Institut Galilée **2025-2026**
Master 1
Functional Analysis - Operator theory

Exercises sheet 3 : Compact operators

Exercise 1

Let $K : [a,b] \times [a,b] \to \mathbb{C}$ a continuous function. Consider the operator $T : C([a,b],\mathbb{C}) \to C([a,b],\mathbb{C})$ defined by : $\forall u \in C([a,b],\mathbb{C})$,

$$\forall x \in [a,b],\ Tu(x) = \int_a^x K(x,y)u(y)\mathrm{d}y.$$

1. Show that T is compact.
2. Show that $\sigma(T) = \{0\}$.

Exercise 2

Show that the multiplication operator T, defined on $L^2([0,2],\mathbb{C})$ by

$$\forall u \in L^2([0,2],\mathbb{C}), \forall x \in [0,2], (T(u))(x) = xu(x),$$

is bounded and self-adjoint but not compact.

Exercise 3

Let H a Hilbert space. Let $(\lambda_n)_{n\in\mathbb{N}}$ a sequence of non-zero complex numbers which tends to 0 and, for every $n \in \mathbb{N}$, P_n an orthogonal projector of finite rank with $P_m P_n = 0$ if $m \neq n$.
1. Show that $\sum \lambda_n P_n$ converge for the operator norm to some operator $T \in \mathcal{B}_\infty(H)$.
2. If moreover the λ_n are real numbers, show that T is self-adjoint.

Exercise 4

Let $(E, || \ ||)$ a Banach space, $x_0 \in E$ and f a linear form on E such that $f(x_0) \neq 0$. Let $T \in \mathcal{L}(E)$ defined by :

$$\forall x \in E,\ Tx = f(x)x_0.$$

1. Show that T is a projector if and only if $f(x_0) = 1$.
2. Determine $\sigma(T)$.
3. Compute the resolvant of T.

Exercise 5 - Hilbert-Schmidt's operators

Let H a Hilbert space. An operator T on H is said Hilbert-Schmidt if there exists $M \geq 0$ such that, for every orthonormal family $(e_n)_{n\in\mathbb{N}}$ of H,

$$\forall N \geq 0,\ \sum_{n=0}^{N} ||Te_n||^2 \leq M.$$

Set $\|T\|_{HS}$ the smallest M satisfying this inequality. Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt's operators.
1. Show that $T \in \mathcal{B}_2(H)$ if and only if tr $(T^*T) < \infty$.
2. Let $T \in \mathcal{B}_2(H)$, $\varepsilon > 0$ and $\{e_0, \dots, e_N\}$ an orthonormal family of H such that

$$\sum_{n=0}^{N} \|Te_n\|^2 \geq \|T\|_{HS} - \varepsilon^2.$$

If P_N is the orthogonal projector on $V = \mathrm{Vect}(e_0, \dots, e_N)$, show that $\|T - TP_N\|_{\mathcal{L}(H)} \leq \varepsilon$.
3. Deduce that T is compact.
4. If $H = L^2(X,\mathbb{C})$, let $K \in L^2(X \times X, \mathbb{C})$. Show that T_K defined on H by

$$\forall u \in H,\ \forall x \in X,\ T_K u(x) = \int_X K(x,y)u(y)dy$$

is Hilbert-Schmidt.

Remark : one can show the reciprocal : if $T : H \to H$ is in $\mathcal{B}_2(H)$, there exists $K \in L^2(X \times X, \mathbb{C})$ such that $T = T_K$.

Exercise 6 - Mercer's theorem

Let K a complex-valued continuous function on $[0,1] \times [0,1]$. Let T_K the element of $\mathcal{L}(L^2([0,1],dx))$ defined by: $\forall f \in L^2([0,1])$,

$$\forall x \in [0,1],\ T_K f(x) = \int_0^1 K(x,y)f(y)dy$$

Assume that the operator T_K is self-adjoint and positive *i.e* $(T_K f|f) \geq 0$ for every $f \in L^2([0,1])$.

1. Show that for any interval $I \subset [0,1]$,

$$\int_I \int_I K(x,y)dxdy \in \mathbb{R}_+.$$

Deduce that, for any $x \in [0,1]$, $K(x,x) \in \mathbb{R}_+$.
2. Let (λ_n) the sequence of nonzero eigenvalues of T_K repeated according to their multiplicity and let (φ_n) an Hilbert basis of the orthogonal of $\ker(T_K)$ satisfying for every n :

$$T_K\varphi_n = \lambda_n\varphi_n$$

Hence we have the identity, for any $f \in L^2([0,1])$,

$$T_K f = \sum_n \lambda_n (f|\varphi_n)\varphi_n$$

the convergence being in $L^2([0,1])$.
a. Show that each φ_n is continuous.
b. Applying question 1 to a good kernel K_N, show that for every N and every $x \in [0,1]$,

$$K(x,x) \geq \sum_{n\leq N} \lambda_n |\varphi_n(x)|^2$$

3. a. Show that $\sum \lambda_n$ converge.
b. Show that $\sum \lambda_n (f|\varphi_n)\varphi_n$ converge uniformly in $x \in [0,1]$. What is its sum?
c. Show that for every $x \in [0,1]$, $\sum \lambda_n \varphi_n(x)\overline{\varphi_n(y)}$ converge uniformly in $y \in [0,1]$. What is its sum?
d. Compute the sum of the λ_n in terms of K.

Exercice 7

Consider the Hilbert space $H = L^2([0,1],\mathbb{C})$ endowed with the scalar product

$$\forall f,g \in H,\ (f|g) = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x$$

and the associated norm $||\cdot||_2$. Let T the operator on H defined by:

$$\forall f \in H,\ \forall x \in [0,1],\ (Tf)(x) = \int_0^1 K(x,t)f(t)\mathrm{d}t$$

where the kernel K is defined by:

$$K(x,t) = \begin{cases} x(1-t) & \text{if} \quad 0 \leq x \leq t \leq 1 \\ t(1-x) & \text{if} \quad 0 \leq t \leq x \leq 1 \end{cases}$$

1. Show that T is a bounded operator.
2. Show that T is self-adjoint.
3. Show that the range of T is included in the space of continuous functions on $[0,1]$.
4. Show that T is a compact operator.
5. Let $\lambda \in \mathbb{C}\setminus\{0\}$. Show that the equation in $f \in H$, $Tf = \lambda f$ is equivalent to

$$\begin{cases} f'' + \frac{1}{\lambda}f = 0 \\ f(0) = f(1) = 0. \end{cases}$$

6. Show that the set of nonzero eigenvalues of T is:

$$\{(n\pi)^{-2}\ ;\ n \in \mathbb{N}^*\}.$$

7. Compute $\sigma(T)$.
8.a. Show that the norm of T is equal to its spectral radius.
b. Deduce the norm of T.
9.a. Let P an orthogonal projector in H. Show that P is a positive operator.
b. Deduce that T is a positive operator.
10. Using Mercer's theorem, show that

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$