Institut Galilée **2023-2024**
Master 1
Functional analysis

Exercises sheet 2 : Hilbert spaces

Exercise 1

Consider $\mathcal{H} = L^2([0,1],\mathbb{R})$ which, with the usual scalar product, is a Hilbert space. We consider

$$\varphi : \begin{array}{ccc} [0,1] & \to & \mathcal{H} \\ t & \mapsto & \mathbf{1}_{[0,t]}. \end{array}$$

where $\mathbf{1}_{[0,t]}$ is the characteristic function of the interval $[0,t]$, which is 1 on $[0,t]$ and 0 everywhere else.

1. Show that φ is continuous. Show that φ is nowhere differentiable.

2. Show that if $0 \leq s < s' \leq t < t' \leq 1$, then $\varphi(s') - \varphi(s)$ is orthogonal to $\varphi(t') - \varphi(t)$.

Exercise 2

We define an application $\varphi : \mathbb{C}[X] \times \mathbb{C}[X] \to \mathbb{C}$ by

$$\varphi(P,Q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{P(\mathrm{e}^{i\theta})}Q(\mathrm{e}^{i\theta})\,\mathrm{d}\theta$$

1. Show that φ is a Hermitian scalar product on $\mathbb{C}[X]$.

2. Show that the family $(X^k)_{k\in\mathbb{N}}$ is an orthonormal basis for the previous scalar product.

3. Let $Q = X^n + a_{n-1}X^{n-1} + \cdots + a_0$. Compute $\|Q\|^2$.

4. We set
$$M = \sup_{|z|=1} |Q(z)|$$
Show that $M \geqslant 1$ and study the case of equality.

Exercise 3

We define an application $\varphi : \mathbb{R}[X] \times \mathbb{R}[X] \to \mathbb{R}$ by

$$\varphi(P,Q) = \int_0^{+\infty} P(t)Q(t)\mathrm{e}^{-t}\,\mathrm{d}t.$$

1. Show that φ defines a scalar product on $\mathbb{R}[X]$.

2. Let p and q be two integers. Calculate $\varphi(X^p, X^q)$.

3. Determine

$$\inf_{(a,b)\in\mathbb{R}^2} \int_0^{+\infty} \mathrm{e}^{-t}(t^2 - (at+b))^2\,\mathrm{d}t.$$

Exercise 4

Let $\ell^2(\mathbb{N},\mathbb{C})$ be the space of summable square sequences, with the norm defined by :

$$\forall (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{C}),\ ||(x_n)||_2 = \left(\sum_{n=0}^{+\infty} |x_n|^2\right)^{\frac{1}{2}}.$$

For any integer $N \in \mathbb{N}$ fixed, let M_N be the vector subspace of $\ell^2(\mathbb{N},\mathbb{C})$ composed of the sequences $(x_n)_{n\in\mathbb{N}}$ such that $\sum_{n=0}^{N} x_n = 0$.

1. Show that the application $T : (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{N} x_n$ is a continuous linear form on $\ell^2(\mathbb{N},\mathbb{C})$. What can we deduce about M_N?

2. Justify that $\ell^2(\mathbb{N},\mathbb{C}) = M_N \overset{\perp}{\oplus} M_N^{\perp}$.

3. Let $E = \{(y_n)_{n\in\mathbb{N}} \mid \forall 0 \leq i < j \leq N,\ y_i = y_j \text{ and } \forall n > N,\ y_n = 0\}$.

a. Show that $E \subset M_N^{\perp}$.

b. Show that $M_N^{\perp} = E$. *Hint: Note that for $0 \leq i < j \leq N$, the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_i = 1$, $x_j = -1$ and $x_n = 0$ for $n \neq i,j$ belongs to M_N.*

Exercise 5

Let Ω be a bounded open set of $\mathbb{R}^d$ and $H = L^2(\Omega)$. Let $A = \left\{f \in H;\ \int_\Omega f(x)dx \geq 1\right\}$.

1. Show that A is a closed convex of H.

2. Let $g \in H$. Show that the minimum $\min_{f\in A} \|g - f\|_H$ is reached in a single element $\bar{g} \in A$.

3. Calculate $\bar{g}$ for $g = 0$ (use projection characterization).

Exercise 6

Let $(H, \langle\cdot|\cdot\rangle)$ be a Hilbert space and A be a continuous endomorphism of H.
1. Let $y \in H$ fixed.
a. Show that the linear form $\phi_y : x \mapsto \langle Ax|y\rangle$ is continuous.
b. Deduce that there is a vector A^*y such that :

$$\forall x \in H, \quad \langle Ax|y\rangle = \langle x|A^*y\rangle.$$

2. Show that the application of H in H, $y \mapsto A^*y$ is a continuous endomorphism of H. Call A^* the adjoint of A.
3. Verify that $(A^*)^* = A$ and that $|||A^*||| = |||A|||$.
4. Let $H = \mathbb{C}^n$ endowed with the scalar product

$$\langle x|y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Compute the matrix of A^* in the canonical basis as a function of A.
5. Let T be the linear application defined on $\ell^2(\mathbb{Z}, \mathbb{C})$ by

$$\forall u \in \ell^2(\mathbb{Z}, \mathbb{C}),\ \forall n \in \mathbb{Z},\ (Tu)_n = u_{n+1}.$$

a. Justify that T is continuous.
b. Compute the adjoint T^* of T.

Exercise 7

Let Ω be an open set of the complex plane $\mathbb{C}$. We define the Bergman space of Ω by :

$$\mathcal{A}^2(\Omega) = \{f \in L^2(\Omega, \mathrm{d}\lambda) \mid f \text{ is holomorphic in } \Omega\}.$$

where $L^2(\Omega, \mathrm{d}\lambda)$ denotes the space of square integrable functions on Ω for $\mathrm{d}\lambda$ the Lebesgue measure in $\mathbb{C}$ identified to $\mathbb{R}^2$. Let $L^2(\Omega, \mathrm{d}\lambda)$ be the usual Hermitian scalar product $(f, g) \mapsto \int_\Omega f\bar{g}\mathrm{d}\lambda$ and the associated norm $||\cdot||_2$. We also provide $\mathcal{A}^2(\Omega)$ with the restrictions of this scalar product and norm.

For $z \in \mathbb{C}$ and $r > 0$ we set $\overline{B}(z, r) = \{w \in \mathbb{C}\ |w - z| \le r\}$.
1.a. Let $z \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{N}$. Show that

$$\int_{\overline{B}(z,r)} (w - z)^n \mathrm{d}\lambda(w) = \begin{cases} 0 & \text{if } \ n \neq 0 \\ \pi r^2 & \text{if } \ n = 0 \end{cases}$$

b. Let $z \in \mathbb{C}$ and $r > 0$. It is assumed that $\overline{B}(z, r) \subset \Omega$. Show that for any $f \in \mathcal{A}^2(\Omega)$,

$$f(z) = \frac{1}{\pi r^2} \int_{\overline{B}(z,r)} f(w)\mathrm{d}\lambda(w).$$

c. Let $z \in \mathbb{C}$. Let $d(z, \Omega^c)$ be the distance from z to the complement of Ω. Show that :

$$\forall f \in \mathcal{A}^2(\Omega),\ \forall z \in \Omega,\ \forall r > 0,$$

$$\left(d(z, \Omega^c) > r \ \Rightarrow\ |f(z)| \le \frac{1}{r\sqrt{\pi}} ||f||_2 \right).$$

2.a. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $\mathcal{A}^2(\Omega)$. Let K be a compact of Ω. Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on K.
b. Deduce that there exists f holomorphic on Ω such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on any compact Ω hence pointwise to f on Ω.
c. Show that $\mathcal{A}^2(\Omega)$ endowed with the scalar product of $L^2(\Omega)$ is a Hilbert space.
3. For $z \in \Omega$ we set

$$\delta_z \ :\ \begin{array}{ccc} \mathcal{A}^2(\Omega) & \to & \mathbb{C} \\ f & \mapsto & f(z) \end{array}$$

a. Show that for any $z \in \Omega$, δ_z is a continuous linear form on $\mathcal{A}^2(\Omega)$.
b. For any $z \in \Omega$, show the existence of $K_z \in \mathcal{A}^2(\Omega)$ such that

$$\forall f \in \mathcal{A}^2(\Omega),\ f(z) = \int_\Omega f(w)\overline{K_z(w)}\mathrm{d}\lambda(w).$$

4. We note for all $(z, w) \in \Omega^2$, $K_\Omega(w, z) = K_z(w)$.
a. Show that, for all $(z, w) \in \Omega^2$, $K_\Omega(z, w) = \overline{K_\omega(w, z)}$.
b. Let $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathcal{A}^2(\Omega)$. Let $z \in \Omega$. Show that

$$K_z = \sum_{n=0}^{+\infty} \overline{e_n(z)} e_n$$

with convergence of the infinite sum in $\mathcal{A}^2(\Omega)$.
5. We now choose the example of the open unit disk. We take $\Omega = D = \{z \in \mathbb{C} \mid |z| < 1\}$.
a. For all $n \in \mathbb{N}$ and $z \in D$, we set :

$$e_n(z) = \sqrt{\frac{n+1}{\pi}} z^n.$$

Show that $(e_n)_{n\in\mathbb{N}}$ is a Hilbert basis of $\mathcal{A}^2(D)$.
Hint: we can use the equality case in Bessel's inequality to show the total character of the family: if $(e_n)_{n\in\mathbb{N}}$ *is an orthonormal family of a Hilbert space* $(\mathcal{H}, (\cdot|\cdot))$, *then*

$$x \in \overline{\mathrm{Vect}((e_n)_{n\in\mathbb{N}})} \ \Leftrightarrow\ ||x||^2 = \sum_{n\in\mathbb{N}} |(x|e_n)|^2.$$

b. Compute the function $(w, z) \mapsto K_D(w, z)$.