Exercises sheet 1 : Duality - Weak convergence

Exercise 1
Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space.
1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of H and let $x \in H$. Show that $x_n \rightharpoonup x$ if and only if

$$\forall y \in H,\ \langle x_n, y\rangle \xrightarrow[n\to+\infty]{} \langle x, y\rangle.$$

2. Let $(x_n)_{n\in\mathbb{N}}$ be an orthonormal family in H. Show that $x_n \rightharpoonup 0$.

Exercise 2
Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of H and let $x \in H$.
Assume that $x_n \rightharpoonup x$ and $||x_n|| \xrightarrow[n\to+\infty]{} ||x||$. Show that $x_n \to x$.

Exercise 3
Let $(H, \langle\cdot|\cdot\rangle)$ be a complex Hilbert space. Denote by $||\cdot||$ the norm on H associated with the inner product.
1. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized sequence in H, *i.e.*, for every $n \in \mathbb{N}$, $||x_n|| = 1$. Justify that one can extract from $(x_n)_{n\in\mathbb{N}}$ a weakly convergent subsequence in H.
2. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized sequence in H that converges weakly to $x \in H$. Show that:

$$\lim_{n\to+\infty} ||x_n - x||^2 = 1 - ||x||^2.$$

3. Let $T \in \mathcal{L}(H)$ be a bounded operator on H such that no normalized vector $x \in H$ satisfies $||T||_{\mathcal{L}(H)} = ||Tx||$ (that is, the norm of T is not attained).
a. Show that there exists a normalized sequence $(x_n)_{n\in\mathbb{N}}$, weakly convergent, such that:

$$\lim_{n\to+\infty} ||Tx_n|| = ||T||_{\mathcal{L}(H)}.$$

b. Show that this normalized sequence converges weakly to the zero vector of H.

Exercise 4
Consider the Banach space $(C([0,1]), ||\ ||_\infty)$. Let $E = C([0,1])$. Define $u \in E'$ by

$$\forall f \in E,\ u(f) = \int_0^1 f(x)\mathrm{d}x$$

and for all $n \geq 1$, define $u_n \in E'$ by

$$\forall f \in E,\ u_n(f) = \frac{1}{n}\sum_{k=1}^{n} f(\tfrac{k}{n}).$$

1. Compute $||u||_{E'}$ and for all $n \geq 1$, $||u_n||_{E'}$.
2. Show that

$$\forall f \in E,\ u_n(f) \xrightarrow[n\to+\infty]{} u(f),$$

but that, for every $n \in \mathbb{N}$, $||u_n - u||_{E'} = 2$.

Exercise 5
Let $E = C([0,1], \mathbb{R})$ be the space of continuous real-valued functions on $[0,1]$, endowed with the sup norm: $\forall f \in E,\ ||f||_\infty = \sup_{x\in[0,1]} |f(x)|$.

Recall Riesz's theorem, which states that the set of continuous, *positive* linear forms on E can be identified with the set of positives Borel measures μ on $[0,1]$. More precisely, we set:

$$\forall f \in E,\ \mu(f) = \int_0^1 f\mathrm{d}\mu.$$

Let $(f_n)_{n\geq1}$ be the sequence of functions defined by:

$$\forall x \in [0,1],\ f_n(x) = \begin{cases} 1 - nx & \text{if} \quad x \in [0, \frac{1}{n}] \\ 0 & \text{if} \quad x \in [\frac{1}{n}, 1] \end{cases}$$

1. Show that for every Borel measure μ on $[0,1]$, $(\mu(f_n))_{n\geq1}$ is convergent.
2. Show that $(f_n)_{n\geq1}$ is not weakly convergent in E.
Hint: one may consider the continuous linear functionals of Dirac type: for every $t \in [0,1]$,

$$\delta_t\ :\ \begin{array}{ccc} E & \to & \mathbb{R} \\ f & \mapsto & f(t) \end{array}.$$

We say that E is not *weakly sequentially complete.*

Exercise 6 - Geometric Hahn-Banach Theorem

Let E be a real vector space and $C \subset E$ a convex subset with nonempty interior in E. We denote by $Int(C)$ the interior of C.

The goal of this exercise is to show that:

> if $x \notin Int(C)$, there exists a nonzero linear form $\ell : E \to \mathbb{R}$ and a real number α such that $\ell(x) = \alpha$ and $\ell(y) < \alpha$ for every $y \in Int(C)$.

Note that the linear form and the real number α depend on x.

We then say that the hyperplane $\ell(y) = \alpha$ *separates the point x and the convex set C*.

For every convex set K whose interior contains 0 (think of K as a translate of C), we define the *gauge of the convex set K* as the mapping

$$J_K : \begin{array}{ccc} E & \to & \mathbb{R}_+ \\ x & \mapsto & \inf\{a > 0, |, \frac{x}{a} \in K\} \end{array}$$

1. Show that J_K is a sublinear functional on E.

2. Show that, for every $y \in E$, $y \in Int(K)$ if and only if $J_K(y) < 1$.

3. Conclude using the Hahn-Banach theorem.

Exercise 7

Let $\Omega \subset \mathbb{R}^d$ be an open set. We will show that the space $L^1(\Omega)$ is not reflexive. We assume here that the topological dual of $L^1(\Omega)$ is $L^\infty(\Omega)$ in the sense that:

$$\forall u \in (L^1(\Omega))',\ \exists! g \in L^\infty(\Omega),\ \forall f \in L^1(\Omega),\ u(f) = \int_\Omega fg.$$

For simplicity, assume that $0 \in \Omega$. For all $n \geq 1$, define

$$f_n = \frac{1}{\mathrm{vol}(B(0, \frac{1}{n}))} \mathbb{1}_{B(0,\frac{1}{n})}.$$

1. Show that, for all $n \geq 1$, $||f_n||_{L^1} = 1$.

2. Show that if $L^1(\Omega)$ were reflexive, there would exist a subsequence $(f_{n_k})_{k\geq 0}$ of (f_n) and a function $f \in L^1(\Omega)$ such that, for every function $g \in L^\infty(\Omega)$,

$$\int_\Omega f_{n_k} g \xrightarrow[k\to+\infty]{} \int_\Omega fg.$$

3. Show that if $g \in C_c^0(\Omega \setminus \{0\})$, then there exists an open ball centered at 0 on which $g = 0$.

4. Deduce that

$$\forall g \in C_c^0(\Omega \setminus \{0\}),\ \int_\Omega fg = 0.$$

5. Show that $f = 0$ almost everywhere on Ω.

6. Deduce a contradiction with Question 2 and conclude.

Exercise 8 - A Runge Theorem

Let D be a bounded simply connected open subset of $\mathbb{C}$. Let K be a simply connected compact subset included in D and set $R = \max |\xi|$, $\xi \in K$.

1. Let $z \in D$, $|z| > R$. Show that $\xi \mapsto (z - \xi)^{-1}$ is the uniform limit of polynomial functions in ξ on K.

2. Deduce that for every $z \in D \setminus K$, $\xi \mapsto (z - \xi)^{-1}$ is the uniform limit of polynomial functions in ξ on K.

3. Using Cauchy's integral formula, prove that every analytic function f on D, $\xi \mapsto f(\xi)$, is the uniform limit on every compact subset of D of polynomial functions in ξ.