2022-2023
M1
Option Operator Theory

Exercises sheet 1

Exercise 1:

1. We want to verify that the integral cv and that for $u \in L^2(X, \mathbb{C})$, $T_K u$ is also in $L^2(X, \mathbb{C})$.

If $u \in L^2(X, \mathbb{C})$ and $x \in X$,

$$|T_K u(x)| = \left| \int_X K(x,y) u(y)\, dy \right|$$

$$\overset{CS}{\rightarrow} \leq \left(\int_X |K(x,y)|^2 dy \right)^{1/2} \underbrace{\left(\int_X |u(y)|^2 dy \right)^{1/2}}_{\|u\|_{L^2(X)}} < +\infty$$

Since $K \in L^2(X \times X)$ by Fubini, for a.e x, $\int_X |K(x,y)|^2 dy < +\infty$.

For $u \in L^2(X)$ and $x \in X$,

$$|T_K u(x)|^2 \leq \int_X |K(x,y)|^2 dy \times \|u\|^2_{L^2(X)}$$

and $$\int_X |T_K u(x)|^2 dx \leq \underbrace{\int_X \left(\int_X |K(x,y)|^2 dy \right) dx}_{\|K\|^2_{L^2(X\times X)}} \times \|u\|^2_{L^2(X)}$$

So $T_K u \in L^2(X)$ and we already have that:

$\forall u \in L^2(X)$, $\|T_K u\|_{L^2(X)} \leq \|K\|_{L^2(X\times X)} \|u\|_{L^2(X)}$

2. It shows that T_K is bounded on $L^2(X)$ and $\|T_K\|_{\mathcal{L}(L^2(X))} \leq \|K\|_{L^2(X\times X)}$

3. To compute T_K^* we take any $u, v \in L^2(X)$ and we try to write:

$$(T_K u | v) = (u | \underbrace{?}_{\text{linear in } v \,:\, (T_K)^* v})$$

$$(T_K u | v) = \int_X T_K u(x) \overline{v(x)}\, dx$$

$$= \int_X \left(\int_X K(x,y)\, u(y)\, dy \right) \overline{v(x)}\, dx$$

Fubini

$$\overset{\downarrow}{=} \int_X u(y) \left(\int_X K(x,y)\, \overline{v(x)}\, dx \right) dy$$

$$= \int_X u(y) \overline{\left(\underbrace{\int_X \overline{K(x,y)}\, v(x)\, dx}_{(T_{K^*} v)(y) \text{ with } K^*(x,y) = \overline{K(y,x)}} \right)}\, dy$$

$$= \int_X u(y)\, \overline{T_{K^*} v(y)}\, dy$$

$$= (u \mid T_{K^*} v)$$

and $(T_K)^* = T_{K^*}$ with $K^*(x,y) = \overline{K(y,x)}$

Remark: This is the analog of ${}^t\bar{A} = {}^t\left(\overline{(a_{ij})_{i,j}}\right)$

$$= (\overline{a_{ji}})_{i,j}$$

Exercise 3:

Recall: U unitary means that $UU^* = U^*U = Id_{\mathcal{H}}$.

1. Hint: show that it is enough to prove that

$$\operatorname{Ker}(U^* - I) = \operatorname{Ker}(U - I)$$

We show that $\ker(U-I) = \ker(U^*-I)$.

$\ker(U-I) \subset \ker(U^*-I)$: let $x \in \ker(U-I)$:

$Ux = x$: applying U^* : $U^*Ux = U^*x$

$\Leftrightarrow x = U^*x$

$\Leftrightarrow x \in \ker(U^*-I)$.

$\ker(U^*-I) \subset \ker(U-I)$: let $x \in \ker(U^*-I)$:

$U^*x = x$: applying U : $UU^*x = Ux$

$\Leftrightarrow x = Ux$

$\Leftrightarrow x \in \ker(U-I)$.

Then: $\ker(U-I) = \ker(U^*-I) = \ker((U-I)^*)$

$= \operatorname{Im}(U-I)^\perp$

and $\left(\operatorname{Ker}(U-I)\right)^{\perp} = \overline{\operatorname{Im}(U-I)}$

Since U si bounded, $\operatorname{Ker}(U-I)$ is closed in $\mathcal{H}$

and:

$$\mathcal{H} = \operatorname{Ker}(U-I) \overset{\perp}{\oplus} \left(\operatorname{Ker}(U-I)\right)^{\perp}$$
$$= \operatorname{Ker}(U-I) \overset{\perp}{\oplus} \overline{\operatorname{Im}(U-I)}$$

2. If $u \in \mathcal{H}$: $\exists! (x,y) \in \operatorname{Ker}(U-I) \times \overline{\operatorname{Im}(U-I)}$,

$$u = x + y$$

By definition $Pu = x$.

We have to prove that: $\forall n \in \mathbb{N}$, $S_n u \xrightarrow[n \to +\infty]{} x = Pu$

$\forall n \in \mathbb{N}$, $S_n u = S_n x + S_n y$. We prove (i): $S_n x \xrightarrow[n \to +\infty]{} x$

(ii) $S_n y \xrightarrow[n \to +\infty]{} 0$

(i) We have $x \in \mathrm{Ker}(U-I)$:

$$Ux = x \quad \text{and} : \forall n \geq 1, \; U^n x = x$$

So : $$S_n x = \frac{1}{n+1}(I + U + \dots + U^n)x$$

$$= \frac{1}{n+1}(\underbrace{x + \dots + x}_{n+1 \text{ times}}) = x$$

So : $\forall n \geq 1$, $S_n x = x$ hence $S_n x \xrightarrow[n \to +\infty]{} x$.

(ii) We have $y \in \overline{\mathrm{Im}(U-I)}$.

Let $\varepsilon > 0$: $\exists y_\varepsilon \in \mathrm{Im}(U-I)$, $\|y - y_\varepsilon\|_H \leq \varepsilon$

Since $y_\varepsilon \in \mathrm{Im}(U-I)$: $\exists z_\varepsilon \in H$, $y_\varepsilon = (U-I)z_\varepsilon$

Then: $\forall n \geq 1$, $S_n y_\varepsilon = \frac{1}{n+1}(I + U + \cdots + U^n)\, y_\varepsilon$

$$= \frac{1}{n+1}(I+U+\cdots+U^n)(U z_\varepsilon - z_\varepsilon)$$

$$= \frac{1}{n+1}\left(\cancel{U z_\varepsilon} - z_\varepsilon + \cancel{U^2 z_\varepsilon} - \cancel{U z_\varepsilon} + \cancel{-} \cdots \cancel{+} U^{n+1} z_\varepsilon - \cancel{U^n z_\varepsilon}\right)$$

$$= \frac{1}{n+1}\left(U^{n+1} z_\varepsilon - z_\varepsilon\right)$$

Hence: $\forall n \geq 1$, $\|S_n y_\varepsilon\|_{\mathcal{H}} \leq \frac{1}{n+1}\left(\|U^{n+1} z_\varepsilon\|_{\mathcal{H}} + \|z_\varepsilon\|_{\mathcal{H}}\right)$

$$\leq \frac{1}{n+1}\Big(\underbrace{\|U\|_{\mathcal{L}(\mathcal{H})}}_{=1 \text{ since } U \text{ is unitary}}^{n+1} \|z_\varepsilon\|_{\mathcal{H}} + \|z_\varepsilon\|_{\mathcal{H}}\Big)$$

$$= \frac{2}{n+1}\|z_\varepsilon\|_{\mathcal{H}} \xrightarrow[n \to +\infty]{} 0$$

i.e. $\exists N_0 \in \mathbb{N}$, $\forall n \geq N_0$, $\|S_n y_\varepsilon\|_{\mathcal{H}} \leq \varepsilon$.

Then: $\forall n \geq 1$, $\|S_n y\|_H = \|S_n y - S_n y_\varepsilon + S_n y_\varepsilon\|_H$

$$= \|S_n(y - y_\varepsilon) + S_n y_\varepsilon\|_H$$

$$\leq \|S_n\|_{\mathcal{L}(H)} \underbrace{\|y - y_\varepsilon\|_H}_{\leq \varepsilon} + \underbrace{\|S_n y_\varepsilon\|_H}_{\leq \varepsilon \text{ for } n \geq N_0}$$

But: $\|S_n\|_{\mathcal{L}(H)} \leq \dfrac{1}{n+1}\Big(\underbrace{\|I\|_{\mathcal{L}(H)}}_{=1} + \underbrace{\|U\|_{\mathcal{L}(H)}}_{=1} + \dots + \underbrace{\|U\|^n_{\mathcal{L}(H)}}_{=1}\Big)$

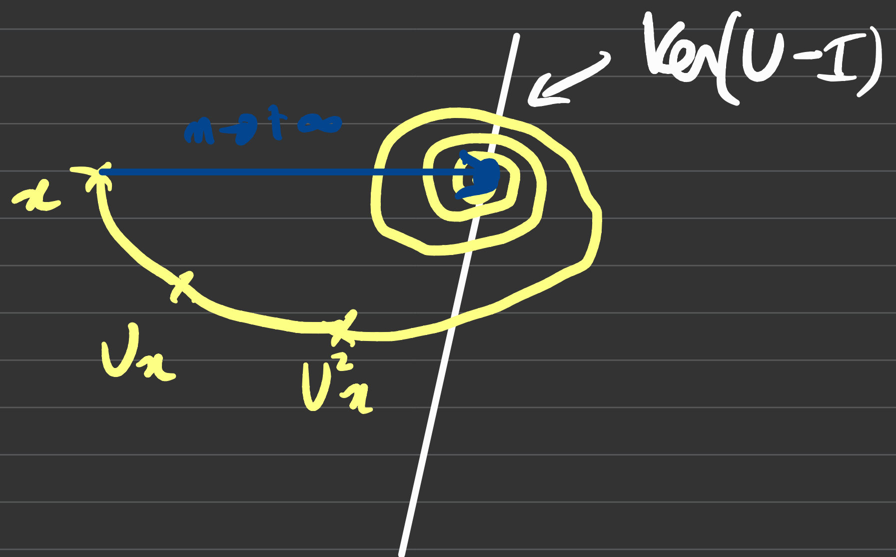
$$= \frac{n+1}{n+1} = 1$$

Finally for $n \geq N_0$:

$\|S_n y\|_H \leq 1 \times \varepsilon + \varepsilon = 2\varepsilon$. and $S_n y \xrightarrow[n \to +\infty]{} 0$.

$S_n u = S_n x + S_n y \xrightarrow[n \to +\infty]{} x + 0 = x = Pu$.

Example: for U a rotation in $\mathbb{R}^3$: $\mathrm{Ker}\,(U-I)$ is the axis of the rotation (the set of fixed points).



$$U = r_\theta$$
$$U^n = r_{n\theta}$$