2023 - 2024
M1
Option Operator Theory

Exercises sheet 2 - Corrections

Exercise 2:

1

- T is the multiplication operator by the real-valued and bounded function $\varphi: \begin{array}{l}[0,2] \to \mathbb{R} \\ x \mapsto x\end{array}$ hence it is self-adjoint and bounded with: $\|T\|_{\mathcal{L}(L^2([0,2]))} = \|\varphi\|_\infty = 2$.

- Assume that T is compact. Then, by the Fredholm alternative, either $(I-T)^{-1}$ exists and is bounded or $Tu = u$ admits a non-zero solution u.

But: $Tu = u \iff \forall x \in [0,2]\,,\ x u(x) = u(x)$

2

$$\iff \forall x \in [0,2], \quad (x-1)\, u(x) = 0.$$

Hence : $\forall x \in [0,2] \setminus \{1\}$, $u(x) = 0$ and since $Leb(\{1\}) = 0$, $u = 0$ in $L^2([0,2])$.

Hence $Tu = u$ does not admit any non-zero solution in $L^2([0,2])$ and $(I-T)^{-1}$ exists and is bounded.

But: $\forall u, v \in L^2([0,2])$, $(I-T)^{-1} v = u \iff v = (I-T)u$

$$\iff \forall x \in [0,2], \; v(x) = u(x) - x\, u(x) = (1-x)\, u(x)$$

i.e. $\forall x \in [0,2] \setminus \{1\}$, $u(x) = \frac{1}{1-x} v(x)$

and $(I-T)^{-1} v(x) = \frac{1}{1-x} v(x)$

But $(I-T)^{-1}$ defined this way is not bounded. Indeed, for $v \equiv \frac{1}{\sqrt{2}}$ on $[0,2]$, $\|v\|_{L^2} = 1$ but $\left\| \frac{1}{1-x} v \right\|_{L^2} = \frac{1}{\sqrt{2}} \int_0^2 \frac{dx}{|1-x|^2} = +\infty$

3

Hence T does not verify the Fredholm alternative, therefore it is not compact.

Exercise 3:

1. Let $m, n \in \mathbb{N}$, $m < n$.

By Pythagore : $\forall u \in H, \left\| \sum_{j=m}^{n} \lambda_j P_j u \right\|_H^2 = \sum_{j=m}^{n} \|\lambda_j P_j u\|_H^2 \leq \sup_{j \geq m} |\lambda_j|^2 \|u\|_H^2$

Since $\lambda_n \xrightarrow[n \to +\infty]{} 0$, the sequence $\left(\sum_{n=0}^{N} \lambda_n P_n \right)_{N \geq 0}$ is Cauchy in the Banach space $\mathcal{L}(H)$, hence it converges.

Its limit is a limit of finite rank operators, hence it is compact.

4

2. Assume that moreover the λ_n are real numbers. The $\lambda_n P_n$ are therefore self-adjoint operators and T is self-adjoint as a strong limit of self-adjoint operators: indeed, since $\lambda_n \in \mathbb{R}$ for every $n \in \mathbb{N}$, $(\lambda_n P_n)^* = \lambda_n P_n^* = \lambda_n P_n$ and:

$$\|(\lambda_n P_n)^* - T^*\|_{\mathcal{L}(H)} = \|(\lambda_n P_n - T)^*\|_{\mathcal{L}(H)} = \|\lambda_n P_n - T\|_{\mathcal{L}(H)} \xrightarrow[n \to \infty]{} 0$$

Hence: $(\lambda_n P_n)^* \xrightarrow[n\to\infty]{} T^*$ and $(\lambda_n P_n)^* = \lambda_n P_n \xrightarrow[n\to\infty]{} T$.

By uniqueness of the limit: $T = T^*$.

⑤

Exercise 5

1. Let $(e_n)_{n\in\mathbb{N}}$ an orthonormal family (which is completed in a Hilbert basis if necessary). Then: $\forall n\in\mathbb{N}$, $\|Te_n\|^2 = (Te_n|Te_n) = (\underbrace{T^*T}_{\geq 0} e_n|e_n)$.

$\Rightarrow$ If $T\in\mathcal{B}_2(H)$, there exists $M>0$,

$$\forall N\geq 0, \quad \sum_{n=0}^{N} \|Te_n\|^2 \leq M$$

Hence: $\forall N\geq 0, \quad \sum_{n=0}^{N}(T^*Te_n|e_n)\leq M$

Since for every $n\in\mathbb{N}$, $(T^*Te_n|e_n)\geq 0$, the sequence $\left(\sum_{n=0}^{N}(T^*Te_n|e_n)\right)_{N\geq 0}$ is increasing and bounded by M hence it converges.

Then: $$\mathrm{Tr}(T^*T) = \sum_{n=0}^{+\infty}(T^*Te_n|e_n)\leq M < +\infty$$

6

($\Leftarrow$) If $Tr(T^*T) < +\infty$ then by positivity:

$$\forall N \geq 0, \quad \sum_{n=0}^{N} (T^*Te_n | e_n) \leq Tr(T^*T) := M$$

$$\Leftrightarrow \forall N \geq 0, \quad \sum_{n=0}^{N} \|Te_n\|_H^2 \leq Tr(T^*T)$$

2. Soient $T \in \mathcal{B}_2(H)$, $\varepsilon > 0$ and $\{e_0, \dots, e_n\}$ an orthonormal family such that:

$$\sum_{n=0}^{N} \|Te_n\|^2 \geq \|T\|_{HS}^2 - \varepsilon^2$$

Let $V = \mathrm{Vect}(e_0, \dots, e_N)$. Then $H = V \overset{\perp}{\oplus} V^\perp$ since V is finite dimensional hence closed.

Let $u \in H$, $\|u\|_H = 1$. $\exists! (u_V, u_{V^\perp}) \in V \times V^\perp$, $u = u_V + u_{V^\perp}$

7

and using Pythagore, $\|u\|_{\mathcal{H}}^2 = \|u_V\|_{\mathcal{H}}^2 + \|u_{V^\perp}\|_{\mathcal{H}}^2 = 1$

In particular, $\|u_V\|_{\mathcal{H}} \leq 1$ and $\|u_{V^\perp}\|_{\mathcal{H}} \leq 1$.

One has:

$$(T - TP_N)u = (T-TP_N)u_V + (T-TP_N)u_{V^\perp}$$

$$= T(I-P_N)u_V + T(I-P_N)u_{V^\perp}$$

But $I-P_N$ is the orthogonal projection on $V^\perp$ and since $\mathrm{Im} P_N = V$,

$$(I-P_N)u_V = 0 \quad \text{and} \quad (I-P_N)u_{V^\perp} = u_{V^\perp}$$

Therefore: $\|(T-TP_N)u\|_H = \|Tu_{V^\perp}\|_H$

Moreover, $\left(e_0, \dots, e_N, \frac{u_{V^\perp}}{\|u_{V^\perp}\|}\right)$ is an orthonormal family

hence by definition of $\| \ \|_{HS}$:

8

$$\|T\|_{HS}^2 - \varepsilon^2 + \frac{1}{\|u_{V\perp}\|^2}\|Tu_{V\perp}\|^2 \leq \sum_{n=0}^{N}\|Te_n\|^2 + \left\|T\left(\frac{u_{V\perp}}{\|u_{V\perp}\|}\right)\right\|^2 \leq \|T\|_{HS}^2$$

In particular: $\cancel{\|T\|_{HS}^2} - \varepsilon^2 + \frac{1}{\|u_{V\perp}\|^2}\|Tu_{V\perp}\|^2 \leq \cancel{\|T\|_{HS}^2}$

$$\Rightarrow \quad \|Tu_{V\perp}\|^2 \leq \varepsilon^2 \|u_{V\perp}\|^2 \leq \varepsilon^2$$

Therefore, $\|(T - TP_N)u\| = \|Tu_{V\perp}\| \leq \varepsilon$

which implies that $\|T - TP_N\|_{\mathcal{L}(H)} \leq \varepsilon$

3. We deduce that T is the limit of the finite rank operators TP_N, hence it is a compact operator.

9

4. Let $(e_n)_{n\in\mathbb{N}}$ an orthonormal family of $L^2(X)$ and let $N\geq 1$.

Then: $$\sum_{n=0}^{N} \|T_K e_n\|^2 = \sum_{n=0}^{N} \int_X \Big| \underbrace{\int_X K(x,y)\, e_n(y)\,dy}_{=(e_n \mid \overline{K(x,\cdot)}) = (K(x,\cdot)\mid \overline{e_n})} \Big|^2 dx$$

$$= \sum_{n=0}^{N} \int_X |(\overline{K(x,\cdot)} \mid e_n)|^2 dx$$

$$= \int_X \sum_{n=0}^{N} |(e_n \mid \overline{K(x,\cdot)})|^2 dx$$

But, using Bessel inequality:

$$\sum_{n=0}^{N} |(e_n \mid \overline{K(x,\cdot)})|^2 \leq \sum_{n=0}^{N} |(e_n \mid \overline{K(x,\cdot)})|^2 \leq \|K(x,\cdot)\|^2_{L^2(X)}$$

Hence: $$\sum_{n=0}^{N} \|T_K e_n\|^2 \leq \int_X \|K(x,\cdot)\|^2_{L^2(X)} dx \underset{\text{Fubini}}{=} \|K\|^2_{L^2(X\times X)}$$

En particulier, $T_K \in \mathcal{B}_2(H)$.

Exercise 6:

1. Let $\mathbb{1}_I$ be the characteristic function of the interval I.
Then $\mathbb{1}_I \in L^2([0,1])$ and applying the definition of the positivity of T_k:

$$0 \leq (T_k \mathbb{1}_I | \mathbb{1}_I) = \int_0^1 T_k \mathbb{1}_I(x) \overline{\mathbb{1}_I(x)}\, dx = \int_I (T_k \mathbb{1}_I)(x)\, dx$$

$$= \int_I \left(\int_0^1 k(x,y) \mathbb{1}_I(y)\, dy \right) dx$$

$$= \int_I \int_I k(x,y)\, dx\, dy$$

In particular for $x \in]0,1[$ and $n \geq 1$ such that $I_n = [x - \frac{1}{n}, x + \frac{1}{n}] \subset [0,1]$

(11) $\forall n \geq 1$, $\int_{I_n \times I_n} K(u,v)\,du\,dv \in \mathbb{R}_+$

Using uniform continuity of K, one has as n tends to $+\infty$:

$\forall x \in]0,1[$, $\int_{I_n \times I_n} K(u,v)\,du\,dv \xrightarrow[n \to +\infty]{} K(x,x) \in \mathbb{R}_+$

By C^0 of K one also has: $\forall x \in [0,1]$, $K(x,x) \in \mathbb{R}_+$.

2. a. Each φ_n satisfies: $\forall x \in [0,1]$, $\int_0^1 K(x,y)\varphi_n(y)\,dy = \lambda_n \varphi_n(x)$

To prove continuity of φ_n it suffices to show that

$x \mapsto \int_0^1 K(x,y)\,\varphi_n(y)\,dy$ is continuous.

12

One has, for $x, x' \in [0,1]$,

$$\left| \int_0^1 K(x,y)\varphi_n(y)\,dy - \int_0^1 K(x',y)\varphi_n(y)\,dy \right| \leq \int_0^1 |K(x,y)-K(x',y)|\,|\varphi_n(y)|\,dy$$

$$\underset{\text{Cauchy-Schwarz}}{\longrightarrow} \leq \left(\int_0^1 |K(x,y)-K(x',y)|^2\,dy \right)^{1/2} \underbrace{\|\varphi_n\|_{L^2}}_{=1}$$

But, using uniform C^0 of K on $[0,1]^2$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $|x-x'| < \delta$, $\sup_{y \in [0,1]} |K(x,y)-K(x',y)| \leq \varepsilon$

Hence, for $|x-x'| < \delta$,

$$\left| \int_0^1 K(x,y)\varphi_n(y)\,dy - \int_0^1 K(x',y)\varphi_n(y)\,dy \right| \leq \varepsilon.$$

(13) Hence $x \mapsto \int_0^1 K(x,y)\varphi_n(y)\,dy$ is uniformly continuous hence continuous and so is φ_n since $\lambda_n \neq 0$.

b. Let $N \geq 1$. We set for every $x, y \in [0,1]$,

$$K_N(x,y) = K(x,y) - \sum_{n=0}^{N} \lambda_n \varphi_n(x)\overline{\varphi_n(y)}$$

Let us prove that T_{K_N} is a positive operator.

Let $f \in L^2([0,1])$.

$$(T_{K_N} f | f) = \int_0^1 \left(\int_0^1 K_N(x,y) f(y)\,dy\right)\overline{f(x)}\,dx$$

$$= \int_0^1 \left(\int_0^1 K(x,y) f(y)\,dy - \int_0^1 \sum_{n=0}^{N} \lambda_n \varphi_n(x)\overline{\varphi_n(y)} f(y)\,dy\right)\overline{f(x)}\,dx$$

$$= (T_K f | f) - \sum_{n=0}^{N} \lambda_n \int_0^1 \varphi_n(x) \Big(\underbrace{\int_0^1 \overline{\varphi_n(y)} f(y)\,dy}_{(f|\varphi_n)}\Big)\overline{f(x)}\,dx$$

(14)

$$= (T_K f|f) - \sum_{n=0}^{N} \lambda_n (f|\varphi_n) \int_0^1 \varphi_n(x) \overline{f(x)}\, dx$$

$$= (T_K f|f) - \sum_{n=0}^{N} \lambda_n (f|\varphi_n)(\varphi_n|f)$$

$$= \sum_{n=0}^{+\infty} \lambda_n (f|\varphi_n)(\varphi_n|f) - \sum_{n=0}^{N} \lambda_n (f|\varphi_n)(\varphi_n|f)$$

$$= \sum_{n=N+1}^{+\infty} \lambda_n |(f|\varphi_n)|^2 \geq 0. \text{ since the } \lambda_n \geq 0.$$

Hence T_{K_N} is positive. Using question 1, for every $x \in [0,1]$, $K_N(x,x) \geq 0$.

i.e. $\forall x \in [0,1], \quad K(x,x) \geq \sum_{n=0}^{N} \lambda_n |\varphi_n(x)|^2$

(15)

3. a. Intregating the inequality obtained at Q2(b) on $[0,1]$:

$$\int_0^1 k(x,x)\,dx \geq \sum_{n=0}^{N} \lambda_n \underbrace{\int_0^1 |\varphi_n(x)|^2 dx}_{\|\varphi_n\|_{L^2}^2 = 1} = \sum_{n=0}^{N} \lambda_n .$$

Since the λ_n are positive, $\left(\sum_{n=0}^{N} \lambda_n\right)_{N\in\mathbb{N}}$ is an increasing sequence which is upper bounded by $\int_0^1 k(x,x)\,dx$, hence $\sum \lambda_n$ cv.

b. One has, for every $x \in [0,1]$,

$$\left| \int_0^1 k(x,y) f(y)\,dy - \sum_{n=0}^{N} \lambda_n (f|\varphi_n)\varphi_n(x) \right| = \left| T_k f(x) - \sum_{n=0}^{N} (f|\varphi_n) T_k \varphi_n(x) \right|$$

$$= \left| T_k f(x) - T_k\left(\sum_{n=0}^{N} (f|\varphi_n)\varphi_n\right)(x) \right| = \left| \int_0^1 k(x,y)\left(f(y) - \sum_{n=0}^{N} (f|\varphi_n)\varphi_n(y)\right) dy \right|$$

16

$$\underset{\text{Cauchy-Schwarz}}{\leq} \|K(x,\cdot)\|_{L^2([0,1])} \left\| f - \sum_{n=0}^{N} (f|\varphi_n)\varphi_n(\cdot) \right\|_{L^2([0,1])}$$

$$\leq \|K\|_\infty \left\| f - \sum_{n=0}^{N} (f|\varphi_n)\varphi_n(\cdot) \right\|_{L^2([0,1])} \xrightarrow[N\to+\infty]{} 0 \text{ independently of } x$$

since (φ_n) is an Hilbert basis

Hence the uniform convergence of $\sum \lambda_n (f|\varphi_n)\varphi_n$ to $T_K f$.

c. Let $p<q$ be two integers. One has, for every $x, y \in [0,1]$,

$$\left| \sum_{n=p}^{q} \lambda_n \varphi_n(x) \overline{\varphi_n(y)} \right| \underset{CS}{\leq} \left(\sum_{n=p}^{q} \lambda_n |\varphi_n(x)|^2 \right)^{\frac12} \left(\sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2 \right)^{\frac12}$$

$$\leq \left(\sum_{n=0}^{+\infty} \lambda_n |\varphi_n(x)|^2 \right)^{\frac12} \left(\sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2 \right)^{\frac12}$$

But $\Big(\underbrace{\sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2}_{\geq 0} \Big)_{q \geq p}$ cv uniformly by Dini's theorem since

17

for any y, $\sum \lambda_n |\varphi_n(y)|^2$ cv (it is $\uparrow$ and $\leq K(y,y)$).

Hence: $\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \forall q > p \geq N_\varepsilon, \sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2 \leq \varepsilon$ uniformly in y.

Using the uniform Cauchy criterion, it implies that the serie $\sum \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$ cv uniformly in y on $[0,1]$ for any fixed $x \in [0,1]$.

It remains to compute the limit. We apply 3.6. with $f_p = \mathbb{1}_{[y-\frac{1}{p}, y+\frac{1}{p}]}$ for $y \in]0,1[$, $f_p = \mathbb{1}_{[0,\frac{1}{p}]}$ if $y=0$ and $f_p = \mathbb{1}_{[1-\frac{1}{p}, 1]}$ if $y=1$.

18

Then, $(f_p | \varphi_n) = \int_0^1 f_p(t) \overline{\varphi_n(t)}\, dt \xrightarrow[p \to +\infty]{} \overline{\varphi_n(y)}$

Hence: $\forall x, y \in [0,1]$, $\sum_{n \geq 0} \lambda_n (f_p | \varphi_n) \varphi_n(x) \xrightarrow[p \to +\infty]{} \sum_{n \geq 0} \lambda_n \overline{\varphi_n(y)} \varphi_n(x)$

by uniform convergence.

But $\sum_{n \geq 0} \lambda_n (f_p | \varphi_n) \varphi_n(x) = \int_0^1 K(x,t) f_p(t)\, dt \xrightarrow[p \to +\infty]{} K(x,y)$

Hence: $\sum_{n \geq 0} \lambda_n \overline{\varphi_n(y)} \varphi_n(x) = K(x,y)$.

d. Since it cv uniformly in y, one can let y tend to x in the previous equality:

$$K(x,x) = \sum_{n=0}^{+\infty} \lambda_n |\varphi_n(x)|^2$$

19

Then: $$\int_0^1 K(x,x)\,dx = \int_0^1 \sum_{n\geq 0} \lambda_n |\varphi_n(x)|^2 dx \underset{\text{Beppo-Levi}}{=} \sum_{n\geq 0} \lambda_n \underbrace{\int_0^1 |\varphi_n(x)|^2 dx}_{=1}$$

$$= \sum_{n\geq 0} \lambda_n$$

Finally: $$\sum_{n\geq 0} \lambda_n = \int_0^1 K(x,x)\,dx \quad \text{(Mercer's formula)}.$$