2023-2024
M1 Maths
Functional Analysis

Exercise Sheet 3 - L^p spaces - Solutions

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Exercise 1:

$$\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$$

Let $f \in L^p(\mathbb{R})$, $p \in]1, +\infty[$.

1.2. Let $x \geq 0$. Using Hölder with $\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{p}{p-1}$, one has:

$$\left|\int_0^x f(t)\,dt\right| = \left|\int_0^x f(t) \times 1\,dt\right| \leq \left(\int_0^x |f(t)|^p\right)^{\frac{1}{p}} \times \left(\int_0^x 1^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$= \|f\|_{L^p}\, x^{\frac{p-1}{p}}$$

Hence $F(x)$ is well defined and: $F(x) = O\left(x^{\frac{p-1}{p}}\right)$.

② 3. One has: $\left(\int_a^{+\infty} |f(t)|^p dt\right)^{1/p} = \left(\int_{\mathbb{R}} |f(t)|^p \mathbb{1}_{[a,+\infty[}(t)\, dt\right)^{1/p}$

But; $\forall t \in \mathbb{R}$, $\lim_{a \to +\infty} |f(t)|^p \mathbb{1}_{[a,+\infty[}(t) = 0$ (it suffices to take $a > t$ to obtain 0 ...)

Moreover: $\forall t \in \mathbb{R}, \forall a \in \mathbb{R}$, $\left| |f(t)|^p \mathbb{1}_{[a,+\infty[}(t) \right| \leq \underbrace{|f(t)|^p}_{\text{ind of } a}$

and in $L^1(\mathbb{R})$ since $f \in L^p(\mathbb{R})$.

Applying dominated conv theorem, one gets:

$$\left(\int_a^{+\infty} |f(t)|^p dt\right)^{1/p} \xrightarrow[a \to +\infty]{} 0$$

i.e: $\forall \varepsilon > 0$, $\exists a > 0$, $\forall a' \geq a$, $\left(\int_{a'}^{+\infty} |f(t)|^p\right)^{1/p} \leq \varepsilon$

and we take $a' = a$.

③ 4. Let $a \geq 0$ given at question 3. Let $x \geq a$.

Then: $|F(x) - F(a)| = \left|\int_a^x f(t)\,dt\right| \leq \int_a^x |f(t)|\,dt$

Hölder $\rightarrow \leq \left(\int_a^x |f(t)|^p\,dt\right)^{1/p} \left(\int_a^x 1^{\frac{p}{p-1}}\,dt\right)^{\frac{p-1}{p}}$

positivity $\rightarrow \leq \left(\int_a^{+\infty} |f(t)|^p\,dt\right)^{1/p} \left(\int_0^x dt\right)^{\frac{p-1}{p}}$

3. $\rightarrow \leq \varepsilon\, x^{\frac{p-1}{p}}$

Then, using triangular inequality: $\forall x \geq a$, $\dfrac{|F(x)|}{x^{\frac{p-1}{p}}} \leq \varepsilon + \dfrac{|F(a)|}{x^{\frac{p-1}{p}}}$

But $\dfrac{|F(a)|}{x^{\frac{p-1}{p}}} \xrightarrow[x \to +\infty]{} 0$, hence: $\exists x_0 \geq a$, $\forall x \geq x_0$, $\dfrac{|F(a)|}{x^{\frac{p-1}{p}}} \leq \varepsilon$.

Finally: $\forall x \geq x_0$, $\dfrac{|F(x)|}{x^{\frac{p-1}{p}}} \leq 2\varepsilon$ hence the desired result.

Exercise 2

Let $p \in [1, +\infty[$. Let $f \in C_0^0(\mathbb{R})$: continuous and compactly supported. Let $A > 0$ such that $\operatorname{supp} f \subset [-A, A]$.

Since f is C^0, equal to zero outside of $[-A, A]$ and C^0 on the compact $[-A, A]$, it is uniformly continuous on $\mathbb{R}$:

$\forall \varepsilon > 0, \exists \eta_\varepsilon > 0, \forall x, y \in \mathbb{R}, |x-y| < \eta_\varepsilon \Rightarrow |f(x) - f(y)| \leq \varepsilon.$

Let $x \in \mathbb{R}$ and a such that $|a| < \eta_\varepsilon$. Then : $|(x-a) - x| < \eta_\varepsilon$ and $|f(x-a) - f(x)| < \varepsilon$.

Moreover, by eventually taking η_ε smaller, one can assume $|a| < 1$ and in this case, if $x \notin [-A-1, A+1]$, $x \notin [-A, A]$ and $x - a \notin [-A, A]$.

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Hence $f(x-a) = f(x) = 0$ and $|f(x-a) - f(x)| = 0$.

Therefore: $\forall a$, $|a| < \min(1, \eta_\varepsilon)$,

$$\|\tau_a(f) - f\|_p^p = \int_{\mathbb{R}} |f(x-a) - f(x)|^p dx$$

$$= \int_{-A-1}^{A+1} |f(x-a) - f(x)|^p dx$$

$$\leq \int_{-A-1}^{A+1} \varepsilon^p dx = (2A+2)\varepsilon^p$$

By definition: $\lim\limits_{a \to 0} \|\tau_a(f) - f\|_p = 0$.

The result is therefore proven for $f \in C_0^0(\mathbb{R})$.

Let $f \in L^p(\mathbb{R})$ and let $\varepsilon > 0$. Since $C_0^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $\|\ \|_p$, there exists $g_\varepsilon \in C_0^0(\mathbb{R})$ such

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that $\|f-g_\varepsilon\|_p \leq \varepsilon$.

By change of variable $y = x-a$ we also get that:

$$\|\tau_a(f) - \tau_a(g_\varepsilon)\|_p \leq \varepsilon.$$

But, $g_\varepsilon \in C^0_c(\mathbb{R})$ hence there exists $\eta_\varepsilon > 0$ such that if $|a| < \eta_\varepsilon$, $\|\tau_a(g_\varepsilon) - g_\varepsilon\|_p \leq \varepsilon$. Then, for $|a| < \eta_\varepsilon$:

$$\|\tau_a(f) - f\|_p \leq \|\tau_a(f) - \tau_a(g_\varepsilon)\|_p + \|\tau_a(g_\varepsilon) - g_\varepsilon\|_p + \|g_\varepsilon - f\|_p$$
$$\leq 3\varepsilon$$

hence the wanted result.

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Exercise 3:

1. Let $f \in C_0^0(\mathbb{R})$. Let $A>0$, $\operatorname{supp} f \subset [-A, A]$.

then: (i) $\forall x \in \mathbb{R}$, $y \mapsto f(x-y)\mathbb{1}_{[0,1]}(y) \in L^1(\mathbb{R})$ since $f \in L^1(\mathbb{R})$

(ii) $\forall y \in [0,1]$, $x \mapsto f(x-y)\mathbb{1}_{[0,1]}(y)$ is C^0

(iii) $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $|f(x-y)\mathbb{1}_{[0,1]}(y)| \leq \underbrace{M \mathbb{1}_{[0,1]}(y)}_{\in L^1(\mathbb{R}) \text{ and ind of } x}$

where M is an upper bound of f on $\mathbb{R}$

Using the theorem of continuity under the integral, Tf is C^0.

2. The sequence $(f_n)_{n\in\mathbb{N}}$ exists by density of $C_0^0(\mathbb{R})$ in $L^1(\mathbb{R})$ for $\| \|_1$.

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Then : $\forall x \in \mathbb{R}$, $|Tf_n(x) - Tf(x)| \leq \int_0^1 |f_n(x-y) - f(x-y)|\,dy$

$$\leq \|f_n - f\|_1$$

and $\|Tf_n - Tf\|_\infty \leq \|f_n - f\|_1 \xrightarrow[n\to+\infty]{} 0$. Then $(Tf_n)_{n\in\mathbb{N}}$ CV uniformly to Tf on $\mathbb{R}$. Since each Tf_n is C^0 on $\mathbb{R}$ by 1., we deduce that Tf is also C^0 on $\mathbb{R}$.

3. Remark that: $\forall f \in L^1(\mathbb{R})$, $Tf = f * \mathbb{1}_{[0,1]}$.

Assume by the absurd that there exists $u \in L^1(\mathbb{R})$ such that: $\forall f \in L^1(\mathbb{R})$, $f * u = f$. Then : $Tu = u * \mathbb{1}_{[0,1]} = \mathbb{1}_{[0,1]}$.

But $u \in L^1(\mathbb{R})$, hence by 2. Tu is C^0 and $\mathbb{1}_{[0,1]}$ would be C^0 on $\mathbb{R}$!

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Exercise 4:

1. One has $\|f\|_p = \left(\int_{-1}^{1} \left(\mathbb{1}_{]-1,0[}(x)\right)^p dx\right)^{1/p} = \left(\int_{-1}^{0} dx\right)^{1/p} = 1$

Similarly $\|g\|_p = 1$. Then:

$$\|f+g\|_p = \int_{-1}^{1} \left|\mathbb{1}_{]-1,0[}(x) + \mathbb{1}_{]0,1[}(x)\right|^p dx\Big)^{1/p} = \left(\int_{-1}^{1} dx\right)^{1/p} = 2^{1/p}$$

and $\|f-g\|_p = \left(\int_{-1}^{1} \left|\mathbb{1}_{]-1,0[}(x) - \mathbb{1}_{]0,1[}(x)\right|^p dx\right)^{1/p} = 2^{1/p}$.

2. Assume that for $p \neq 2$, $L^p([-1,1])$ is an Hilbert space.

Applying parallelogram identity:

$\forall f, g \in L^p([-1,1])$, $2(\|f\|_p^2 + \|g\|_p^2) = \|f-g\|_p^2 + \|f+g\|_p^2$

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For f and g as in question 1 we would obtain:

$$2 \times (1+1) = 2^{2/p} + 2^{2/p} \iff 4 = 2 \times 2^{2/p}$$

$$\iff 2 = 2^{2/p} \iff p = 2.$$

Hence for $p \neq 2$, $L^p([-1,1])$ is not an Hilbert space.

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Exercise 5

1. Let $f \in L^\infty(\Omega)$. Since Ω is of finite measure:

$$\forall p \in [1,+\infty[,\ \|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p} \leq \|f\|_\infty \left(\int_\Omega dx\right)^{1/p}$$

$$= \|f\|_\infty \mu(\Omega)^{1/p} < +\infty$$

and $f \in L^p(\Omega)$.

Moreover, taking the limsup in the previous inequality,

$$\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty \underbrace{\limsup_{p\to+\infty} \mu(\Omega)^{1/p}}_{= \lim_{p\to+\infty} \mu(\Omega)^{1/p} = 1}$$

Hence $\limsup_{p\to+\infty} \|f\|_p \leq \|f\|_\infty$.

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Then : $\forall \varepsilon > 0$, $\exists\, \Omega_0 \subset \Omega$, $Leb(\Omega_0) > 0$, $\forall x \in \Omega_0$,

$$|f(x)| \geq \|f\|_\infty - \varepsilon$$

by definition of $\| \, \|_\infty$ in $L^\infty(\Omega)$. Then, by positivity,

$$\left(\int_\Omega |f(x)|^p dx\right)^{1/p} = \left(\int_{\Omega_0} |f(x)|^p dx\right)^{1/p} \geq \left(\int_{\Omega_0} (\|f\|_\infty - \varepsilon)^p dx\right)^{1/p} = (\|f\|_\infty - \varepsilon)\, \mu(\Omega_0)^{1/p}$$

i.e $\|f\|_p \geq (\|f\|_\infty - \varepsilon)\, \mu(\Omega_0)^{1/p}$ and :

$$\liminf_{p \to +\infty} \|f\|_p \geq (\|f\|_\infty - \varepsilon) \liminf_{p \to +\infty} \mu(\Omega_0)^{1/p} = \|f\|_\infty - \varepsilon$$

We deduce that : $\forall \varepsilon > 0$, $\|f\|_\infty \geq \limsup_{p \to +\infty} \|f\|_p \geq \liminf_{p \to +\infty} \|f\|_p \geq \|f\|_\infty - \varepsilon$

hence $\limsup_{p \to +\infty} \|f\|_p = \liminf_{p \to +\infty} \|f\|_p = \|f\|_\infty$ and $\lim_{p \to +\infty} \|f\|_p = \|f\|_\infty$.

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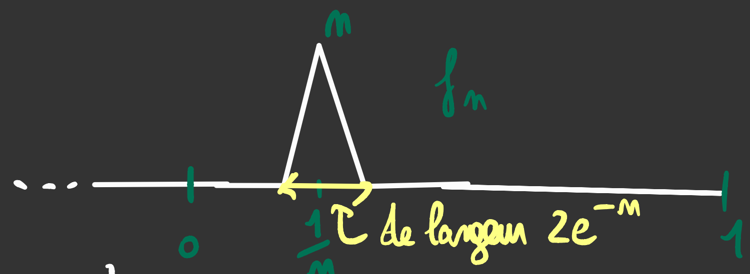
2. Let $t \in [0, \|f\|_\infty)$. By definition of $\|f\|_\infty \in [0,+\infty]$, the set $A = \{x \in \Omega \mid |f(x)| \geq t\}$ is of positive measure.

Then: $\forall p \geq 1$, $\|f\|_p \geq \left(\int_A |f(x)|^p \, d\mu(x)\right)^{1/p} \geq \left(t^p \mu(A)\right)^{1/p} = t\mu(A)^{1/p}$

Since $0 < \mu(A) < +\infty$, $\mu(A)^{1/p} \xrightarrow[p\to+\infty]{} 1$ and: $t \leq \liminf \|f\|_p \leq C$

Since t is arbitrary: $\|f\|_\infty \leq C$ and $f \in L^\infty(\Omega)$.

3. Take for example the function f_n for $n \geq 1$:



$\Omega = [0,1]$

Set $f = \sum_{n\geq 2} f_n$

Then $f \notin L^\infty([0,1])$

But if $p \in [1,+\infty[$, $\|f\|_p \leq \sum_{n\geq 2} n^p e^{-n} < +\infty$.

and $f \in \bigcap_{1\leq p<\infty} L^p(\Omega)$.

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1. Let $(v_n)_{n\in\mathbb{N}}$ which cv to v in $L^1(\Omega)$ and such that $(\phi(v_n))_{n\in\mathbb{N}}$ cv in $L^1(\Omega)$ to w.

Let us show that $w=\phi(v)=uv$.

Since $(v_n)_{n\in\mathbb{N}}$ cv to v in $L^1(\Omega)$, one can extract from $(v_n)_{n\in\mathbb{N}}$ a subsequence $(v_{\varphi(n)})_{n\in\mathbb{N}}$ which cv almost everywhere to v in Ω. Then, since $(\phi(v_{\varphi(n)}))_{n\in\mathbb{N}}$, cv to w in $L^1(\Omega)$ (comme suite extraite) one can extract a subsequence $(\phi(v_{\varphi\circ\psi(n)}))_{n\in\mathbb{N}}$ which cv to w a.e. on Ω.

Then $(v_{\varphi\circ\psi(n)})_{n\in\mathbb{N}}$ cv also a.e. to v on Ω as a

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subsequence of $(v_{\varphi(n)})_{n\in\mathbb{N}}$. Hence $(u\, v_{\varphi\circ\psi(n)})_{n\in\mathbb{N}}$ cv a.e. to uv on Ω. By uniqueness of the limit, $uv = w$ a.e. on Ω hence in $L^1(\Omega)$.

We have just proven that the graph of ϕ is closed.

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2. Using the closed graph theorem, the linear map

$\phi : \begin{array}{l} L^1(\Omega) \to L^1(\Omega) \\ v \mapsto uv \end{array}$ is continuous, hence:

$\exists C>0, \forall v \in L^1(\Omega), \|uv\|_1 \leq C\|v\|_1$.

Let $x \in \Omega$. Let $\varepsilon>0$, $B(x,\varepsilon) \subset \Omega$ and let ρ_ε C^∞ compactly supported function with $\operatorname{supp} \rho_\varepsilon \subset B(x,\varepsilon)$, $\rho_\varepsilon \geq 0$ and $\int_\Omega \rho_\varepsilon = 1$. Then $\rho_\varepsilon \in L^1(\Omega)$ and $y \mapsto \rho_\varepsilon(x-y)$ is also for y such that $x-y \in \Omega$.

We then have: $\forall \varepsilon>0, \|u\rho_\varepsilon\|_1 \leq C\|\rho_\varepsilon\|_1$

But $\|\rho_\varepsilon\|_1 = 1$ and $\|u\rho_\varepsilon\|_1 = (|u| * \rho_\varepsilon)(x) \xrightarrow[\varepsilon \to 0^+]{} |u(x)|$.

Hence : $|u(x)| \leq C$ and $u \in L^\infty(\Omega)$.