M 1

2023-2024

Functional Analysis

Exercises sheet 1 - Answers

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Exercise 1:

Let $\{r_0, \ldots, r_n, \ldots\}$ be an enumeration of the elements of $\mathbb{Q}$. For each $n \in \mathbb{N}$, let $O_n = \mathbb{R} \setminus \{r_n\}$. Then each O_n is an open dense set of $\mathbb{R}$ and $\bigcap_{n \in \mathbb{N}} O_n = \mathbb{R} \setminus \mathbb{Q}$ which is not open but of course dense in $\mathbb{R}$ (which does not contradict Baire's theorem).

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Exercise 2

Let $\varepsilon > 0$. First, for every $p \in \mathbb{N}$,

$$F_p = \bigcap_{n \geq p} \{ a \geq 0, \ |f(na)| \leq \varepsilon \}$$ is a closed set as an intersection of closed sets, by continuity of f.

Moreover, $\mathbb{R}_+^* \subset \bigcup_{p \geq 0} F_p$.

Indeed, if $a \in \mathbb{R}_+^*$, $\exists p \in \mathbb{N}, \forall n \geq p, |f(na)| \leq \varepsilon$ by hypothesis. It means exactly that for every $a \in \mathbb{R}_+^*$, there exists $p \in \mathbb{N}$, $a \in F_p$, i.e. $a \in \bigcup_{p \geq 0} F_p$.

In particular, since $\mathbb{R}_+^* \subset \bigcup_{p \geq 0} F_p$, $\bigcup_{p \geq 0} F_p$ is of

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non-empty interior. By Baire's theorem at least one of the F_p's is of non-empty interior.

Indeed, any countable union of closed sets of empty interiors is of empty interior.

Hence: $\exists p_0 \geq 0$, $\mathrm{Int}(F_{p_0}) \neq \emptyset$ and:

$$\exists a \geq 0, \exists \delta > 0, \;]a-\delta, a+\delta[\subset F_{p_0} \subset \mathbb{R}_+.$$

By definition of F_{p_0}: $\forall x \in]a-\delta, a+\delta[$, $\forall n \geq p_0$,

$$|f(nx)| \leq \varepsilon.$$

Let $p_1 \in \mathbb{N}$ such that: $\forall n \geq p_1$, $a < (2n+1)\delta$, and let $N_0 = \max(p_0, p_1)$.

Then $$\bigcup_{n \geq N_0}]n(a-\delta), n(a+\delta)[=]N_0(a-\delta), +\infty[$$

④ If $y \geq \underbrace{N_0(a-\delta)}_{>0}$, $|f(y)| \leq \varepsilon$.

Indeed, there exists $n \geq N_0$, $y \in]n(a-\delta), n(a+\delta)[$

i.e. $y/n \in]a-\delta, a+\delta[$ and $|f(n \frac{y}{n})| \leq \varepsilon$.

Finally, for every $\varepsilon > 0$, there exists $A = N_0(a-\delta) > 0$,

$\forall y \geq A$, $|f(y)| \leq \varepsilon$ i.e. $f(y) \xrightarrow[y \to +\infty]{} 0$.

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Exercise 3

For $n \in \mathbb{N}$, let $F_n = \{z \in \mathbb{C},\ f^{(n)}(z) = 0\}$. Then F_n is closed in $\mathbb{C}$.

By hypothesis : $\forall z \in \mathbb{C},\ \exists n_z \in \mathbb{N},\ f^{(n_z)}(z) = 0$.

Hence : $\bigcup_{n \in \mathbb{N}} F_n = \mathbb{C}$ since if $z \in \mathbb{C}$, $\exists n_z \in \mathbb{N},\ z \in F_{n_z}$.

i.e $\mathbb{C} \subset \bigcup_{n \in \mathbb{N}} F_n$.

In particular, $\bigcup_{n \in \mathbb{N}} F_n$ is of non-empty interior and at least one of the F_n's, say F_{n_0} is of non-empty interior.

Then: $\exists z_0 \in F_{n_0}, \exists r_0 > 0,\ D(z_0, r_0) \subset F_{n_0}$ and:

$$\forall z \in D(z_0, r_0),\ f^{(n_0)}(z) = 0.$$

By analytic continuation, $f^{(n_0)} = 0$. Hence f is a polynomial function.

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Exercise 4:

1. This is exactly Baire's theorem.

2. Since V_n is an open set of $\mathbb{R}$, $W_n = V_n \setminus \{x_0, \dots, x_n\}$ is also an open set of $\mathbb{R}$.

Let U be a non-empty open set of $\mathbb{R}$:

$$W_n \cap U = \left(V_n \setminus \{x_0, \dots, x_n\}\right) \cap U$$
$$= V_n \cap \underbrace{(U \setminus \{x_0, \dots, x_n\})}_{\text{non-empty open set of } \mathbb{R}} \neq \emptyset \text{ by density of } V_n.$$

Hence W_n is dense in $\mathbb{R}$.

3. By Baire's theorem, $\bigcap_{n \in \mathbb{N}} W_n$ is dense in $\mathbb{R}$ hence it is non-empty.

But $\bigcap_{n \in \mathbb{N}} W_n = V \setminus \{x_n \mid n \in \mathbb{N}\} \neq \emptyset$. Since any countable subset A of $\mathbb{R}$ can be written as $\{x_n \mid n \in \mathbb{N}\}$, $V \setminus A \neq \emptyset$ i.e. $V \neq A$.

Exercise 5

1. One writes for each $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$,

$$A_{m,n} = \bigcap_{l \geq m} \left\{ x \in E \mid |f_m(x) - f_l(x)| \leq \frac{1}{n} \right\}$$

and each set $\left\{ x \in E \mid |f_m(x) - f_l(x)| \leq \frac{1}{n} \right\}$ is a closed set by continuity of f_m and f_l.

Hence $A_{m,n}$ is a closed set as intersection of closed sets.

2. Let $n \in \mathbb{N}^*$ fixed. Since $(f_m)_{m \in \mathbb{N}}$ converges pointwise to f, for each $x \in E$, $(f_m(x))_{m \in \mathbb{N}}$ is a Cauchy sequence of real numbers.

In particular, since $\frac{1}{n} > 0$, there exists $m_x \in \mathbb{N}$, $\forall l \geq m_x$,

$$|f_{m_x}(x) - f_l(x)| \leq \frac{1}{n} \quad \text{i.e} \quad x \in A_{m_x, n}.$$

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We did just prove that $E \subset \bigcup_{m \in \mathbb{N}} A_{m,n}$ and since for each $m \in \mathbb{N}$, $A_{m,n} \subset E$, $\bigcup_{m \in \mathbb{N}} A_{m,n} \subset E$.

Hence
$$\bigcup_{m \in \mathbb{N}} A_{m,n} = E.$$

3. For each $n \in \mathbb{N}^*$ fixed, O_n is an open set as a union of open sets, the $O_{m,n}$.

We aim at showing that O_n is dense in E. Let U be an open non-empty set of E. Let $\tilde{A}_{m,n} = A_{m,n} \cap U$. for each n, m.

Then $(\tilde{A}_{m,n})_{m \in \mathbb{N}}$ is a countable family of closed sets of U whose union is U since $\bigcup_{m \in \mathbb{N}} A_{m,n} = E$.

Hence, by Baire's theorem, at least one of the $\tilde{A}_{m,n}$ has non-empty interior for the induced topology on U. Let $m_0 \in \mathbb{N}$ be such that $\mathrm{Int}_U(\tilde{A}_{m_0,n}) \neq \emptyset$.

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Since U is an open set of E, $\mathrm{Int}_U(\tilde{\hat{A}}_{m_0,n})$ is also an open set of E and since $\mathrm{Int}_U(\tilde{\hat{A}}_{m_0,n}) \subset \tilde{\hat{A}}_{m_0,n}$, $\mathrm{Int}_U(\tilde{\hat{A}}_{m_0,n}) \subset \mathrm{Int}(A_{m_0,n})$

Since $\mathrm{Int}_U(\tilde{\hat{A}}_{m_0,n}) \subset U$ one has : $\mathrm{Int}_U(\tilde{\hat{A}}_{m_0,n}) \subset \mathrm{Int}(A_{m_0,n}) \cap U$

Hence $$\emptyset \neq \mathrm{Int}_U(\tilde{\hat{A}}_{m_0,n}) \subset \left(\bigcup_{m \in \mathbb{N}} \mathrm{Int}(A_{m,n})\right) \cap U = O_n \cap U$$

which is non-empty.

Hence O_n is dense in E.

4. Let $G = \bigcap_{n \in \mathbb{N}^*} O_n$, $a \in G$ and $\varepsilon > 0$.

a. Let $n \in \mathbb{N}^*$, $\frac{1}{n} \leq \varepsilon$. Since $a \in G$, $a \in O_n$ and there exists $m \in \mathbb{N}$, $a \in O_{m,n}$. Since $O_{m,n} \subset A_{m,n}$, for each $x \in O_{m,n}$, if $l \geq m$,

$$|f_m(x) - f_l(x)| \leq \frac{1}{n} \leq \varepsilon$$

Letting l tends to $+\infty$, $|f_m(x) - f(x)| \leq \varepsilon$.

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In particular, for $x = a \in O_{m,m}$, $|f_m(a) - f(a)| \leq \varepsilon$.

b. By continuity of f_m at point a, there exists V neighborhood of a such that: $\forall x \in V$, $|f_m(x) - f_m(a)| \leq \varepsilon$.

c. Let $x \in V$. Then:

$$|f(x) - f(a)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(a)| + |f_m(a) - f(a)|$$

$$\leq \quad \varepsilon \quad + \quad \varepsilon \quad + \varepsilon = 3\varepsilon$$

Hence the continuity of f at point a.

5. Recall that a residual set of a metric space is a set which contains an intersection of open dense sets. Since the set of continuity points of f contains G, it is a residual set of E.

6. It is not since it is nowhere continuous in $\mathbb{R}$.

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Exercise 6.

1. Let $x \in E$. One has:

$$\forall T_n \in A, \quad \|T_n(x)\|_{\ell^2}^2 = \sum_{i=1}^{+\infty} |(T_n(x))_i|^2 = n^2 |x_n|^2$$

This implies that: $\forall T_n \in A, \quad \|T_n(x)\|_{\ell^2} = n|x_n|$

But $x \in E$ and the x_n are all equal to 0 except for a finite number of them. Hence: $\exists n_0 \geq 1, \ \forall n \geq n_0, \quad n|x_n| = 0$ and the sequence $(n|x_n|)_{n \geq 1}$ is bounded, say by $C > 0$.

Then: $\forall T_n \in A, \quad \|T_n(x)\|_{\ell^2} \leq C$

and A_x is bounded by C in $\ell^2(\mathbb{N}^*)$.

2. Let $n \geq 1$. One has: $\forall x \in E$ such that $\|x\|_{\ell^2} = 1$,

$$\|T_n(x)\|_{\ell^2} = n|x_n| \leq n\|x\|_{\ell^2} \leq n$$

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Indeed for each $n \geq 1$, $|x_n|^2 \leq \sum_{i=1}^{+\infty} |x_i|^2$ hence $|x_n| \leq \|x\|_{\ell^2}$.

Hence: $|||T_n||| \leq n$

But, if for each $n \geq 1$, e_n is the element of $\ell^2(\mathbb{N}^*)$ such that for each $i \neq n$, $(e_n)_i = 0$ and $(e_n)_n = 1$, then $\|e_n\|_{\ell^2} = 1$ and : $\|T_n e_n\|_{\ell^2} = n$. Hence $|||T_n||| \geq n$ and finally, $\forall n \geq 1$, $|||T_n||| = n$ and A is not bounded.

3. If E was a Banach space, since each A_x is bounded, using Banach-Steinhaus theorem, A should also be bounded.

Hence E is not a Banach space.

Actually, it can be proven directly by constructing a Cauchy sequence in E which does not converge in E.

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For $n \geq 1$, let:

$$\forall i \geq 1, \quad (f_n)_i = \begin{cases} \frac{1}{i} & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

Then $f_n \in E$ but:

$\forall p > q \geq 1$, $\|f_p - f_q\|_{\ell^2}^2 = \sum_{i=1}^{p} \frac{1}{i^2} - \sum_{i=1}^{q} \frac{1}{i^2} = \sum_{i=q+1}^{p} \frac{1}{i^2}$. Since $\left(\sum \frac{1}{i^2}\right)$ CV, by the Cauchy criterion: $\forall \varepsilon > 0, \exists N_\varepsilon \geq 0, \forall p > q \geq N_\varepsilon$, $\sum_{i=q+1}^{p} \frac{1}{i^2} \leq \varepsilon$

and $\|f_p - f_q\|_{\ell^2}^2 \leq \varepsilon$.

Hence $(f_n)_{n \geq 1}$ is a Cauchy sequence in E.

But: $f_n \xrightarrow[n \to +\infty]{} \left(\frac{1}{i}\right)_{i \geq 1}$ which is not in E since it is not equal to 0 after some i_0.

Indeed, if $f = \left(\frac{1}{i}\right)_{i \geq 1}$: $\forall n \geq 1$, $\|f_n - f\|_{\ell^2}^2 = \sum_{i=n+1}^{+\infty} \frac{1}{i^2} \xrightarrow[n \to +\infty]{} 0$.

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Exercise 7

1. • Let $u = (1)_{n\in\mathbb{N}}$. Then $u \in \ell^\infty(\mathbb{N})$ but:

$\forall v \in A$, $\|u - v\|_\infty \geq 1$ since an infinite number of terms of v are equal to 0.

Hence A is not dense in $(\ell^\infty(\mathbb{N}), \|\ \|_\infty)$

• Let $u \in \ell^1(\mathbb{N})$. For every $N \in \mathbb{N}$, let

$v_N = (u_0, \dots, u_N, 0, \dots) \in A$.

One has:

$$\forall N \in \mathbb{N}, \quad \|u - v_N\|_1 = \sum_{k=N+1}^{+\infty} |u_k| \xrightarrow[N \to +\infty]{} 0 \quad \text{since } u \in \ell^1(\mathbb{N}).$$

Hence A is dense in $(\ell^1(\mathbb{N}), \|\ \|_1)$.

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2. $(\ell^1(\mathbb{N}), \|\ \|_1)$ and $(\ell^\infty(\mathbb{N}), \|\ \|_\infty)$ are both Banach spaces.

Assume that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$(a_n u_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}) \Leftrightarrow (u_n) \in \ell^\infty(\mathbb{N}).$$

Then, introduce $T : \begin{array}{l} \ell^\infty(\mathbb{N}) \longrightarrow \ell^1(\mathbb{N}) \\ (u_n)_{n\in\mathbb{N}} \longmapsto (a_n u_n)_{n\in\mathbb{N}} \end{array}$.

T is linear and bijective of explicit inverse:

$$T^{-1} : \begin{array}{l} \ell^1(\mathbb{N}) \longrightarrow \ell^\infty(\mathbb{N}) \\ (u_n)_{n\in\mathbb{N}} \longmapsto (a_n^{-1} u_n)_{n\in\mathbb{N}} \end{array}$$

Moreover, if one takes $u = (1)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$ then

$$(a_n)_{n\in\mathbb{N}} = (a_n \underbrace{u_n}_{1})_{n\in\mathbb{N}} \in \ell^1(\mathbb{N}) \quad \text{i.e } \sum a_n \text{ cv.}$$

Hence: $\forall u \in \ell^\infty(\mathbb{N})$, $\|Tu\|_1 = \sum_{n=0}^{+\infty} a_n |u_n| \leq \|u\|_\infty \sum_{n\in\mathbb{N}} a_n$

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and T is continuous. By the isomorphism theorem of Banach, T^{-1} is also continuous.

Note that since all the a_n are different from 0, $A = TA$.

and A being dense in $(\ell^1(\mathbb{N}), \| \|_1)$ and T being an homeomorphism, $\underset{\underset{A}{\shortparallel}}{TA}$ should be dense in $(\ell^\infty(\mathbb{N}), \| \|_\infty)$.

But it is not!

Hence $(a_n)_{n \in \mathbb{N}}$ cannot exist and there is no smaller rate of cv for the absolutely converging series.

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Exercise 8

1. If $f \in L^1(\mathbb{T})$ then $(c_n(f))_{n\in\mathbb{Z}}$ tends to 0 at $\pm\infty$ by Riemann-Lebesgue's theorem. i.e $(c_n(f))_{n\in\mathbb{Z}} \in F$.

Hence T is well-defined and it is linear by linearity of the integral.

Moreover: $\forall f \in L^1(\mathbb{T})$, $\forall n \in \mathbb{Z}$, $|c_n(f)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x)| \underbrace{|e^{-inx}|}_{=1} dx$

$= \|f\|_1$

Hence: $\forall f \in L^1(\mathbb{T})$, $\|(c_n(f))_{n\in\mathbb{Z}}\|_\infty \leq \|f\|_1$

and T is continuous.

We admit that it is injective. To prove this one could use the Fejer kernel$_N$:

$$\forall N \in \mathbb{N},\quad f_N = \sum_{k=-N}^{N} \left(1 - \frac{|k|}{N+1}\right) c_k(f) e^{ik\cdot} \xrightarrow[N\to+\infty]{\|\ \|_\infty} f$$

If for every $k \in \mathbb{Z}$, $c_k(f) = 0$ then for every N, $f_N = 0$ and $f = 0$. Hence $\operatorname{Ker} T = \{0\}$.

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2. If one assume that T is also onto, then T is an isomorphism and T^{-1} is also continuous:

$$\exists \delta > 0, \ \forall f \in L^1(\mathbb{T}), \ \|f\|_{L^1} \leq \delta \sup_{n \in \mathbb{Z}} |c_n(f)|.$$

3. Let $N \geq 0$. For each $n \in \mathbb{Z}$ one sets:

$$\alpha_n = \begin{cases} 0 & \text{if } c_n(g) = 0 \\ \dfrac{c_n(g)}{|c_n(g)|} & \text{if } c_n(g) \neq 0. \end{cases}$$

Then: $\forall n \in \mathbb{Z}$, $\overline{\alpha_n}\, c_n(g) = |c_n(g)|$.

Therefore: $$\sum_{|n| \leq N} |c_n(g)| = \sum_{|n| \leq N} \overline{\alpha_n}\, c_n(g) = \sum_{|n| \leq N} \frac{1}{2\pi} \int_0^{2\pi} \overline{\alpha_n}\, g(x) e^{-inx} dx$$

$$= \sum_{|n| \leq N} \frac{1}{2\pi} \int_0^{2\pi} g(x)\, \overline{\alpha_n e^{inx}}\, dx = \frac{1}{2\pi} \int_0^{2\pi} g(x)\, \overline{f_N(x)}\, dx$$

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Hence: $\forall N \geq 0, \quad \sum_{|n| \leq N} |c_n(g)| \leq \|g\|_\infty \|f_N\|_1 \leq \|g\|_\infty \delta \sup_{n \in \mathbb{Z}} |c_n(f_N)|$

But: $\forall N, \forall n \in \mathbb{Z}, \quad |c_n(f_N)| = |a_n| = 1$ if $|n| \leq N$

and $c_n(f_N) = 0$ if $|n| \geq N+1$.

In particular: $\sup_{n \in \mathbb{Z}} |c_n(f_N)| = 1$ and

$$\sum_{|n| \leq N} |c_n(g)| \leq \delta \|g\|_\infty .$$

4. Since $N \mapsto \sum_{|n| \leq N} |c_n(g)|$ is an increasing sequence with an upper bound it converges. Hence, $\sum_{n \in \mathbb{Z}} c_n(g) e^{in\cdot}$ cv normally on $[0, 2\pi]$ to the continuous function $x \mapsto \sum_{n \in \mathbb{Z}} c_n(g) e^{inx}$ since every term in the sum is continuous.

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Since $[0,2\pi]$, by the dominated cv theorem, the cv of the serie also takes place in $L^2([0,2\pi])$ with the same limit.

Since $g \in L^2([0,2\pi])$ (it is bounded on $\mathbb{R}$ and 2π-periodic) using Parseval's equality, $g(x) = \sum_{n\in\mathbb{Z}} c_n(g) e^{inx}$ a.e.

Hence, g being any 2π-periodic L^∞ function would be equal to a continuous function a.e! Of course it is not true hence T cannot be onto.

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Exercise 9

1. Let $(f_n)_{n\in\mathbb{N}} \in F^{\mathbb{N}}$ a sequence which cv to $f \in E_\alpha$ for $\|\ \|_\alpha$.
Since $E_\alpha \subset E$ and $\|f_n - f\|_\alpha \geq \|f_n - f\|_\infty$ for every n, the sequence $(f_n)_{n\in\mathbb{N}}$ also cv to f in $(E, \|\ \|_\infty)$. But F is closed in E hence $f \in F$, i.e. F is closed in $(E_\alpha, \|\ \|_\alpha)$.

2. F is closed in $(E, \|\ \|_\infty)$ which is a Banach space and is also closed in $(E_\alpha, \|\ \|_\alpha)$ which is also a Banach space.
Hence, $(F, \|\ \|_\infty)$ and $(F, \|\ \|_\alpha)$ are both Banach spaces.
Let $i : \begin{array}{l}(F, \|\ \|_\alpha) \to (F, \|\ \|_\infty) \\ f \longmapsto f\end{array}$. Then i is a linear map which is a bijection. Moreover it is continuous since: $\forall f \in F, \|f\|_\infty \leq \|f\|_\alpha$

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By the isomorphism theorem of Banach, i^{-1} is also continuous and:

$$\exists C>0, \forall f \in F, \|f\|_\alpha \leq C\|f\|_\infty .$$

3. Let B be the unit ball of $(F, \| \|_\infty)$.

Let $f \in B$: $\|f\|_\infty \leq 1$ and $\|f\|_\alpha \leq C$.

Then: $\forall \varepsilon > 0, \ \forall x, y \in [0,1], \ |x-y| < \varepsilon^{1/\alpha},$

$$\forall f \in B, \ |f(x)-f(y)| \leq \|f\|_\alpha |x-y|^\alpha \leq C\varepsilon$$

and B is equicontinuous in $C([0,1], \mathbb{C})$.

Moreover: $\forall x \in [0,1], \forall f \in B, |f(x)| \leq \|f\|_\infty \leq 1$.

and for each $x \in [0,1]$, $\{f(x)\}_{f \in B}$ is bounded.

Applying Ascoli's theorem, B is relatively compact, hence

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compact. By Riesz's Theorem, F is of finite dimension.

<u>Ascoli</u>: • $A \subset C(E,F)$ is equicontinuous in $x \in E$ if:

$\forall \varepsilon > 0, \exists \alpha > 0, \forall y \in E, d_E(y,x) < \alpha, d_F(f(x), f(y)) \leq \varepsilon, \forall f \in A$

A is equicontinuous if it is equicontinuous at each $x \in E$.

• If $A \subset C(\underset{\text{complete}}{E, F})$ is equicontinuous and if for each $x \in E$, $A_x = \{f(x)\}_{f \in A}$ is relatively compact in F, then A is relatively compact.

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Exercise 10

1. Let $(x_n)_{n\in\mathbb{N}}$ which cv strongly to x in E and let $u\in E'$. Then:

$$|u(x_n)-u(x)| \leq \underbrace{\|u\|_{E'}\,\|x_n-x\|}_{\xrightarrow[n\to+\infty]{} 0} \xrightarrow[n\to+\infty]{} 0$$

and $x_n \rightharpoonup x$.

2. Assume that $x_n \rightharpoonup x_0$ in E and that $x_n \rightharpoonup x_1$ in E. Then: $\forall u\in E'$, $u(x_1) = \lim\limits_{n\to+\infty} u(x_n) = u(x_0)$.

Recall the corollary of Hahn-Banach's theorem:

$\forall x\in E$, $\|x\| = \sup\{|u(x)| \mid u\in E' \text{ and } \|u\|_{E'} \leq 1\}$ and this supremum is attained.

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Then: $\exists v \in E'$, $\|v\|_{E'} = 1$, $\|x_1 - x_0\| = v(x_1 - x_0)$
$= v(x_1) - v(x_0) = 0$

Hence $x_0 = x_1$ and the weak limit is unique.

3. Let $(x_n)_{n \in \mathbb{N}}$ which cv weakly to $x \in E$.

Let $u \in E'$. Then $(u(x_n))_{n \in \mathbb{N}}$ cv to $u(x)$ and in particular it is bounded by some constant $M_u > 0$.

Then, for each $n \in \mathbb{N}$, consider the linear form:

$$l_n : \begin{array}{l} E' \to \mathbb{K} \\ u \mapsto u(x_n) \end{array}$$

One has: $\forall n \in \mathbb{N}, \forall u \in E', |l_n(u)| \leq M_u$. Since E' and $\mathbb{K}$ are Banach spaces, one can apply Banach-Steinhaus theorem to get the existence of $M > 0$ such that:

$$\forall u \in E', \ \|u\|_{E'} = 1, \quad \sup_{n \in \mathbb{N}} |f_n(u)| \leq M.$$

But using again the duality theorem, for each $n \in \mathbb{N}$, there exists $u_n \in E'$, $\|u_n\|_{E'} = 1$, $|f_n(u_n)| = \|x_n\|$.

One deduces that:

$$\forall n \in \mathbb{N}, \ \|x_n\| = |f_n(u_n)| \leq M$$

and $(x_n)_{n \in \mathbb{N}}$ is a bounded sequence.

4. Assume that $x_n \rightharpoonup x \in E$

Using the duality theorem, there exists $u \in E'$, $\|u\|_{E'} = 1$ and $|u(x)| = \|x\|$. For this $u \in E'$ and for every $n \in \mathbb{N}$,

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$$|u(x_n)| \leq \|u\|_{E'} \|x_n\| = \|x_n\|.$$

Since $|u(x_n)| \xrightarrow[n\to+\infty]{} |u(x)| = \|x\|$ one gets:

$$\|x\| = \lim_{n\to+\infty} |u(x_n)| = \liminf |u(x_n)| \leq \liminf \|x_n\|.$$

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Exercise 11

1. By Riesz theorem, since E is of infinite dimension, its unit ball is not compact.

2. a. Let $x \in [0,1]$. Let $u \in F_{k,M}$.

Then: $|u(x)| \leq |u(x) - u(0) + u(0)| \leq |u(x) - u(0)| + |u(0)|$
$$\leq k\underbrace{x}_{\in[0,1]} + M \leq k + M.$$

Hence, $\{u(x)\}_{u \in F_{k,M}}$ is bounded by $k+M$.

b. Let $\varepsilon > 0$ and let $x \in [0,1]$. Let also $\delta = \frac{\varepsilon}{k}$.

Then: $\forall y \in [0,1]$, $|x-y| < \delta$ one has,

$\forall u \in F_{k,M}$, $|u(x) - u(y)| \leq k|x-y| \leq k\frac{\varepsilon}{k} = \varepsilon$

Hence, $F_{k,M}$ is equicontinuous on $[0,1]$.

c. Using Ascoli's theorem, $\overline{F_{k,M}}$ is compact. Since $F_{k,M}$ is closed, $F_{k,M}$ is compact.

Exercise 12:

1. For every $t \in [0,1]$ and $n \in \mathbb{N}$, let $F_n(t) = \int_0^t f_n'(s)\,ds$.

Since (f_n') cv uniformly to g on $[0,1]$,

$$\lim_{n\to\infty} F_n(t) = \int_0^t \lim_{n\to\infty} f_n'(s)\,ds = \int_0^t g(s)\,ds.$$

But: $\forall t \in [0,1]$, $F_n(t) = f_n(t) - f_n(0)$ hence, using uniqueness of the limit,

$$\forall t \in [0,1],\quad f(t) - f(0) = \int_0^t g(s)\,ds.$$

Moreover, since the f_n' are C^0 on $[0,1]$ and cv uniformly to g on $[0,1]$, g is also C^0 on $[0,1]$. Hence $t \mapsto \int_0^t g(s)\,ds$ is C^1 on $[0,1]$ and f is C^1 on $[0,1]$ with:

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$$\forall t \in [0,1],\ f'(t) = g(t).$$

Hence the wanted result.

2. a. Let (f_n) a sequence of elements of $(E, \| \|_\infty)$ which cv to $f \in E$ and such that $(T(f_n))_{n \in \mathbb{N}}$ cv to g in $(F, \| \|_\infty)$.

i.e (f_n) cvu to f on $[0,1]$ and (f_n') cvu to g on $[0,1]$.

By 1., f is C^1 on $[0,1]$ and $f' = g$ i.e. $g = T(f)$.

Hence the graph of T is closed and since $(E, \| \|_\infty)$ and $(F, \| \|_\infty)$ are Banach space (as F is closed in the Banach space E), T being linear, it is continuous.

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b. Let B_F the unit closed ball of F.

Let $\varepsilon > 0$ and $t, t' \in [0,1]$. Then:

$$\forall f \in F, \quad |f(t) - f(t')| = \left| \int_{t'}^{t} f'(s)\,ds \right| \leq |t-t'|\, \|f'\|_\infty$$

$$= |t-t'|\, \|T(f)\|_\infty \leq |t-t'|\, C \underbrace{\|f\|_\infty}_{\leq 1} \leq C|t-t'|$$

where $C > 0$ is such that: $\forall f \in E, \ \|T(f)\|_\infty \leq C \|f\|_\infty$ which exists by 2.a. and the C^0 of T.

For $\delta = \frac{1}{C}\varepsilon > 0$ one gets:

$$\forall t, t' \in [0,1], \ |t-t'| < \delta, \ \forall f \in F, \ |f(t) - f(t')| \leq C \frac{1}{C}\varepsilon = \varepsilon.$$

Hence B_F is equicontinuous.

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Moreover : $\forall t \in [0,1], \forall f \in B_F, |f(t)| \leq \|f\|_\infty \leq 1$

Hence, $\{f(x)\}_{f \in B_F}$ is bounded for every $x \in [0,1]$.

Using Ascoli's theorem, $\overline{B_F} = B_F$ is compact in $E = C([0,1], \mathbb{R})$.

$\subseteq$: Since B_F is compact, by Riesz theorem, F is of finite dimension.

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Exercise 13

1. Since $\ell^\infty(\mathbb{N})$ is of infinite dimension, by Riesz theorem, its unit ball is not compact.

2. a. Let $(v_n) \in K^{\mathbb{N}}$ which cv to some $\ell \in [0,1]$. Then either (v_n) takes only a finite number of different values and in this case (v_n) is stationary to some $k \in K$ for n large enough and $\ell = k$; either (v_n) is a subsequence of $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ which tends to 0 at infinity hence $\ell = 0$.

In both cases $\ell \in K$ and K is closed in $[0,1]$. Since it is bounded by 1, it is a compact set of $\mathbb{R}$.

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b. Let us show first that $G \subset C(K, \mathbb{R})$. Let $f \in G$.
Let $x_0 \in K$. Let $(x_n) \in K^{\mathbb{N}}$ such that $x_n \xrightarrow[n \to +\infty]{} x_0$.

Let us show that $f(x_n) \xrightarrow[n \to +\infty]{} f(x_0)$ to prove that f is continuous at x_0.

- If $x_0 = 0$ then since : $\forall n \in \mathbb{N}, |f(x_n)| \leq x_n \xrightarrow[n \to +\infty]{} 0$.
  hence $f(x_n) \xrightarrow[n \to +\infty]{} 0$.
  But since $|f(0)| \leq 0$, $f(0) = 0$ and $f(x_n) \xrightarrow[n \to +\infty]{} f(0) = f(x_0)$.
- If $x_0 \neq 0$ since $x_0 \in K$, $\exists n_0 \geq 1$, $x_0 = \frac{1}{n_0}$.
  Since $\frac{1}{n_0}$ is an isolated point of K and (x_n) cv to x_0,
  $\exists N_0 \in \mathbb{N}$, $\forall n \geq N_0$, $x_n = \frac{1}{n_0}$ and : $f(x_n) = f\left(\frac{1}{n_0}\right) = f(x_0)$
  Hence, again: $f(x_n) \xrightarrow[n \to +\infty]{} f(x_0)$.

Hence f is continuous at each $x_0 \in K$ hence, $f \in C(K, \mathbb{R})$.

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Let (f_n) a sequence of elements of G which converges to f for $\| \|_K$.

In particular (f_n) cv pointwise to f on K and since:

$\forall n \in \mathbb{N}, \forall x \in K, |f_n(x)| \leq x$,

and letting n tends to infinity: $\forall x \in K, |f(x)| \leq x$.

Hence, $f \in G$ and G is a closed set in $(C(K, \mathbb{R}), \| \|_K)$.

c. For $u, v \in F$,

$$\|T(u) - T(v)\|_K = \sup_{x \in K} |T(u)(x) - T(v)(x)|$$

$$= \max\left(0, \sup_{n \geq 1} \left|T(u)\left(\frac{1}{n}\right) - T(v)\left(\frac{1}{n}\right)\right|\right)$$

$$= \sup_{n \geq 1} |u_n - v_n| = \|u - v\|_\infty.$$

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Hence T is an isometry from $(F, \|\ \|_\infty)$ to $(G, \|\ \|_K)$.
In particular, T is continuous.
Moreover, T is a bijection of reciprocal map,

$$T^{-1} : \begin{array}{l} G \longrightarrow F \\ f \longmapsto u_n \end{array} \quad \text{where : } \forall n \in \mathbb{N},\ u_n = \begin{cases} 0 = f(0) & \text{if } n = 0 \\ f\left(\frac{1}{n}\right) & \text{if } n \geq 1 \end{cases}$$

Then : $\forall f, g \in G,$

$$\|T^{-1}(f) - T^{-1}(g)\|_\infty = \max\left(0, \sup_{n \in \mathbb{N}^*} \left|f\left(\tfrac{1}{n}\right) - g\left(\tfrac{1}{n}\right)\right|\right)$$

$$= \sup_{x \in K} |f(x) - g(x)| \quad \text{(since for } f, g \in G, f(0) = 0 \text{ and } g(0) = 0\text{)}$$

$$= \|f - g\|_K.$$

Hence T^{-1} is also an isometry and therefore it is continuous.

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Finally, T is C^0, T is a bijection and T^{-1} is C^0:

T is an homeomorphism from $(F, \| \|_\infty)$ to $(G, \| \|_K)$.

d. If F is compact, $T(F)$ is also compact by C^0 of T and $T(F) = G$ hence G is compact.

If G is compact, $F = T^{-1}(G)$ is compact by C^0 of T^{-1}.

3.a. Let $x \in K$. Then:

$\forall f \in G$, $|f(x)| \leq x \leq 1$ and $\{f(x)\}_{f \in G}$ is bounded by 1.

b. Let $x \in K$, and let $\varepsilon > 0$.

- If $x = 0$ then for every $y \in K$, $|x - y| \leq \varepsilon$, $|y| \leq \varepsilon$ and: $\forall f \in G$, $|f(x) - f(y)| = |\underbrace{f(0)}_{=0 \text{ since } f \in G} - f(y)| = |f(y)| \leq y \leq \varepsilon$.

G is therefore equicontinuous at 0.

- If $x \in \{\frac{1}{n}\}_{n \geq 1}$ then there exists $k \geq 1$, $x = \frac{1}{k}$.

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Let $\delta = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2k(k+1)} > 0$. Then for every $y \in K$, $|x-y| < \delta$, $x = y$ and $|f(x) - f(y)| = 0 \leq \varepsilon$.

Hence G is equicontinuous at x.

Since $K = \{0\} \cup \{\frac{1}{n}\}_{n \geq 1}$, G is equicontinuous on K.

$\therefore$ Using Ascoli's theorem, $\overline{G}$ is compact. But since G is closed, G is compact and therefore F is compact.