# Lifshitz Tails for Continuous Matrix-Valued Anderson Models

**Hakim Boumaza & Hatem Najar**

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# Lifshitz Tails for Continuous Matrix-Valued Anderson Models

**Hakim Boumaza**[1] · **Hatem Najar**[2]

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**Abstract** This paper is devoted to the study of Lifshitz tails for a continuous matrix-valued Anderson-type model acting on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^D$, for arbitrary $d \geq 1$ and $D \geq 1$. We prove that, under a hypothesis of non-degeneracy of the bottom of the spectrum, the integrated density of states of the model has a Lifshitz behaviour at the bottom of the spectrum. We obtain a Lifshitz exponent equal to $-d/2$ and this exponent is independent of $D$. It shows that the behaviour of the integrated density of states at the bottom of the spectrum of a quasi-d-dimensional Anderson model is the same as its behaviour for a d-dimensional Anderson model. For $d = 1$, we prove that the bottom of the spectrum is always non-dege nerate, for any matrix-valued periodic background potential, and thus each quasi-one-dimensional Anderson model has a Lifshitz exponent equal to $-1/2$.

**Keywords** Lifshitz tails · Random operators · Anderson localization · Integrated density of states

## 1 Introduction

### 1.1 A General Model

We study the Lifshitz tails behaviour of the integrated density of states (IDS for short) of random Schrödinger operators of the form:

✉ Hakim Boumaza
boumaza@math.univ-paris13.fr

Hatem Najar
hatem.najar@ipeim.rnu.tn

1 Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS, UMR 7539, 93430 Villetaneuse, France

2 Département de Mathématiques, Faculté des Sciences de Monastir, Avenue de l'environnement, 5019 Monastir, Tunisie

$$H_0(\omega) = -\Delta_d \otimes I_{\mathrm{D}} + \sum_{n\in\mathbb{Z}^d} V_\omega^{(n)}(x-n), \tag{1.1}$$

acting on the Hilbert space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$, where $d\geq 1$ and $D\geq 1$ are integers, $\Delta_d$ is the d-dimensional continuous Laplacian and $I_{\mathrm{D}}$ is the identity matrix of order $D$.

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a complete probability space and let $\omega\in\Omega$. We assume that, for every $n\in\mathbb{Z}^d$, the functions $x\mapsto V_\omega^{(n)}(x)$ take values in the space $\mathrm{S}_{\mathrm{D}}(\mathbb{R})$ of real symmetric matrices of order $D$ and that these functions are supported on $[-\frac{1}{2},\frac{1}{2}]^d$ and bounded uniformly on $x$, $n$ and $\omega$. The sequence $(V_\omega^{(n)})_{n\in\mathbb{Z}^d}$ is a sequence of independent and identically distributed (*i.i.d.* for short) random variables on $(\Omega,\mathcal{A})$. We finally assume that the sequence $(V_\omega^{(n)})_{n\in\mathbb{Z}^d}$ is such that the family of random operators $\{H_0(\omega)\}_{\omega\in\Omega}$ is $\mathbb{Z}^d$-ergodic. An operator like (1.1) is also called a quasi-$d$-dimensional Anderson model.

The vector space $L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$ is endowed with the usual scalar product:

$$< f, g >_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}} = \int_{\mathbb{R}^d} < f(x), g(x) >_{\mathbb{C}^{\mathrm{D}}} \mathrm{d}x = \sum_{i=1}^{\mathrm{D}} \int_{\mathbb{R}^d} f_i(x)\overline{g_i(x)}\mathrm{d}x,$$

where $f=(f_1,\ldots,f_{\mathrm{D}})$, $g=(g_1,\ldots,g_{\mathrm{D}})$ and $f_i\in L^2(\mathbb{R}^d)\otimes\mathbb{C}$, $g_i\in L^2(\mathbb{R}^d)\otimes\mathbb{C}$ are the $i$-th components of $f$ and $g$.

As the functions $x\mapsto V_\omega^{(n)}(x)$, for every $\omega\in\Omega$ and every $n\in\mathbb{Z}^d$, take values in $\mathrm{S}_{\mathrm{D}}(\mathbb{R})$, the operator $H_0(\omega)$ is self-adjoint on the Sobolev space $H^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$, for every $\omega\in\Omega$ (see [21]). Thus, its spectrum $\sigma(H_0(\omega))$ is included in $\mathbb{R}$. Moreover, due to the hypothesis of $\mathbb{Z}^d$-ergodicity of the family $\{H_0(\omega)\}_{\omega\in\Omega}$, there exists a set $\Sigma_0\subset\mathbb{R}$ with the property: for $\mathsf{P}$-almost-every $\omega\in\Omega$, $\sigma(H_0(\omega))=\Sigma_0$ (see [5]). This set $\Sigma_0$ is called the almost-sure spectrum of $\{H_0(\omega)\}_{\omega\in\Omega}$.

## 1.2 Existence of the IDS

We want to study the asymptotic behaviour of the IDS associated to $H_0(\omega)$ near the bottom of the almost-sure spectrum of $H_0(\omega)$. The IDS of $H_0(\omega)$ is the repartition function of energy levels, per unit volume, of $H_0(\omega)$. To define it properly, we first need to restrict the operator $H_0(\omega)$ to boxes of finite volume. We set, for $L\geq 1$ an integer,

$$C_L = \left[-\frac{2L+1}{2}, \frac{2L+1}{2}\right]^d. \tag{1.2}$$

Then, we consider $H_{0,C_L}(\omega)$ the restriction of $H_0(\omega)$ to the Hilbert space $L^2(C_L)\otimes\mathbb{C}^{\mathrm{D}}$ with Dirichlet boundary conditions on the border $\partial C_L$ of $C_L$. To define the IDS, we now consider, for every $E\in\mathbb{R}$, the following thermodynamical limit:

$$N_0(E) = \lim_{L\to+\infty} \frac{1}{(2L+1)^d} \#\left\{\lambda\leq E \mid \lambda\in\sigma(H_{0,C_L}(\omega))\right\}. \tag{1.3}$$

We have already proved in [2] that, for the general model (1.1), for every $E\in\mathbb{R}$, the limit (1.3) exists and is $\mathsf{P}$-almost-surely independent of $\omega\in\Omega$ (see [2, Corollary1]). The question of the existence of (1.3) involves two problems to solve. First we had to prove that, for every $E\in\mathbb{R}$ and every $\omega\in\Omega$, the cardinal $\#\{\lambda\leq E \mid \lambda\in\sigma(H_{0,C_L}(\omega))\}$ is finite. Then we had to prove the existence of the limit when $L$ tends to infinity. Both solutions to these two problems rely strongly on the fact that the semigroup $(e^{-tH_{0,C_L}(\omega)})_{t>0}$ has an $L^2$-kernel, which is given through a matrix-valued Feynman–Kac formula (see [2, Proposition 1]). Once we obtain that the cardinal $\#\left\{\lambda \leq E |\, \lambda \in \sigma\left(H_{0,C_L}(\omega)\right)\right\}$ is finite, we prove the convergence, as $L$ tends to infinity, of the sequence of Laplace transforms of the counting measures of the spectral values of $H_{0,C_L}(\omega)$ smaller than $E$. We prove the convergence of this sequence by using Birkhoff's ergodic theorem which leads to the existence of a Borel measure $\mathfrak{n}_0$ on $\mathbb{R}$, independent of $\omega$, which is the desired limit. We finally set

$$\forall E \in \mathbb{R},\ \ N_0(E) = \mathfrak{n}_0((-\infty, E]), \tag{1.4}$$

the distribution function of $\mathfrak{n}_0$. The measure $\mathfrak{n}_0$ is called the density of states of $H_0(\omega)$.

### 1.3 A Particular Model

After this review of existence result of the IDS for the general model $H_0(\omega)$, we may consider a particular example of such model for which we will be able to prove precise results on Lifshitz tails of the IDS at the bottom of the spectrum.

We consider

$$H_\omega = -\Delta_d \otimes I_{\mathrm{D}} + W + \sum_{n\in\mathbb{Z}^d} \begin{pmatrix} \omega_1^{(n)} V_1(x-n) & & 0 \\ & \ddots & \\ 0 & & \omega_D^{(n)} V_D(x-n) \end{pmatrix}, \tag{1.5}$$

acting on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$, where $d \geq 1$ and $D \geq 1$ are integers, $\Delta_d$ and $I_{\mathrm{D}}$ are as in (1.1). We set:

$$H = -\Delta_d \otimes I_{\mathrm{D}} + W \quad \text{and} \quad V_\omega(x) = \sum_{n\in\mathbb{Z}^d} \begin{pmatrix} \omega_1^{(n)} V_1(x-n) & & 0 \\ & \ddots & \\ 0 & & \omega_D^{(n)} V_D(x-n) \end{pmatrix}. \tag{1.6}$$

For the model (1.5), we make the assumptions:

**(H1)** $W : \mathbb{R}^d \to \mathrm{S}_{\mathrm{D}}(\mathbb{R})$ is $\mathbb{Z}^d$-periodic, measurable and bounded.

**(H2)** $V_1, \ldots, V_{\mathrm{D}}$ are non-negative, bounded, measurable, real-valued functions supported on $[-\frac{1}{2}, \frac{1}{2}]^d$. Moreover, we assume that, for every $i \in \{1, \ldots, \mathrm{D}\}$, there exists a non-empty cube $\mathcal{C}_i \subset [-\frac{1}{2}, \frac{1}{2}]^d$ which is not reduced to a single point such that $V_i > \mathbf{1}_{\mathcal{C}_i}$, where $\mathbf{1}_{\mathcal{C}_i}$ is the characteristic function of $\mathcal{C}_i$.

**(H3)** For every $i \in \{1, \ldots, \mathrm{D}\}$, $(\omega_i^{(n)})_{n\in\mathbb{Z}^d}$ is a family of *i.i.d.* random variables on a complete probability space $(\widetilde{\Omega}_i, \widetilde{\mathcal{A}}_i, \widetilde{\mathsf{P}}_i)$, which are bounded, and whose support of their common law $\nu_i$ contains zero and is not reduced to this single point. Moreover, we assume that,

$$\forall i \in \{1, \ldots, \mathrm{D}\},\ \limsup_{\varepsilon\longrightarrow 0^+} \frac{\log|\log \widetilde{\mathsf{P}}_i(\omega_i^{(0)} \leq \varepsilon)|}{\log \varepsilon} = 0. \tag{1.7}$$

**(H4)** The bottom of the spectrum of $H$ is non-degenerate in the following sense : let $E_0(\theta)$ be the first Floquet eigenvalue of $H$ as defined in Sect. 2 and let $\theta^0 \in \mathbb{T}^*$ be a minimum of $\theta \mapsto E_0(\theta)$. We say that the bottom of the spectrum of $H$ is non-degenerate if there exist $\delta > 0$, $C_1 > 0$ and $C_2 > 0$ such that

$$\forall \theta \in \mathbb{T}^*,\ |\theta-\theta^0| < \delta \Rightarrow C_1|\theta-\theta^0|^2 \leq E_0(\theta) - E_0(\theta^0) \leq C_2|\theta-\theta^0|^2. \tag{1.8}$$

In particular, we can take Bernoulli random variables for the $\omega_i^{(n)}$'s and $(H3)$ is still satisfied. By adding a suitable constant diagonal matrix to the periodic background $W$, we may always assume that the $\omega_i^{(n)}$'s are non-negative (because of their boundedness). If we set

$$(\Omega, \mathcal{A}, \mathsf{P}) = \left( \bigotimes_{n\in\mathbb{Z}^d} (\widetilde{\Omega}_1 \otimes \cdots \otimes \widetilde{\Omega}_{\mathrm{D}}), \bigotimes_{n\in\mathbb{Z}^d} (\widetilde{\mathcal{A}}_1 \otimes \cdots \otimes \widetilde{\mathcal{A}}_{\mathrm{D}}), \bigotimes_{n\in\mathbb{Z}^d} (\widetilde{\mathsf{P}}_1 \otimes \cdots \otimes \widetilde{\mathsf{P}}_{\mathrm{D}}) \right),$$

then $(\Omega, \mathcal{A}, \mathsf{P})$ is a complete probability space and $\{H_\omega\}_{\omega\in\Omega}$ is $\mathbb{Z}^d$-ergodic because of the non-overlapping of the random variables $\omega_i^{(n)}$. We denote by $\Sigma$ the almost-sure spectrum of $\{H_\omega\}_{\omega\in\Omega}$. By adding a suitable scalar matrix $\lambda I_{\mathrm{D}}$ to the periodic potential $W$, we may always assume that $\inf \Sigma = 0$.

The model (1.5) is a particular case of (1.1) for which the potential split into a deterministic periodic part $W$ and a random part $V_\omega$ which appears as a diagonal matrix. We will denote by $N : E \to N(E)$ the IDS of $H_\omega$.

*Remark 1* If we assume that, at least for one $x \in \mathbb{R}^d$, $W(x)$ is not a diagonal matrix, then we cannot write $H_\omega$ as a direct sum $\otimes_{i=1}^{\mathrm{D}} H_{\omega,i}$ of scalar-valued operators $H_{\omega,i}$ acting on $L^2(\mathbb{R}^d) \otimes \mathbb{C}$ and for which all the results we will present here are already known.

*Remark 2* The hypothesis $(H2)$ on the boundedness of the functions $V_i$ and the boundedness of their support implies in particular that each $V_i$ is in $L^p(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ with $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p > \frac{d}{2}$ if $d \geq 5$. These are the assumptions made in [12].

For $d = 1$, matrix-valued operators as (1.5) are also called quasi-one-dimensional Anderson models. Localization results in both dynamical and spectral senses for such models, for particular simple choices of $W$ and $V_1, \ldots, V_{\mathrm{D}}$, are obtained in [3] and [4]. These quasi-one-dimensional models are of physical interest as they can be considered as partially discrete approximations of Anderson models on a two-dimensional continuous strip. Such a two-dimensional Anderson model on a continuous strip can, by example, modelize electronic transport in nanotubes. Indeed, a two dimensional continuous Anderson model is defined by:

$$H_{cs}(\omega) = -\Delta_2 + \sum_{n\in\mathbb{Z}} \omega^{(n)} V(x - n, y), \tag{1.9}$$

acting on $L^2(\mathbb{R} \times [0, 1]) \otimes \mathbb{C}$ with Dirichlet boundary conditions on $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. The $\omega^{(n)}$'s are *i.i.d.* random variables and $V$ is supported in $[0, 1]^2$. As the continuous strip $\mathbb{R}\times[0, 1]$ has one finite length dimension ($[0, 1]$) and one infinite length dimension ($\mathbb{R}$), we can physically consider this strip as a quasi-one-dimensional nanotube. The spectral properties of $H_{cs}(\omega)$ describe properties of the electronic transport in the nanotube $\mathbb{R} \times [0, 1]$.

### 1.4 The Behaviour of the IDS

The main result of this paper is about Lifshitz tails for the IDS $N(E)$ of $H_\omega$ at the bottom of the spectrum. In 1963, Lifshitz (see [14]) had conjecture that, for a continuous random Schrödinger operator of IDS $N(E)$, there exist $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic:

$$N(E) \simeq c_1 \exp\left(-c_2 (E - E_0)^{-\frac{d}{2}}\right), \tag{1.10}$$

as $E$ tends to $E_0$, where $E_0$ is the bottom of the spectrum of the considered Schrödinger operator. The behaviour (1.10) is known as Lifshitz tails (for more details, see part IV.9.A of [20]) and the exponent $-d/2$ is called the Lifshitz exponent of the operator. The principal results known on Lifshitz tails are mainly for Schrödinger operators, in both continuous and discrete cases (see [8,9,12,15,19,23] and others) and for Schrödinger operators with magnetic fields (see [13,16] ). Up to our knowledge, all studied examples of Schrödinger operators are for scalar-valued operators, and no adaptation of the known results to matrix-valued operators like (1.5) has been done yet.

In a previous article of one of the author (see [2]), we had already obtain a result of Hölder continuity of the IDS for a particular example of model (1.5), in dimension $d = 1$. For $d = 1$, we can use the formalism of transfer matrices and define Lyapunov exponents for $\{H_\omega\}_{\omega\in\Omega}$ and, in this case, the sum of the positive Lyapunov exponents is harmonically conjugated to the IDS of $\{H_\omega\}_{\omega\in\Omega}$, through a so-called Thouless formula (see [2, Theorem3]). It allows us to prove that under some assumptions on the group generated by the transfer matrices, the Fürstenberg group of $\{H_\omega\}_{\omega\in\Omega}$, we have positivity of the Lyapunov exponents, their Hölder continuity with respect to the energy parameter and thus, by the Thouless formula, the same Hölder regularity for the IDS. This assumptions are hard to verify for a general $D$ (where $D$ is the size of the matrix-valued potential), but we were able to verify them for a very particular example of $\{H_\omega\}_{\omega\in\Omega}$ in dimension $d = 1$, where the periodic and random potentials $W$ and $V_\omega$ acts like constant functions. In [4], we studied the following continuous matrix-valued Anderson operator:

$$H_\ell(\omega) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{D}} + V + \sum_{n\in\mathbb{Z}} \begin{pmatrix} c_1\omega_1^{(n)}\mathbf{1}_{[0,\ell]}(x-\ell n) & & 0 \\ & \ddots & \\ 0 & & c_{\mathrm{D}}\omega_{\mathrm{D}}^{(n)}\mathbf{1}_{[0,\ell]}(x-\ell n) \end{pmatrix}, \tag{1.11}$$

acting on $L^2(\mathbb{R})\otimes\mathbb{C}^{\mathrm{D}}$, where $D \geq 1$ is an integer, $I_{\mathrm{D}}$ is the identity matrix of order $D$ and $\ell > 0$ is a real number. The matrix $V$ is a real $\mathrm{D}\times\mathrm{D}$ symmetric matrix. The constants $c_1, \dots, c_{\mathrm{D}}$ are non-zero real numbers. The random variables $\omega_i^{(n)}$ are like in model (1.5). As we can see, $H_\ell(\omega)$ is a particular example of $H_\omega$ with $W$ constant and, for every $i \in \{1, \dots, \mathrm{D}\}$, $V_i = \mathbf{1}_{[0,\ell]}$, where $\mathbf{1}_{[0,\ell]}$ is the characteristic function of $[0, \ell]$. For this operator we had obtained the following regularity result:

**Proposition 3** [4, Proposition 6.2] *For Lebesgue-almost every $V \in \mathrm{S}_{\mathrm{D}}(\mathbb{R})$, there exist a finite set $\mathcal{S}_V \subset \mathbb{R}$ and a real number $\ell_C := \ell_C(\mathrm{D}, V) > 0$ such that, for every $\ell \in (0, \ell_C)$, there exists a compact interval $I(\mathrm{D}, V, \ell) \subset \mathbb{R}$ such that, if $I \subset I(\mathrm{D}, V, \ell)\setminus\mathcal{S}_V$ is an open interval, then the integrated density of states of $H_\ell(\omega)$, $E \mapsto N_\ell(E)$, is Hölder continuous on $I$.*

*Remark 4* We actually proved even more : in such an open interval $I$ with $\Sigma \cap I \neq \emptyset$, we have Anderson localization in both spectral and dynamical senses.

The Proposition 3 is interesting in itself but does not give any information about the behaviour of the IDS at the bottom of the spectrum and, until now, it was not clearly stated that it has a Lifshitz behaviour. One of the motivation of the present article is to fill this lack of information of the IDS for quasi-one-dimensional operators and in particular those like $H_\ell(\omega)$ we have studied before from the localization point of view. It will be obtained through Proposition 2 and Theorem 5.

### 1.5 The Result

We can now state the main result of the present article.

**Theorem 5** *Let $\{H_\omega\}_{\omega\in\Omega}$ be the family of operators defined by (1.5) and let $N$ be its IDS. We assume hypothesis $(H1)$, $(H2)$, $(H3)$ and $(H4)$ and we also assume that* $\inf \Sigma = 0$. *Then,*

$$\lim_{E\to 0^+} \frac{\log|\log\left(N(E)-N(0^+)\right)|}{\log(E)} = -\frac{d}{2}. \tag{1.12}$$

*In particular, this limit does not depend on D.*

*Remark 6* (1) Under some assumption on the behaviour of the integrated density of states of the background operator $H$, it might be possible to obtain a result for internal bands.
(2) Theorem 5 could be used to give a different proof of localization than the one provided in [3] (see [6,17,18,24]).

It is important to insist on the fact that the Lifshitz exponent $-d/2$ obtained here does not depend on the integer $D \geq 1$. It means that, looking only at the Lifshitz behaviour of the IDS at the bottom of the spectrum of the considered operator, we cannot distinguish a quasi-$d$-dimensional Anderson model like (1.5) from a $d$-dimensional Anderson model (for $D = 1$).

One of the motivations in considering matrix-valued Anderson models is that we could expect, as $D$ tends to infinity, that we could obtain information about a $(d+1)$-dimensional Anderson model from a quasi-$d$-dimensional Anderson model. In particular, by obtaining a localization result for (1.11) for an arbitrary $D \geq 1$, we could have expect to obtain a similar localization result for the continuous strip (1.9). Such a strategy has been recently used to obtain a precise lower bound of the sum of Lyapunov exponents for a discrete matrix-valued Anderson model in dimension $d = 1$ (see [1]). The presence of the Lifshitz tails behaviour of the IDS is usually a strong sign of the presence of localization at the bottom of the spectrum. If we wanted to use the Lifshitz tails behaviour of the IDS to prove localization (like in [6]) and at the same time following the idea of approaching a $(d+1)$-dimensional Anderson model by a quasi-$d$-dimensional Anderson model, we would have expected a Lifshitz exponent depending on $D$ in a way such that this exponent would tend to $-(d+1)/2$ as $D$ tends to infinity. But, Theorem 5 contradicts this. So, we actually obtained an argument in favor of the idea that we cannot really get a proof of localization in dimension 2 (or more generally in dimension $d+1$) by an approximation procedure using quasi-one-dimensional Anderson models. This, at least if we follow a localization proof based upon the Lifshitz tails behaviour of the IDS.

## 2 Matrix-Valued Floquet Theory

In this section, we review the main results of the Floquet theory for the deterministic operator

$$H = -\Delta_d \otimes I_{\mathrm{D}} + W \tag{2.1}$$

acting on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$, with periodic potential $W$, and we adapt them to the matrix-valued setting. More precisely, we assume here that $W$ is a $\mathbb{Z}^d$-periodic function in $L^p(\mathbb{R}^d)\otimes \mathrm{S}_{\mathrm{D}}(\mathbb{R})$ with $p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p > \frac{d}{2}$ if $d \geq 5$. If $W$ is $\mathbb{Z}^d$-periodic, measurable and bounded as in model (1.5), it is in such an $L^p(\mathbb{R}^d)\otimes \mathrm{S}_{\mathrm{D}}(\mathbb{R})$ space. Then, $H$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$ (the space of compactly supported function, $\mathbb{C}^{\mathrm{D}}$-valued, of class $C^\infty$) with domain the Sobolev space $H^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$ [21].

First of all, let us notice that the formalism and all the general results about constant fiber direct integrals are still valid in our setting of matrix-valued operators. We refer to [22, Section XIII.16] for a complete presentation of these results.

For $y \in \mathbb{R}^d$, we denote by $\tau_y$ the operator of translation by $y$ which is defined, for $u \in L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$ and $x \in \mathbb{R}^d$, by $(\tau_y u)(x) = u(x-y)$. Then, because $W$ is $\mathbb{Z}^d$-periodic, the operator $H$ is invariant by conjugation by $\tau_n$, for every $n \in \mathbb{Z}^d$: $\forall n \in \mathbb{Z}^d,\ \tau_n \circ H \circ \tau_n^* = \tau_n \circ H \circ \tau_{-n} = H$.

Thus, $H$ is a $\mathbb{Z}^d$-periodic operator. Let $(e_1, \dots, e_d)$ be the canonical basis of $\mathbb{R}^d$. We recall that the box $C_0$ can be considered as the fundamental cell of the lattice $\mathbb{Z}^d$,

$$C_0 = \left\{ x_1 e_1 + \cdots + x_d e_d \;\middle|\; \forall j \in \{1, \dots, d\}, -\frac{1}{2} \leq x_j \leq \frac{1}{2} \right\}.$$

If $C_0^*$ is the fundamental cell of the dual lattice $(\mathbb{Z}^d)^* \simeq 2\pi\mathbb{Z}^d$, then $C_0^*$ is identified to the torus $\mathbb{T}^* = \mathbb{R}^d/2\pi\mathbb{Z}^d$. Let $\theta \in \mathbb{T}^*$. We denote by $\mathcal{D}_\theta'$ the space of $\mathbb{C}^{\mathrm{D}}$-valued, $\theta$-quasiperiodic distributions in $\mathbb{R}^d$, which is the space of distributions $u \in \mathcal{D}'(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ such that, for any $n \in \mathbb{Z}^d$, $\tau_n u = \mathrm{e}^{-\mathrm{i}n.\theta} u$. Let $\mathcal{H}_\theta = (L^2_{\mathrm{loc}}(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}) \cap \mathcal{D}_\theta'$, endowed with the norm on $L^2(C_0) \otimes \mathbb{C}^{\mathrm{D}}$. We also define, for $k \in \mathbb{Z}$, the spaces $\mathcal{H}_\theta^k = (H^k_{\mathrm{loc}}(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}) \cap \mathcal{D}_\theta'$, where $H^k_{\mathrm{loc}}(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ is the space of distributions that locally belong to the Sobolev space $H^k(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$. In order to define the Fourier decomposition we will use later, it remains to define the space:

$$\mathcal{H} = \Big\{ u \in \left(L^2_{\mathrm{loc}}(\mathbb{R}^d) \otimes L^2(\mathbb{T}^*)\right) \otimes \mathbb{C}^{\mathrm{D}} \;\Big|\; \forall (x, \theta, n) \in \mathbb{R}^d \times \mathbb{T}^* \times \mathbb{Z}^d,\ u(x+n, \theta) = \mathrm{e}^{\mathrm{i}n.\theta} u(x, \theta) \Big\},$$

endowed with the norm:

$$\forall u \in \mathcal{H},\ ||u||_{\mathcal{H}} = \frac{1}{\mathrm{vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} ||u(., \theta)||_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}}\, \mathrm{d}\theta.$$

For $\theta \in \mathbb{R}^d$ and $u \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$, we define $Uu \in \mathcal{H}$ by

$$\forall x \in \mathbb{R}^d,\ \forall \theta \in \mathbb{T}^*,\ (Uu)(x, \theta) = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} (\tau_n u)(x) = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} u(x - n). \tag{2.2}$$

Actually, the expression (2.2) is well-defined for $u \in \mathcal{S}(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$, the Schwartz space, and by Parseval theorem, this expression can be extended as an isometry from $L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ to $\mathcal{H}$. For every $v \in \mathcal{H}$, we can define $U^*$, the inverse of $U$ by:

$$\forall x \in \mathbb{R}^d,\ (U^* v)(x) = \frac{1}{\mathrm{vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} v(x, \theta)\, \mathrm{d}\theta. \tag{2.3}$$

Indeed, we have, for $v \in \mathcal{H}$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}(UU^* v)(x) &= \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} (U^* v)(x-n) = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} \frac{1}{\mathrm{vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} v(x-n, \theta)\, \mathrm{d}\theta \\ &= \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} \frac{1}{\mathrm{vol}(\mathbb{T}^*)} \int_{\mathbb{T}^*} \mathrm{e}^{-\mathrm{i}n.\theta} v(x, \theta)\, \mathrm{d}\theta = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} \hat{v}_n(x) = v(x, \theta).\end{aligned}$$

Thus, $U^*$ is a left inverse for $U$ which is an isometry from $L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ to $\mathcal{H}$, therefore $U$ is unitary and $U^* = U^{-1}$. To obtain a Floquet decomposition for the operator $H$, it remains to prove that the operators $U$ and $H$ commute. As, for any $j \in \{1, \dots, d\}$ and any $n \in \mathbb{Z}^d$, the partial derivation $\partial_j$ commute with the translation $\tau_n$, we have $[\partial_j, U] = 0$. Thus, $[-\Delta_d \otimes I_{\mathrm{D}}, U] = 0$. Then, using the $\mathbb{Z}^d$-periodicity of $W$, we also have, for every $u \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$, every $x \in \mathbb{R}^d$ and every $\theta \in \mathbb{T}^*$,

$$(U \circ W)(u)(x,\theta) = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} W(x-n)u(x-n) = \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} W(x)u(x-n)$$
$$= W(x) \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n.\theta} u(x-n) = W(x)(Uu)(x,\theta) = (W \circ U)(u)(x,\theta),$$

as at $x$ fixed, the multiplication by $W(x) \in \mathrm{S_D}(\mathbb{R})$ is continuous. Thus, $[W, U] = 0$ and we finally have $[H, U] = 0$. As we can see, even in the matrix-valued case we still have that $H$ and $U$ commute. Following [22], we deduce that $H$ admits the Floquet decomposition:

$$UHU^* = \int_{\mathbb{T}^*}^{\oplus} H_\theta \, \mathrm{d}\theta, \tag{2.4}$$

where $H_\theta$ is the selfadjoint operator $H$ acting on $\mathcal{H}_\theta$ with domain $\mathcal{H}_\theta^2$. Having this Floquet decomposition, we can continue to follow [22] to obtain that $H_\theta$ has a compact resolvent. It is a consequence on the assumptions made on the $L^p$-regularity of $W$ which ensure that $H$ is elliptic. As $H_\theta$ has a compact resolvent, its spectrum is discrete and we denote by

$$E_0(\theta) \leq E_1(\theta) \leq \cdots \leq E_j(\theta) \leq \cdots$$

its eigenvalues, called the *Floquet eigenvalues* of $H$. Moreover, the functions $\theta \mapsto E_j(\theta)$, for $j \in \mathbb{N}$, are continuous and, if $j$ tends to infinity, then $E_j(\theta)$ tends to $+\infty$, uniformly in $\theta$. Actually, as $H_\theta$ depends analytically on $\theta$, we also have that $\theta \mapsto E_j(\theta)$ is an analytic function in the neighborhood of any point $\theta^0 \in \mathbb{T}^*$ such that $E_j(\theta^0)$ is an eigenvalue of multiplicity one of $H_{\theta^0}$.

The set $E_j(\mathbb{T}^*)$ is a closed interval called the $j$-th spectral band of $H$ and the spectrum of $H$ is given by

$$\sigma(H) = \bigcup_{j\in\mathbb{N}} E_j(\mathbb{T}^*).$$

If $d \geq 2$, the bands can overlap, but it is not the case in dimension 1 (except maybe at an edge point). The difference between the usual scalar-valued case ($D = 1$) and the matrix-valued case is that they are "$D$ times" more Floquet eigenvalues in the matrix-valued case and thus the multiplicities of the $E_j(\theta)$'s are *a priori* bigger in the matrix-valued case than in the scalar-valued case. This phenomenon can be clearly observed for $W$ constant as we will see in the proof of Proposition 1.

## 3 Non-degeneracy of the Bottom of the Spectrum for $d = 1$

In this section we will prove that the hypothesis $(H4)$ is always verified for the operator (1.5) in dimension $d = 1$, and for every periodic background potential $W$. We will start by proving the result for a constant periodic background $W$ and then we will prove the general result which contains the constant case. We separate the constant case because the proof is very simple and completely explicit in this case. The proof in the constant case also allows us to picture our previous remark about multiplicities of the Floquet eigenvalues.

**Proposition 1** *Let $H$ be defined by* (1.6) *and assume that $W \in \mathrm{S_D}(\mathbb{R})$ is constant. Let $\theta^0 \in \mathbb{T}^*$ be a minimum of $\theta \mapsto E_0(\theta)$ where $E_0(\theta)$ is the first Floquet eigenvalue of $H$. Then,*

$$\forall \theta \in \mathbb{T}^*, \; E_0(\theta) - E_0(\theta^0) = |\theta - \theta^0|^2.$$

*In particular, the bottom of the spectrum of $H$ is non-degenerate.*

*Proof* In order to simplify the computations, we will write the Dirichlet boundary conditions defining $H_\theta$ in real coordinates:

$$D(H_\theta) = \left\{ \psi \in H^2([0,1]) \otimes \mathbb{R}^{\mathrm{D}} \,,\ \begin{pmatrix} \psi(1) \\ \psi'(1) \end{pmatrix} = \begin{pmatrix} \cos(\theta) I_{\mathrm{D}} & -\sin(\theta) I_{\mathrm{D}} \\ \sin(\theta) I_{\mathrm{D}} & \cos(\theta) I_{\mathrm{D}} \end{pmatrix} \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} \right\}. \tag{3.1}$$

We want to determine the values of $E \in \mathbb{R}$ for which the second order differential system $H\psi = E\psi$ admits a non-zero solution $\psi \in D(H_\theta)$. First, we rewrite it as a first order system:

$$H\psi = E\psi \ \Leftrightarrow \ \begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W - EI_{\mathrm{D}} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}.$$

Since $W$ is constant, using the Dirichlet boundary conditions,

$$\begin{pmatrix} \cos(\theta) I_{\mathrm{D}} & -\sin(\theta) I_{\mathrm{D}} \\ \sin(\theta) I_{\mathrm{D}} & \cos(\theta) I_{\mathrm{D}} \end{pmatrix} \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} = \exp\left( \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W - EI_{\mathrm{D}} & 0 \end{pmatrix} \right) \times \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix}. \tag{3.2}$$

The Eq. (3.2) will give us the Floquet eigenvalues. First, we have to compute the exponential that appears in this equation. Since $W$ is real symmetric, we can diagonalize it in an orthonormal basis. We denote by $\lambda_0 \leq \cdots \leq \lambda_{\mathrm{D}-1}$ its real eigenvalues. Then, there exists $P \in \mathrm{O}_{\mathrm{D}}(\mathbb{R})$, the orthogonal group of order $D$, such that

$$W - EI_{\mathrm{D}} = P^{-1} \begin{pmatrix} \lambda_0 - E & & 0 \\ & \ddots & \\ 0 & & \lambda_{\mathrm{D}-1} - E \end{pmatrix} P.$$

Thus, Eq. (3.2) has solutions only for $E \geq \lambda_{D-1}$ and we have

$$\exp\left( \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W - EI_{\mathrm{D}} & 0 \end{pmatrix} \right) = \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \cdot \begin{pmatrix} C & S_1 \\ S_2 & C \end{pmatrix} \cdot \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix},$$

where, for $j \in \{0, \ldots, \mathrm{D}-1\}$, $r_j = \sqrt{E - \lambda_j}$ and

$$C = \begin{pmatrix} \cos(r_0) & & 0 \\ & \ddots & \\ 0 & & \cos(r_{\mathrm{D}-1}) \end{pmatrix}, S_1 = \begin{pmatrix} \frac{\sin(r_0)}{r_0} & & 0 \\ & \ddots & \\ 0 & & \frac{\sin(r_{\mathrm{D}-1})}{r_{\mathrm{D}-1}} \end{pmatrix}$$

$$\text{and } S_2 = \begin{pmatrix} -r_0 \sin(r_0) & & 0 \\ & \ddots & \\ 0 & & -r_{\mathrm{D}-1} \sin(r_{\mathrm{D}-1}) \end{pmatrix}.$$

Then, by taking functions $\psi_j$ such that, for $j \in \{0, \ldots, \mathrm{D}-1\}$, $\psi_j(0) = {}^t(0, \ldots, 1, \ldots, 0)$ with a 1 at the $j$-th place and $\psi_j'(0) = 0$ in Eq. (3.2), one obtains, for every $j \in \{0, \ldots, \mathrm{D}-1\}$, $r_j = \theta$. Thus, the Floquet eigenvalues for $W$ constant are the $E_j(\theta) = \theta^2 + \lambda_j$, for $j \in \{0, \ldots, \mathrm{D}-1\}$, and we have : $E_0(\theta) \leq \ldots \leq E_{D-1}(\theta)$. The first Floquet eigenvalue is $E_0(\theta)$ and its minimum is reached in $\theta^0 = 0$ and equal to $\lambda_0$. We finally have

$$E_0(\theta) - E_0(\theta^0) = E_0(\theta) - E_0(0) = \theta^2 + \lambda_0 - \lambda_0 = \theta^2 = |\theta - \theta^0|^2.$$

This ends the proof. □

From this proof, we deduce that the Floquet eigenvalues have the same multiplicities as the $\lambda_j$ considered as eigenvalues of $W$. We can now prove a more general result which contains the particular case of $W$ constant.

**Proposition 2** *Let $H$ be defined by (1.6) for $d = 1$. Let $\theta^0 \in \mathbb{T}^*$ be a minimum of $\theta \mapsto E_0(\theta)$ where $E_0(\theta)$ is the first Floquet eigenvalue of $H$. Then, there exist $\delta > 0$, $C_1 > 0$ and $C_2 > 0$ such that*

$$\forall \theta \in \mathbb{T}^*,\ |\theta - \theta^0| < \delta \Rightarrow C_1|\theta - \theta^0|^2 \leq E_0(\theta) - E_0(\theta^0) \leq C_2|\theta - \theta^0|^2. \tag{3.3}$$

*Proof* In order to simplify some of the computations, we will restrict ourselves to real-valued functions and we write the Dirichlet boundary conditions in a real setting. We will denote by $< \cdot, \cdot >$ the euclidian scalar product on $\mathbb{R}^{\mathrm{D}}$ and by $|| \cdot ||$ the associated norm. We will also denote by $< \cdot, \cdot >_{\mathrm{D}}$ the euclidian scalar product on $L^2([0,1]) \otimes \mathbb{R}^{\mathrm{D}}$ and by $|| \cdot ||_{\mathrm{D}}$ the associated norm.

Let $\psi_1, \ldots, \psi_{2\mathrm{D}} \in H^2([0,1]) \otimes \mathbb{R}^{\mathrm{D}}$ be solutions of $H_{\theta^0}\psi = E_0(\theta^0)\psi$ with Dirichlet boundary conditions as in (3.1) and initial conditions : $\psi_j(0) = {}^t(0, \ldots, 1, \ldots, 0)$ with a 1 at the $j$-th place and $\psi_j'(0) = 0$ if $j \in \{1, \ldots, \mathrm{D}\}$, $\psi_j(0) = 0$ and $\psi_j'(0) = {}^t(0, \ldots, 1, \ldots, 0)$ with a 1 at the $j$-th place if $j \in \{\mathrm{D}+1, \ldots, 2\mathrm{D}\}$. Let

$$\mathcal{S}(\cdot, \theta^0) = \left( \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} \cdots \begin{pmatrix} \psi_{2\mathrm{D}} \\ \psi_{2\mathrm{D}}' \end{pmatrix} \right) \in \mathrm{GL}_{\mathrm{D}}(\mathbb{R})$$

be the fundamental matrix of the first order differential system associated to $H_{\theta^0}\psi = E_0(\theta^0)\psi$ :

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(x) - E_0(\theta^0)I_{\mathrm{D}} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix}.$$

One has :

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{S}(x, \theta^0) = \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(x) - E_0(\theta^0)I_{\mathrm{D}} & 0 \end{pmatrix} \mathcal{S}(x, \theta^0). \tag{3.4}$$

We introduce a notation for the four blocks of $\mathcal{S}(\cdot, \theta^0)$ : $\mathcal{S}(\cdot, \theta^0) = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2' \end{pmatrix}$. Then one has:

$$\Phi_1'' = (W(x) - E_0(\theta^0))\Phi_1, \text{ and } \Phi_2'' = (W(x) - E_0(\theta^0))\Phi_2. \tag{3.5}$$

We use the matrix $\mathcal{S}(\cdot, \theta^0)$ to define a new scalar product on the space $\mathcal{H}^1_{\theta^0} \otimes \mathbb{R}^2$ by the formula:

$$\forall f, g \in \mathcal{H}^1_{\theta^0} \otimes \mathbb{R}^2,\ < f, g >_{\theta^0} = \int_0^1 < \mathcal{S}(x, \theta^0) f(x), \mathcal{S}(x, \theta^0) g(x) >_{\mathbb{R}^{2\mathrm{D}}} \mathrm{d}x. \tag{3.6}$$

For $f \in \mathcal{H}^1_\theta \otimes \mathbb{R}^2$, we set

$$\overline{H}_\theta f = \frac{\mathrm{d}}{\mathrm{d}x} f - \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(x) - E_0(\theta^0)I_{\mathrm{D}} & 0 \end{pmatrix} f \tag{3.7}$$

and we define

$$\forall u \in \mathcal{H}^2_{\theta^0},\ \tilde{H}_\theta \begin{pmatrix} u \\ u' \end{pmatrix} = \mathcal{S}(\cdot, \theta^0)^{-1} \cdot \overline{H}_\theta \cdot \mathcal{S}(\cdot, \theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}. \tag{3.8}$$

Let $u \in \mathcal{H}^2_{\theta^0}$. Using (3.5), we get:

$$\begin{aligned}
&\overline{H}_\theta \cdot \mathcal{S}(x,\theta^0)\begin{pmatrix} u \\ u' \end{pmatrix} \\
&= \begin{pmatrix} (\Phi_1 u + \Phi_2 u')' \\ (\Phi_1' u + \Phi_2' u')' \end{pmatrix} - \begin{pmatrix} \Phi_1' u + \Phi_2' u' \\ (W(x) - E_0(\theta))(\Phi_1 u + \Phi_2 u') \end{pmatrix} \\
&= \begin{pmatrix} \Phi_1 u' + \Phi_2 u'' \\ (\Phi_1'' - (W(x) - E_0(\theta))\Phi_1) u + (\Phi_2'' - (W(x) - E_0(\theta))\Phi_2) u' + \Phi_1' u' + \Phi_2' u'' \end{pmatrix} \\
&= \mathcal{S}(x,\theta^0)\begin{pmatrix} u' \\ u'' \end{pmatrix} + \begin{pmatrix} 0 \\ \Big(\big(W(x) - E_0(\theta^0)\big) - \big(W(x) - E_0(\theta)\big)\Big)\big(\Phi_1 u + \Phi_2 u'\big) \end{pmatrix} \\
&= \mathcal{S}(x,\theta^0)\begin{pmatrix} u \\ u' \end{pmatrix}' + \begin{pmatrix} 0 \\ (E_0(\theta) - E_0(\theta^0))(\Phi_1 u + \Phi_2 u') \end{pmatrix}.
\end{aligned}$$

We deduce from this expression that

$$\begin{aligned}
\forall u \in \mathcal{H}^2_{\theta^0},\ \left\langle \begin{pmatrix} u \\ u' \end{pmatrix}', \tilde{H}_\theta \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\theta^0} &= \left\| \begin{pmatrix} u \\ u' \end{pmatrix}' \right\|^2_{\theta^0} + \big(E_0(\theta) - E_0(\theta^0)\big) \int_0^1 \langle \Phi_1' u' \\
&\quad + \Phi_2' u'', \Phi_1 u + \Phi_2 u' \rangle \mathrm{d}x. \qquad (3.9)
\end{aligned}$$

To get a better expression, we will now use the fact that, since $\mathcal{S}(\cdot,\theta^0)$ is the fundamental matrix of an Hamiltonian first order system, it is a symplectic matrix :

$$ {}^t\mathcal{S}(\cdot,\theta^0) J \mathcal{S}(\cdot,\theta^0) = J, \quad \text{with } J = \begin{pmatrix} 0 & -I_{\mathrm{D}} \\ I_{\mathrm{D}} & 0 \end{pmatrix}.$$

Identifying block by block we obtain the following relations on the blocks of $\mathcal{S}(\cdot,\theta^0)$:

$$ {}^t\Phi_1'\Phi_1 = {}^t\Phi_1\Phi_1', \quad {}^t\Phi_2'\Phi_1 = I_{\mathrm{D}} + {}^t\Phi_2\Phi_1', \quad {}^t\Phi_1'\Phi_2 = -I_{\mathrm{D}} + {}^t\Phi_1\Phi_2', \quad {}^t\Phi_2'\Phi_2 = {}^t\Phi_2\Phi_2'. \qquad (3.10)$$

With these relations we can compute the scalar product in (3.9):

$$\begin{aligned}
\langle \Phi_1' u' + \Phi_2' u'', \Phi_1 u + \Phi_2 u' \rangle &= {}^t(\Phi_1' u' + \Phi_2' u'')(\Phi_1 u + \Phi_2 u') \\
&= {}^t u'\, {}^t\Phi_1'\Phi_1 u + {}^t u'\, {}^t\Phi_1'\Phi_2 u' + {}^t u''\, {}^t\Phi_2'\Phi_1 u + {}^t u''\, {}^t\Phi_2'\Phi_2 u \\
&= {}^t(\Phi_1 u')\Phi_1' u + {}^t(\Phi_1 u')\Phi_2' u - {}^t u' u' + {}^t u'' u \\
&\quad + {}^t(\Phi_2 u'')\Phi_1' u + {}^t(\Phi_2 u'')\Phi_2' u' \\
&= \langle \Phi_1 u' + \Phi_2 u'', \Phi_1' u + \Phi_2' u' \rangle - ||u'||^2 + \langle u'', u \rangle \\
&= \langle (\Phi_1 u + \Phi_2 u')', \Phi_1 u + \Phi_2 u' \rangle - ||\Phi_1' u \\
&\quad + \Phi_2' u'||^2 - ||u'||^2 + \langle u'', u \rangle.
\end{aligned}$$

Integrating between 0 and 1, using integration by parts and using (3.5) we get

$$\begin{aligned}
\int_0^1 \langle (\Phi_1 u + \Phi_2 u')', \Phi_1 u + \Phi_2 u' \rangle \mathrm{d}x &= -\int_0^1 \langle \Phi_1 u + \Phi_2 u', (\Phi_1 u + \Phi_2 u')' \rangle \mathrm{d}x \\
&= -\int_0^1 \langle \Phi_1' u' + \Phi_2' u'', \Phi_1 u + \Phi_2 u' \rangle \mathrm{d}x - \int_0^1 \langle \Phi_1 u + \Phi_2 u', (W(x) \\
&\quad - E_0(\theta^0))(\Phi_1 u + \Phi_2 u') \rangle \mathrm{d}x.
\end{aligned}$$

Since $\langle u'', u\rangle_{\mathrm{D}} = -||u'||^2_{\mathrm{D}}$,

$$\begin{aligned}2\int_0^1 &\langle \Phi_1' u' + \Phi_2' u'', \Phi_1 u + \Phi_2 u'\rangle \mathrm{d}x \\ &= -\int_0^1 \langle \Phi_1 u + \Phi_2 u', (W(x) - E_0(\theta^0))(\Phi_1 u + \Phi_2 u')\rangle \mathrm{d}x \\ &\quad -\int_0^1 \langle \Phi_1' u + \Phi_2' u', \Phi_1' u + \Phi_2' u'\rangle \mathrm{d}x - 2||u'||^2_{\mathrm{D}}. \end{aligned} \tag{3.11}$$

But, using (3.4),

$$\begin{aligned}\langle \Phi_1 u \; &+\Phi_2 u', (W(x) - E_0(\theta^0))(\Phi_1 u + \Phi_2 u')\rangle + \langle \Phi_1' u + \Phi_2' u', \Phi_1' u + \Phi_2' u'\rangle \\ &= \left\langle \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2' \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(x) - E_0(\theta^0) I_{\mathrm{D}} & 0 \end{pmatrix} \times \begin{pmatrix} \Phi_1 & \Phi_2 \\ \Phi_1' & \Phi_2' \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\mathbb{R}^{2\mathrm{D}}} \\ &= \left\langle \mathcal{S}(x,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(x) - E_0(\theta^0) I_{\mathrm{D}} & 0 \end{pmatrix} \times \mathcal{S}(x,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\mathbb{R}^{2\mathrm{D}}} \\ &= \left\langle \mathcal{S}(x,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}, \frac{\mathrm{d}}{\mathrm{d}x} (\mathcal{S}(x,\theta^0)) \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\mathbb{R}^{2\mathrm{D}}} \\ &= \left\langle \mathcal{S}(x,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}, \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathcal{S}(x,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}\right) \right\rangle_{\mathbb{R}^{2\mathrm{D}}} \\ &\quad - \left\langle \mathcal{S}(x,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}, \mathcal{S}(x,\theta^0) \begin{pmatrix} u' \\ u'' \end{pmatrix} \right\rangle_{\mathbb{R}^{2\mathrm{D}}}. \end{aligned}$$

Putting this expression in (3.11), we get

$$2\int_0^1 < \Phi_1' u' + \Phi_2' u'', \Phi_1 u + \Phi_2 u' > \mathrm{d}x = -\frac{1}{2} \left|\left| \begin{pmatrix} u \\ u' \end{pmatrix} \right|\right|^2_{\theta^0} + \left\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}' \right\rangle_{\theta^0} - 2||u'||^2_{\mathrm{D}}.$$

And finally, (3.9) rewrites, for every $u \in \mathcal{H}^2_{\theta^0}$,

$$\begin{aligned}\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}', \tilde{H}_\theta \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\theta^0} &= \left|\left| \begin{pmatrix} u \\ u' \end{pmatrix}' \right|\right|^2_{\theta^0} - (E_0(\theta) - E_0(\theta^0)) \\ &\quad \times \left( \frac{1}{4} \left|\left| \begin{pmatrix} u \\ u' \end{pmatrix} \right|\right|^2_{\theta^0} - \frac{1}{2} \left\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}' \right\rangle_{\theta^0} + ||u'||^2_{\mathrm{D}} \right). \end{aligned} \tag{3.12}$$

We will now prove the upper bound in (3.3). In order to prove this upper bound, we will take the function $u = {}^t(\cos(\sqrt{|\theta - \theta^0|}\cdot), 0, \ldots, 0)$ in (3.12). To simplify the notations we set $b = \sqrt{|\theta - \theta^0|}$. Then,

$$\begin{aligned}\mathcal{S}(\cdot,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix} &= \begin{pmatrix} \cos(b\cdot)\psi_1 - b\sin(b\cdot)\psi_{D+1} \\ \cos(b\cdot)\psi_1' - b\sin(b\cdot)\psi_{D+1}' \end{pmatrix} \quad \text{and} \\ \mathcal{S}(\cdot,\theta^0) \begin{pmatrix} u \\ u' \end{pmatrix}' &= \begin{pmatrix} -b\sin(b\cdot)\psi_1 - b^2\cos(b\cdot)\psi_{D+1} \\ -b\sin(b\cdot)\psi_1' - b^2\cos(b\cdot)\psi_{D+1}' \end{pmatrix}. \end{aligned} \tag{3.13}$$

By direct computations we get:

$$\left\| \begin{pmatrix} u \\ u' \end{pmatrix}' \right\|_{\theta^0}^2 = b^2 \int_0^1 \sin^2(bx)(||\psi_1(x)||^2 + ||\psi_1'(x)||^2) + 2b\cos(bx)\sin(bx)(\langle \psi_1(x), \psi_{D+1}(x)\rangle + \langle \psi_1'(x), \psi_{D+1}'(x)\rangle) + b^2\cos^2(bx)(||\psi_{D+1}(x)||^2 + ||\psi_{D+1}'(x)||^2)\mathrm{d}x, \tag{3.14}$$

$$\left\| \begin{pmatrix} u \\ u' \end{pmatrix} \right\|_{\theta^0}^2 = \int_0^1 \cos^2(bx)(||\psi_1(x)||^2 + ||\psi_1'(x)||^2) - 2b\cos(bx)\sin(bx)(\langle \psi_1(x), \psi_{D+1}(x)\rangle + \langle \psi_1'(x), \psi_{D+1}'(x)\rangle) + b^2\sin^2(bx)(||\psi_{D+1}(x)||^2 + ||\psi_{D+1}'(x)||^2)\mathrm{d}x, \tag{3.15}$$

$$\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}' \right\rangle_{\theta^0} = \int_0^1 b^2(\sin^2(bx) - \cos^2(bx))(\langle \psi_1(x), \psi_{D+1}(x)\rangle + \langle \psi_1'(x), \psi_{D+1}'(x)\rangle) + \cos(bx)\sin(bx)\left(-b(||\psi_1(x)||^2 + ||\psi_1'(x)||^2) + b^3(||\psi_{D+1}(x)||^2 + ||\psi_{D+1}'(x)||^2)\right)\mathrm{d}x \tag{3.16}$$

and

$$||u'||_{\mathrm{D}}^2 = b^2\left(\frac{1}{2} + \frac{\sin(2b)}{4b}\right). \tag{3.17}$$

We are interested in the behaviour of these quantities for $\theta$ close to $\theta^0$ which is equivalent to look at $b$ close to 0. We have:

$$\left\| \begin{pmatrix} u \\ u' \end{pmatrix}' \right\|_{\theta^0}^2 = b^4 \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} + \begin{pmatrix} \psi_2 \\ \psi_2' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + \mathcal{O}(b^6)$$

and

$$\frac{1}{4}\left\| \begin{pmatrix} u \\ u' \end{pmatrix} \right\|_{\theta^0}^2 - \frac{1}{2}\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}' \right\rangle_{\theta^0} + ||u'||_{\mathrm{D}}^2 = \frac{3}{4}\left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + b^2 - \frac{1}{3}b^4 + \mathcal{O}(b^5).$$

In particular, since $\psi_1 \neq 0$, we deduce that for $\theta$ close to $\theta^0$,

$$\frac{1}{4}\left\| \begin{pmatrix} u \\ u' \end{pmatrix} \right\|_{\theta^0}^2 - \frac{1}{2}\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}' \right\rangle_{\theta^0} + ||u'||_{\mathrm{D}}^2 > 0. \tag{3.18}$$

Since $E_0(\cdot)$ is not constant and $\theta^0$ is a minimum of this function, by Taylor expansion there exist $p \geq 2$ an integer and $\alpha > 0$ such that

$$E_0(\theta) - E_0(\theta^0) = \alpha|\theta - \theta^0|^p + o(|\theta - \theta^0|^p) = \alpha b^{2p} + o(b^{2p}).$$

If $p = 2$, we directly have (3.3) from this Taylor expansion and the proof is finished. Thus, we can assume that $p \geq 3$ and we have:

$$\begin{aligned}\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}', \tilde{H}_\theta \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\theta^0} &= b^4 \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} + \begin{pmatrix} \psi_2 \\ \psi_2' \end{pmatrix} \right\|_{2\mathrm{D}}^2 \\ &\quad + \mathcal{O}(b^6) - (\alpha b^{2p} + o(b^{2p})) \left( \frac{3}{4} \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + \mathcal{O}(b^2) \right) \\ &= -\frac{3}{4}\alpha \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} \right\|_{2\mathrm{D}}^2 b^{2p} + b^4 \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} + \begin{pmatrix} \psi_2 \\ \psi_2' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + \mathcal{O}(b^6) \\ &= b^4 \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} + \begin{pmatrix} \psi_2 \\ \psi_2' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + \mathcal{O}(b^6).\end{aligned}$$

In particular, we deduce that for $\theta$ close to $\theta^0$,

$$\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}', \tilde{H}_\theta \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{\theta^0} \geq 0. \tag{3.19}$$

Combining (3.12) and (3.19) and using (3.18), we get the upper bound for $E_0(\theta) - E_0(\theta^0)$,

$$\begin{aligned}\exists C_2 > 0,\ E_0(\theta) - E_0(\theta^0) &\leq \frac{\left\| \begin{pmatrix} u \\ u' \end{pmatrix}' \right\|_{\theta^0}^2}{\frac{1}{4} \left\| \begin{pmatrix} u \\ u' \end{pmatrix} \right\|_{\theta^0}^2 - \frac{1}{2} \left\langle \begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}' \right\rangle_{\theta^0} + ||u'||_{\mathrm{D}}^2} \\ &= \frac{b^4 \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} + \begin{pmatrix} \psi_2 \\ \psi_2' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + \mathcal{O}(b^6)}{\frac{3}{4} \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} \right\|_{2\mathrm{D}}^2 + b^2 - \frac{1}{3} b^4 + \mathcal{O}(b^5)} \\ &= b^4 \frac{\left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} + \begin{pmatrix} \psi_2 \\ \psi_2' \end{pmatrix} \right\|_{2\mathrm{D}}^2}{\frac{3}{4} \left\| \begin{pmatrix} \psi_1 \\ \psi_1' \end{pmatrix} \right\|_{2\mathrm{D}}^2} + \mathcal{O}(b^6) \leq C_2 b^4 = C_2 |\theta - \theta^0|^2.\end{aligned} \tag{3.20}$$

It remains to prove the lower bound in (3.3) to finish the proof. We start by noticing that the formula (3.12) is valid for any background potential $W$. In particular, we can write (3.12) for $W = 0$. In this case, the computations done in the proof of Proposition 1 give us that $E_0(\theta) = \theta^2$, $\theta^0 = 0$ and $\mathcal{S}(\cdot, \theta^0) = I_{2\mathrm{D}}$. We denote by $\tilde{H}_{\theta,0}$ the operator $\tilde{H}_\theta$ corresponding to $W = 0$. Then we have:

$$\begin{aligned}\left\langle \begin{pmatrix} u \\ u' \end{pmatrix}', \tilde{H}_{\theta-\theta^0,0} \begin{pmatrix} u \\ u' \end{pmatrix} \right\rangle_{2\mathrm{D}} &= \int_0^1 \left\langle \begin{pmatrix} u' \\ u'' \end{pmatrix}, \begin{pmatrix} 0 \\ u'' + (\theta - \theta^0)^2 u \end{pmatrix} \right\rangle_{\mathbb{R}^{2\mathrm{D}}} \mathrm{d}x \\ &= \int_0^1 ||u''||^2 + |\theta - \theta^0|^2 \langle u'', u \rangle \mathrm{d}x \\ &= ||u''||_{\mathrm{D}}^2 - \frac{1}{2} |\theta - \theta^0|^2 \cdot ||u'||_{\mathrm{D}}^2.\end{aligned} \tag{3.21}$$

Putting this expression in (3.12), we get

$$||u''||_{\mathrm{D}}^2 - \frac{1}{2}|\theta-\theta^0|^2||u'||_{\mathrm{D}}^2 = \left\|\begin{pmatrix} u \\ u' \end{pmatrix}'\right\|_{2\mathrm{D}}^2 - |\theta-\theta^0|^2\left(\frac{1}{4}\left\|\begin{pmatrix} u \\ u' \end{pmatrix}\right\|_{2\mathrm{D}}^2 - \frac{1}{2}\left\langle\begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}'\right\rangle_{2\mathrm{D}} + ||u'||_{\mathrm{D}}^2\right). \tag{3.22}$$

Since the matrix $\mathcal{S}(\cdot,\theta^0)$ satisfies (3.4) and is equal to $I_{2\mathrm{D}}$ at $x = 0$ due to the initial conditions, we have

$$\begin{aligned} \forall x \in [0,1],\ \mathcal{S}(x,\theta^0) &= I_{2\mathrm{D}} \cdot \exp_{\mathrm{ord}}\left(\begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(x)-E_0(\theta^0)I_{\mathrm{D}} & 0 \end{pmatrix}\right) \\ &= I_{2\mathrm{D}} + \int_0^x \begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(t)-E_0(\theta^0)I_{\mathrm{D}} & 0 \end{pmatrix} \\ &\quad \cdot \exp_{\mathrm{ord}}\left(\begin{pmatrix} 0 & I_{\mathrm{D}} \\ W(t)-E_0(\theta^0)I_{\mathrm{D}} & 0 \end{pmatrix}\right) \mathrm{d}t \\ &:= I_{2\mathrm{D}} + R_{\theta^0}, \end{aligned} \tag{3.23}$$

where $\exp_{\mathrm{ord}}$ denote the time-ordered exponential (see [7]). We remark that the matrix $R_{\theta^0}$ is positive which allows us to prove the following inequality:

$$\begin{aligned} \left\|\begin{pmatrix} u \\ u' \end{pmatrix}'\right\|_{\theta^0}^2 &= \int_0^1 \left\|(I_{2\mathrm{D}}+R_{\theta^0})\begin{pmatrix} u' \\ u'' \end{pmatrix}\right\|^2 \mathrm{d}x = \int_0^1 \left\|\begin{pmatrix} u' \\ u'' \end{pmatrix}\right\|^2 \\ &\quad + 2\left\langle R_{\theta^0}\begin{pmatrix} u' \\ u'' \end{pmatrix}, \begin{pmatrix} u' \\ u'' \end{pmatrix}\right\rangle + \left\|R_{\theta^0}\begin{pmatrix} u' \\ u'' \end{pmatrix}\right\|^2 \mathrm{d}x \\ &= ||u''||_{\mathrm{D}}^2 + ||u'||_{\mathrm{D}}^2 + \int_0^1 2\left\langle R_{\theta^0}\begin{pmatrix} u' \\ u'' \end{pmatrix}, \begin{pmatrix} u' \\ u'' \end{pmatrix}\right\rangle + \left\|R_{\theta^0}\begin{pmatrix} u' \\ u'' \end{pmatrix}\right\|^2 \mathrm{d}x \geq ||u''||_{\mathrm{D}}^2. \end{aligned} \tag{3.24}$$

From (3.24) and using the upper bound (3.20) already obtained, one deduce that for $\theta$ close to $\theta^0$,

$$||u''||_{\mathrm{D}}^2 - \frac{1}{2}|\theta-\theta^0|^2||u'||_{\mathrm{D}}^2 \leq \left\|\begin{pmatrix} u \\ u' \end{pmatrix}'\right\|_{\theta^0}^2 - \left(E_0(\theta)-E_0(\theta^0)\right)\left(\frac{1}{4}\left\|\begin{pmatrix} u \\ u' \end{pmatrix}\right\|_{\theta^0}^2 - \frac{1}{2}\left\langle\begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}'\right\rangle_{\theta^0} + ||u'||_{\mathrm{D}}^2\right). \tag{3.25}$$

With (3.22) we get:

$$\begin{aligned} &\left\|\begin{pmatrix} u \\ u' \end{pmatrix}'\right\|_{2\mathrm{D}}^2 - |\theta-\theta^0|^2\left(\frac{1}{4}\left\|\begin{pmatrix} u \\ u' \end{pmatrix}\right\|_{2\mathrm{D}}^2 - \frac{1}{2}\left\langle\begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}'\right\rangle_{2\mathrm{D}} + ||u'||_{\mathrm{D}}^2\right) \\ &\leq \left\|\begin{pmatrix} u \\ u' \end{pmatrix}'\right\|_{\theta^0}^2 - (E_0(\theta)-E_0(\theta^0))\left(\frac{1}{4}\left\|\begin{pmatrix} u \\ u' \end{pmatrix}\right\|_{\theta^0}^2 - \frac{1}{2}\left\langle\begin{pmatrix} u \\ u' \end{pmatrix}, \begin{pmatrix} u \\ u' \end{pmatrix}'\right\rangle_{\theta^0} + ||u'||_{\mathrm{D}}^2\right). \end{aligned} \tag{3.26}$$

Let $a$ be a non-zero real number. By taking $u = {}^t(a, 0, \dots, 0)$ in (3.26), we get the lower bound: for $\theta$ close to $\theta^0$

$$E_0(\theta) - E_0(\theta^0) \geq \frac{\frac{1}{4}a^2}{\frac{1}{4}\left|\left|\begin{pmatrix} a \\ 0 \end{pmatrix}\right|\right|_{\theta^0}^2} |\theta - \theta^0|^2 = \frac{1}{||\psi_1||_{\mathrm{D}}^2} |\theta - \theta^0|^2. \tag{3.27}$$

Setting $C_1 = \frac{1}{||\psi_1||_{\mathrm{D}}^2}$ which is independent of $\theta$ (but depends on $\theta^0$), with (3.20) and (3.27) we have finally proven (3.3). This ends the proof. □

*Remark 7* It is not clear if the non-degeneracy of the bottom of the spectrum is still true for arbitrary $d \geq 1$ and $D \geq 1$. Although it was shown in any dimension $d \geq 1$ in the scalar-valued case (see [10]), the proof of Kirsch and Simon relies strongly on the existence of a positive ground state which can thus be inverted. In the matrix-valued case and in dimension 1, the fact that we can deal with a first order differential system allows us to define a fundamental matrix which is invertible and which plays the role of the positive ground state of [10]. In dimension $d \geq 2$, this approach is no longer possible.

## 4 Wannier Basis

We recall concepts used in [12,16]. Let $\mathcal{E} \subset L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ be a closed subspace, invariant by $\mathbb{Z}^d$-translations, *i.e.*, for every $n \in \mathbb{Z}^d$, $\Pi^{\mathcal{E}} = \tau_n^* \Pi^{\mathcal{E}} \tau_n$, where $\Pi^{\mathcal{E}}$ is the orthogonal projection on $\mathcal{E}$.

As $\Pi^{\mathcal{E}}$ is $\mathbb{Z}^d$-periodic, it admits a Floquet decomposition similar to the one of $H$ and, using the orthogonality, one gets:

$$\Pi^{\mathcal{E}} = \int_{\mathbb{T}^*}^{\oplus} \Pi_\theta^{\mathcal{E}} \, \mathrm{d}\theta,$$

where $\Pi_\theta^{\mathcal{E}}$ is the operator $\Pi^{\mathcal{E}}$ acting on $\mathcal{H}_\theta$. The operator $\Pi_\theta^{\mathcal{E}}$ is therefore an orthogonal projection acting on $L^2(C_0) \otimes \mathbb{C}^{\mathrm{D}}$. As for $(H_\theta)_{\theta \in \mathbb{T}^*}$, the family $(\Pi_\theta^{\mathcal{E}})_{\theta \in \mathbb{T}^*}$ is continuous in $\theta$ and thus is of constant rank. If we fix $\theta \in \mathbb{T}^*$, we can find an orthonormal system $(w_{m,0})_{m \in M}$, with $M \subset \mathbb{N}$ a set of indices independent of $\theta$, that spans the range of $\Pi_\theta^{\mathcal{E}}$. Taking the image by $U^*$ of this orthonormal system, one gets an orthonormal system $(\widetilde{w}_{m,0})_{m \in M}$. If we set, for $n \in \mathbb{Z}^d$, $\widetilde{w}_{m,n} = \tau_n(\widetilde{w}_{m,0})$, then $(\widetilde{w}_{m,n})_{(m,n) \in M \times \mathbb{Z}^d}$ is an orthonormal basis of $\mathcal{E}$. Such a system is called a *Wannier basis* of $\mathcal{E}$. The vectors $(\widetilde{w}_{m,0})_{m \in M}$ are called the *Wannier generators* of $\mathcal{E}$.

Let $\mathcal{E} \subset L^2(\mathbb{R}^d) \otimes \mathbb{C}^{\mathrm{D}}$ be a space which is invariant by $\mathbb{Z}^d$-translations. The closed subspace $\mathcal{E}$ is said to be *of finite energy for* $H$ if $\Pi^{\mathcal{E}} H \Pi^{\mathcal{E}}$ is a bounded operator. In this case, $\mathcal{E}$ admits a finite set of Wannier generators. We now assume that $\mathcal{E}$ is of finite energy for $H$.

Let $J_0$ be the set of indices of the Floquet eigenvalues of $H$ which take the value 0 for some values of $\theta \in \mathbb{T}^*$. We identify $J_0$ to $\{1, \dots, n_0\}$. Let $Z$ be the set of $\theta \in \mathbb{T}^*$ for which there exists $j \in J_0$ such that, $E_j(\theta) = 0$. When $\theta^0$ is a non-degenerate minimum of $E_j$, $Z$ is a set of isolated points (see [12]). It occurs when the density of states $\mathfrak{n}$ has a non-degenerate behaviour at 0 (see [11]). For $j \in J_0$, we define $Z_j = \{\theta \in \mathbb{T}^* ;\ E_j(\theta) = 0\}$. The sequence $(Z_j)_{j \in J_0}$ is decreasing for the inclusion and $Z_1 = Z$. For $\theta^0 \in Z$, $M_{\theta^0} \subset \mathbb{N}$ is the set of indices such that $E_j(\theta^0) = 0$.

We will denote by $w_j(\cdot,\theta)$ a Floquet eigenvector associated with the Floquet eigenvalue $E_j(\theta)$ of $H$. For $(\theta,\theta')\in(\mathbb{T}^*)^2$, we define $T_{\theta\to\theta'}:\mathcal{H}_\theta\to\mathcal{H}_{\theta'}$ by:

$$\forall v\in\mathcal{H}_\theta,\ \forall x\in\mathbb{R}^d,\ (T_{\theta\to\theta'}v)(x)=\mathrm{e}^{\mathrm{i}x\cdot(\theta'-\theta)}v(x).$$

**Lemma 8** *There exists $(v_j(\cdot,\theta))_{j\in J_0}$, a family of functions on $\mathcal{H}_\theta$, such that:*

*(1) for $\theta^0\in Z$ and $j\in M_{\theta^0}$, there exists $V_{\theta^0}$ a neighborhood of $\theta^0$ in $\mathbb{T}^*$ such that the map $\theta\in V_{\theta^0}\mapsto v_j(\cdot,\theta)\in\mathcal{H}_\theta$ is real analytic (i.e. $\theta\mapsto T_{\theta\to\theta^0}v_j(\cdot,\theta)$ is analytic as a function from $V_{\theta^0}$ to $\mathcal{H}_{\theta^0}$) and, for $\theta\in V_{\theta^0}$, $\mathrm{span}\langle(v_j(\cdot,\theta))_{j\in M_{\theta^0}}\rangle=\mathrm{span}\langle(w_j(\cdot,\theta))_{j\in M_{\theta^0}}\rangle$.*

*(2) For $\theta\in\mathbb{T}^*$, the system $(v_j(\cdot,\theta))_{j\in M_{\theta^0}}$ is orthonormal in $\mathcal{H}_\theta$ and $\mathrm{span}\langle(w_j(\cdot,\theta))_{j\in J_0}\rangle=\mathrm{span}\langle(v_j(\cdot,\theta))_{j\in J_0}\rangle$.*

*Proof* We refer to [12,16], Lemma 3.1. □

In the next section, we will use this notion of Wannier basis and the notations we have just introduce to reduce our problem on estimating $N(E)-N(0^+)$ to a discrete problem.

## 5 Reduction of the Problem

The goal of this section is to give an estimate of $N(E)-N(0^+)$ for an energy $E$ close to 0. This will be accomplished by means of the IDS of certain reference operators, which are discrete operators. In this section we will use the notations introduced in Sect. 2.

### 5.1 Reduction to a Discrete Problem

The reduction procedure consists into decomposing the operator $H_\omega$ according to various translation-invariant subspaces. The random operators thus obtained are what we consider as reference operators. They will be used to prove the upper bound on the IDS.

We denote by $\Pi_0(\theta)$ the orthogonal projection in $\mathcal{H}_\theta$ on the vector space generated by $(w_j(\cdot,\theta))_{j\in J_0}$. One defines

$$\Pi_0=U^{-1}\left(\int_{\mathbb{T}^*}\Pi_0(\theta)\mathrm{d}\theta\right)U\quad:\quad L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}\to L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}.\tag{5.1}$$

$\Pi_0$ is an orthogonal projection on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$ and, for every $n\in\mathbb{Z}^d$, we have $\tau_n^*\Pi_0\tau_n=\Pi_0$. Thus, $\Pi_0$ is $\mathbb{Z}^d$-periodic. We set $\mathcal{E}_0=\Pi_0(L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}})$. This space is translation-invariant because of the $\mathbb{Z}^d$-periodicity of $\Pi_0$. Moreover $\mathcal{E}_0$ is of finite energies for $H$ as defined in (1.6). The main result justifying this reduction procedure is the following theorem which compares $E\mapsto N(E)$, the IDS of $H_\omega$, to $E\mapsto N_{\mathcal{E}_0}(E)$, the IDS of the discretize operator $H_\omega^0=\Pi_0H_\omega\Pi_0$.

**Theorem 9** *Let $H_\omega$ be defined by (1.5) with the assumptions $(H1)$, $(H2)$, $(H3)$ and $(H4)$. There exist $\varepsilon>0$ and $C>1$ such that, for $0\le E\le\varepsilon$ we have*

$$0\le N(E)-N(0^+)\le N_{\mathcal{E}_0}(C\cdot E),\tag{5.2}$$

*where $N_{\mathcal{E}_0}$ is the IDS of the discretized operator $H_\omega^0=\Pi_0H_\omega\Pi_0$.*

*Proof* See Theorem 4.1 in [12]. □

### 5.2 Periodic Approximations

In order to get bounds on the density of states of $\Pi_0 H_\omega \Pi_0$, we will now define periodic approximations of the operator $H_\omega$. For these approximations, we will be able to control the density of state near 0 by comparing it to some reduced operators. Then by taking a limit on the density of state of the reduced operators, we can get bounds on the density of states of $\Pi_0 H_\omega \Pi_0$ and thus on the density of states of $H_\omega$ itself by using Theorem 9. Let $k$ an integer larger than 1. We define the following periodic operator

$$H_{\omega,k} = -\Delta_d \otimes I_{\mathrm{D}} + W + \sum_{n \in C_k \cap \mathbb{Z}^d} \sum_{\beta \in (2k+1)\mathbb{Z}} \times \begin{pmatrix} \omega_1^{(n)} V_1(x-(n+\beta)) & & 0 \\ & \ddots & \\ 0 & & \omega_{\mathrm{D}}^{(n)} V_{\mathrm{D}}(x-(n+\beta)) \end{pmatrix}. \tag{5.3}$$

The operator $H_{\omega,k}$ is $(2k+1)\mathbb{Z}^d$-periodic and essentially selfadjoint. It is an $H$-bound perturbation of $H$ with relative bound zero. Because of the $(2k+1)\mathbb{Z}^d$-periodicity, we introduce the torus $\mathbb{T}_k^* = \mathbb{R}^d/(2(2k+1)\pi\mathbb{Z}^d)$. We also define $N_{\omega,k}$, the IDS of $H_{\omega,k}$ by

$$N_{\omega,k}(E) = \frac{1}{(2\pi)^d} \sum_{j \in \mathbb{N}} \int_{\{\theta \in \mathbb{T}_k^*,\ E_{\omega,k,j}(\theta) \le E\}} \mathrm{d}\theta. \tag{5.4}$$

where $E_{\omega,k,j}$ is the $j$-th Floquet eigenvalue of the periodic operator $H_{\omega,k}$. Let $dN_{\omega,k}$ be the derivative of $N_{\omega,k}$, in the distribution sense. As $E \mapsto N_{\omega,k}(E)$ is an increasing function, $dN_{\omega,k}$ is a positive measure, it is the density of states of $H_{\omega,k}$. Then, by [12,22], for every $\varphi \in C_0^\infty(\mathbb{R})$, the distribution $dN_{\omega,k}$ verifies

$$\langle \varphi, dN_{\omega,k} \rangle = \frac{1}{(2\pi)^d} \int_{\theta \in \mathbb{T}_k^*} \mathrm{tr}_{\mathcal{H}_\theta} \big(\varphi(H_{\omega,k,\theta})\big) \mathrm{d}\theta = \frac{1}{\mathrm{vol}(C_k)} \mathrm{tr}\Big(\mathbf{1}_{C_k} \varphi(H_{\omega,k}) \mathbf{1}_{C_k}\Big), \tag{5.5}$$

where $\mathrm{tr}(A)$ is the trace of a trace-class operator $A$. We index this trace by $\mathcal{H}_\theta$ if the trace is taken in $\mathcal{H}_\theta$ and here, the operator $\mathbf{1}_{C_k}\varphi(H_{\omega,k})$ is a trace-class operator. The proof of (5.5) is given in [12, Proposition 5.1].

We want to take a limit on the density of states $dN_{\omega,k}$ of the periodic approximations in order to recover properties of the density of states of $H_\omega$ from properties of $dN_{\omega,k}$. The following theorem ensure that it is possible.

**Theorem 10** (1) *For any $\varphi \in C_0^\infty(\mathbb{R})$ and for almost every $\omega \in \Omega$, we have*

$$\lim_{k \to \infty} \langle \varphi, dN_{\omega,k} \rangle = \langle \varphi, dN \rangle.$$

(2) *For any $E \in \mathbb{R}$ a continuity point for $N$, we have* $\lim_{k \to \infty} \mathbb{E}(N_{\omega,k}(E)) = N(E)$.

*Proof* The result of Theorem 10 is close to that of Theorem 5.1 of [12]. The proof is also similar and is based on functional analysis. □

## 6 Proof of Theorem 5

We will proceed in two steps. First, we will prove a lower bound and then an upper bound.

### 6.1 Lower Bound

In this subsection we prove

**Theorem 11** *Let $H_\omega$, be the operator defined by (1.5) with the assumptions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$. *We have*

$$\liminf_{E\longrightarrow 0^+} \frac{\log\left|\log\left(N(E)-N(0^+)\right)\right|}{\log E} \geq -\frac{d}{2}. \tag{6.1}$$

*Proof* As 0 is the bottom of the spectrum, for $\varepsilon > 0$ we have $N(\varepsilon)-N(0) = N(\varepsilon)-N(-\varepsilon)$. To prove Theorem 11, we will lower bound $N(\varepsilon) - N(-\varepsilon)$. Then, for $L$ large, we will show that $H_{\omega,C_L}$ (we recall that $H_{\omega,C_L}$ is $H_\omega$ restricted to $C_L$ with Dirichlet boundary conditions) has a large number of eigenvalues in $[-\varepsilon, \varepsilon]$ with a large probability. To do this we will construct a family of approximate eigenvectors associated to approximate eigenvalues of $H_{\omega,C_L}$ in $[-\varepsilon, \varepsilon]$. These functions will be constructed from an eigenvector of $-\Delta_d \otimes I_{\mathrm{D}} + W$ associated to 0. Locating this eigenvector in $\theta$ and imposing to $\omega_1^{(n)}$ to be small for $n$ in some well chosen box, one obtains an approximate eigenfunction of $H_{\omega,C_L}$. Locating the eigenfunction in $x$ in several disjointed places, we get several eigenfunctions two by two orthogonal.

In order to simplify the notations, we assume in what follows that $\theta^0 = 0$ is a point where $E_0(\theta)$ reaches 0. From the same arguments as in [16] and using hypothesis $(H4)$, there exists $C > 0$ such that, for $\tilde{f}(\cdot,\theta) := (f_1(\cdot,\theta), \cdots, f_D(\cdot,\theta)) = v_1(\ldots,\theta)$ in $L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$, ($v_1$ is the vector constructed in Lemma 8) one has

$$||(-\Delta_d \otimes I_{\mathrm{D}} + W)\tilde{f}(\cdot,\theta)||_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} \leq C|\theta|^2. \tag{6.2}$$

This is due to the fact that locally near $\theta = 0$, we can reduce the study of $-\Delta_d \otimes I_D + W$ which is analytic in $\theta$ and is equal to 0 when $\theta = 0$ and to the use of (1.8).

We assume, without loss of generality, that $f_1 \neq 0$ and we set

$$f(\cdot,\theta) = \frac{f_1(\cdot,\theta)}{|\theta_1|}(1, 0, \ldots, 0). \tag{6.3}$$

Let $0 < \xi < 1$ be a small constant. Let $\chi \in C_0^\infty(\mathbb{R})$ being positive, supported in $[\frac{\xi}{2}, \xi]$ and such that $\displaystyle\int_{[\frac{\xi}{2},\xi]} \chi(t)^2 dt = 2$.

For $\varepsilon > 0$, we define

$$\mathcal{W}_\varepsilon(\theta) = \varepsilon^{-d/4}\prod_{j=1}^{d}\chi(\varepsilon^{-\frac{1}{2}}\theta_j) \in L^2(\mathbb{T}^*) \quad\text{and}\quad \mathcal{W}_\varepsilon^f(\cdot,\theta) = \mathcal{W}_\varepsilon(\theta)\cdot f(\cdot,\theta) \in L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}. \tag{6.4}$$

Now let us estimate $||(-\Delta_d \otimes I_{\mathrm{D}} + W)\mathcal{W}_\varepsilon^f||^2_{\mathcal{H}}$. We have

$$\begin{aligned}&||(-\Delta_d \otimes I_{\mathrm{D}} + W)\mathcal{W}_\varepsilon^f||^2_{\mathcal{H}}\\ &\quad= \frac{1}{\mathrm{vol}(\mathbb{T}^*)}\int_{\mathbb{T}^*} ||(-\Delta_d \otimes I_{\mathrm{D}} + W)(\theta) f(\cdot,\theta)||^2_{L^2(C_0)\otimes\mathbb{C}^D}|\mathcal{W}_\varepsilon(\theta)|^2 \mathrm{d}\theta.\end{aligned}$$

So using (6.2) and (6.4), we get

$$||(-\Delta_d \otimes I_{\mathrm{D}} + W)\mathcal{W}_\varepsilon^f||_{\mathcal{H}}^2 \leq C^2 \int_{\mathbb{T}^*} |\theta|^4 |\mathcal{W}_\varepsilon(\theta)|^2 \mathrm{d}\theta \leq C^2 \varepsilon^2 \int_{[\frac{\xi}{2},\xi]^d} |\theta|^4 \prod_{j=1}^{d} \chi^2(\theta_j) \mathrm{d}\theta \leq \frac{\varepsilon^2}{8}, \tag{6.5}$$

if $\xi$ is small enough. For $\beta \in \mathbb{Z}^d$, we define

$$\mathcal{W}_{\varepsilon,\beta}^f(\cdot,\theta) = e^{-i\beta\cdot\theta}\mathcal{W}_\varepsilon^f(\cdot,\theta) \quad \text{and} \quad \mathcal{W}_{\alpha,\varepsilon,\beta,\zeta}^f(\cdot,\theta) = e^{-i\beta\cdot\theta}(\Pi_{\Lambda_\alpha(\zeta)}\mathcal{W}_\varepsilon^f)(\cdot,\theta),$$

where $\Lambda_\alpha(\zeta)$ is the cube defined by

$$\Lambda_\alpha(\zeta) = \left\{ n \in \mathbb{Z}^d \;\middle|\; \text{for } 1 \leq j \leq d,\ |n_j| \leq \zeta^{-(\frac{1}{2}+\alpha)} \right\}$$

and $\Pi_{\Lambda_\alpha(\zeta)}$ is the orthogonal projection on $\Lambda_\alpha(\zeta)$, i.e it is the operator of orthogonal projection on $L^2(\mathbb{T}^*)$ on the space spanned by vectors $\theta \to e^{i\gamma\theta}$, $\gamma \in \Lambda_\alpha(\zeta)$.

We set

$$\mathcal{U}_{\varepsilon,\beta}^f(x) = \int_{\mathbb{T}^*} \mathcal{W}_{\varepsilon,\beta}^f(x,\theta)\mathrm{d}\theta \quad \text{and} \quad \mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f(x) = \int_{\mathbb{T}^*} \mathcal{W}_{\alpha,\varepsilon,\beta,\zeta}^f(x,\theta)\mathrm{d}\theta.$$

For $L$ large and $\beta$ and $(\omega_1^{(n)})_{n\in\mathbb{Z}^d}$ well chosen, $\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f$ will be an approximate eigenfunction of $H_{\omega,C_L}$ associated to an approximate eigenvalue in the interval $[-\varepsilon,\varepsilon]$.

We notice that $\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f \in L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}$ and $\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^{f_1} \in L^2(\mathbb{R}^d)$. As in [16] one gets that

$$||\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f||_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}} \geq ||\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^{f_1}||_{L^2(\mathbb{R}^d)} > C > 0.$$

Now we have to look to the conditions under which we have

$$\left|\left|\Big(-\Delta_d \otimes I_{\mathrm{D}} + W\Big)\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f\right|\right|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}}^2 \leq \varepsilon^2. \tag{6.6}$$

□

Note that

$$\begin{aligned}\left|\left|H_{\omega,C_L}\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f\right|\right|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}}^2 &\leq \left|\left|H_\omega\cdot\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f\right|\right|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}}^2 \\ &\leq 2\left|\left|\Big(-\Delta_d \otimes I_{\mathrm{D}} + W\Big)\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f\right|\right|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}}^2 \\ &\quad + 2\left|\left|V_\omega\mathcal{U}_{\alpha,\varepsilon,\beta,\zeta}^f\right|\right|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}}^2.\end{aligned} \tag{6.7}$$

Eq. (6.6) give the bound on the first member of (6.7). It just remains to control the second term. To do so, one needs the following lemma

**Lemma 12** *Let $\zeta = \varepsilon$. There exists $K > 0$, such that*

$$\left|\left|V_\omega\cdot\mathcal{U}_{\alpha,\varepsilon,\beta,\varepsilon}^f\right|\right|_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^{\mathrm{D}}}^2 \leq \varepsilon^4 + K\cdot\left(\sup_{n\in\beta+2\Lambda_\alpha(\varepsilon)} \omega_1^{(n)}\right)^2. \tag{6.8}$$

Before proving this lemma let us use it to finish the proof of Theorem 11.

Taking (6.6) and (6.7) into account, we get that there exists $K > 0$ such that

$$\left|\left|H_\omega\cdot\mathcal{U}_{\alpha,\varepsilon,\beta,\varepsilon}^f\right|\right|^2 \leq 4\varepsilon^2 + K\left(\sup_{n\in\beta+2\Lambda_\alpha(\varepsilon)} \omega_1^{(n)}\right)^2. \tag{6.9}$$

Now, for $L$ large, we may divide $C_L$ into $L(\varepsilon)$ disjoints cubes of size $2\Lambda_\alpha(\varepsilon)$. For $\alpha < \frac{1}{2}$, there exists $C > 0$ such that $L(\varepsilon)$ satisfies

$$L(\varepsilon) \simeq \frac{(2L)^d}{\varepsilon^{-d(\frac{1}{2}+\alpha)}} \geq \frac{(L\varepsilon)^d}{C}. \tag{6.10}$$

We can find $\beta_1, \dots, \beta_{L(\varepsilon)}$ in $\mathbb{Z}^d$ such that :

$$\bigcup_{j=1}^{L(\varepsilon)} (\beta_j + 2\Lambda_\alpha(\varepsilon)) \subset C_L \quad \text{and, for } j \neq j', \quad (\beta_j + 2\Lambda_\alpha(\varepsilon)) \cap (\beta_{j'} + 2\Lambda_\alpha(\varepsilon)) = \emptyset.$$

In particular, for $j \neq j'$, $\mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon}$ and $\mathcal{U}^f_{\alpha,\varepsilon,\beta_{j'},\varepsilon}$ are orthogonal. Then,

$$\begin{aligned} &\mathbb{E}\Big(\#\Big\{\text{eigenvalues of } \Pi_{C_L} H_\omega \Pi_{C_L} \text{ in } [-\varepsilon, \varepsilon]\Big\}\Big) \\ &\geq \mathbb{E}\Big(\#\Big\{j \in \{1, \dots, L(\varepsilon)\} \;\Big|\; ||H_\omega \mathcal{U}^f_{\alpha,\varepsilon,\beta_j,\varepsilon}||_{L^2(\mathbb{R}^d)\otimes\mathbb{C}^D} \leq \varepsilon\Big\}\Big) \\ &\geq \mathbb{E}\left(\sum_{j=1}^{L(\varepsilon)} B_j(\omega)\right), \end{aligned} \tag{6.11}$$

where

$$B_j(\omega) = \begin{cases} 1 & \text{if } K \cdot \left(\sup_{n\in\beta_j+2\Lambda_\alpha(\varepsilon)} \omega_1^{(n)}\right)^2 \leq \frac{\varepsilon^2}{2}. \\ 0 & \text{if not.} \end{cases}$$

The $(B_j)_{1\leq j\leq L(\varepsilon)}$ are *i.i.d.* Bernoulli random variables. So Eqs. (6.11) and (6.10) imply that there exists $C > 0$ such that one has

$$\begin{aligned} &\frac{1}{(2L+1)^d}\mathbb{E}\Big(\#\Big\{ \text{eigenvalues of } \Pi_{C_L} H_\omega \Pi_{C_L} \text{ in } [-\varepsilon, \varepsilon]\Big\}\Big) \\ &\geq \frac{L(\varepsilon)}{(2L+1)^d}\mathsf{P}(B_1 = 1) \geq \frac{1}{C}\varepsilon^d \mathsf{P}(B_1 = 1). \end{aligned}$$

Hence, taking the limit $L \to \infty$, we get that, for $\varepsilon > 0$ small,

$$N(\varepsilon) - N(-\varepsilon) \geq \frac{1}{C}\varepsilon^d \mathsf{P}(B_1 = 1). \tag{6.12}$$

It just remains to estimate $\mathsf{P}(B_1 = 1)$. If, for $1 \leq j \leq L(\varepsilon)$ and $n \in \beta_j + 2\Lambda_\alpha(\varepsilon)$, one has $\omega_1^{(n)} \leq \frac{\varepsilon \cdot \varepsilon}{2K}$, then for $\varepsilon$ rather small

$$K\left(\sup_{n\in\beta_j+2\Lambda_\alpha(\varepsilon)} \omega_1^{(n)}\right)^2 \leq \frac{\varepsilon^2}{2}.$$

As the random variables are *i.i.d.*, one has the estimate

$$\mathsf{P}(B_1 = 1) \geq \tilde{\mathsf{P}}_1\left(\omega_1^{(0)} \leq \frac{\varepsilon}{2K}\right)^{\#(2\Lambda_\alpha(\varepsilon))}.$$

Hence, taking the double logarithm of (6.12), using assumption (H3) and the fact that $\#(2\Lambda_\alpha(\varepsilon)) \simeq \varepsilon^{-(\frac{d}{2}+d\cdot\alpha)}$, we get that

$$\lim_{\varepsilon\to 0^+} \frac{\log\left|\log\left(N(\varepsilon) - N(0)\right)\right|}{\log \varepsilon} \geq -\frac{d}{2} - d\alpha. \tag{6.13}$$

The Eq. (6.13) is true for any $\alpha > 0$, by letting $\alpha$ tend to 0, we end the proof of Theorem 11.

It remains to prove Lemma 12 to finish this section on the lower bound.

*The proof of Lemma 12* We have :

$$\left\| V_\omega \cdot \mathcal{U}^{f}_{\alpha,\varepsilon,\beta,\varepsilon} \right\|^2_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} \lesssim \left\| \sum_{n\in\mathbb{Z}^d} \omega_1^{(n)} V_1(x-n)\mathcal{U}^{f_1}_{\alpha,\varepsilon,\beta,\varepsilon} \right\|^2_{L^2(C_0)}. \tag{6.14}$$

Then,

$$\left\| \sum_{n\in\mathbb{Z}^d} \omega_1^{(n)} V_1(x-n)\mathcal{U}^{f_1}_{\alpha,\varepsilon,\beta,\varepsilon} \right\|^2 \lesssim \varepsilon^5 + \int_{\mathbb{R}^d} \left( \sum_{n\in\mathbb{Z}^d} \omega_1^{(n)} V_1(x-n) \right)^2 \left| \mathcal{U}^{f_1}_{\varepsilon,\beta}(x) \right|^2 \mathrm{d}x. \tag{6.15}$$

Here we used the fact that $\mathcal{U}^{f_1}_{\alpha,\varepsilon,\beta,\varepsilon}$ and $\mathcal{U}^{f_1}_{\varepsilon,\beta}$ areclose to each other.

We set

$$S_{\beta,\varepsilon} \leq K \sum_{\eta\in\mathbb{Z}^d} \left( \sum_{n\in\eta+\Lambda_\alpha(\varepsilon)} \omega_1^{(n)} \right)^2 \times \int_{C_0} \left| \mathcal{U}^{f_1}_{\varepsilon,\eta-\beta}(x) \right|^2 \mathrm{d}x. \tag{6.16}$$

So for our choice of $f_1$ and using the fact that $V_1$ is supported in $C_0$, we deal with a simple quantity to control using the non-stationary phase and which was already estimated in [12,16].

To finish the proof of Theorem 5, it remains to prove the upper bound. □

### 6.2 Upper Bound

We start this section by recalling that, as we deal with the bottom of the spectrum, we have non-degeneracy of the first Floquet eigenvalue at the bottom of the spectrum as stated in $(H4)$. Using this, we prove the following theorem:

**Theorem 13** *Let $H_\omega$ be the operator defined by* (1.5) *with the assumptions* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$. *Then*

$$\limsup_{E\longrightarrow 0^+} \frac{\log|\log(N(E)-N(0^+))|}{\log E} \leq -\frac{d}{2}.$$

*Proof* To prove the upper bound, it is enough to prove the same upper bound on $N_{\mathcal{E}_0}$ (as defined in Theorem 9). To do this, we show that $N_{\mathcal{E}_0}$ (and so $N$) may be compared to the IDS of some well chosen discrete Anderson model whose behaviour of its IDS is already known.

We begin by isolating the contributions from the various points for which $E_j(\theta)$ take the value 0. We recall that the band at 0 is generated by $(E_j(\theta))_{1\leq j\leq n_0}$. For $1 \leq j \leq n_0$, $Z_j = \{\theta \in \mathbb{T}^*;\, E_j(\theta) = 0\}$. The sequence $(Z_j)_{1\leq j\leq n_0}$ is decreasing $(Z_{j+1} \subset Z_j)$. Let $\theta^0 \in Z$. We set $j(\theta^0) = \sup M_{\theta^0}$ with $M_{\theta^0} = \{j\ ;\ 1 \leq j \leq n_0,\, E_j(\theta^0) = 0\}$. We replace the Floquet eigenvectors $(w_j(\cdot,\theta))_{1\leq j\leq j(\theta^0)}$ associated to $(E_j(\theta))_{1\leq j\leq j(\theta^0)}$ by the vectors $(v_j(\cdot,\theta))_{1\leq j\leq j(\theta^0)}$ constructed in Lemma 8. They are analytic in a neighborhood $V_{\theta^0}$ of $\theta^0$. Let $\theta$ be close to $\theta^0$. The operator $H^0(\theta) = \Pi_0 H(\theta)\Pi_0$ is unitarily equivalent to the multiplication operator by a function on $L^2(\mathbb{T}^*)$ with values in $\mathcal{M}_{n_0}(\mathbb{C})$. This matrix-valued function takes the following block diagonal form :

$$\begin{pmatrix} B_{j(\theta^0)}(\theta) & 0 & 0 & \dots & 0 \\ 0 & E_{j(\theta^0)+1}(\theta) & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & E_{n_0}(\theta) \end{pmatrix},$$

where the matrix $B_{j(\theta^0)}(\theta)$ is of size $j(\theta^0) \times j(\theta^0)$ and is given by

$$\begin{pmatrix} \langle v_1(\cdot,\theta), H(\theta)v_1(\cdot,\theta)\rangle_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} & \dots & \langle v_1(\cdot,\theta), H(\theta)v_{j(\theta^0)}(\cdot,\theta)\rangle_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} \\ \vdots & \ddots & \vdots \\ \langle v_{j(\theta^0)}(\cdot,\theta), H(\theta)v_1(\cdot,\theta)\rangle_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} & \dots & \langle v_{j(\theta^0)}(\cdot,\theta), H(\theta)v_{j(\theta^0)}(\cdot,\theta)\rangle_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} \end{pmatrix}.$$

The matrix $B_{j(\theta^0)}(\theta)$ has $(E_j(\theta))_{1\leq j\leq j(\theta^0)}$ for eigenvalues. The operator $V_\omega^0 = \Pi_0 V_\omega \Pi_0$ is unitarily equivalent to the multiplication operator by the matrix with entries $(\langle V_\omega v_i, v_j\rangle)_{1\leq i,j\leq n_0}$.

For $u \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}}$, $\Pi_0 u = \sum_{i=1}^{n_0}\langle u, v_i\rangle_{L^2(C_0)\otimes\mathbb{C}^{\mathrm{D}}} v_i$. For $\theta^0 \in Z$, we set

$$\varpi_{\theta^0}(\theta) = \sum_{j=1}^{d}\Big(1-\cos(\theta_j-\theta_j^0)\Big).$$

We recall that the eigenvalues $(E_j(\theta))_{1\leq j\leq j(\theta^0)}$ are non-degenerate at 0. So there exists $\widetilde{V}_{\theta^0}$ (an open neighborhood of $\theta^0$) and $C > 1$ such that, for $\theta \in \widetilde{V}_{\theta^0}$, we have, for $1 \leq j \leq j(\theta^0)$, $CE_j(\theta) \geq \varpi_{\theta^0}(\theta)$ and, for $j \geq j(\theta^0)$, $CE_j(\theta) \geq 2d$. We remark that the neighborhood $\widetilde{V}_{\theta^0}$ can be chosen such that $V_{\theta^0} \subset \widetilde{V}_{\theta^0}$, where $V_{\theta^0}$ was defined in Lemma 8.

Let $H^b_{\theta^0}(\theta)$ be the $n_0 \times n_0$ diagonal matrix with identical diagonal entries equal to $\varpi_{\theta^0}$. For $\theta \in \widetilde{V}_{\theta^0}$, we have

$$H^b_{\theta^0}(\theta) \leq C \times H^0(\theta). \tag{6.17}$$

Finally, we note that $(\widetilde{V}_{\theta^0})_{\theta^0\in Z}$ can be chosen so that they cover $\mathbb{T}^*$, (i.e. $\cup_{\theta^0\in Z}\widetilde{V}_{\theta^0} = \mathbb{T}^*$) and such that each one of them contains only one point of $Z$ (i.e. for $\theta \in Z, \theta' \in Z$ such that $\theta \neq \theta'$, we have $\theta' \notin \overline{\widetilde{V}_\theta}$). We order the points in $Z = \{\theta^k;\, 1 \leq k \leq m_0\}$, where $m_0 = \#Z$. Let $(\chi_k)_{1\leq k\leq m_0}$ be functions in $C^\infty(\mathbb{T}^*)$ which form a partition of the unity on $\mathbb{T}^*$ such that, for $1 \leq k \leq m_0$, supp $(\chi_k) \subset \widetilde{V}_{\theta^k}$, $0 \leq \chi_k \leq 1$ and $\chi_k \equiv 1$ in a neighborhood of $\theta^k$.

So there exists $C > 1$ such that, for any $\theta \in \mathbb{T}^*$, we have,

$$\frac{1}{m_0} \leq \sum_{k=1}^{m_0}\chi_k^2 \leq 1 \quad \text{and} \quad \sum_{k=1}^{m_0} H^b_{\theta^k}(\theta)\chi_k^2 \leq CH^0(\theta). \tag{6.18}$$

For $t \in (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0}$, we set $t = (t_{j,k})_{1\leq j\leq n_0;1\leq k\leq m_0}$. We consider $t$ as a system of $m_0$ columns denoted by $(t_{\cdot,k})_{1\leq k\leq m_0}$. Each column belongs to $(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}$. We endow $(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0}$ with the scalar product generating the following Euclidean norm:

$$\left\|t\right\|^2_{(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0}} = \sum_{k=1}^{m_0}\left\|t_{\cdot,k}\right\|^2_{(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}} = \sum_{1\leq j\leq n_0, 1\leq k\leq m_0}\left\|t_{j,k}\right\|^2_{L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}}}.$$

We define the mapping $S : (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0} \longrightarrow (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0}$ by

$$S(t) = (\chi_k t)_{1\leq k\leq m_0} = (\chi_k t_j)_{1\leq j\leq n_0,\ 1\leq k\leq m_0}, \quad \text{if } t = (t_j)_{1\leq j\leq n_0} \in (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}.$$

Here, for any $1 \leq j \leq n_0$, $t_j = (t_{ij})_{1\leq i\leq D} \in L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}}$.

The adjoint of $S$, $S^* : (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0} \longrightarrow (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}$ is defined by

$$S^*(t) = \left(\sum_{1\leq k\leq m_0}\chi_k t_{j,k}\right)_{1\leq j\leq n_0} \qquad \text{for } t = (t_{j,k})_{1\leq j\leq n_0;1\leq k\leq m_0} \in (L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0}.$$

Here, for any $1 \leq j \leq n_0$ and any $1 \leq k \leq m_0$, we have $t_{j,k} = (t_{i,j,k})_{1\leq i\leq D}$. According to Eq. (6.18) we have $\frac{1}{m_0} I \leq S^* \circ S \leq I$, (here $I$ is the identity in $(L^2(\mathbb{T}^*) \otimes \mathbb{C}^{\mathrm{D}}) \otimes \mathbb{C}^{n_0}$), thus $S$ is one to one. Using the boundedness assumption on the $V_i$ and on the support of the $\omega_i^{(n)}$, one shows the following lemma:

**Lemma 14** *There exists $C > 0$ such that, for $t \in (L^2(\mathbb{T}^*) \otimes \mathbb{C}^{\mathrm{D}}) \otimes \mathbb{C}^{n_0}$, we have*

$$\langle H_\omega^a S(t), S(t)\rangle_{(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}\otimes\mathbb{C}^{m_0}} \leq C\langle H_\omega^0 t, t\rangle_{(L^2(\mathbb{T}^*)\otimes\mathbb{C}^{\mathrm{D}})\otimes\mathbb{C}^{n_0}},$$

*where the operator $H_\omega^a$ acting on $(L^2(\mathbb{T}^*) \otimes \mathbb{C}^{\mathrm{D}}) \otimes \mathbb{C}^{n_0} \otimes \mathbb{C}^{m_0}$ is defined by*

$$H_\omega^a t = \left(H_k^a t_{i,j,k} + V_{\omega,i}^a t_{i,j,k}\right)_{1\leq i\leq D,\ 1\leq j\leq n_0,\ 1\leq k\leq m_0}.$$

*Here, $H_k^a$ is the multiplication by $\varpi_{\theta^k}$ acting as a multiplication operator on $L^2(\mathbb{T}^*)$, $V_{\omega,i}^a = \sum_{n\in\mathbb{Z}^d} \omega_i^{(n)} \Pi_n$, where $\Pi_n$ is the orthogonal projection on the vector $\theta \mapsto e^{in\theta}$ in $L^2(\mathbb{T}^*)$, and $H_\omega^0$ is defined in* Sect. 5.1. *For $A = (a_{i,j})_{1\leq i,j\leq n_0}$, and $t \in (L^2(\mathbb{T}^*) \otimes \mathbb{C}^{\mathrm{D}}) \otimes \mathbb{C}^{n_0}$, $At \in (L^2(\mathbb{T}^*) \otimes \mathbb{C}^{\mathrm{D}}) \otimes \mathbb{C}^{n_0}$, with $(At)_j = \sum_{i=1}^{n_0} a_{j,i} t_i$*

The proof of this lemma follow the same steps as Lemma 5.5 in [16]. We use it to end the proof of Theorem 13. Let us first notice that the operator $H_\omega^a$ could be written as a direct sum of $n_0$ copies of $m_0 \times \mathrm{D}$ random scalar-valued continuous Anderson models. Indeed, we can write

$$(L^2(\mathbb{T}^*) \otimes \mathbb{C}^{\mathrm{D}}) \otimes \mathbb{C}^{n_0} \otimes \mathbb{C}^{m_0} = \bigoplus_{1\leq i\leq D,\ 1\leq j\leq n_0,\ 1\leq k\leq m_0} L^2(\mathbb{T}^*) \otimes \tilde{\mathbb{C}^i} \otimes \tilde{\mathbb{C}^j} \otimes \tilde{\mathbb{C}^k}.$$

Here, for $1 \leq j \leq l$, we use the notation $\tilde{\mathbb{C}^j} = \{0\}^{j-1} \times \mathbb{C} \times \{0\}^{l-j}$. So $H_\omega^a$ is unitarily equivalent to

$$\bigoplus_{1\leq i\leq D,\ 1\leq j\leq n_0,\ 1\leq k\leq m_0} H_{\omega,i,k}^{\mathrm{And}},$$

Here $H_{\omega,i,k}^{\mathrm{And}}$ acts on $L^2(\mathbb{T}^*) \otimes \tilde{\mathbb{C}^i} \otimes \tilde{\mathbb{C}^j} \otimes \tilde{\mathbb{C}^k}$. Using the discrete Fourier transformation, we get that for every $k \in \{1, \ldots, m_0\}$, $H_{\omega,i,k}^{\mathrm{And}}$ is unitarily equivalent to $h_{\omega,i}^{\mathrm{And}}$, where $h_{\omega,i}^{\mathrm{And}}$ acts on $\ell^2(\mathbb{Z}^d)$ and is defined by

$$h_{\omega,i}^{\mathrm{And}} = -\Delta_{\mathbb{Z}^d} + \sum_{n\in\mathbb{Z}^d} \omega_i^{(n)} \pi_n. \tag{6.19}$$

Here, if $\delta_n$ is the vector $(\delta_m^n)_{\beta\in\mathbb{Z}^d}$ where $\delta_m^n$ is the Kronecker's symbol, then $\pi_n$ is the orthogonal projection on $\delta_n$ and $-\Delta_{\mathbb{Z}^d}$ is the discrete Laplacian defined by :

$$\forall u \in l^2(\mathbb{Z}^d),\ (\Delta_{\mathbb{Z}^d} u)_n = \frac{1}{2} \sum_{|m-n|=1} (u_n - u_m). \tag{6.20}$$

Using the fact that for operators $A$ and $B$, we have $N(A \oplus B, E) = N(A, E) + N(B, E)$ (see [20]), we get that

$$N_{\mathcal{E}_0}(\varepsilon) \leq n_0 \times m_0 \times \sum_{i=1}^{\mathrm{D}} N(h_{\omega,i}^{\mathrm{And}}, C.m_0.\varepsilon). \tag{6.21}$$

To satisfy assumptions of [23], we set, for every $i \in \{1, \ldots, \mathrm{D}\}$, $s_i = \sup_{n\in\mathbb{Z}^d} \omega_i^{(n)}$ and :

$$\forall i \in \{1, \ldots, \mathrm{D}\},\ \tilde{\omega}_i^{(n)} = \begin{cases} 0 & \text{if } \omega_i^{(n)} \in [0, s_i/2] \\ s_i/2 & \text{if } \omega_i^{(n)} \in (s_i/2, s_i] \end{cases}$$

By changing $\omega_i^{(n)}$ into $\tilde{\omega}_i^{(n)}$ in (6.19), we define a new operator which we denote by $\tilde{h}_{\omega,i}^{\mathrm{And}}$. We notice that $\tilde{h}_{\omega,i}^{\mathrm{And}}$ lower bound $h_{\omega,i}^{\mathrm{And}}$ with the same bottom of the spectrum. As it is known that each $\tilde{h}_{\omega,i}^{\mathrm{And}}$ exhibits Lifshitz tails with Lifshitz exponent $-d/2$ (see [8,23]), using Theorem 9 and (6.21), we get that

$$\limsup_{\varepsilon \longrightarrow 0^+} \frac{\log|\log(N(\varepsilon) - N(0^+))|}{\log \varepsilon} \leq -\frac{d}{2}.$$

This ends the proof of Theorem 13. □

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