2023-2024
M1 Maths
Functional Analysis

# Partial Exam - Solution

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## Exercise 1:

1. Let $\{(f_n, Tf_n)\}_{n\in\mathbb{N}}$ a sequence of elements of $G(T)$ which converges to $(f,g)$ in $E\times F$.

By definition of the product topology:

$$\|f_n - f\|_\infty \xrightarrow[n\to+\infty]{} 0 \quad \text{and} \quad \|Tf_n - g\|_\infty \xrightarrow[n\to+\infty]{} 0$$

Hence $(f_n)$ cv uniformly to $f$ on $[0,1]$ and $(Tf_n) = (f_n')$ cv uniformly to $g$ on $[0,1]$.

By a classical result, $g \in E$ and $g = f' = Tf$.

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Hence $(f,g)=(f,Tf)\in G(T)$ and $G(T)$ is closed.

2. Assume that $T$ is continuous. Since it is linear:

$$\exists C\geq 0,\ \forall f\in E,\ \|Tf\|_\infty \leq C\|f\|_\infty.$$

Let for any $n\geq 1$, $f_n: \begin{array}{l}[0,1]\to\mathbb{R}\\ x\mapsto x^n\end{array}$. Then: $\forall n\geq 1$, $f_n\in E$ and:

$$\forall n\in\mathbb{N}^*,\ \|Tf_n\|_\infty = \left\|\begin{array}{l}[0,1]\to\mathbb{R}\\ x\mapsto nx^{n-1}\end{array}\right\|_\infty = n$$

and: $\forall n\in\mathbb{N}^*$, $\|f_n\|_\infty=1$.

Hence $\forall n\geq 1$, $n\leq C\times 1$ which is absurd: $T$ is not continuous.

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3. If both $(E, \| \|_\infty)$ and $(F, \| \|_\infty)$ where Banach spaces, the closed graph theorem would ensure that $T$ is continuous since by 1, $G(T)$ is closed.

Since $(F, \| \|_\infty)$ is a Banach space and $T$ is not continuous, $(E, \| \|_\infty)$ is not a Banach space.

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<u>Exercise 2:</u>

First, $B$ is a closed convex set of $H$ hence $P_B$, the orthogonal projection on $B$ is well defined.

For $x \in B$ it is clear that $P_B(x) = x$.

Assume that $x \notin B$ and set $p(x) = \frac{x}{\|x\|}$ ( $\|x\| > 1$ hence different from $0$).

Let $y \in B$. Then $p = P_B$ if and only if: $\langle y - p(x) | x - p(x) \rangle \leq 0$.

But:

$$\left\langle y - \frac{x}{\|x\|} \middle| x - \frac{x}{\|x\|} \right\rangle = \langle y | x \rangle - \left\langle \frac{x}{\|x\|} \middle| x \right\rangle - \left\langle y \middle| \frac{x}{\|x\|} \right\rangle + \left\langle \frac{x}{\|x\|} \middle| \frac{x}{\|x\|} \right\rangle$$

$$= \langle y | x \rangle - \underbrace{\frac{\langle x | x \rangle}{\|x\|}}_{= \|x\|} - \left\langle y \middle| \frac{x}{\|x\|} \right\rangle + \underbrace{\frac{\langle x | x \rangle}{\|x\|^2}}_{=1}$$

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$$= \|x\| \left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle - \|x\| - \left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle + 1$$

$$= \|x\| \left( \left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle - 1 \right) - \left( \left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle - 1 \right)$$

$$= (\|x\| - 1) \left( \left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle - 1 \right)$$

Since $x \notin B$, $\|x\| - 1 \geq 0$ and since $y \in B$, by Cauchy-Schwarz:

$$\left| \left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle \right| \leq \|y\| \left\| \frac{x}{\|x\|} \right\| = \|y\| \leq 1$$

Hence $\left\langle y \,\middle|\, \frac{x}{\|x\|} \right\rangle - 1 \leq 0$ and finally:

$$\langle y - p(x) \mid x - p(x) \rangle \leq 0$$

We conclude that: $\forall x \notin B$, $P_B(x) = p(x) = \frac{x}{\|x\|}$.

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## Exercise 3:

1. First, since for any $x \in H$, $p_F(x) \in F$, by definition of $d(x,F)$: $\|x - p_F(x)\| \geq d(x,F)$.

Conversely, let $y \in F$ and let $x \in H$. We decompose $x$ in $F \overset{\perp}{\oplus} F^\perp$: $x = p_F(x) + x_{F^\perp}$ where $x_{F^\perp} \in F^\perp$.

Then: $$\|x-y\|^2 = \langle p_F(x) + x_{F^\perp} - y \mid p_F(x) + x_{F^\perp} - y \rangle$$

$$= \langle p_F(x) - y \mid p_F(x) - y \rangle + 2\,\mathrm{Re}\Big(\underbrace{\langle \underbrace{p_F(x) - y}_{\in F} \mid \underbrace{x_{F^\perp}}_{\in F^\perp} \rangle}_{=0}\Big) + \langle x_{F^\perp} \mid x_{F^\perp} \rangle$$

$$= \|p_F(x) - y\|^2 + \|x_{F^\perp}\|^2$$

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But : $\|x_F\|^2 = \| x - p_F(x)\|^2$.

Hence :

$$\forall y \in F, \ \|x-y\|^2 = \|p(x)-y\|^2 + \| x - p_F(x)\|^2 \geq \|x-p_F(x)\|^2$$

and : $\quad d(x,F) \geq \|x-p_F(x)\|$.

Finally : $\quad d(x,F) = \|x-p_F(x)\|$.

2 (a) For $N \geq 0$ fixed, let $u_N : \begin{array}{l} \ell^2(\mathbb{N},\mathbb{C}) \to \mathbb{C} \\ (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{N} x_n \end{array}$

Then $M_N = \operatorname{Ker} u_N$. But $u_N$ is a continuous linear form.

Indeed : $\forall x \in \ell^2(\mathbb{N},\mathbb{C})$, $\left|\sum_{n=0}^{N} x_n\right| \underset{\text{Cauchy-Schwarz}}{\leq} \sqrt{N+1}\left(\sum_{n=0}^{N}|x_n|^2\right)^{\frac{1}{2}} \leq \sqrt{N+1}\ \|x\|_{\ell^2}$

Hence $M_N = \operatorname{Ker} u_N$ is a closed subspace of $\ell^2(\mathbb{N},\mathbb{C})$.

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(b) Soit $G = \mathrm{Vect}\left(\left(\underbrace{1,\dots,1}_{N+1 \text{ times}},0\dots\right)\right)$

Then: $\forall x \in M_N, \forall y \in G, \langle x \mid y \rangle = \langle x \mid \lambda(1,\dots,1,0\dots) \rangle$
$$= \lambda \sum_{n=0}^{N} x_n = 0$$

Hence $G = M_N^{\perp}$.

(c) Let $e_0 = (1,0,\dots)$

Then: $e_0 = p_{M_N}(e_0) + p_{M_N^{\perp}}(e_0) = p_{M_N}(e_0) + \lambda_{e_0}(1,\dots,1,0\dots)$

with $\lambda_{e_0} \in \mathbb{C}$. To determine $\lambda_{e_0}$ one compute the sum of the $n+1$ coordinatinates of the vectors in the decomposition of $e_0$:

$$1+0\dots+0 = \underbrace{0}_{\text{since } p_{M_N}(e_0) \in M_N} + \lambda_{e_0} + \dots + \lambda_{e_0} = (N+1)\lambda_{e_0}$$

Hence $\lambda_{e_0} = \dfrac{1}{N+1}$

9 But:
$$d(e_0, M_N) = \|e_0 - P_{M_N}(e_0)\|_{\ell^2} = \|\lambda_{e_0}(1, \dots 1, 0 \dots\dots)\|_{\ell^2}$$
$$= \frac{1}{N+1}\|(1, \dots, 1, 0 \dots)\|_{\ell^2} = \frac{\sqrt{N+1}}{N+1} = \frac{1}{\sqrt{N+1}}$$

Finally:
$$d(e_0, M_N) = \frac{1}{\sqrt{N+1}}.$$

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## Exercise 4:

1. We use Ascoli's theorem.

- Let $x \in [0,1]$ and $f \in F$:

$$|f(x)| = |f(x) - f(0) + \underbrace{f(0)}_{=0}| = |f(x) - f(0)| \leq 2|x-0| \leq 2$$

Hence $F$ is punctually bounded.

- Let $\varepsilon > 0$. For every $x, y \in [0,1]$, $|x-y| \leq \varepsilon/2$, for every $f \in F$:

$$|f(x) - f(y)| \leq 2|x-y| \leq 2\,\varepsilon/2 = \varepsilon$$

Hence $F$ is equicontinuous.

By Ascoli's theorem, $F$ is relatively compact in $C([0,1],\mathbb{R})$.

11 But, if $(f_n)$ is a sequence of elements of $F$ which cv for $\| \|_\infty$ to $f \in C([0,1],\mathbb{R})$, then:

$$\forall x,y \in [0,1], \quad |f(x)-f(y)| = \lim_{n\to+\infty} |f_n(x)-f_n(y)| \leq 2|x-y|$$

and $f$ is 2-Lipschitzian.

Moreover: $f(0) = \lim_{n\to+\infty} f_n(0) = 0$ and $f(1) = \lim_{n\to+\infty} f_n(1) = 0$.

Indeed, cv for $\| \|_\infty$ implies pointwise cv on $[0,1]$.

Hence, $F$ is closed in $(C([0,1],\mathbb{R}), \| \|_\infty)$. Since it is relatively compact, it is compact.

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2. Let 
$$u : \begin{array}{rcl} C([0,1],\mathbb{R}) & \longrightarrow & \mathbb{R} \\ g & \longmapsto & \int_0^1 g \end{array}.$$

Then:

$$\forall g \in C([0,1],\mathbb{R}),\ |u(g)| = \left|\int_0^1 g\right| \leq (1-0)\|g\|_\infty = \|g\|_\infty$$

and $u$ is continuous.

In particular, $u$ is continuous on the compact set $F$, hence it is bounded on $F$ and it attained its supremum on $F$:

$$\exists f \in F,\quad u(f) = \sup\{u(g) \mid g \in F\}$$

$$\text{i.e } \exists f \in F,\ \int_0^1 f = \sup\left\{\int_0^1 g \mid g \in F\right\}.$$