tag{27}$$

then

$$\dim V_1 + \dim V_2 \leq 2L. \tag{28}$$

Indeed, V_1 and $\mathcal{L}V_2$ are orthogonal for W.

We set

$$D_\pm = \{\varphi \in V_{\mathrm{sol}}(z) \mid \varphi \text{ decays exponentially at } \pm\infty\}.$$

Since $z \in Z_j$, exactly $2j$ among the Lyapunov exponents vanish at z. Thus, by Oseledets theorem,

$$\dim D_\pm = L - j.$$

We also have $D_+ \cap D_- \subset V_{\ell^2}(z)$, so

$$\begin{aligned}\dim(D_+ + D_-) &= \dim D_+ + \dim D_- - \dim(D_+ \cap D_-)\\ &\geq 2L - 2j - \dim V_{\ell^2}(z).\end{aligned} \tag{29}$$

Moreover, if we take $\varphi \in V_{\mathrm{P}}(z)$ and $\psi \in D_+ + D_-$, then, by direct domination, we have

$$\lim_{|n|\to+\infty} \begin{pmatrix}\varphi_n\\ \varphi_{n+1}\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_n\\ \psi_{n+1}\end{pmatrix} = 0. \tag{30}$$

But, one can actually show that the sequence $\left(\begin{pmatrix}\varphi_n\\ \varphi_{n+1}\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_n\\ \psi_{n+1}\end{pmatrix}\right)_{n\in\mathbb{Z}}$ is constant when we choose φ and ψ in $V_{\mathrm{sol}}(z)$. Indeed, since $|z| = 1$, $T(z,\tau^n(\omega)) \in \mathrm{U}(L,L)$ for every $n \in \mathbb{Z}$. Thus, for every $n \in \mathbb{Z}$,

$$\begin{aligned}\begin{pmatrix}\varphi_{n+1}\\ \varphi_{n+2}\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_{n+1}\\ \psi_{n+2}\end{pmatrix} &= \left(T(z,\tau^{n+1}(\omega)) \begin{pmatrix}\varphi_n\\ \varphi_{n+1}\end{pmatrix}\right)^* \mathcal{L} T(z,\tau^{n+1}(\omega)) \begin{pmatrix}\psi_n\\ \psi_{n+1}\end{pmatrix}\\ &= \begin{pmatrix}\varphi_n\\ \varphi_{n+1}\end{pmatrix}^* T(z,\tau^{n+1}(\omega))^* \mathcal{L} T(z,\tau^{n+1}(\omega)) \begin{pmatrix}\psi_n\\ \psi_{n+1}\end{pmatrix}\\ &= \begin{pmatrix}\varphi_n\\ \varphi_{n+1}\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_n\\ \psi_{n+1}\end{pmatrix}.\end{aligned}$$

In particular,

$$\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = \lim_{|n|\to+\infty} \begin{pmatrix}\varphi_n\\ \varphi_{n+1}\end{pmatrix}^* \mathcal{L} \begin{pmatrix}\psi_n\\ \psi_{n+1}\end{pmatrix} = 0. \tag{31}$$

Thus, if $\varphi \in V_{\mathrm{P}}(z)$ and $\psi \in D_+ + D_-$, $W(\varphi,\psi) = 0$. Then, by (28),

$$\dim V_{\mathrm{P}}(z) + \dim (D_+ + D_-) \leq 2L.$$

Combining this inequality with (29),

$$\dim V_{\mathrm{P}}(z) \leq 2j + \dim V_{\ell^2}(z),$$

which proves the proposition. □

Let $\mathrm{P}_{\mathrm{bdd}}$ be the set of all the complex numbers z of modulus 1 such that the equation $\mathbb{U}_\omega \varphi = z\varphi$ admits a non-trivial polynomially bounded solution. Since operator $\mathbb{U}_\omega$ has band structure, we can use results from Ref. 7. In particular, Ref. 7 (Lemma 5.4) and Ref. 7 (Corollary 5.2) applied to (2) lead to the following result.

Proposition 4. For $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\sigma(\mathbb{U}_\omega) = \overline{\mathrm{P}_{\mathrm{bdd}}} = \overline{\{z \in \mathbb{S}^1 \mid \exists \varphi \in V_{\mathrm{P}}(z) \setminus \{0\}\}}$$

and $\mathrm{E}_{\mathbb{S}^1\setminus \mathrm{P}_{\mathrm{bdd}}}(\mathbb{U}_\omega) = 0$, where $\mathrm{E}_{\mathbb{S}^1\setminus \mathrm{P}_{\mathrm{bdd}}}(\mathbb{U}_\omega)$ is the spectral projector on $\mathbb{S}^1\setminus \mathrm{P}_{\mathrm{bdd}}$ associated to the unitary operator $\mathbb{U}_\omega$.

With this proposition, we can prove the main result of this section, an Ishii-Pastur theorem for scattering zippers.

Theorem 4. *For every $\omega \in \Omega_{\mathrm{Lyap}}$, the multiplicity of the absolutely continuous spectrum of $\mathbb{U}_\omega$ on Z_j is at most $2j$.*

Proof. Let $\omega \in \Omega_{\mathrm{Lyap}}$. For Δ a Borelian subset of $\mathbb{S}^1$, we denote by $\mathrm{E}_\Delta(\mathbb{U}_\omega)$ the spectral projection on Δ and by $\mathrm{E}^{\mathrm{ac}}_\Delta(\mathbb{U}_\omega)$ the spectral projection on the absolutely continuous part of $\sigma(\mathbb{U}_\omega)$ in Δ.

To prove the theorem, we have to prove that

$$\mathrm{rk}\ \mathrm{E}^{\mathrm{ac}}_{Z_j}(\mathbb{U}_\omega) \leq 2j.$$

Since by Proposition 4, $\mathrm{E}_{\mathbb{S}^1\setminus \mathrm{P}_{\mathrm{bdd}}}(\mathbb{U}_\omega) = 0$, we have

$$\mathrm{E}^{\mathrm{ac}}_{Z_j}(\mathbb{U}_\omega) = \mathrm{E}^{\mathrm{ac}}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}}(\mathbb{U}_\omega) = \mathrm{E}^{\mathrm{ac}}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}\cap S}(\mathbb{U}_\omega) + \mathrm{E}^{\mathrm{ac}}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}\cap S^c}(\mathbb{U}_\omega),$$

where $S = \{z \in \mathbb{S}^1 \mid \exists \varphi \in V_{\mathrm{P}}(z) \cap V_{\ell^2}(z),\ \varphi \neq 0\}$. If $z \in S$, then z is an eigenvalue of $\mathbb{U}_\omega$. Since there are only countably many ℓ^2-eigenvectors for $\mathbb{U}_\omega$, the Haar measure of S is zero. Thus,

$$\mathrm{E}^{\mathrm{ac}}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}\cap S}(\mathbb{U}_\omega) = 0 \quad \text{and} \quad \mathrm{E}^{\mathrm{ac}}_{Z_j}(\mathbb{U}_\omega) = \mathrm{E}^{\mathrm{ac}}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}\cap S^c}(\mathbb{U}_\omega). \tag{32}$$

But, since $z \in Z_j$, we can apply Proposition 3 to get directly that

$$\mathrm{rk}\ \mathrm{E}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}\cap S^c}(\mathbb{U}_\omega) \leq 2j$$

which implies

$$\mathrm{rk}\ \mathrm{E}^{\mathrm{ac}}_{Z_j\cap \mathrm{P}_{\mathrm{bdd}}\cap S^c}(\mathbb{U}_\omega) \leq 2j. \tag{33}$$

So, combining (32) and (33), we have proven that

$$\mathrm{rk}\ \mathrm{E}^{\mathrm{ac}}_{Z_j}(\mathbb{U}_\omega) \leq 2j,$$

which achieves the proof of the theorem. □

Recall that ν_1 denotes the Haar measure on $\mathbb{S}^1$. Theorem 4 implies the following corollary.

Corollary 1. If for ν_1-almost every $z \in \mathbb{S}^1$, $\gamma_1(z) \geq \cdots \geq \gamma_L(z) > 0$, then $\Sigma_{\mathrm{ac}} = \emptyset$.

Proof. If $z \in \mathbb{S}^1$ is such that $\gamma_L(z) > 0$, then no Lyapunov exponent vanishes at z, which means that $z \in Z_0$. By Theorem 4, $\mathrm{E}^{\mathrm{ac}}_{Z_0}(\mathbb{U}_\omega) = 0$. Now, if Δ is a Borelian subset of $\mathbb{S}^1$, since by hypothesis $\nu_1(Z_0) = 1$,

$$\mathrm{E}^{\mathrm{ac}}_\Delta(\mathbb{U}_\omega) = \mathrm{E}^{\mathrm{ac}}_{\Delta\cap Z_0}(\mathbb{U}_\omega) = 0, \text{ for } \mathbb{P}\text{-a.e } \omega \in \Omega.$$

It implies $\Sigma_{\mathrm{ac}} = \emptyset$ by definition of Σ_{ac}. □

Finally, from Corollary 1 and Theorem 1, we deduce Theorem 2: the random scattering zipper $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ has no absolutely continuous spectrum.

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