Institut Galilée **2023-2024**
Master 1
Functional Analysis

Exercises sheet 1 : Theorems of Functional Analysis

1. Baire's theorem

Exercise 1

Find a sequence $(\mathcal{O}_n)_{n\in\mathbb{N}}$ of dense open sets of $\mathbb{R}$ such that $\bigcap_{n\in\mathbb{N}}\mathcal{O}_n$ is not not an open set.

Exercise 2

Let $f : \mathbb{R}_+ \to \mathbb{R}$ a continuous function such that for every $a > 0$, $\lim\limits_{n\to+\infty} f(na) = 0$. Let $\varepsilon > 0$. Applying Baire's theorem to the closed sets $F_p = \{a \geq 0\ ,\ \forall n \geq p,\ |f(na)| \leq \varepsilon\}$, show that f tends to 0 at $+\infty$.

Exercise 3

Let f an entire function, *i.e.* an holomorphic function on the whole complex plan $\mathbb{C}$. Show that if for each $z \in \mathbb{C}$ there exists $n \in \mathbb{N}$ such that $f^{(n)}(z) = 0$, then f is a polynomial function.
Hint : use the closed sets

$$F_n = \{z \in \mathbb{C},\ f^{(n)}(z) = 0\}.$$

Exercise 4

Let $(V_n)_{n\in\mathbb{N}}$ a family of open dense sets of $\mathbb{R}$.
1. Recall why $V = \cap_{n\in\mathbb{N}} V_n$ is dense in $\mathbb{R}$.
2. Show that if $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers, then, for every $n \in \mathbb{N}$, $W_n = V_n \setminus \{x_0, \ldots, x_n\}$ is an open dense set of $\mathbb{R}$.
3. Deduce that V cannot be finite or countable.

Exercise 5

Let (E, d) a complete metric space and let $(f_n)_{n\in\mathbb{N}}$ a sequence of continuous functions from E to $\mathbb{R}$ which converges pointwise to a function f.
1. For $m \in \mathbb{N}$ et $n \in \mathbb{N}^*$, let

$$A_{m,n} = \{x \in E,\ |f_m(x) - f_l(x)| \leq \tfrac{1}{n},\ \forall l \in \mathbb{N},\ l \geq m\}.$$

Show that $A_{m,n}$ is closed in E.
2. Let $n \in \mathbb{N}^*$ fixed. Show that $E = \cup_{m\in\mathbb{N}} A_{m,n}$.
3. Let $O_{m,n} = \mathrm{Int}(A_{m,n})$. Show that $O_n = \cup_{m\in\mathbb{N}} O_{m,n}$ is an open dense set of E.
4. We will now show that f is continuous on every point of the set $G = \cap_{n\in\mathbb{N}^*} O_n$. Let $a \in G$ and $\varepsilon > 0$.
a. Show that there exist $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$ such that, for every $x \in O_{m,n}$, $|f_m(x) - f(x)| \leq \varepsilon$.
b. For this integer m, show that there exists a neighborhood V of a such that for every $x \in V$, $|f_m(x) - f_m(a)| \leq \varepsilon$.
c. Deduce from previous questions that f is continuous at point a.
5. Show that the set of continuity points of f is a residual set of E.
6. Is the characteristic function of $\mathbb{Q}$, $\mathbf{1}_{\mathbb{Q}}$, the pointwise limit on $\mathbb{R}$ of a sequence of continuous functions?

2. Banach-Steinhaus and the open map theorems

Exercice 6

Let

$$\ell^2(\mathbb{N}^*) = \left\{ (x_i)_{i\in\mathbb{N}^*} \;\middle|\; \sum_{i=1}^{+\infty} |x_i|^2 < +\infty \right\}$$

endowed with the norm $||\cdot||_{\ell^2}$ defined by

$$\forall x \in \ell^2(\mathbb{N}^*),\ ||x||_{\ell^2} = \left(\sum_{i=1}^{+\infty} |x_i|^2 \right)^{\frac{1}{2}}.$$

Let

$$E = \{x \in \ell^2(\mathbb{N}^*) \mid x_i = 0 \text{ except for a finite number of } i\ \}.$$

For $n \in \mathbb{N}^*$, consider the linear map $T_n : E \to \ell^2(\mathbb{N}^*)$ defined by :

$$\forall x \in E,\ \forall i \in \mathbb{N}^*,\ (T_n(x))_i = \begin{cases} 0 & \text{if} \quad i \neq n \\ n x_n & \text{if} \quad i = n \end{cases}$$

Finally, let $A = \{T_n \mid n \in \mathbb{N}^*\}$ and for every $x \in E$, $A_x = \{T_n(x) \mid T_n \in A\}$.
1. Show that for every $x \in E$, A_x is bounded in $\ell^2(\mathbb{N}^*)$.

2. Let $\mathcal{L}(E, \ell^2(\mathbb{N}^*))$ the space of bounded linear maps from E to $\ell^2(\mathbb{N}^*)$ endowed with the norm $|||\cdot|||$ defined by :

$$\forall T \in \mathcal{L}(E, \ell^2(\mathbb{N}^*)),\ |||T||| = \sup_{x\in E,\ ||x||_{\ell^2}=1} ||T(x)||_{\ell^2}.$$

Show that A is not bounded in $\mathcal{L}(E, \ell^2(\mathbb{N}^*))$.
3. Explain why the Banach-Steinhaus theorem does not apply here.

Exercise 7

Let $\ell^1(\mathbb{N})$ the space of real-valued sequences $(u_n)_{n\in\mathbb{N}}$ such that $||u||_1 = \sum_{n=0}^{\infty} |u_n| < +\infty$ and $\ell^\infty(\mathbb{N})$ the space of bounded real-valued sequences, endowed with the norm $||u||_\infty = \sup_{n\in\mathbb{N}} |u_n|$.
1. Let $A = \{u \in \ell^\infty(\mathbb{N}) \mid u_n = 0 \text{ except for a finite number of } n\}$. Show that A is dense in $(\ell^1(\mathbb{N}), ||\ ||_1)$ but not in $(\ell^\infty(\mathbb{N}), ||\ ||_\infty)$.
2. Show that there is no sequence of positive real numbers $(a_n)_{n\in\mathbb{N}}$ such that

$$(a_n u_n) \in \ell^1(\mathbb{N}) \iff (u_n) \in \ell^\infty(\mathbb{N}).$$

Exercise 8

Let $E = L^1(\mathbb{T})$ the space of locally integrable and 2π-periodic functions on $\mathbb{R}$, endowed with the norm :

$$||f||_1 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|dx.$$

Let $F = c_0(\mathbb{Z})$ the space of families $(x_n)_{n\in\mathbb{Z}}$ of complex numbers which tends to 0 at infinity, endowed with the norm $||x||_\infty = \sup_{n\in\mathbb{Z}} |x_n|$.
We aim at proving that the following application is not onto :

$$T : \begin{array}{ccl} E & \to & F \\ f & \mapsto & (c_n(f))_{n\in\mathbb{Z}},\ c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \end{array}$$

1. Recall briefly why T is well-defined, linear and continuous. It is assumed to be injective.

2. Show that if T is surjective, there exists $\delta > 0$ such that, for all $f \in L^1(\mathbb{T})$,

$$||f||_{L^1} \leq \delta \sup_{n\in\mathbb{Z}} |c_n(f)|$$

3. For every $g \in L^\infty(\mathbb{R})$, 2π-periodic, we choose a sequence $(\alpha_n)_{n\in\mathbb{Z}}$ of complex numbers of modulus 1 such that, for all $n \in \mathbb{Z}$, $\bar{\alpha} c_n(g) = |c_n(f)|$. Applying question 2 to

$$f_N = \sum_{|n|\leq N} \alpha_n e^{inx},$$

show that, if T is surjective, for all $N \geq 0$,

$$\sum_{|n|\leq N} |c_n(g)| \leq \delta ||g||_\infty$$

4. Conclude.

Exercise 9

Let E denote the Banach space of continuous functions on the interval $[0, 1]$, with complex values, endowed with the norm $||\ ||_\infty$. Let α be such that $0 < \alpha < 1$ and let E_α be the subspace of E consisting of functions f such that there exists a constant $A > 0$ for which :

$$\forall x \in [0, 1],\ \forall y \in [0, 1],\ |f(x) - f(y)| \leq A|x - y|^\alpha$$

Define the following norm on E_α:

$$||f||_\alpha = ||f||_\infty + \sup_{0\leq x\neq y\leq 1} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

Then $(E_\alpha, ||\ ||_\alpha)$ is a Banach space. Let F be a closed subspace of $(E, ||\ ||_\infty)$. We assume that F is contained in E_α and we aim at showing that F is finite-dimensional.

1. Show that F is closed in $(E_\alpha, ||\ ||_\alpha)$.

2. Show that there is a constant $C > 0$ such that, for any element f of F, $||f||_\alpha \leq C||f||_\infty$.

3. Conclude by studying the unit ball of $(F, ||\ ||_\infty)$.

Exercise 10

Let $(E, ||\ ||)$ a normed vector space over $\mathbb{K}$ and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of E. It is said that $(x_n)_{n\in\mathbb{N}}$ *converges weakly* to $x \in E$ when for every $u \in E' = \mathcal{L}_c(E, \mathbb{K})$, $\lim_{n\to+\infty} u(x_n) = u(x)$. We denote it by $x_n \rightharpoonup x$.

1. Show that if $(x_n)_{n\in\mathbb{N}}$ converges strongly to $x \in E$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to x.

2. Show that the weak limit of a weakly convergent sequence is unique.

3. Show that if $x_n \rightharpoonup x$ then $(x_n)_{n\in\mathbb{N}}$ is strongly bounded.

4. Show that if $x_n \rightharpoonup x$ then

$$||x|| \leq \liminf\ ||x_n||.$$

3. Ascoli's theorem

Exercise 10

Let $E = C([0,1],\mathbb{R})$ the space of continuous real-valued functions on $[0,1]$ endowed with the norm $||u||_\infty = \sup_{x\in[0,1]} |u(x)|$.

1. Show that the unit ball of $(E, || \, ||_\infty)$ is not compact.

2. For $k \in \mathbb{R}_+$ and $M > 0$, let

$$F_{k,M} = \{u \in E \mid |u(0)| \leq M \text{ and } u \text{ } k\text{−Lipschitzian}\}.$$

a. Show that for every $x \in [0,1]$, $\{u(x) \mid u \in F_{k,M}\}$ is bounded.

b. Show that $F_{k,M}$ is equicontinuous.

c. Deduce that $F_{k,M}$ is compact.

Exercise 11

1. Let f, g be continuous functions from $[0,1]$ into $\mathbb{R}$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of class C^1 functions from $[0,1]$ into $\mathbb{R}$. Show that if (f_n) converges uniformly to f and (f_n') converges uniformly to g, then f is of class C^1 and $f' = g$. (Hint: use an integral).

2. Let $E = C([0,1],\mathbb{R})$ endowed with the norm $\|\cdot\|_\infty$. Let F be a closed vector subspace of E. It is assumed that all the elements of F are of class C^1.

a. Let the application $T : F \to E$ be defined by $T(f) = f'$. Using question 1 and the closed graph theorem, show that T is continuous.

b. Deduce that the unit ball of F is compact.

c. Deduce that F is finite-dimensional.

Exercise 12

Let $\ell^\infty(\mathbb{N})$ be the vector space of bounded sequences. It is endowed with the norm $||u||_\infty = \sup_{n\in\mathbb{N}} |u_n|$ for $u = (u_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. Then $(\ell^\infty(\mathbb{N}), || \, ||_\infty)$ is a Banach space.

1. Show that the unit ball of $(\ell^\infty(\mathbb{N}), || \, ||_\infty)$ is not compact.

2. Let $F = \{u \in \ell^\infty(\mathbb{N}) \mid \forall n \in \mathbb{N},\ |u_n| \leq \frac{1}{n}\}$, endowed with the norm $|| \, ||_\infty$ induced by the one on $\ell^\infty(\mathbb{N})$.

a. Let $K = \{\frac{1}{n}\}_{n\in\mathbb{N}} \cup \{0\}$. Show that K is a compact of $\mathbb{R}$.

b. Let $G = \{f : K \to \mathbb{R} \mid \forall x \in K,\ |f(x)| \leq x\}$, endowed with the norm $|| \, ||_K$ defined by, for every $f \in G$, $||f||_K = \sup_{x\in K} |f(x)|$.
Show that G is closed in $C(K,\mathbb{R})$, the space of continuous functions from K into $\mathbb{R}$, for the norm $|| \, ||_K$.

c. Consider the application

$$T : \begin{array}{ccc} F & \to & G \\ u & \mapsto & \left(f : \begin{array}{ccc} K & \to & \mathbb{R} \\ \frac{1}{n} & \mapsto & u_n \\ 0 & \mapsto & 0 \end{array} \right) \end{array}$$

Show that T is a homeomorphism of $(F, || \, ||_\infty)$ into $(G, || \, ||_K)$.

d. Deduce that F is compact if and only if G is compact.

3.a Show that, for all $x \in K$, $\{f(x) \mid f \in G\}$ is bounded.

b. Show that G is equicontinuous.

c. Deduce that G is compact and that F is compact.