2022-2023
Master 1

Functional Analysis - Bounded operators
Final exam / Solution

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1. Let $f \in H$. Then:

$$\|Tf\|_2^2 = \int_0^{2\pi} \left| \int_0^{2\pi} \cos(x-t)\, f(t)\, dt \right|^2 dx$$

$$\xrightarrow{\text{Cauchy-Schwarz}} \leq \int_0^{2\pi} \left(\int_0^{2\pi} \underbrace{|\cos(x-t)|^2}_{\leq 1} dt \right) \underbrace{\left(\int_0^{2\pi} |f(t)|^2 dt \right)^2}_{\|f\|_2^2} dx$$

$$\leq 4\pi^2 \|f\|_2^2$$

Hence: $\forall f \in H, \quad \|Tf\|_2 \leq 2\pi \|f\|_2 \quad (1)$ and T is bounded.

Note that T is clearly linear by linearity of the integral.

(2) 2.

Let $f, g \in H$.

$$(Tf \mid g) = \int_0^{2\pi} Tf(x)\,\overline{g(x)}\,dx = \int_0^{2\pi}\left(\int_0^{2\pi} \cos(x-t) f(t)\,dt\right)\overline{g(x)}\,dx$$

Fubini

$$= \int_0^{2\pi}\left(\int_0^{2\pi} \cos(x-t)\,\overline{g(x)}\,dx\right) f(t)\,dt$$

$$= \int_0^{2\pi} f(t)\,\overline{\left(\int_0^{2\pi} \cos(t-x)\, g(x)\,dx\right)}\,dt$$

si cos is real-valued and even

$$= \int_0^{2\pi} f(t)\,\overline{Tg(t)}\,dt = (f \mid Tg)$$

By uniqueness of the adjoint, $T^* = T$ and T is self-adjoint.

3 **3.** Let $f \in H$. Show that Tf is a continuous function on $[0,2\pi]$

Let $\varepsilon > 0$. The function $(x,t) \mapsto \cos(x-t)$ is continuous on the compact set $[0,2\pi]^2$ hence it is uniformly continuous. There exists $\eta > 0$ such that:

$$\forall (x,x') \in [0,2\pi]^2, \quad |x-x'| < \eta \Rightarrow \sup_{t\in[0,2\pi]} |\cos(x-t) - \cos(x'-t)| \leq \varepsilon$$

Let $x, x' \in [0,2\pi]$, $|x-x'| < \eta$. Then,

$$|Tf(x) - Tf(x')| = \left| \int_0^{2\pi} (\cos(x-t) - \cos(x'-t)) f(t)\,dt \right|$$

$$\leq \int_0^{2\pi} |\cos(x-t) - \cos(x'-t)|\,|f(t)|\,dt$$

$$\leq \left(\sup_{t\in[0,2\pi]} |\cos(x-t) - \cos(x'-t)| \right) \times \int_0^{2\pi} 1 \times |f(t)|\,dt$$

$$\leq \varepsilon \times \left(\int_0^{2\pi} 1^2\,dt \right)^{1/2} \times \left(\int_0^{2\pi} |f(t)|^2\,dt \right)^{1/2} = 2\pi \|f\|_2\, \varepsilon \qquad (2)$$

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Hence, Tf is also uniformly continuous on $[0,2\pi]$ and $\operatorname{Im} T$ is included in the set of continuous functions on $[0,2\pi]$.

4. Since $\operatorname{Im} T \subset C^0([0,2\pi],\mathbb{C})$, to show that T is a compact operator, one applies Ascoli's theorem to $T(B_H)$ where B_H is the unit ball of H.

Equicontinuity: Let $\varepsilon > 0$ and η as in question 3. For $x, x' \in [0,2\pi]$, $|x-x'| < \eta$ and for any $f \in B_H$, using (2),

$$|Tf(x) - Tf(x')| \leq 2\pi \underbrace{\|f\|_2}_{\leq 1} \varepsilon \leq 2\pi\varepsilon$$

Hence $T(B_H)$ is equicontinuous.

Boundedness: Let $f \in B_H$ and $x \in [0,2\pi]$.

$$|Tf(x)| \leq \int_0^{2\pi} \overbrace{|\cos(x-t)|}^{\leq 1} |f(t)| dt \leq \int_0^{2\pi} |f(t)| dt \leq 2\pi \|f\|_2 \underset{f \in B_H}{\leq} 2\pi$$

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Hence, $T(B_H)$ is equicontinuous and uniformly bounded. By Ascoli's theorem, $T(B_H)$ is relatively compact for the infinite norm on $C^0([0,2\pi])$. But since for any $f \in H$, $\|Tf\|_2 \leq \sqrt{2\pi}\,\|Tf\|_\infty$, it is also relatively compact for the L^2 norm on H.

Hence, T is a compact operator on H.

5. Let $f \in H$, $f = \sum_{n \in \mathbb{Z}} c_n(f)\, e_n$.

Then, since T is a bounded operator,

$$Tf = \sum_{n \in \mathbb{Z}} (f \,|\, e_n)\, Te_n$$

But, for any $n \in \mathbb{Z}$ and any $x \in [0, 2\pi]$,

$$(Te_n)(x) = \int_0^{2\pi} \frac{1}{2}\left(e^{i(x-t)} + e^{-i(x-t)}\right) \frac{e^{int}}{\sqrt{2\pi}}\, dt$$

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$$= \frac{1}{\sqrt{2\pi}} \times \frac{1}{2} \left(\int_0^{2\pi} e^{ix} e^{i(n-1)t} dt + \int_0^{2\pi} e^{-ix} e^{i(n+1)t} dt \right)$$

$$= \frac{1}{2} \frac{1}{\sqrt{2\pi}} \left(e^{ix} \times \begin{cases} 2\pi & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases} + e^{-ix} \times \begin{cases} 2\pi & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases} \right)$$

$$= \pi \; e_1(x) \times \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases} + \pi e_{-1}(x) \times \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

Hence :

$$\forall x \in [0, 2\pi], \; Tf(x) = \pi \left(c_1(f) e_1(x) + c_{-1}(f) e_{-1}(x) \right)$$

and : $\forall f \in H, \; Tf = \pi (c_1(f) e_1 + c_{-1}(f) e_{-1}).$

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6. Using question 5. and Pythagore's theorem:

$\forall f \in H, \quad \|Tf\|_2^2 = \pi^2 \left(|c_1(f)|^2 + |c_{-1}(f)|^2\right)$

since $e_1 \perp e_{-1}$ and $\|e_1\|_2^2 = \|e_{-1}\|_2^2 = 1$.

Using Parseval equality: $\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |c_n(f)|^2 \geq |c_1(f)|^2 + |c_{-1}(f)|^2$

Hence:

$\forall f \in H, \quad \|Tf\|_2^2 \leq \pi^2 \sum_{n\in\mathbb{Z}} |c_n(f)|^2 = \pi^2 \|f\|_2^2$

and: $\quad \forall f \in H, \quad \|Tf\|_2 \leq \pi \|f\|_2$

which improves the upper bounded obtained at question 1.

It implies that:

$$\|T\|_{\mathcal{L}(H)} \leq \pi.$$

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But $Te_1 = \pi e_1$ using the computation for Te_n done at question 5, hence,

$$\|Te_1\|_2 = \pi \quad \text{and} \quad \|e_1\|_2 = 1.$$

Hence $\|T\|_{\mathcal{L}(H)} \geq \pi$.

Finally: $\|T\|_{\mathcal{L}(H)} = \pi$.

7. Let $\lambda \in \mathbb{R}$ and $f \in H$, $f = \sum_{n\in\mathbb{Z}} c_n(f)\, e_n$. Then, using 5,

$$Tf = \lambda f \iff \pi c_{-1}(f) e_{-1} + \pi c_1(f) e_1 = \sum_{n\in\mathbb{Z}} \lambda c_n(f)\, e_n$$

$$\iff (\pi - \lambda) c_{-1}(f)\, e_{-1} + (\pi-\lambda) c_1(f)\, e_1 + \sum_{n\in\mathbb{Z}\setminus\{\pm 1\}} \lambda c_n(f)\, e_n = 0$$

Since $(e_n)_{n\in\mathbb{Z}}$ is an Hilbert basis of H, this last equality is

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equivalent to: $\begin{cases} (\pi-\lambda)\, c_{-1}(f) = 0 \\ (\pi-\lambda)\, c_1(f) = 0 \\ \forall n \in \mathbb{Z}\setminus\{-1,1\},\ \lambda\, c_n(f) = 0 \end{cases}$ (2)

8. From system (2), one deduces that the only possible eigenvalues of T are 0 and π. Hence: $\sigma_p(T) \subset \{0, \pi\}$.

<u>If $\lambda = \pi$:</u> One has $Te_1 = \pi e_1$ and $Te_{-1} = \pi e_{-1}$ and $\mathrm{Vect}(e_1, e_{-1}) \subset \mathrm{Ker}(T-\pi)$.

By (2), if $f \in H$ is such that $Tf = \pi f$, then

$\forall n \in \mathbb{Z}\setminus\{-1,1\}$, $c_n(f) = 0$.

i.e. $\forall n \in \mathbb{Z}\setminus\{-1,1\}$, $(f \mid e_n) = 0$

and $f \in \left(\mathrm{Vect}\left((e_n)_{n \in \mathbb{Z}\setminus\{\pm 1\}}\right)\right)^{\perp} = \mathrm{Vect}(e_1, e_{-1})$

Hence: $\mathrm{Ker}(T-\pi) = \mathrm{Vect}(e_1, e_{-1}) \neq \{0\}$

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<u>If $\lambda = 0$:</u> Then, by (2), if $f \in H$ is such that $Tf = \lambda f$,

$c_1(f) = c_{-1}(f) = 0$. Hence $f \in \left(\mathrm{Vect}(e_{-1}, e_1)\right)^{\perp}$

and $f \in \mathrm{Vect}\left((e_n)_{n \in \mathbb{Z} \setminus \{\pm 1\}}\right)$

Hence : $\ker T \subset \mathrm{Vect}\left((e_n)_{n \in \mathbb{Z} \setminus \{\pm 1\}}\right)$.

Conversely, if $n \in \mathbb{Z} \setminus \{-1, 1\}$, $Te_n = 0$ and $e_n \in \ker T$.

Hence : $\ker T = \mathrm{Vect}\left((e_n)_{n \in \mathbb{Z} \setminus \{\pm 1\}}\right) \neq \{0\}$

Finally : $\{0, \pi\} \subset \sigma_p(T)$ and : $\sigma_p(T) = \{0, \pi\}$

and we already compute the eigenspaces associated respectively to 0 and π.

Remark also that since T is self-adjoint, $\sigma_p(T) \subset \mathbb{R}$.

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9. Since T is self-adjoint, $\sigma(T) \subset \mathbb{R}$. Moreover, since T is compact and H is infinite dimensional, its spectrum is equal to:

$$\sigma(T) = \{0\} \cup \sigma_p(T) = \{0, \pi\} = \sigma_p(T).$$