Institut Galilée **2023-2024**
Master 1
Functional Analysis

Exercises sheet 3 : L^p spaces

Exercise 1 Let $f \in L^p(\mathbb{R})$ with $1 < p < +\infty$.

1. Show that we can define, for any $x \geq 0$, $F(x) = \int_0^x f(t)\mathrm{d}t$.
2. Justify that $F(x) =_{+\infty} \mathcal{O}(x^{(p-1)/p})$.
3. Let $\varepsilon > 0$. Show that there exists $a > 0$ such that
$$\left(\int_a^{+\infty} |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} \leq \varepsilon.$$
4. Deduce that $F(x) =_{+\infty} o(x^{(p-1)/p})$.

Exercise 2 For $1 \leq p < +\infty$, let $\tau_a : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be defined, for all $f \in L^p(\mathbb{R})$ and all $x \in \mathbb{R}$, by $\tau_a(f)(x) = f(x-a)$.
Show that, for any$f \in L^p(\mathbb{R})$,
$$\lim_{a \to 0} ||\tau_a(f) - f||_p = 0.$$

Hint: Start with the case where f is a continuous function with compact support.

Exercise 3 Let $f \in L^1(\mathbb{R})$. Consider the application
$$Tf \ : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \int_0^1 f(x-y)\mathrm{d}y \end{array}.$$

1. Show that, if f is continuous with compact support, Tf is continuous.
2. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions with compact support that converges to f in $L^1(\mathbb{R})$. Show that $(Tf_n)_{n\in\mathbb{N}}$ converges to Tf uniformly on $\mathbb{R}$. Deduce that Tf is continuous on $\mathbb{R}$.
3. Deduce that the convolution product on $L^1(\mathbb{R})$ admits no unitary element.

Exercise 4 Let $f, g :]-1,1[\to \mathbb{R}$ given by $f = \mathbf{1}_{]-1,0[}$ and $g = \mathbf{1}_{]0,1[}$

1. Let $p \in [1, \infty[$. Compute $\|f\|_p$, $\|g\|_p$, $\|f+g\|_p$ and $\|f-g\|_p$.
2. Deduce that if $p \neq 2$, then $L^p(]-1,1[)$ is not a Hilbert space.

Exercise 5 Let $\Omega \subset \mathbb{R}^d$ be of finite measure.

1. Show that for all $f \in L^\infty(\Omega)$,
$$\lim_{p \to +\infty} ||f||_p = ||f||_\infty.$$
Hint: we can show that the limsup is inferior to $||f||_\infty$ and that for any $\varepsilon > 0$, the liminf is inferior to $||f||_\infty - \varepsilon$.
2. Let
$$f \in \bigcap_{1 \leq p < \infty} L^p(\Omega).$$
It is assumed that there exists $C > 0$ such that for all $p \in [1, +\infty[$, $||f||_p \leq C$. Show that $f \in L^\infty(\Omega)$.
3. Find $f \in \bigcap_{1 \leq p < \infty} L^p(\Omega)$ such that $f \notin L^\infty(\Omega)$.

Exercise 6 Let Ω be an open set of $\mathbb{R}^d$, $u : \Omega \to \mathbb{R}$ a measurable function. It is assumed that for all $v \in X = L^1(\Omega)$, we have $uv \in L^1(\Omega)$.

1. Let $\phi : X \to X$ be defined by $\phi(v) = uv$. Show that the graph of ϕ is closed in $X \times X$.
2. Deduce that $u \in L^\infty(\Omega)$. (Hint: apply the closed graph theorem, then proceed by the absurd).