2023-2024
M1
Option Operator Theory

Exercises sheet 1 - Corrections

Exercise 3:

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Recall: U unitary means that $UU^* = U^*U = Id_{\mathcal{H}}$.

1. Hint: show that it is enough to prove that

$$\mathrm{Ker}(U^* - I) = \mathrm{Ker}(U - I)$$

We show that $\mathrm{Ker}(U - I) = \mathrm{Ker}(U^* - I)$.

$\mathrm{Ker}(U - I) \subset \mathrm{Ker}(U^* - I)$: let $x \in \mathrm{Ker}(U - I)$:

$Ux = x$: applying U^*: $U^*Ux = U^*x$

$\Leftrightarrow \quad x = U^*x$

$\Leftrightarrow \quad x \in \mathrm{Ker}(U^* - I)$.

2. $\ker(U^*-I) \subset \ker(U-I)$: let $x \in \ker(U^*-I)$:

$U^* x = x$: applying U: $UU^*x = Ux$

$\Leftrightarrow x = Ux$

$\Leftrightarrow x \in \ker(U-I)$.

Then: $\ker(U-I) = \ker(U^*-I) = \ker((U-I)^*)$

$= \mathrm{Im}(U-I)^{\perp}$

and $(\ker(U-I))^{\perp} = \overline{\mathrm{Im}(U-I)}$

Since U si bounded, $\ker(U-I)$ is closed in H

and:

$$\mathcal{H} = \ker(U-I) \overset{\perp}{\oplus} (\ker(U-I))^{\perp}$$

$$= \ker(U-I) \overset{\perp}{\oplus} \overline{\mathrm{Im}(U-I)}$$

3 **2.** If $u \in H$: $\exists ! (x,y) \in \mathrm{Ker}(U-I) \times \overline{\mathrm{Im}(U-I)}$,

$$u = x + y$$

By definition $Pu = x$.

We have to prove that: $\forall n \in \mathbb{N}$, $S_n u \xrightarrow[n\to+\infty]{} x = Pu$

$\forall n \in \mathbb{N}$, $S_n u = S_n x + S_n y$. We prove (i): $S_n x \xrightarrow[n\to+\infty]{} x$

(ii) $S_n y \xrightarrow[n\to+\infty]{} 0$

(i) We have $x \in \mathrm{Ker}(U-I)$:

$$Ux = x \quad \text{and: } \forall n \geq 1,\ U^n x = x$$

So :

$$S_n x = \frac{1}{n+1}(I + U + \dots + U^n)x$$

$$= \frac{1}{n+1}(\underbrace{x + \dots + x}_{n+1 \text{ times}}) = x$$

So: $\forall n \geq 1$, $S_n x = x$ hence $S_n x \xrightarrow[n\to+\infty]{} x$.

4 (ii) We have $y \in \overline{\mathrm{Im}(U-I)}$.

Let $\varepsilon > 0$: $\exists\, y_\varepsilon \in \mathrm{Im}(U-I)$, $\|y - y_\varepsilon\|_{H} \leq \varepsilon$

Since $y_\varepsilon \in \mathrm{Im}(U-I)$: $\exists\, z_\varepsilon \in H$, $y_\varepsilon = (U-I) z_\varepsilon$

Then: $\forall n \geq 1$, $S_n y_\varepsilon = \frac{1}{n+1}(I + U + \cdots + U^n)\, y_\varepsilon$

$$= \frac{1}{n+1}(I + U + \cdots + U^n)(U z_\varepsilon - z_\varepsilon)$$

$$= \frac{1}{n+1}(\cancel{U z_\varepsilon} - z_\varepsilon + \cancel{U^2 z_\varepsilon} - \cancel{U z_\varepsilon} + \cancel{\cdots} + U^{n+1} z_\varepsilon - \cancel{U^n z_\varepsilon})$$

$$= \frac{1}{n+1}(U^{n+1} z_\varepsilon - z_\varepsilon)$$

⑤ Hence: $\forall n \geq 1$, $\|S_n y_\varepsilon\|_{\mathcal{H}} \leq \frac{1}{n+1}\left(\|U^{n+1} z_\varepsilon\|_{\mathcal{H}} + \|z_\varepsilon\|_{\mathcal{H}}\right)$

$$\leq \frac{1}{n+1}\Big(\underbrace{\|U\|^{n+1}_{\mathcal{L}(\mathcal{H})}}_{=1 \text{ since } U \text{ is unitary}} \|z_\varepsilon\|_{\mathcal{H}} + \|z_\varepsilon\|_{\mathcal{H}}\Big)$$

$$= \frac{2}{n+1}\|z_\varepsilon\|_{\mathcal{H}} \xrightarrow[n\to+\infty]{} 0$$

i.e. $\exists N_0 \in \mathbb{N}$, $\forall n \geq N_0$, $\|S_n y_\varepsilon\|_{\mathcal{H}} \leq \varepsilon$.

Then: $\forall n \geq 1$, $\|S_n y\|_{\mathcal{H}} = \|S_n y - S_n y_\varepsilon + S_n y_\varepsilon\|_{\mathcal{H}}$

$$= \|S_n(y - y_\varepsilon) + S_n y_\varepsilon\|_{\mathcal{H}}$$

$$\leq \|S_n\|_{\mathcal{L}(\mathcal{H})} \underbrace{\|y - y_\varepsilon\|_{\mathcal{H}}}_{\leq \varepsilon} + \underbrace{\|S_n y_\varepsilon\|_{\mathcal{H}}}_{\leq \varepsilon \text{ for } n \geq N_0}$$

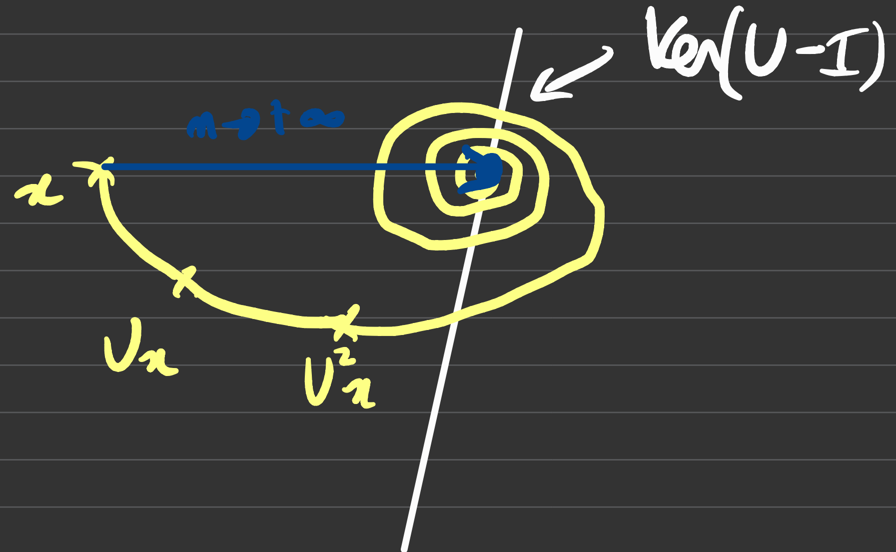
But: $\|S_n\|_{\mathcal{L}(\mathcal{H})} \leq \frac{1}{n+1}\Big(\underbrace{\|I\|_{\mathcal{L}(\mathcal{H})}}_{=1} + \underbrace{\|U\|_{\mathcal{L}(\mathcal{H})}}_{=1} + \dots + \underbrace{\|U\|^n_{\mathcal{L}(\mathcal{H})}}_{=1}\Big) = \frac{n+1}{n+1} = 1$

⑥ Finally for $n \geq N_0$:

$$\|S_n y\|_H \leq 1 \times \varepsilon + \varepsilon = 2\varepsilon, \text{ and } S_n y \xrightarrow[n\to+\infty]{} 0,$$

$$S_n u = S_n x + S_n y \xrightarrow[n\to+\infty]{} x + 0 = x = Pu.$$

<u>Example</u>: for U a rotation in $\mathbb{R}^3$: $\mathrm{Ker}(U-I)$ is the axis of the rotation / the set of fixed points /.



$U = r_\theta$

$U^n = r_{n\theta}$

Exercise 8

1. a. Since A is self-adjoint, for any $\phi \in \mathcal{H}$, $(\phi|A\phi) \in \mathbb{R}$.

Hence, for any $x < 0$,

$$\|(A-x)\phi\|^2 = \underbrace{\|A\phi\|^2}_{\geq 0} + x^2\|\phi\|^2 \underbrace{-2\underbrace{x}_{<0}\underbrace{(\phi|A\phi)}_{\geq 0}}_{\geq 0} \geq x^2\|\phi\|^2$$

b. If ϕ is such that $(A-x)\phi = 0$ then using 1. a. $x^2\|\phi\|^2 = 0$

But $x<0$ hence different from zero and $\|\phi\| = 0 \Rightarrow \phi = 0$.

Hence $A-x$ is injective.

c. Let $\psi \in (\mathrm{Im}(A-x))^{\perp}$. Then: $\forall \phi \in \mathcal{H}, ((A-x)\phi|\psi) = 0$

Since A is self-adjoint, so is $A-x$ and:

$$\forall \phi \in \mathcal{H}, (\phi|(A-x)\psi) = 0 \Rightarrow (A-x)\psi = 0.$$

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Then: $(A\psi|\psi) = (x\psi|\psi) = x\|\psi\|^2$.

But $(A\psi|\psi) \geq 0$ and $x < 0$. It implies that $\|\psi\|^2 = 0$ and $\psi = 0$. Hence $(\mathrm{Im}(A-x))^{\perp} = \{0\}$ and

$$\overline{\mathrm{Im}(A-x)} = ((\mathrm{Im}(A-x))^{\perp})^{\perp} = \{0\}^{\perp} = \mathcal{H}.$$

which proves that $\mathrm{Im}(A-x)$ is dense in $\mathcal{H}$.

d. We prove that $\mathrm{Im}(A-x)$ is closed.

Let $(v_n)_{n\in\mathbb{N}}$ a sequence in $\mathrm{Im}(A-x)$ which converges to $v \in \mathcal{H}$. Then: $\forall n \in \mathbb{N}$, $\exists u_n \in \mathcal{H}$, $v_n = (A-x)u_n$ and using question 1, a:

$$\forall n, m \in \mathbb{N}, \quad \|v_n - v_m\| = \|(A-x)u_m - (A-x)u_n\|$$

$$= \|(A-x)(u_m - u_n)\| \geq x\|u_m - u_n\|.$$

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Since (w_n) cv it is a Cauchy sequence which implies that (u_n) is a Cauchy sequence using the previous inequality. Hence (u_n) cv to some $u \in H$ and by C^0 of $A-x$ (or the fact that since it is self-adjoint, it is a closed operator),

$$\forall n \in \mathbb{N} \quad v_n = (A-x)\, u_n \overset{n \to \infty}{\Longrightarrow} v = (A-x)\, u \in \operatorname{Im}(A-x)$$

and $\operatorname{Im}(A-x)$ is closed. Since it is dense in H, $\operatorname{Im}(A-x) = H$ and $A-x$ is surjective. Since by 1.(b) it is injective, it is bijective.

e. Let $\psi \in H$ and $\phi = R_A(x)\psi \in H$. Then $(A-x)\phi = \psi$ and by 1. a. :

$$x^2 \, \| R_A(x)\psi \|^2 \leq \|\psi\|^2 \Rightarrow \| R_A(x)\psi \| \leq \frac{1}{|x|} \|\psi\|$$

Hence $R_A(x)$ is bounded and : $\|R_A(x)\|_{\mathcal{L}(H)} \leq \frac{1}{|x|}$

(f) Since A is self-adjoint, $\sigma(A) \subset \mathbb{R}$.

But, question 2(e) shows that any $x < 0$ is in $\rho(A)$

hence $\sigma(A) \subset \mathbb{R}_+$.

2. We prove the converse. Assume that $\sigma(A) \subset \mathbb{R}_+$.

Since A is self-adjoint, by the spectral theorem, there exists a unique spectral family E such that:

$$A = \int_{\sigma(T)} t \, dE(t)$$

But since $\sigma(A) \subset [0, +\infty[$, $A = \int_0^{+\infty} t \, dE(t)$

(11)

Let $B = \int_0^{+\infty} \sqrt{t}\, dE(t) = \sqrt{A}$. Then $B^2 = A$ and B is self-adjoint hence:

$$\forall \phi \in \mathcal{H},\ (\phi | A\phi) = (\phi | B^2 \phi) = (\phi | \underbrace{B^*}_{=B} B\phi) = (B\phi | B\phi)$$

$$= \|B\phi\|^2 \geq 0$$

Hence A is positive.

Exercise 9

1. Let $\phi \in \ell^2(\mathbb{N})$. Then if $(\Delta_{disc}\phi)_n = \phi_{n+1} + \phi_{n-1}$

$$\|H\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq \|\Delta_{disc}\|_{\mathcal{L}(\ell^2(\mathbb{N}))} + \|V\|_{\mathcal{L}(\ell^2(\mathbb{N}))}$$

But we proved that $\|\Delta_{disc}\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq 2$ and we have $\|V\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = \|v\|_\infty$

Hence: $\|H\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq 2 + \|v\|_\infty$ and H is bounded.

To show that it is self-adjoint it remains to show that it is symmetric.

(13) Let $\phi, \psi \in \ell^2(\mathbb{N})$.

$$(H\phi|\psi) = \sum_{n=0}^{+\infty} (H\phi)_n \overline{\psi_n}$$

$$= (-\phi_1 + v_0\phi_0)\overline{\psi_0} + \sum_{n=1}^{+\infty} (-\phi_{n+1} - \phi_{n-1} + v_n\phi_n)\overline{\psi_n}$$

$$= (-\phi_1 + v_0\phi_0)\overline{\psi_0} - \sum_{n=1}^{+\infty} \phi_{n+1}\overline{\psi_n} - \sum_{n=1}^{+\infty} \phi_{n-1}\overline{\psi_n} + \sum_{n=1}^{+\infty} v_n\phi_n\overline{\psi_n}$$

$$= (-\phi_1 + v_0\phi_0)\overline{\psi_0} - \sum_{n=2}^{+\infty} \phi_n\overline{\psi_{n-1}} - \sum_{n=0}^{+\infty} \phi_n\overline{\psi_{n+1}} + \sum_{n=1}^{+\infty} \phi_n\left(\overline{v_n\psi_n}\right)$$

car $v_n \in \mathbb{R}$

$$= \cancel{-\phi_1\overline{\psi_0}} + \phi_0\left(\overline{v_0\psi_0}\right) + \cancel{\phi_1\overline{\psi_0}} - \sum_{n=1}^{+\infty} \phi_n\overline{\psi_{n-1}} - \phi_0\overline{\psi_1} - \sum_{n=1}^{+\infty} \phi_n\overline{\psi_{n+1}} + \sum_{n=1}^{+\infty} \phi_n\left(\overline{v_n\psi_n}\right)$$

$$= \phi_0\Big(\underbrace{\overline{-\psi_1 + v_0\psi_0}}_{\overline{(H\psi)_0}}\Big) + \sum_{n=1}^{+\infty} \phi_n\Big(\underbrace{\overline{-\psi_{n-1} - \psi_{n+1} + v_n\psi_n}}_{\overline{(H\psi)_n}}\Big)$$

$$= (\phi|H\psi).$$

Hence H is self-adjoint.

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2. Since H is self-adjoint, $\sigma(H) \subset \mathbb{R}$ and $\rho(H) \supset \mathbb{C}\setminus\mathbb{R}$.
If $z \in \mathbb{C}\setminus\mathbb{R}$, $H-z$ is invertible and $G(z)$ is well-defined.
One has:

$$G(z) - G(\bar{z}) = (z-\bar{z})\, G(z) G(\bar{z}) = 2i \,\mathrm{Im}\, z\; G(z) G(\bar{z})$$

But $G(\bar{z}) = (H-\bar{z})^{-1} \overset{H=H^*}{=} \left((H-z)^*\right)^{-1} = \left((H-z)^{-1}\right)^* = (G(z))^*$

hence $\|G(z)\| = \|G(\bar{z})\|$.

Then:

$$2\,|\mathrm{Im}\, z|\; \|G(z)G(\bar{z})\| = \|G(z) - G(\bar{z})\| \leq \|G(z)\| + \|G(\bar{z})\|$$
$$\leq 2\|G(z)\|$$

Hence $|\mathrm{Im}\, z|\; \|G(z)G(\bar{z})\| \leq \|G(z)\|$

But: $\|G(z)G(\bar{z})\| = \|(G(z))(G(z))^*\| = \|G(z)\|^2$

Hence: $|\mathrm{Im}\, z|\; \|G(z)\|^2 \leq \|G(z)\|$.

Since $G(z)$ is invertible, $G(z) \neq 0$ and $\|G(z)\| \neq 0$.

Hence: $|\mathrm{Im}\, z| \, \|G(z)\| \leq 1$ and $\|G(z)\| \leq \frac{1}{|\mathrm{Im}\, z|}$

3. One has, using spectral theorem for H self-adjoint,

$$G(z) = (H-z)^{-1} = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, dE(\lambda)$$

Hence: $\forall n, m$, $(\delta_m | G(z) \delta_n) = \int_{\mathbb{R}} \frac{1}{\lambda - z} \, d(\delta_m | E(\lambda) \delta_n)$

$$= \int_{\mathbb{R}} \frac{1}{\lambda - z} \psi_m(\lambda) \psi_n(\lambda) \, d\rho(\lambda)$$

Finally: $$G_{m,n}(z) = \int_{\mathbb{R}} \frac{\psi_n(\lambda) \psi_m(\lambda)}{\lambda - z} \, d\rho(\lambda)$$

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4. One has: $\forall n \geq 0$, $\operatorname{Im} G_{n,n}(i) = \operatorname{Im} \int_{\mathbb{R}} \frac{\psi_n(\lambda)\psi_n(\lambda)}{\lambda - i} \, d\rho(\lambda)$

$$= \int_{\mathbb{R}} (\psi_n(\lambda))^2 \operatorname{Im}\left(\frac{1}{\lambda - i}\right) d\rho(\lambda)$$

But: $\operatorname{Im} \frac{1}{\lambda - i} = \operatorname{Im} \frac{\lambda + i}{\lambda^2 + 1} = \frac{1}{1 + \lambda^2}$

Therefore: $\operatorname{Im} G_{n,n}(i) = \int_{\mathbb{R}} \frac{(\psi_n(\lambda))^2}{\lambda^2 + 1} \, d\rho(\lambda)$

But: $\operatorname{Im} G_{n,n}(i) \leq |G_{n,n}(i)| = |(\delta_n, G(i)\delta_n)|$

$$\leq \|\delta_n\|_{\ell^2} \|G(i)\delta_n\|_{\ell^2}$$

$$\leq \underbrace{\|\delta_n\|_{\ell^2}}_{=1} \underbrace{\|\delta_n\|_{\ell^2}}_{=1} \|G(i)\|_{\mathcal{L}(\ell^2(\mathbb{N}))}$$

$$= \|G(i)\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \underset{2.}{\leq} \frac{1}{|\operatorname{Im} i|} = 1$$

Hence: $\int_{\mathbb{R}} \frac{(\psi_n(\lambda))^2}{\lambda^2 + 1} \, d\rho(\lambda) \leq 1.$

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5. Let $\varepsilon > 0$.

$$\int_{\mathbb{R}} \theta(\lambda)\, d\rho(\lambda) = \int_{\mathbb{R}} \sum_{n=0}^{+\infty} \frac{(\psi_n(\lambda))^2}{(\lambda^2+1)(1+n^{\frac{1}{2}+\varepsilon})^2}\, d\rho(\lambda)$$

$$\text{Beppo-Levi} \rightarrow = \sum_{n=0}^{+\infty} \frac{1}{(1+n^{\frac{1}{2}+\varepsilon})^2} \int_{\mathbb{R}} \frac{(\psi_n(\lambda))^2}{\lambda^2+1}\, d\rho(\lambda)$$

But, using 4.,

$$\left| \frac{1}{(1+n^{\frac{1}{2}+\varepsilon})^2} \int_{\mathbb{R}} \frac{(\psi_n(\lambda))^2}{\lambda^2+1}\, d\rho(\lambda) \right| \leq \frac{1}{(1+n^{\frac{1}{2}+\varepsilon})^2} \times 1 \leq \frac{1}{n^{1+2\varepsilon}}$$

and $\sum \frac{1}{n^{1+2\varepsilon}}$ cv

Hence $\int_{\mathbb{R}} \theta(\lambda)\, d\rho(\lambda) < +\infty$ and $\theta(\lambda) < +\infty$ for ρ-a.e. λ.

6. Since for ρ-a.e λ, $\theta(\lambda) < +\infty$, the serie which defines $\theta(\lambda)$ is cv for ρ-a.e λ and for such a λ:

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$$\frac{(\psi_n(t))^2}{(1+t^2)(1+n^{\frac{1}{2}+\varepsilon})^2} \xrightarrow[n\to+\infty]{} 0$$

Then: $\exists N_0 \in \mathbb{N},\ \forall n \geq N_0,\ |\psi_n(t)|^2 \leq (1+t^2)(1+n^{\frac{1}{2}+\varepsilon})^2$

and: $\forall n \geq N_0\ ,\ |\psi_n(t)| \leq \sqrt{1+t^2}\,(1+n^{\frac{1}{2}+\varepsilon})$

and $\psi_n(t)$ is polynomially bounded in n.

(more precisely: $\forall n \geq 0, |\psi_n(t)| \leq \mathrm{Max}\left(\frac{|\psi_0(t)|}{1}, \frac{|\psi_1(t)|}{1+1^{\frac{1}{2}+\varepsilon}}, \dots, \frac{|\psi_{N_0-1}(t)|}{1+(N_0-1)^{\frac{1}{2}+\varepsilon}}, \sqrt{1+t^2}\right)(1+n^{\frac{1}{2}+\varepsilon})$)