Institut Galilée **2024-2025**
Master 1
Functional Analysis - Operator theory

Exercises sheet 1 : Bounded operators

Exercise 1

Let $H = L^2(X, \mathbb{C})$ and let $K \in L^2(X \times X, \mathbb{C})$. Let T_K be the operator defined on H by

$$\forall u \in H,\ \forall x \in X,\ T_K u(x) = \int_X K(x,y)u(y)dy.$$

1. Show that T_K is well defined.
2. Show that $\|T_K\|_{\mathcal{L}(H)} \leq \|K\|_{L^2(X\times X,\mathbb{C})}$.
3. Compute the adjoint of T_K.

Exercise 2

Let H be a Hilbert space. Show that if $T \in \mathcal{L}(H)$ is self-adjoint, then $r(T) = \|T\|$ where $r(T)$ is the spectral radius of T.

Exercise 3

Let H be a Hilbert space and let U be a unitary operator on H.
1. Show that $H = \mathrm{Ker}(U - I) \oplus \overline{\mathrm{Im}\ (U - I)}$.
2. Let P be the orthogonal projector on $\mathrm{Ker}\ (U-I)$. Let, for every $n \geq 1$,

$$S_n = \frac{I + U + \ldots + U^n}{n+1}.$$

Show that, for every $u \in H$, $S_n u \to Pu$ when n tends to infinity.

Exercise 4 - Trace of a positive operator

Let H a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ a Hilbert basis of H.
Let T a positive operator on H, *i.e.* T is bounded, self-adjoint and for every $u \in H$, $(Tu|u) \in \mathbb{R}_+$.
We admit the existence of a unique positive operator S such that $S^2 = T$. We denote it by $T^{\frac{1}{2}}$.
We set

$$\mathrm{tr}\ T = \sum_{n=1}^{\infty} (Te_n|e_n) \in [0, +\infty].$$

The real number tr T is called the trace of the operator T.

1. Show that the trace is independent of the choice of the Hilbert basis of H.
2. Show that, for every positives operators T and S, $\mathrm{tr}\ (T+S) = \mathrm{tr}\ T + \mathrm{tr}\ S$ and that, for every $\lambda \geq 0$, $\mathrm{tr}\ (\lambda T) = \lambda\ \mathrm{tr}\ T$.
3. Show that, if T and S are two positive operators such that $T - S$ is also positive, then $\mathrm{tr}\ S \leq \mathrm{tr}\ T$.
4. Show that, for every unitary operator U, $\mathrm{tr}\ (UTU^{-1}) = \mathrm{tr}\ (U^{-1}TU) = \mathrm{tr}\ T$.

Exercise 5

Let $\varphi \in L^\infty(\mathbb{R}, \mathbb{C})$. Show that $\sigma(M_\varphi) = \mathrm{Im\ ess}\ \varphi$ where M_φ is the multiplication operator by φ on $L^2(\mathbb{R})$ and

$$\mathrm{Im\ ess}\ \varphi = \{\lambda \in \mathbb{C},\ \forall \varepsilon > 0,\ \mathrm{Leb}(\varphi^{-1})(D(\lambda, \varepsilon)) > 0\}$$

is the essential range of φ. Deduce that if $\varphi : [a, b] \to \mathbb{R}$ is a continuous function, then $\sigma(M_\varphi) = \varphi([a, b])$.

Exercise 6

Denote by E the space $\ell^\infty(\mathbb{Z})$ of the complex-valued bounded sequences $u = (u_n)_{n\in\mathbb{Z}}$, endowed with the norm :

$$\|u\|_\infty = \sup_{n\in\mathbb{Z}} |u_n|.$$

Let T by defined on E by :

$$\forall u \in E,\ \forall n \in \mathbb{Z},\ (Tu)_n = u_{n+1}.$$

1. Compute the norm of T.
2. Show that every complex number of modulus 1 is an eigenvalue of T.
3. Let $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$. Let $u \in E$. We set $(T - \lambda)u = f$. For every $p \geq 1$, give the expression of u_n in terms of $f_{n-1}, \ldots, f_{n-p}$ and u_{n-p}. What happens when p tends to $+\infty$? Deduce that λ does not belong to the spectrum of T.
4. Using the previous questions, determine $\sigma(T)$.

Consider the closed subspace F of E whose elements are the sequences u such that $u_n = 0$ for every $n > 0$. Then, $T(F) \subset F$ and we set T_F the restriction of T on F.
5. Let $\lambda \in \mathbb{C}$ be such that $|\lambda| < 1$ and $\lambda \neq 0$. Using the expression of $(T - \lambda)^{-1}$ computed at question 3, show that λ belongs to $\sigma(T_F)$.

6. Using questions 1 and 5, determine $\sigma(T_F)$. Compare to the result obtained at question 4.

Exercise 7

Let E be the Banach space of complex-valued continuous functions on $[0,1]$, endowed with $|| \, ||_\infty$. Let T_0 be the operator defined on E by :

$$\forall u \in E, \ \forall x \in [0,1], \ T_0 u(x) = x u(x).$$

Let $f \in E$. We define L by :

$$\forall u \in E, \ Lu = \int_0^1 f(x) u(x) dx.$$

Finally, if $g \in E$ we define T by :

$$\forall u \in E, \ Tu = T_0 u + (Lu) g.$$

1. Let $\lambda \in [0,1]$. Show that if $g(\lambda) = 0$, then $T - \lambda$ is not onto. If $g(\lambda) \neq 0$, show that the function $h \ : \ x \mapsto \sqrt{|x - \lambda|}$ is not in the range of $T - \lambda$. Deduce that $[0,1] \subset \sigma(T)$.
2. Show that $\sigma(T) \setminus [0,1]$ contains only simple eigenvalues. Characterize these eigenvalues as solutions of an equation $F(\lambda) = 0$ where F is an holomorphic function on $\mathbb{C} \setminus [0,1]$. Give the expression of F in terms of f and g. Deduce that $\mathbb{C} \setminus [0,1]$ is a discrete set.
3. Assume that for every $x \in [0,1]$, $f(x) = 1$ and $g(x) = \alpha$, α being a nonzero real number. Determine $\sigma(T)$.

Exercise 8

Let A be a bounded self-adjoint operator on an Hilbert space $\mathcal{H}$.
1. Assume that $A \geq 0$ in the sense :

$$\forall \phi \in \mathcal{H}, (\phi, A\phi) \geq 0$$

a. Let $x < 0$. Show that :

$$\forall \phi \in \mathcal{H}, \ ||(A - x)\phi||^2 \geq x^2 ||\phi||^2$$

Deduce that $A - x \ : \ \mathcal{H} \to \mathcal{H}$ is injective.
b. Show that the range of A is dense in $\mathcal{H}$. Hint : compute the orthogonal of Im A.
c. Deduce that $A - x$ is onto, hence one-to-one.
Let $R_A(x) = (A - x)^{-1} \ : \ \mathcal{H} \to \mathcal{H}$.
d. Give an upper bound for $||R_A(x)||$.
e. Show that $A \geq 0 \Rightarrow \sigma(A) \subset [0, +\infty)$.
2. Using spectral theorem, show the reciprocal.

Exercice 9

Discrete version of Schnol's lemma.

Let $\ell^2(\mathbb{N})$ be the Hilbert space of complex-valued square-summable sequences $\phi = (\phi_n)_{n \geq 0}$. Consider on $\ell^2(\mathbb{N})$ the multiplication operator V by a real-valued sequence $v = (v_n)_{n \geq 0}$ and the Schrödinger operator $H = H_0 + V$ defined by :

$$(H\phi)_n = \begin{cases} -\phi_{n+1} - \phi_{n-1} + v_n \phi_n & \text{if } n \geq 1 \\ -\phi_1 + v_0 \phi_0 & \text{if } n = 0 \end{cases}$$

Let $G(z) = (H - z)^{-1}$ be the resolvant of H at $z \in \mathbb{C} \setminus \mathbb{R}$ and let, for $m, n \in \mathbb{N}$:

$$G_{m,n} = (\delta_m | G(z) \delta_n)$$

where $\delta_m = (\delta_{m,k})_{k \geq 0}$ with $\delta_{m,k} = 1$ if $k = m$ and $\delta_{m,k} = 0$ if $k \neq m$. Assume that v is a bounded sequence.

1. Show that H is a bounded self-adjoint operator.
2. Show that :

$$\forall z \in \mathbb{C} \setminus \mathbb{R}, \ ||G(z)|| \leq \frac{1}{|\text{Im} z|} \tag{1}$$

3. For $\lambda \in \mathbb{R}$, let $\psi(\lambda) = (\psi_n(\lambda))$ be the solution of $H\psi = \lambda\psi$ with $\psi_0 = 1$. Let $E(\lambda)$ the spectral family associated to H. Set $\mathrm{d}\rho(\lambda)$ the spectral measure of H defined by :

$$\forall m, n \in \mathbb{N}, \ (\delta_m | E(\lambda) \delta_n) = \psi_m(\lambda) \psi_n(\lambda) \mathrm{d}\rho(\lambda).$$

Write $G_{m,n}(z)$ with an integral of the type $\displaystyle\int_{\mathbb{R}} f(\lambda) \mathrm{d}\rho(\lambda)$.
4. Show that :

$$\text{Im } G_{n,n}(\mathrm{i}) = \int_{\mathbb{R}} \frac{\psi_n^2(\lambda)}{\lambda^2 + 1} \mathrm{d}\rho(\lambda) \leq 1 \tag{2}$$

5. Let $\varepsilon > 0$ and:

$$\theta(\lambda) = \sum_{n=0}^{+\infty} \frac{\psi_n^2(\lambda)}{(\lambda^2 + 1)(1 + n^{\frac{1}{2} + \varepsilon})^2}$$

Show that $\theta(\lambda) < +\infty$ for ρ-a.e. λ.
Hint : Estimate $\displaystyle\int_{\mathbb{R}} \theta(\lambda) \mathrm{d}\rho(\lambda)$ *using (2).*
6. Conclude: for any $\varepsilon > 0$, there exists for ρ-a.e. $\lambda \in \mathbb{R}$ a constant $C_{\lambda,\varepsilon}$ such that for every $n \geq 0$:

$$|\psi_n(\lambda)| \leq C_{\lambda,\varepsilon} (1 + n)^{\frac{1}{2} + \varepsilon}$$

Hence, any generalized eigenfunction $\psi(\lambda)$ is polynomially bounded in n for ρ-a.e. $\lambda \in \mathbb{R}$.