Functional Analysis - Duality and Operator theory

Exercises sheet 1: Duality - Weak convergence

Exercise 1

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of H and let $x\in H$. Show that $x_n\rightharpoonup x$ if and only if

$$\forall y \in H, \ \langle x_n, y \rangle \xrightarrow[n \to +\infty]{} \langle x, y \rangle.$$

2. Let $(x_n)_{n\in\mathbb{N}}$ be an orthonormal family in H. Show that $x_n \to 0$.

Exercise 2

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of H and let $x \in H$.

Assume that $x_n \rightharpoonup x$ and $||x_n|| \xrightarrow[n \to +\infty]{} ||x||$. Show that $x_n \to x$.

Exercise 3

Let $(H, \langle \cdot | \cdot \rangle)$ be a complex Hilbert space. Denote by $||\cdot||$ the norm on H associated with the inner product.

- 1. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized sequence in H, *i.e.*, for every $n\in\mathbb{N}$, $||x_n||=1$. Justify that one can extract from $(x_n)_{n\in\mathbb{N}}$ a weakly convergent subsequence in H.
- **2.** Let $(x_n)_{n\in\mathbb{N}}$ be a normalized sequence in H that converges weakly to $x\in H$. Show that:

$$\lim_{n \to +\infty} ||x_n - x||^2 = 1 - ||x||^2.$$

- **3.** Let $T \in \mathcal{L}(H)$ be a bounded operator on H such that no normalized vector $x \in H$ satisfies $||T||_{\mathcal{L}(H)} = ||Tx||$ (that is, the norm of T is not attained).
- **a.** Show that there exists a normalized sequence $(x_n)_{n\in\mathbb{N}}$, weakly convergent, such that:

$$\lim_{n \to +\infty} ||Tx_n|| = ||T||_{\mathcal{L}(H)}.$$

b. Show that this normalized sequence converges weakly to the zero vector of H.

Exercise 4

Consider the Banach space $(C([0,1]), || ||_{\infty})$. Let E = C([0,1]). Define $u \in E'$ by

$$\forall f \in E, \ u(f) = \int_0^1 f(x) \mathrm{d}x$$

and for all $n \geq 1$, define $u_n \in E'$ by

$$\forall f \in E, \ u_n(f) = \frac{1}{n} \sum_{k=1}^n f(\frac{k}{n}).$$

- **1.** Compute $||u||_{E'}$ and for all $n \geq 1$, $||u_n||_{E'}$.
- 2. Show that

$$\forall f \in E, \ u_n(f) \xrightarrow[n \to +\infty]{} u(f),$$

but that, for every $n \in \mathbb{N}$, $||u_n - u||_{E'} = 2$.

Exercise 5

Let $E = C([0,1], \mathbb{R})$ be the space of continuous realvalued functions on [0,1], endowed with the sup norm: $\forall f \in E, ||f||_{\infty} = \sup_{x \in [0,1]} |f(x)|$.

Recall Riesz's theorem, which states that the topological dual of E can be identified with the set of Radon measures μ on [0,1] (that is, measures on the Borel σ -algebra of [0,1] that take finite values on compact sets). More precisely, we set:

$$\forall f \in E, \ \mu(f) = \int_0^1 f \mathrm{d}\mu.$$

Let $(f_n)_{n\geq 1}$ be the sequence of functions defined by:

$$\forall x \in [0,1], \ f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0,\frac{1}{n}] \\ 0 & \text{if } x \in [\frac{1}{n},1] \end{cases}$$

- **1.** Show that for every Radon measure μ on [0,1], $(\mu(f_n))_{n\geq 1}$ is a Cauchy sequence.
- **2.** Show that $(f_n)_{n\geq 1}$ is not weakly convergent in E.

Hint: one may consider the continuous linear functionals of Dirac type: for every $t \in [0, 1]$,

$$\delta_t : \begin{array}{ccc} E & \to & \mathbb{R} \\ f & \mapsto & f(t) \end{array}$$

We say that E is not weakly sequentially complete.

Exercise 6 - Geometric Hahn-Banach Theorem

Let E be a real vector space and $C \subset E$ a convex subset with nonempty interior in E. We denote by Int(C) the interior of C.

The goal of this exercise is to show that:

if $x \notin Int(C)$, there exists a nonzero linear form $\ell: E \to \mathbb{R}$ and a real number α such that $\ell(x) = \alpha$ and $\ell(y) < \alpha$ for every $y \in Int(C)$.

Note that the linear form and the real number α depend on x.

We then say that the hyperplane $\ell(y) = \alpha$ separates the point x and the convex set C.

For every convex set K whose interior contains 0 (think of K as a translate of C), we define the gauge of the convex set K as the mapping

$$J_K: \begin{array}{ccc} E & \to & \mathbb{R}_+ \\ x & \mapsto & \inf\{a > 0, |, \frac{x}{a} \in K\} \end{array}$$

- **1.** Show that J_K is a sublinear functional on E.
- **2.** Show that, for every $y \in E$, $y \in Int(K)$ if and only if $J_K(y) < 1$.
- 3. Conclude using the Hahn-Banach theorem.

Exercise 7

Let $\Omega \subset \mathbb{R}^d$ be an open set. We will show that the space $L^1(\Omega)$ is not reflexive. We assume here that the topological dual of $L^1(\Omega)$ is $L^{\infty}(\Omega)$ in the sense that:

$$\forall u \in (L^1(\Omega))', \ \exists ! g \in L^\infty(\Omega), \ \forall f \in L^1(\Omega), \ u(f) = \int_{\Omega} fg.$$

For simplicity, assume that $0 \in \Omega$. For all $n \geq 1$, define

$$f_n = \frac{1}{\text{vol}(B(0, \frac{1}{n}))} \mathbb{1}_{B(0, \frac{1}{n})}.$$

- **1.** Show that, for all $n \ge 1$, $||f_n||_{L^1} = 1$.
- **2.** Show that if $L^1(\Omega)$ were reflexive, there would exist a subsequence $(f_{n_k})_{k\geq 0}$ of (f_n) and a function $f\in L^1(\Omega)$ such that, for every function $g\in L^\infty(\Omega)$,

$$\int_{\Omega} f_{n_k} g \xrightarrow[k \to +\infty]{} \int_{\Omega} f g.$$

- **3.** Show that if $g \in C_c^0(\Omega \setminus \{0\})$, then there exists an open ball centered at 0 on which g = 0.
- 4. Deduce that

$$\forall g \in C_c^0(\Omega \setminus \{0\}), \ \int_{\Omega} fg = 0.$$

- **5.** Show that f = 0 almost everywhere on Ω .
- ${f 6.}$ Deduce a contradiction with Question 2 and conclude.

Exercise 8 - A Runge Theorem

Let D be a bounded simply connected open subset of \mathbb{C} . Let K be a simply connected compact subset included in D and set $R = \max |\xi|, \xi \in K$.

- 1. Let $z \in D$, |z| > R. Show that $\xi \mapsto (z \xi)^{-1}$ is the uniform limit of polynomial functions in ξ on K
- **2.** Deduce that for every $z \in D \setminus K$, $\xi \mapsto (z \xi)^{-1}$ is the uniform limit of polynomial functions in ξ on K.
- **3.** Using Cauchy's integral formula, prove that every analytic function f on D, $\xi \mapsto f(\xi)$, is the uniform limit on every compact subset of D of polynomial functions in ξ .