2023-2024
M1 Maths
Functional Analysis

Exercises sheet 2 : Hilbert spaces
Solutions.

(1) **Exercise 1:**

1. • Let $\varepsilon > 0$, $t, t' \in [0,1]$. Then:

$$\|\varphi(t) - \varphi(t')\|_{\mathcal{H}}^2 = \int_0^1 |\mathbb{1}_{[0,t]}(x) - \mathbb{1}_{[0,t']}(x)|^2 dx$$

$$= \left| \int_t^{t'} dx \right| = |t' - t|.$$

For $\delta = \varepsilon^2$, if $|t' - t| \leq \delta$ then $\|\varphi(t) - \varphi(t')\|_{\mathcal{H}} \leq \varepsilon$
and φ is continuous.

• Let $0 \leq t' < t \leq 1$. Then: $\dfrac{\varphi(t) - \varphi(t')}{t - t'} = \dfrac{1}{t - t'} \mathbb{1}_{[t', t]}$

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and: $\left\| \frac{\varphi(t)-\varphi(t')}{t-t'} \right\|_H = \frac{1}{|t-t'|}\sqrt{|t'-t|} = \frac{1}{\sqrt{|t'-t|}} \xrightarrow[t\to t']{} +\infty$

Hence φ is not differentiable in t' hence it is nowhere differentiable.

2. Let $0 \leq s < s' \leq t < t' \leq 1$.

Then: $(\varphi(s')-\varphi(s) \mid \varphi(t')-\varphi(t)) = (\mathbb{1}_{[s,s']} \mid \mathbb{1}_{[t,t']})$

$$= \int_0^1 \underbrace{\mathbb{1}_{[s,s']}(x)\, \mathbb{1}_{[t,t']}(x)}_{=0 \text{ since } [s,s']\cap[t,t']=\emptyset}\, dx$$

$$= \int_0^1 0\, dx = 0.$$

Hence $\varphi(s')-\varphi(s) \perp \varphi(t')-\varphi(t)$

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Exercise 2

1. It is clear that for every $P, P', Q, Q' \in \mathbb{C}[X]$ and every $\lambda, \mu \in \mathbb{C}$,

- $\varphi(P, Q) = \overline{\varphi(Q, P)}$ and
- $\varphi(\lambda P + \mu P', Q) = \bar{\lambda}\varphi(P, Q) + \bar{\mu}\varphi(P', Q)$,

  $\varphi(P, \lambda Q + \mu Q') = \lambda\varphi(P, Q) + \mu\varphi(P, Q')$.
- Moreover: $\varphi(P, P) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta \geq 0$ and if

  $\varphi(P, P) = 0$ then by positivity and continuity of $\theta \mapsto |P(e^{i\theta})|^2$: $\forall \theta \in [-\pi, \pi]$, $|P(e^{i\theta})|^2 = 0$

  Hence every $z \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a root of P.

This set being infinite, $P = 0$.

2. Let $k, l \in \mathbb{N}$. One has:

$$\varphi(X^k, X^l) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} e^{il\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(l-k)\theta}\, d\theta$$

$$= \begin{cases} \left[\frac{1}{i(k-l)} e^{i(l-k)\theta}\right]_{-\pi}^{\pi} & \text{if } k \neq l \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i0\theta}\, d\theta & \text{if } k = l \end{cases}$$

$$= \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases}$$

Hence, $(X^k)_{k \in \mathbb{N}}$ is an orthonormal basis of $\mathbb{C}[X]$ for φ.

3. One has:

$$\|Q\|^2 = \varphi(Q,Q) = \sum_{k,l=0}^{n-1} \bar{a}_k a_l \varphi(X^k, X^l) + \varphi(X^n, X^n)$$

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$$= 1 + \sum_{k=0}^{n-1} |a_k|^2 = 1 + |a_0|^2 + \dots + |a_{n-1}|^2 \geq 1.$$

4. On the other side, $\|Q\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(e^{i\theta})|^2 d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} M^2 d\theta = M^2$

and $\|Q\| \leq M$. But $\|Q\| \geq 1$ hence $M \geq 1$.

If $M=1$ using Q3. : $|a_{n-1}| = \dots = |a_0| = 0$ and $Q = X^n$

Reciprocally if $Q = X^n$,

$$M = \sup_{|z|=1} |z^n| = \sup_{|z|=1} |z|^n = 1.$$

Hence : $M = 1 \iff Q = X^n$.

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Exercise 3:

1. It is clear that for every $P, P', Q, Q' \in \mathbb{R}[X]$ and every $\lambda, \mu \in \mathbb{R}$,

- $\varphi(P,Q) = \varphi(Q,P)$ and
- $\varphi(\lambda P + \mu P', Q) = \lambda \varphi(P,Q) + \mu \varphi(P',Q)$,

  $\varphi(P, \lambda Q + \mu Q') = \lambda \varphi(P,Q) + \mu \varphi(P,Q')$.

- If $P \in \mathbb{R}[X]$, $\varphi(P,P) = \int_0^{+\infty} \underbrace{(P(t))^2}_{\geq 0} e^{-t} dt \geq 0$.

and if $\varphi(P,P) = 0$ by C^0 and positivity of $t \mapsto (P(t))^2 e^{-t}$,

$\forall t \in [0, +\infty[$, $(P(t))^2 e^{-t} = 0 \Rightarrow \forall t \geq 0$, $P(t) = 0$.

Hence every positive real number is a root of P which has

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therefore an infinite number of roots: $P=0$.

2. Let $p,q\in\mathbb{N}$.

$$\varphi(X^p,X^q)=\int_0^{+\infty}t^{p+q}e^{-t}\,dt=\left[t^{p+q}(-e^{-t})\right]_0^{+\infty}+(p+q)\int_0^{+\infty}t^{p+q-1}e^{-t}\,dt$$

$$=0+0+(p+q)\int_0^{+\infty}t^{p+q-1}e^{-t}\,dt$$

Using a recurrence:

$$\varphi(X^p,X^q)=(p+q)!\int_0^{+\infty}e^{-t}\,dt=(p+q)!$$

3. This infimum is the square of the distance between X^2 and the plan $\mathbb{R}_1[X]\subset\mathbb{R}[X]$ for the scalar product defined by φ.

Hence, if $p_{\mathbb{R}_1[X]}(X^2)$ is the orthogonal projection of X^2 onto $\mathbb{R}_1[X]$

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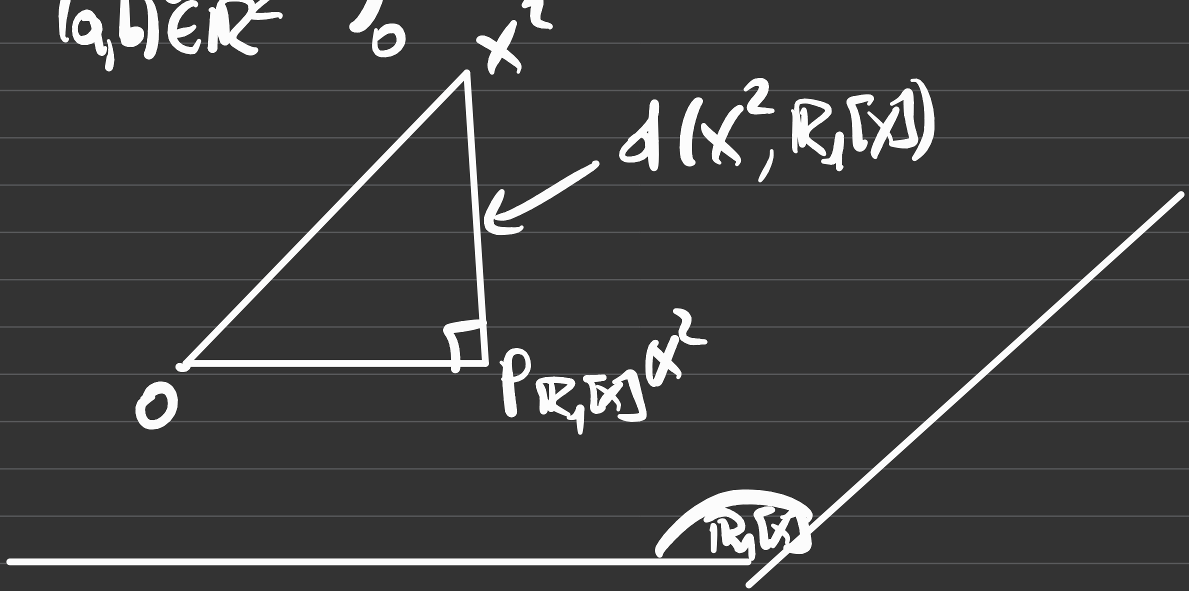
$$\inf_{(a,b)\in\mathbb{R}^2} \int_0^{+\infty} e^{-t}\left(t^2-(at+b)\right)^2 dt = \| X^2 - p_{\mathbb{R}_1[X]}(X^2)\|_\varphi^2$$



First, we need to construct an orthonormal basis of $\mathbb{R}_1[X]$, orthonormal for φ.

We apply Gram-Schmidt to the canonical basis $(1, X)$.

We have: $\varphi(1,1) = (0+0)! = 1$

Then we search $\lambda, \mu \in \mathbb{R}$, $1 \perp \lambda + \mu X$

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$$\varphi(1, \lambda+\mu X) = \varphi(1,\lambda) + \varphi(1,\mu X)$$
$$= \lambda\, \varphi(1,1) + \mu\, \varphi(1,X)$$
$$= \lambda + \mu\,(0+1)! = \lambda+\mu.$$

One can choose $\lambda = 1$ and $\mu = -1$ to get $\varphi(1, 1-X) = 0$ and $1 \perp 1-X$.

It remains to renormalize $1-X$:

$$\|1-X\|_\varphi^2 = \varphi(1-X, 1-X) = \varphi(1,1) - 2\varphi(1,X) + \varphi(X,X)$$
$$= 1 - 2\times 1! + 2! = 1 - 2 + 2 = 1.$$

Hence $\|1-X\|_\varphi = 1.$

Finally, $(1, 1-X)$ is an orthonormal basis of $\mathbb{R}_1[X]$ for φ.

Then: $p_{\mathbb{R}_1[X]}(X^2) = \varphi(X^2, 1) \times 1 + \varphi(X^2, 1-X) \times (1-X)$

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$$= (2+0)! \times 1 + \left((2+0)! - (2+1)!\right)(1-x)$$

$$= 2 - 4(1-x) = 4x - 2 = 2(2x-1)$$

Then: $\left(d(X^2, \mathbb{R}_1[X])\right)^2 = \| x^2 - (4x-2) \|_\varphi^2$

$$= \varphi\left(X^2 - 4X + 2, X^2 - 4X + 2\right)$$

$$= \varphi(x^2, x^2) - 4\varphi(x^2, X) + 2\varphi(X^2, 1) - 4\varphi(X, X^2) + 16\varphi(x, x) - 8\varphi(X, 1) + 2\varphi(1, x^2) - 8\varphi(1, X) + 4\varphi(1,1)$$

$$= \cancel{4}! - 4\cancel{\times}3! + 2\times 2! - 4\times 3! + 16\times 2! - 8\times 1! + 2\times 2! - 8\times 1! + 4\times 0!$$

$$= 4 - 2\cancel{4} + \cancel{32} \cancel{-8} + \cancel{4} \cancel{-8} + \cancel{4} = 4$$

Hence $\displaystyle\inf_{(a,b)\in\mathbb{R}^2} \int_0^{+\infty} e^{-t}\left(t^2 - (at+b)\right)^2 dt = 4$

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Exercise 4.

1. T is clearly linear. Moreover, if $(x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{C})$,

$$\left|T((x_n)_{n\in\mathbb{N}})\right| = \left|\sum_{n=0}^{N} x_n\right| \underset{\text{Cauchy Schwarz}}{\leq} \sqrt{N+1}\left(\sum_{n=0}^{N}|x_n|^2\right)^{\frac{1}{2}}$$

$$\leq \sqrt{N+1}\,\|(x_n)_{n\in\mathbb{N}}\|_2$$

Hence the continuity of T.

Since $M_N = \operatorname{Ker} T$, M_N is closed in $\ell^2(\mathbb{N},\mathbb{C})$.

2. As M_N is a closed subspace of $\ell^2(\mathbb{N},\mathbb{C})$,

$$\ell^2(\mathbb{N},\mathbb{C}) = M_N \overset{\perp}{\oplus} M_N^{\perp}.$$

3. a. Let $(y_n)_{n\in\mathbb{N}} \in E$ and $(x_n)_{n\in\mathbb{N}} \in M_N$

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Then: $\left((x_n)_{n\in\mathbb{N}} \mid (y_n)_{n\in\mathbb{N}}\right) = \sum_{n=0}^{+\infty} x_n \overline{y_n} \underset{\text{since } \forall n>N,\ y_n=0}{=} \sum_{n=0}^{N} x_n \overline{y_n} \underset{\text{by def of } E}{=} \overline{y_0}\sum_{n=0}^{N} x_n$

$\underset{\text{since } (x_n)\in M_N}{=} \overline{y_0}\times 0 = 0$

Hence $(y_n)_{n\in\mathbb{N}} \in M_N^{\perp}$ and $E \subset M_N^{\perp}$.

b. Reciprocally, let us show that $M_N^{\perp} \subset E$.

Let $(y_n)_{n\in\mathbb{N}} \in M_N^{\perp}$. Then: $\forall (x_n)_{n\in\mathbb{N}} \in M_N$, $\left((y_n)_{n\in\mathbb{N}} \mid (x_n)_{n\in\mathbb{N}}\right) = 0$

In particular, if for $0 \leq i < j \leq N$ one sets $x_i = 1$ and $x_j = -1$ and for $k \neq i, j$, $x_k = 0$ then $\sum_{n=0}^{N} x_n = 0$ i.e $(x_n)_{n\in\mathbb{N}} \in M_N$

and: $0 = \sum_{n=0}^{+\infty} y_n \overline{x_n} = y_i x_i + y_j x_j = y_i - y_j$ i.e $y_i = y_j$.

Thus, $(y_n)_{n\in\mathbb{N}} \in E$ and $M_N^{\perp} \subset E$. Finally, $E = M_N^{\perp}$.

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Exercise 5:

1. Let $f, g \in A$ and $\lambda \in [0,1]$. Then

$$\int_\Omega (1-\lambda) f(x) + \lambda g(x)\,dx = (1-\lambda)\int_\Omega f(x)\,dx + \lambda \int_\Omega g(x)\,dx$$

$$\geq (1-\lambda)\times 1 + \lambda \times 1 = 1.$$

Thus $(1-\lambda)f + \lambda g \in A$ and A is convex.

Moreover, if $(f_n) \in A^{\mathbb{N}}$ converge to $f \in H$ then by Cauchy-Schwarz:

$$\left| \int_\Omega f_n(x)\,dx - \int_\Omega f(x) \right| \leq \int_\Omega |f_n(x) - f(x)|\,dx \leq \left(\int_\Omega |f_n(x) - f(x)|^2\,dx \right)^{1/2} \left(\int_\Omega 1^2\,dx \right)^{1/2}$$

$$= \|f_n - f\|_{L^2} \times (\mathrm{Leb}(\Omega))^{1/2} \xrightarrow[n \to +\infty]{} 0$$

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Hence $\int_\Omega f_n(x)\,dx \xrightarrow[n\to+\infty]{} \int_\Omega f(x)\,dx$

But: $\forall n \in \mathbb{N}$, $\int_\Omega f_n(x)\,dx \geq 1$ and letting n tends to infinity,

$\int_\Omega f(x)\,dx \geq 1$ and $f \in A$.

Hence A is a closed convex set of H.

2. Since A is a closed convex set of H, one can apply the orthogonal projection theorem to A since $H = L^2(\Omega)$ is an Hilbert space.

Then $\min_{f \in A} \|g - f\|_2$ is attained at $\bar{g} \in A$, $\bar{g}$ being the orthogonal projection of g on A and only at this point.

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3. By characterization of the orthogonal projection $\bar{g} \in A$ of g on A,

$$\forall f \in A, \mathrm{Re}(g - \bar{g} \mid f - \bar{g})_2 \leq 0.$$

For $g = 0$ it gives

$$\forall f \in A, \quad \mathrm{Re}(-(\bar{g} \mid f - \bar{g})_2) \leq 0 \iff \mathrm{Re}(\bar{g} \mid f - \bar{g})_2 \geq 0, \ \forall f \in A$$

$$\iff \forall f \in A, \mathrm{Re}\left(\int_\Omega f(x)\, \bar{g}(x)\, dx\right) \geq \mathrm{Re} \int_\Omega (\bar{g}(x))^2 dx$$

But, since $f \in A \iff \int_\Omega f(x)\, dx \geq 1$, $\bar{g} \equiv \frac{1}{\mathrm{Leb}(\Omega)}$ on Ω is a possible choice. By uniqueness, one has:

$$\min_{f \in A} \|f\|_H = \left\|\frac{1}{\mathrm{Leb}(\Omega)}\right\|_H = \frac{1}{\mathrm{Leb}(\Omega)}$$

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Exercise 6:

1. Let $y \in H$ fixed.

a. One has: $\forall x \in H$, $|\phi_y(x)| = |\langle Ax | y \rangle| \underset{\text{Cauchy-Schwarz inequality}}{\leq} \|Ax\| \|y\|$

$$\leq \underbrace{|||A||| \, \|y\|}_{\text{constant independent of } x} \|x\|$$

Hence the continuity of ϕ_y.

b. Using Riesz representation theorem of the continuous linear forms in a Hilbert space:

$$\exists u \in H, \ \forall x \in H, \ \phi_y(x) = \langle x | u \rangle$$

Let us set $u = A^* y$. Then: $\forall x \in H$, $\langle Ax | y \rangle = \langle x | A^* y \rangle$

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2. Let $y_1, y_2 \in H$ and $\lambda, \mu \in \mathbb{C}$. One has:

$$\forall x \in H, \ \langle x \mid A^*(\lambda y_1 + \mu y_2) \rangle = \langle Ax \mid \lambda y_1 + \mu y_2 \rangle$$
$$= \bar{\lambda} \langle Ax \mid y_1 \rangle + \bar{\mu} \langle Ax \mid y_2 \rangle$$
$$= \bar{\lambda} \langle x \mid A^* y_1 \rangle + \bar{\mu} \langle x \mid A^* y_2 \rangle$$
$$= \langle x \mid \lambda A^* y_1 + \mu A^* y_2 \rangle$$

Hence: $\forall x \in H, \ \langle x \mid A^*(\lambda y_1 + \mu y_2) - (\lambda A^* y_1 + \mu A^* y_2) \rangle = 0$

and in particular for $x = A^*(\lambda y_1 + \mu y_2) - (\lambda A^* y_1 + \mu A^* y_2)$

one gets: $A^*(\lambda y_1 + \mu y_2) - (\lambda A^* y_1 + \mu A^* y_2) = 0$

i.e $A^*(\lambda y_1 + \mu y_2) = \lambda A^* y_1 + \mu A^* y_2$.

Hence $y \mapsto A^* y$ is linear.

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Moreover: $\forall y \in H$, $\|A^*y\|^2 = (A^*y \mid A^*y) = (y \mid AA^*y)$

$\xrightarrow{\text{Cauchy-Schwarz}} \leq \|y\| \, \|AA^*y\| \leq \|y\| \, |||A||| \, \|A^*y\| \quad (*)$

If $\|A^*y\| = 0$ then: $0 = \|A^*y\| \leq |||A||| \, \|y\|$

Else, dividing by $\|A^*y\|$ the $(*)$ inequality:

$$\|A^*y\| \leq |||A||| \, \|y\| .$$

Hence A^* is continuous and $|||A^*||| \leq |||A|||$.

3. One has: $\forall x, y \in H$, $\langle x \mid (A^*)^*y \rangle = \overline{\langle A^*x \mid y \rangle}$

$= \langle y \mid A^*x \rangle = \overline{\langle Ay \mid x \rangle} = \langle x \mid Ay \rangle$

Hence $(A^*)^* = A$.

Then: $|||(A^*)^*||| \leq |||A^*|||$ by the previous inequality (φ 2.)

i.e. $|||A||| \leq |||A^*|||$. Finally: $|||A^*||| = |||A|||$.

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4. One has: $\forall i,j \in [\![1,n]\!]$, $\langle e_i | A^* e_j \rangle = \langle A e_i | e_j \rangle$
$= \overline{\langle e_j | A e_i \rangle}$

i.e. $a^*_{ij} = \overline{a_{ji}}$ and $A^* = {}^t\overline{A}$.

5. a. For $u \in \ell^2(\mathbb{Z}, \mathbb{C})$,

$$\|Tu\|_{\ell^2} = \left(\sum_{n\in\mathbb{Z}} |u_{n+1}|^2\right)^{1/2} = \left(\sum_{n\in\mathbb{Z}} |u_n|^2\right)^{1/2} = \|u\|_{\ell^2}$$

hence the continuity of T.

b. One has: $\forall u, v \in \ell^2(\mathbb{Z}, \mathbb{C})$, $\langle u | Tv \rangle = \sum_{n\in\mathbb{Z}} u_n \overline{(Tv)_n}$

$$= \sum_{n\in\mathbb{Z}} u_n \overline{v_{n+1}} = \sum_{n\in\mathbb{Z}} u_{n-1} \overline{v_n} = \langle T^* u | v \rangle$$

Hence: $\forall u \in \ell^2(\mathbb{Z}, \mathbb{C})$, $\forall n \in \mathbb{Z}$, $(T^* u)_n = u_{n-1}$.

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Exercise 7:

1. a. To compute this integral, we use polar coordinates:

Let $z \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{N}$. Then:

$$\int_{B(z,r)} (\omega - z)^n \, d\lambda(\omega) \underset{\substack{\uparrow \\ \text{if } \omega \in \bar{B}(z,r),\ \omega - z \in \bar{B}(0,r)}}{=} \int_0^{2\pi}\int_0^r \rho^n e^{in\theta} \rho \, d\rho \, d\theta = \left(\int_0^{2\pi} e^{in\theta} d\theta\right)\left(\int_0^r \rho^{n+1} d\rho\right)$$

But: $\int_0^{2\pi} e^{in\theta} d\theta = \begin{cases} 2\pi & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$ and $\int_0^r \rho^{n+1} d\rho = \frac{r^{n+2}}{n+2}$

Hence: $\int_{B(z,r)} (\omega - z)^n \, d\lambda(\omega) = \begin{cases} 2\pi \frac{r^2}{2} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} = \begin{cases} \pi r^2 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$

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b. Let $f \in \mathcal{A}^2(\Omega)$. Then f is holomorphic on Ω and $\bar{B}(z,r) \subset \Omega$ is connected. Hence:

$$\forall w \in \bar{B}(z,r],\ f(w) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z)}{n!}(w-z)^n$$

with uniform cv on $\bar{B}(z,r)$. In particular one can exchange Σ and $\int$ in:

$$\int_{\bar{B}(z,r)} f(w)\, d\lambda(w) = \int_{B(z,r)} \sum_{n=0}^{+\infty} \frac{f^{(n)}(z)}{n!}(w-z)^n\, d\lambda(w) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(z)}{n!} \int_{B(z,r)} (w-z)^n\, d\lambda(w)$$

$$\overset{1.a.}{=} \frac{f^{(0)}(z)}{0!} \times \pi r^2 = \pi r^2 f(z)$$

Finally: $f(z) = \dfrac{1}{\pi r^2} \displaystyle\int_{\bar{B}(z,r)} f(w)\, d\lambda(w)$.

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c. Let $f \in A^2(\Omega)$, $z \in \Omega$ and $r > 0$. Assume that $d(z, \Omega^c) > r$.

Then $\bar{B}(z,r) \subset \Omega$ and:

$$|f(z)| = \left| \frac{1}{\pi r^2} \int_{\bar{B}(z,r)} f(w)\, d\lambda(w) \right| = \frac{1}{\pi r^2} \left| \int_\Omega \underbrace{f(w)}_{\in L^2(\Omega)} \underbrace{\mathbb{1}_{\bar{B}(z,r)}(w)}_{\in L^2(\Omega)} \, d\lambda(w) \right|$$

$$\xrightarrow{\text{Cauchy-Schwarz}} \leq \frac{1}{\pi r^2} \|f\|_2 \|\mathbb{1}_{\bar{B}(z,r)}\|_2$$

$$= \frac{1}{\pi r^2} \|f\|_2 \left(\int_{\bar{B}(z,r)} d\lambda(w) \right)^{1/2} = \frac{1}{\pi r^2} \|f\|_2 (\pi r^2)^{1/2}$$

$$= \frac{1}{r\sqrt{\pi}} \|f\|_2$$

Hence: $|f(z)| \leq \dfrac{1}{r\sqrt{\pi}} \|f\|_2$

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2. a. Let $(f_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $A^2(\Omega)$

Let $K\subset\Omega$ a compact set. Then, using Borel-Lebesgue property: $\exists r>0$, $K\subset\{z\in\Omega \mid d(z,\Omega^c)>r\}$

Then, if $p,q\in\mathbb{N}$ and if $z\in K$, using 1.c.:

$$|f_p(z)-f_q(z)| = |\underbrace{(f_p-f_q)}_{\in A^2(\Omega)}(z)| \leq \frac{1}{r\sqrt{\pi}}\|f_p-f_q\|_2.$$

Hence $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on K hence it cv uniformly on K.

b. Using Weierstraß theorem, there exists f holomorphic on Ω such that $(f_n)_{n\in\mathbb{N}}$ cv uniformly to f on every compact

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set of Ω and in particular, $(f_n)_{n\in\mathbb{N}}$ cv pointwisely to f on Ω.

c. To show that $A^2(\Omega)$ is complete for $\|\,\|_2$, it remains to prove that f obtained at 2.6 is in $A^2(\Omega)$ and that $(f_n)_{n\in\mathbb{N}}$ cv to f in $A^2(\Omega)$.

Let $\varepsilon>0$. By Cauchy property: $\exists N_\varepsilon\in\mathbb{N}, \forall p,q\geq N_\varepsilon,$

$$\|f_p-f_q\|_2\leq\varepsilon \iff \exists N_\varepsilon\in\mathbb{N}, \forall p,q\geq N_\varepsilon, \int_\Omega |f_p(\omega)-f_q(\omega)|^2 d\lambda(\omega)\leq\varepsilon^2$$

We fix p and use liminf in q to obtain, using Fatou's lemma,

$$(*)\quad \int_\Omega |f_p(\omega)-f(\omega)|^2 d\lambda(\omega)=\int_\Omega \liminf_{q\to+\infty}|f_p(\omega)-f_q(\omega)|^2 d\lambda(\omega)\leq\liminf_{q\to+\infty}\int_\Omega|f_p(\omega)-f_q(\omega)|^2 d\lambda(\omega)\leq\varepsilon^2$$

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Hence $f_p - f \in L^2(\Omega)$ and since $f_p \in L^2(\Omega)$, $f \in L^2(\Omega)$.

Since f is also holomorphic on Ω, $f \in A^2(\Omega)$.

With $f \in A^2(\Omega)$ one rewrites $(*)$:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon \in \mathbb{N},\ \forall p \geq N_\varepsilon\ ,\ \|f_p - f\|_2 \leq \varepsilon$$

It means that $(f_n)_{n\in\mathbb{N}}$ cv to f in $L^2(\Omega)$ hence in $A^2(\Omega)$.

This finally proves that $(A^2(\Omega), (\cdot|\cdot)_2)$ is a Hilbert space.

3. a. δ_z is clearly linear and if $r > 0$ is such that $\bar{B}(z,r) \subset \Omega$ then using 1.(c):

$$\forall f \in A^2(\Omega),\ |\delta_z(f)| = |f(z)| \leq \frac{1}{r\sqrt{\pi}} \|f\|_2.$$

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i.e. δ_z is continuous and its norm as a linear map is smaller than $\frac{1}{r\sqrt{\pi}}$ for every $r>0$ such that $\overline{B}(z,r)\subset\Omega$.

(b) Let $z\in\Omega$. Since $A^2(\Omega)$ is a Hilbert space and δ_z is a continuous linear form on $A^2(\Omega)$, using Riesz representation theorem, there exists a unique $K_z\in A^2(\Omega)$ such that:

$$\forall f\in A^2(\Omega),\quad \delta_z(f)=(f\,|\,K_z)_2=\int_\Omega f\,\overline{K_z}\,d\lambda$$

$$\text{i.e. } \forall f\in A^2(\Omega),\quad f(z)=\int_\Omega f(w)\,\overline{K_z(w)}\,d\lambda(w).$$

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4. (a) Let $\omega, z \in \Omega$. Applying 3(b) with $f = K_z$ and "$z = \omega$":

$$K_\Omega(\omega, z) = K_z(\omega) = (K_z | K_\omega)_2 = \overline{(K_\omega | K_z)_2}$$
$$= \overline{K_\omega(z)} = \overline{K_\Omega(z,\omega)}$$

Hence: $\forall (z,\omega) \in \Omega^2$, $K_\Omega(z,\omega) = \overline{K_\Omega(\omega, z)}$

(b) Since $(e_n)_{n\in\mathbb{N}}$ is a Hilbert basis of $A^2(\Omega)$ one writes the decomposition of $K_z \in A^2(\Omega)$ on it:

$$K_z = \sum_{n=0}^{+\infty} (K_z | e_n)_2 \, e_n \quad \text{with cv in } A^2(\Omega).$$

But: $\forall n \in \mathbb{N}$, $(K_z | e_n)_2 = \overline{(e_n | K_z)_2} = \overline{e_n(z)}$

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Hence: $$K_z = \sum_{n=0}^{+\infty} \overline{e_n(z)}\, e_n \quad \text{with cv in } A^2(\Omega).$$

5. (a) We have to prove that the given family $(e_n)_{n\in\mathbb{N}}$ is orthonormal and is such that: $\overline{\mathrm{Vect}((e_n)_{n\in\mathbb{N}})} = A^2(D)$.

(i) Let $n, p \in \mathbb{N}$:

$$(e_n|e_p)_2 = \int_D \sqrt{\frac{n+1}{\pi}}\, \omega^n \sqrt{\frac{p+1}{\pi}}\, \bar{\omega}^p \, d\lambda(\omega)$$

polar coordinates

$$\Rightarrow = \frac{\sqrt{(n+1)(p+1)}}{\pi} \int_0^{2\pi}\int_0^1 \rho^{n+p} e^{i(n-p)\theta} \rho \, d\rho \, d\theta$$

$$= \frac{\sqrt{(n+1)(p+1)}}{\pi} \left(\int_0^{2\pi} e^{i(n-p)\theta} d\theta\right) \left(\underbrace{\int_0^1 \rho^{n+p+1} d\rho}_{=\frac{1}{n+p+2}}\right)$$

$$= \begin{cases} 0 & \text{if } n \neq p \\ \frac{\sqrt{(n+1)(n+1)}}{\pi} \times 2\pi \times \frac{1}{n+n+2} & \text{if } n = p \end{cases} = \begin{cases} 0 & \text{if } n \neq p \\ 1 & \text{if } n = p \end{cases}$$

Hence $(e_n)_{n\in\mathbb{N}}$ is orthonormal.

(ii) We prove that for any $f \in A^2(D)$, $\|f\|_2^2 = \sum_{n=0}^{+\infty} |(f|e_n)|^2$.

Using equality case in Bessel inequality one gets then that $A^2(D) \subset \overline{\mathrm{Vect}((e_n)_{n\in\mathbb{N}})}$, which implies the equality of the two sets, the other inclusion being always true.

Let $f \in A^2(D)$ and $z \in D$: $f(z) = \sum_{n=0}^{+\infty} f_n z^n$

since f is holomorphic on the connected set D.

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Then: $\|f\|_2^2 = \int_D |f(z)|^2 dz = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 r\,dr\,d\theta$

$$= \int_0^1 r \left(\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right) dr$$

But: $\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \int_0^{2\pi} \left| \sum_{n=0}^{+\infty} f_n r^n e^{in\theta} \right|^2 d\theta$

$$= \int_0^{2\pi} \left(\sum_{n=0}^{+\infty} f_n r^n e^{in\theta} \right) \left(\sum_{p=0}^{+\infty} \overline{f_p} r^p e^{-ip\theta} \right) d\theta \quad (\text{since } |z|^2 = z\overline{z})$$

$$= \int_0^{2\pi} \sum_{n,p \geq 0} f_n \overline{f_p} r^{n+p} e^{i(n-p)\theta} d\theta$$

using uniform cv
on $\overline{D}(0,r)$, $r<1$

$$= \sum_{n,p \geq 0} f_n \overline{f_p} r^{n+p} \int_0^{2\pi} e^{i(n-p)\theta} d\theta$$

$$= \sum_{n=0}^{+\infty} |f_n|^2 r^{2n} \times 2\pi$$

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Remark that we could also have directly apply Parseval's equality.

Then: $\|f\|_2^2 = \int_0^1 r\left(2\pi\sum_{n=0}^{+\infty}|f_n|^2 r^{2n}\right)dr \underset{\text{Beppo-Levi}}{=} 2\pi\sum_{n=0}^{+\infty}|f_n|^2\underbrace{\int_0^1 r^{2n+1}dr}_{=\frac{1}{2n+2}}$

$$= \sum_{n=0}^{+\infty}\frac{\pi}{n+1}|f_n|^2.$$

But, if $n\geq 0$: $(f|e_n)_2 = \int_D f(z)\overline{e_n(z)}\,dz = \int_0^1\int_0^{2\pi} f(re^{i\theta})\sqrt{\frac{n+1}{\pi}}\,r^n e^{-in\theta}\,r\,dr\,d\theta$

$$= \int_0^1\sqrt{\frac{n+1}{\pi}}\,r^{n+1}\left(\int_0^{2\pi}\left(\sum_{p=0}^{+\infty}f_p r^p e^{ip\theta}\right)e^{-in\theta}d\theta\right)dr$$

As before: $\int_0^{2\pi}\sum_{p=0}^{+\infty}f_p r^p e^{i(p-n)\theta}d\theta = \sum_{p=0}^{+\infty}f_p r^p\int_0^{2\pi}e^{i(p-n)\theta}d\theta = 2\pi f_n r^n$

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Hence: $(f|e_n)_2 = \sqrt{\frac{n+1}{\pi}} \times 2\pi f_n \times \int_0^1 r^{2n+1} dr = \sqrt{\frac{n+1}{\pi}} \times \frac{2\pi f_n}{2n+2} = \sqrt{\frac{\pi}{n+1}} f_n$

and:

$$|(f|e_n)_2|^2 = \frac{\pi}{n+1} |f_n|^2$$

Finally: $$\|f\|_2^2 = \sum_{n=0}^{+\infty} |(f|e_n)_2|^2$$

This proves that $A^2(D) = \overline{\mathrm{Vect}((e_n)_{n\in\mathbb{N}})}$

Hence $(e_n)_{n\in\mathbb{N}}$ is a Hilbert basis of $A^2(D)$.

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(b) Using 4(b):

$$\forall \omega \in D, \forall z \in D, \quad K_D(\omega, z) = K_z(\omega) = \sum_{n=0}^{+\infty} \overline{e_n(z)}\, e_n(\omega)$$

$$= \sum_{n=0}^{+\infty} \frac{n+1}{\pi} (\bar{z}\omega)^n$$

Let $u = \bar{z}\omega$. Remark that: $(n+1)u^n = \frac{d}{du}(u^{n+1})$.

Hence: $\forall (\omega, z) \in D^2, \; K_D(\omega, z) = \frac{1}{\pi} \frac{d}{du} \sum_{n=0}^{+\infty} u^{n+1} = \frac{1}{\pi} \frac{d}{du}\left(\frac{1}{1-u} - 1\right)$

uniform cv of derivate series on every compact in the cv disk

$$= \frac{1}{\pi} \frac{1}{(1-u)^2} = \frac{1}{\pi} \frac{1}{(1-\bar{z}\omega)^2}$$

Hence: $\forall (\omega, z) \in D^2, \; K_D(\omega, z) = \frac{1}{\pi} \frac{1}{(1-\bar{z}\omega)^2}$