

Exercises sheet 2 - Corrections

Exercise 1

①

1. For $u \in L^2(X)$ one has:

$$\|T_K u\|_{L^2(X)}^2 = \int_X \left| \int_X K(x, y) u(y) dy \right|^2 dx$$

and:

$$\left| \int_X K(x, y) u(y) dy \right|^2 \underset{\substack{\text{Cauchy} \\ \text{Schwarz}}}{\leq} \left(\int_X |K(x, y)|^2 dy \right) \left(\int_X |u(y)|^2 dy \right)$$

Hence:

$$\int_X \left| \int_X K(x, y) u(y) dy \right|^2 dx \leq \int_X \left(\int_X |K(x, y)|^2 dy \right) \left(\int_X |u(y)|^2 dy \right) dx$$

②

$$= \|u\|_{L^2(X)}^2 \int_X \left(\int_X |K(x,y)|^2 dy \right) dx$$

$$= \|u\|_{L^2(X)}^2 \|K\|_{L^2(X \times X)}^2 < +\infty$$

In particular $T_K u \in L^2(X)$ and $T_K : L^2(X) \rightarrow L^2(X)$ is well-defined.
It is of course linear by linearity of the integral.

2. From question 1, one gets:

$$\forall u \in L^2(X), \|T_K u\|_{L^2(X)} \leq \|K\|_{L^2(X \times X)} \|u\|_{L^2(X)}$$

Hence T_K is a bounded operator and

$$\|T_K\|_{\mathcal{L}(H)} \leq \|K\|_{L^2(X \times X)}$$

3. Let $u, v \in L^2(X)$. One has:

$$(T_K u | v) = \int_X \left(\int_X K(x,y) u(y) dy \right) \overline{v(x)} dx$$

③

$$\begin{aligned}
 & \text{Fubini} \rightarrow \int_X \int_X k(x, y) u(y) \overline{v(x)} dx dy \\
 &= \int_X u(y) \left(\int_X k(x, y) \overline{v(x)} dx \right) dy \\
 &= (u | T^* v) \quad \text{with } \forall y \in X, (T^* v)(y) = \overline{\int_X k(x, y) v(x) dx}
 \end{aligned}$$

$$\text{But: } \overline{\int_X k(x, y) v(x) dx} = \int_X \overline{k(x, y)} v(x) dx$$

let k^* defined by: $\forall x, y \in X, k^*(x, y) = \overline{k(y, x)}$

Then: $\forall u \in H, \forall x \in X, (T_k^* u)(x) = \int_X \overline{k(y, x)} u(y) dy = (T_{k^*} u)(x)$

and $T_k^* = T_{k^*}$.

It is analogous to the case of matrices: $A^* = {}^t \overline{A} = (\overline{a_{ji}})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$.

④

Exercise 2:

We always have : $\|T^*T\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}^2$

Indeed, $\|T^*\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}$ and $\|T^*T\|_{\mathcal{L}(H)} \leq \|T^*\|_{\mathcal{L}(H)} \|T\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}^2$

Reciprocally, for $u \in H$, $\|u\|_H = 1$:

$$\|Tu\|_H^2 = (Tu|Tu) = (T^*Tu|u) \underset{CS}{\leq} \|T^*Tu\| \|u\| \leq \|T^*T\|_{\mathcal{L}(H)} \|u\| \|u\|$$

hence $\|T\|_{\mathcal{L}(H)}^2 \leq \|T^*T\|_{\mathcal{L}(H)}$

If T is self-adjoint, $T^* = T$ and $\|T^2\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}^2$. Iterating

this relation one gets: $\forall n \in \mathbb{N}$, $\|T^{2^n}\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}^{2^n}$

Using the spectral radius formula: $r(T) = \lim_{k \rightarrow \infty} \|T^k\|_{\mathcal{L}(H)}^{1/k} = \lim_{n \rightarrow \infty} \|T^{2^n}\|_{\mathcal{L}(H)}^{1/2^n}$
 $= \lim_{n \rightarrow \infty} \left(\|T\|_{\mathcal{L}(H)}^{2^n} \right)^{1/2^n} = \|T\|_{\mathcal{L}(H)}$

⑤

Exercise 3:

Recall: U unitary means that $UU^* = U^*U = \text{Id}_H$.

1. Hint: show that it is enough to prove that

$$\text{Ker}(U^* - I) = \text{Ker}(U - I)$$

We show that $\text{Ker}(U - I) = \text{Ker}(U^* - I)$.

$\text{Ker}(U - I) \subset \text{Ker}(U^* - I)$: let $x \in \text{Ker}(U - I)$:

$Ux = x$: applying U^* : $U^*Ux = U^*x$

$$\Leftrightarrow x = U^*x$$

$$\Leftrightarrow x \in \text{Ker}(U^* - I).$$

(6) $\text{Ker}(U^* - I) \subset \text{Ker}(U - I)$: let $x \in \text{Ker}(U^* - I)$:

$$U^*x = x : \text{applying } U : UU^*x = Ux$$

$$\Leftrightarrow x = Ux$$

$$\Leftrightarrow x \in \text{Ker}(U - I)$$

$$\text{Then: } \text{Ker}(U - I) = \text{Ker}(U^* - I) = \text{Ker}((U - I)^*) \\ = \overline{\text{Im}(U - I)}^\perp$$

$$\text{and } (\text{Ker}(U - I))^\perp = \overline{\text{Im}(U - I)}$$

Since U is bounded, $\text{Ker}(U - I)$ is closed in H

and:

$$H = \text{Ker}(U - I) \oplus (\text{Ker}(U - I))^\perp \\ = \text{Ker}(U - I) \oplus \overline{\text{Im}(U - I)}$$

⑦ 2. If $u \in V : \exists! (x, y) \in \text{Ker}(U-I) \times \overline{\text{Im}(U-I)}$,

$$u = x + y$$

By definition $pu = x$.

We have to prove that: $\forall n \in \mathbb{N}, S_n u \xrightarrow{n \rightarrow \infty} x = pu$

$\forall n \in \mathbb{N}, S_n u = S_n x + S_n y$. We prove (i): $S_n x \xrightarrow{n \rightarrow \infty} x$

$$(ii) S_n y \xrightarrow{n \rightarrow \infty} 0$$

(i) We have $x \in \text{Ker}(U-I)$:

$$Ux = x \quad \text{and: } \forall n \geq 1, U^n x = x$$

$$\begin{aligned} \text{So: } S_n x &= \frac{1}{n+1} (I + U + \dots + U^n) x \\ &= \frac{1}{n+1} (\underbrace{x + \dots + x}_{n+1 \text{ times}}) = x \end{aligned}$$

So: $\forall n \geq 1, S_n x = x$ hence $S_n x \xrightarrow{n \rightarrow \infty} x$.

⑨ (ii) We have $y \in \overline{\text{Im}(U-I)}$.

Let $\varepsilon > 0$: $\exists y_\varepsilon \in \text{Im}(U-I)$, $\|y - y_\varepsilon\|_H \leq \varepsilon$

Since $y_\varepsilon \in \text{Im}(U-I)$: $\exists z_\varepsilon \in H$, $y_\varepsilon = (U-I)z_\varepsilon$

Then: $\forall n \geq 1$, $S_n y_\varepsilon = \frac{1}{n+1} (I + U + \dots + U^n) y_\varepsilon$

$$= \frac{1}{n+1} (I + U + \dots + U^n) (Uz_\varepsilon - z_\varepsilon)$$

$$= \frac{1}{n+1} (\cancel{Uz_\varepsilon} - z_\varepsilon + \cancel{U^2z_\varepsilon} - \cancel{Uz_\varepsilon} + \dots + \cancel{U^{n+1}z_\varepsilon} - \cancel{U^n z_\varepsilon})$$

$$= \frac{1}{n+1} (U^{n+1}z_\varepsilon - z_\varepsilon)$$

(10) Hence: $\forall n \geq 1, \|S_n y_\varepsilon\|_H \leq \frac{1}{n+1} (\|U^{n+1} z_\varepsilon\|_H + \|z_\varepsilon\|_H)$
 $\leq \frac{1}{n+1} (\|U\|_{\mathcal{L}(H)}^{n+1} \|z_\varepsilon\|_H + \|z_\varepsilon\|_H)$
 $\stackrel{=1}{=} \text{since } U \text{ is unitary}$
 $= \frac{2}{n+1} \|z_\varepsilon\|_H \xrightarrow{n \rightarrow \infty} 0$

i.e. $\exists N_0 \in \mathbb{N}, \forall n \geq N_0, \|S_n y_\varepsilon\|_H \leq \varepsilon.$

Then: $\forall n \geq 1, \|S_n y\|_H = \|S_n y - S_n y_\varepsilon + S_n y_\varepsilon\|_H$
 $= \|S_n (y - y_\varepsilon) + S_n y_\varepsilon\|_H$
 $\leq \|S_n\|_{\mathcal{L}(H)} \|y - y_\varepsilon\|_H + \|S_n y_\varepsilon\|_H$
 $\leq \varepsilon \quad \leq \varepsilon \text{ for } n \geq N_0$

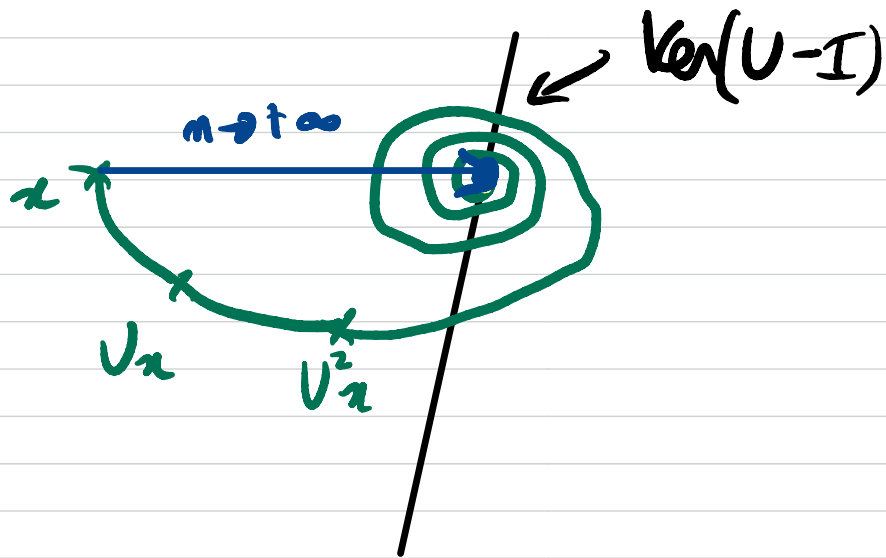
But: $\|S_n\|_{\mathcal{L}(H)} \leq \frac{1}{n+1} (\underbrace{\|I\|_{\mathcal{L}(H)}}_{=1} + \underbrace{\|U\|_{\mathcal{L}(H)}}_{=1} + \dots + \underbrace{\|U\|_{\mathcal{L}(H)}^n}_{=1}) = \frac{n+1}{n+1} = 1$

⑪ Finally for $n \geq N_0$:

$$\|S_n y\|_H \leq 1 \times \varepsilon + \varepsilon = 2\varepsilon, \text{ and } S_n y \xrightarrow{n \rightarrow \infty} 0,$$

$$S_n u = S_n x + S_n y \xrightarrow{n \rightarrow \infty} x + 0 = x = Pu.$$

Example: for U a rotation in $\mathbb{R}^3$: $\text{Ker}(U-I)$ is the axis of the rotation (the set of fixed points).



$$U = r_\theta$$
$$U^n = r_{n\theta}$$

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Exercise 4.

1. let $(f_m)_{m \in \mathbb{N}}$ an other Hilbert basis of H . Then:

$$\begin{aligned} \text{tr}_{(e_n)} T &= \sum_{n=0}^{+\infty} (T e_n | e_n) = \sum_{n=0}^{+\infty} (T^{1/2} T^{1/2} e_n | e_n) = \sum_{n=0}^{+\infty} (T^{1/2} e_n | T^{1/2} e_n) \\ &= \sum_{n=0}^{+\infty} \|T^{1/2} e_n\|^2. \end{aligned}$$

$T^{1/2}$ self-adjoint (s.a)

But, using Parseval's identity with $(f_m)_{m \in \mathbb{N}}$:

$$\forall n \in \mathbb{N}, \|T^{1/2} e_n\|^2 = \sum_{m=0}^{+\infty} |(T^{1/2} e_n | f_m)|^2$$

$$\text{Hence: } \text{tr}_{(e_n)} T = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} |(T^{1/2} e_n | f_m)|^2 = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} |(e_n | T^{1/2} f_m)|^2$$

positive terms
and $T^{1/2}$ s.a

But for any $m \in \mathbb{N}$, using Parseval's identity with $(e_n)_{n \in \mathbb{N}}$:

$$\sum_{n=0}^{+\infty} |(e_n | T^{1/2} f_m)|^2 = \sum_{n=0}^{+\infty} |(T^{1/2} f_m | e_n)|^2 = \sum_{n=0}^{+\infty} |(T^{1/2} f_m | e_n)|^2 = \|T^{1/2} f_m\|^2$$

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Hence: $\text{tr}_{(e)} T = \sum_{m=0}^{+\infty} \|T^{\frac{1}{2}} f_m\|^2 \underset{\text{as previously}}{=} \sum_{m=0}^{+\infty} (T f_m | f_m) = \text{tr}_{(f)} T.$

Finally, $\text{tr} T$ does not depend on the choice of a Hilbert basis for H .

2. One has:

$$\text{tr}(T+S) = \sum_{n=0}^{+\infty} ((T+S)e_n | e_n) = \sum_{n=0}^{+\infty} (Te_n | e_n) + \sum_{n=0}^{+\infty} (Se_n | e_n) = \text{tr} T + \text{tr} S$$

Similarly, for $\lambda \geq 0$ (so that $\lambda T \geq 0$ and $\text{tr}(\lambda T)$ is well defined):

$$\text{tr}(\lambda T) = \sum_{n=0}^{+\infty} (\lambda Te_n | e_n) = \sum_{n=0}^{+\infty} \lambda (Te_n | e_n) = \lambda \text{tr} T$$

3. If $T-S \geq 0$ then:

$$\forall n \in \mathbb{N}, ((T-S)e_n | e_n) \geq 0 \Leftrightarrow \forall n \in \mathbb{N}, (Te_n | e_n) \geq (Se_n | e_n).$$

Taking the sum leads to: $\text{tr} T \geq \text{tr} S.$

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4. If U is unitary and $(e_n)_{n \in \mathbb{N}}$ is a Hilbert basis of H then $(Ue_n)_{n \in \mathbb{N}}$ is also a Hilbert basis of H . Therefore:

$$\text{tr} T = \sum_{n=0}^{+\infty} (T(Ue_n) | Ue_n) = \sum_{n=0}^{+\infty} \underbrace{(U^* T U) e_n | e_n}_{U^{-1}} = \text{tr}(U^{-1} T U)$$

Similarly, show $(U^{-1})^* = U$ and $(U^{-1}e_n)_{n \in \mathbb{N}}$ is also a Hilbert basis of H :

$$\text{tr} T = \sum_{n=0}^{+\infty} (T U^{-1}e_n | U^{-1}e_n) = \sum_{n=0}^{+\infty} (U T U^{-1}e_n | e_n) = \text{tr}(U T U^{-1}).$$

Exercise 5:

Let $\lambda \in \mathbb{C} \setminus \text{Im} \sigma \varphi$. Then: $\exists \varepsilon > 0, \text{leb}(\varphi^{-1}(D(\lambda, \varepsilon))) = 0$.

$\Leftrightarrow \exists \varepsilon > 0$, for Leb a.e. $x \in \mathbb{R}$, $\varphi(x) \notin D(\lambda, \varepsilon)$

$\Leftrightarrow \exists \varepsilon > 0$, for Leb a.e. $x \in \mathbb{R}$, $|\varphi(x) - \lambda| \geq \varepsilon$.

reciprocal image
by φ

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Let $r \in L^2(\mathbb{R})$. We want to solve in $u \in L^2(\mathbb{R})$ the equation
 $(M_\varphi - \lambda) u = r$.

Let u defined by: $\forall x \in \mathbb{R}, u(x) = \begin{cases} \frac{1}{\varphi(x) - \lambda} r(x) & \text{si } \varphi(x) \neq \lambda \\ 0 & \text{sinon} \end{cases}$

Since there exists $\varepsilon > 0$ such that for a.e. $x \in \mathbb{R}$, $|\varphi(x) - \lambda| \geq \varepsilon$,
 $u = \frac{1}{\varphi - \lambda} r$ a.e. on $\mathbb{R}$.

Moreover: $\|u\|^2 \leq \frac{1}{\varepsilon^2} \|r\|^2$ Leb-p.p. since $|\varphi - \lambda| \geq \varepsilon$ Leb-p.p.

In particular $u \in L^2(\mathbb{R})$ and by construction it satisfies $(M_\varphi - \lambda)u = r$
in $L^2(\mathbb{R})$. Hence $\lambda \in \rho(M_\varphi)$.

Finally: $\lambda \in \sigma(\text{Im } \varphi) \Rightarrow \lambda \in \rho(M_\varphi)$ from which
we deduce that: $\sigma(M_\varphi) \subset \text{Im } \varphi$.

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Reciprocally, let us show that $\text{Im} \varphi \subset \sigma(M_\varphi)$.
Again, it suffices to show that $p(M_\varphi) \subset \mathbb{C} \setminus \text{Im} \varphi$.

Let $\lambda \in p(M_\varphi)$. Then: $\exists C > 0, \forall u \in L^2(\mathbb{R}), \| (M_\varphi - \lambda)^{-1} u \|_{L^2} \leq C \| u \|_{L^2}$ (*)

For $\varepsilon > 0$ and $n \in \mathbb{N}$, let: $u_{\varepsilon, n} = \chi_{\varphi^{-1}(\mathcal{D}(\lambda, \varepsilon)) \cap E_{n, n}} \in L^2(\mathbb{R})$.

$$\text{Then: } \| (M_\varphi - \lambda)^{-1} u_{\varepsilon, n} \|_{L^2}^2 = \int_{\mathbb{R}} \left| \frac{u_{\varepsilon, n}(x)}{\varphi(x) - \lambda} \right|^2 dx = \int_{\varphi^{-1}(\mathcal{D}(\lambda, \varepsilon)) \cap E_{n, n}} \frac{1}{|\varphi(x) - \lambda|^2} dx$$

But, if $x \in \varphi^{-1}(\mathcal{D}(\lambda, \varepsilon))$, $\varphi(x) \in \mathcal{D}(\lambda, \varepsilon)$ and $|\varphi(x) - \lambda|^2 \leq \varepsilon^2$.

Hence: $\frac{1}{|\varphi(x) - \lambda|^2} \geq \frac{1}{\varepsilon^2}$. Therefore:

$$\| (M_\varphi - \lambda)^{-1} u_{\varepsilon, n} \|_{L^2}^2 \geq \frac{1}{\varepsilon^2} \text{Leb}(\varphi^{-1}(\mathcal{D}(\lambda, \varepsilon)) \cap E_{n, n})$$

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Using (*): $\frac{1}{\varepsilon^2} \text{Leb}(\phi^{-1}(D(\lambda, \varepsilon)) \cap [-n, n]) \leq C \text{Leb}(\phi^{-1}(D(\lambda, \varepsilon)) \cap [-n, n])$

$$\Leftrightarrow \text{Leb}(\phi^{-1}(D(\lambda, \varepsilon)) \cap [-n, n]) \leq C\varepsilon^2 \text{Leb}(\phi^{-1}(D(\lambda, \varepsilon)) \cap [-n, n])$$

For $\varepsilon > 0$ such that $C\varepsilon^2 < 1$ it implies that

$$\text{Leb}(\phi^{-1}(D(\lambda, \varepsilon)) \cap [-n, n]) = 0.$$

Therefore: $\text{Leb}\left(\bigcup_{n \in \mathbb{N}} (\phi^{-1}(D(\lambda, \varepsilon)) \cap [-n, n])\right) = 0$ and $\text{Leb}(\phi^{-1}(D(\lambda, \varepsilon))) = 0$

i. e. $\lambda \notin \text{Im} \sigma \phi$. Finally, $\mathcal{P}(M_\phi) \subset \mathbb{C} \setminus \text{Im} \sigma \phi$

and $\text{Im} \sigma \phi \subset \sigma(M_\phi)$.

By double inclusion: $\sigma(M_\phi) = \text{Im} \sigma \phi$.

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If φ is defined on a compact interval $[a, b]$ and is assumed to be continuous, it is L^2 and we can prove that:

$$\text{Imess } \varphi = \varphi([a, b]).$$

Indeed, let $y \in \varphi([a, b])$. For every $\varepsilon > 0$, $D(y, \varepsilon)$ is an open disk and by continuity of φ , $\varphi^{-1}(D(y, \varepsilon))$ is an open set of $\mathbb{R}$. In particular:

$$\text{Leb}(\varphi^{-1}(D(y, \varepsilon))) > 0 \quad \text{and} \quad y \in \text{Imess } \varphi.$$

Reciprocally, if $y \in \text{Imess } \varphi$, for every $n \geq 1$, there exists $x_n \in \varphi^{-1}(D(y, \frac{1}{n}))$. In particular: $n \geq 1$, $|\varphi(x_n) - y| < \frac{1}{n}$ and $x_n \in [a, b]$. Since $[a, b]$ is compact we can extract from $(x_n)_{n \geq 1}$ a subsequence $(x_{\ell(n)})_{n \geq 1}$ which cv to some $x \in [a, b]$.

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Then: $\forall n \in \mathbb{N}, |f(x_{q(n)}) - y| < \frac{1}{q(n)}$ which shows,
using continuity of f : $f(x) = y$. Hence $y \in f([a, b])$

We got the second inclusion and we conclude that:

$$\text{Im} f = f([a, b]).$$

Remark: In general we don't have, for $f: \mathbb{R} \rightarrow \mathbb{C}$ continuous and bounded, that $\text{Im} f = f(\mathbb{R})$.

Indeed, $f: \mathbb{R} \rightarrow \mathbb{C}, x \mapsto e^{-|x|}$ is continuous and bounded but

$$\text{Im} f = [0, +\infty[\text{ and } f(\mathbb{R}) =]0, +\infty[!$$

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Exercise 6:

1. One has: $\forall u \in E$, $\|Tu\|_\infty = \sup_{n \in \mathbb{Z}} |u_{n+1}| = \sup_{n \in \mathbb{Z}} |u_n| = \|u\|_\infty$.

Hence T is bounded and it is an isometry: $\|T\|_{\mathcal{L}(E)} = 1$.

2. Let $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Then $(\lambda^n)_{n \in \mathbb{Z}} \in E$ and:

$$\forall n \in \mathbb{Z}, (T(\lambda^n)_{n \in \mathbb{Z}})_n = \lambda^{n+1} = \lambda \lambda^n$$

Hence λ is an eigenvalue of T with associated eigenvector $(\lambda^n)_{n \in \mathbb{Z}}$.

3. Let $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Let $u \in E$. Let $f = (T - \lambda)u$

Then: $\forall n \in \mathbb{Z}, u_{n+1} - \lambda u_n = f_n$

$$\begin{aligned} \text{Hence: } \forall n \in \mathbb{Z}, u_n &= \lambda u_{n-1} + f_{n-1} = \lambda(\lambda u_{n-2} + f_{n-2}) + f_{n-1} \\ &= f_{n-1} + \lambda f_{n-2} + \lambda^2 u_{n-2}. \end{aligned}$$

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By recurrence, one gets: $\forall p \geq 1, u_n = \sum_{k=1}^p \lambda^{k-1} f_{n-k} + \lambda^p u_{n-p}$

Since u is bounded and $|\lambda| < 1$, $\lambda^p u_{n-p} \xrightarrow{p \rightarrow +\infty} 0$.

Hence: $\forall n \in \mathbb{Z}, u_n = \sum_{k=1}^{+\infty} \lambda^{k-1} f_{n-k}$

The infinite sum converges since $|\lambda| < 1$ and (f_n) is bounded since u is bounded.

If for $g \in E$ one sets for every $n \in \mathbb{Z}$,

$$(R_\lambda g)_n = \sum_{k=1}^{+\infty} \lambda^{k-1} g_{n-k}$$

one has $R_\lambda g \in E$ since $\|R_\lambda g\|_\infty \leq \sum_{k=1}^{+\infty} |\lambda|^{k-1} \|g\|_\infty = \frac{1}{1-|\lambda|} \|g\|_\infty$.

Using the previous computations:

$$\forall n \in \mathbb{Z}, (R_\lambda (T-\lambda)g)_n = \sum_{k=1}^{+\infty} \lambda^{k-1} \overbrace{(T-\lambda)g}^{f}_{n-k} = g_n.$$

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It remains: $R_\lambda(T-\lambda) = \text{Id}_E$.

We also have:

$$\begin{aligned} \text{for } n \in \mathbb{Z}, ((T-\lambda)(R_\lambda g))_n &= \sum_{k=1}^{+\infty} \lambda^{k-1} \underbrace{g_{n+1-k}}_{\substack{k-1 \rightarrow k \\ \text{in the first} \\ \text{sum}}} - \sum_{k=1}^{+\infty} \lambda^k \overbrace{g_{n-k}}^{k \rightarrow k-1} \\ &\stackrel{\text{in the first sum}}{\Rightarrow} = \sum_{k=0}^{+\infty} \lambda^k g_{n-k} - \sum_{k=1}^{+\infty} \lambda^k g_{n-k} = \lambda^0 g_{n-0} = g_n. \end{aligned}$$

Hence: $(T-\lambda)R_\lambda = \text{Id}_E$.

Finally: $R_\lambda = (T-\lambda)^{-1}$ and since R_λ is bounded ($\|R_\lambda\|_{X/E} \leq \frac{1}{1-\|\lambda\|}$)

we have $\lambda \notin \sigma(T)$.

If $|\lambda| < 1$, $\lambda \notin \sigma(T)$

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4. By 1., $\sigma(T) \subset \overline{D(0,1)}$ since $\|T\|_{\mathcal{L}(E)} = 1$.

By 2.: $\forall \lambda \in \mathbb{C}, |\lambda| = 1, \lambda \in \sigma(T)$.

By 3.: $\forall \lambda \in D(0,1), \lambda \notin \sigma(T)$.

We deduce that $\sigma(T) = S^1$ the unit circle.

5. Let $\lambda, |\lambda| < 1$. Assume that $\lambda \notin \sigma(T_F)$. Then

$\lambda \in \rho(T_F)$ and $T_F - \lambda$ is bijective.

But $T_F - \lambda: F \rightarrow F$ and by surjectivity,

$(T_F - \lambda)(F) = F$. Similarly $(T_F - \lambda)^{-1}(F) = F$.

Let $g \in F$ defined by:

$g_n = 1$ if $n \leq 0$ and $g_n = 0$ if $n > 0$.

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By the computations done at question 3, valid for every $\lambda \in \mathbb{C}$, $|\lambda| < 1$, we have

$$\begin{aligned} \forall n \in \mathbb{Z}, \left((T_F - \lambda)^{-1} (g) \right)_n &= \sum_{k=1}^{+\infty} \lambda^{k-1} g_{n-k} = \sum_{k=n}^{+\infty} \lambda^{k-1} \times 1 + \sum_{k=1}^{n-1} \lambda^{k-1} \times 0 \\ &= \sum_{k=n}^{+\infty} \lambda^{k-1} = \frac{\lambda^{n-1}}{1-\lambda} \neq 0 \text{ if } \lambda \neq 0. \end{aligned}$$

and if $\lambda \neq 0$, $(T_F - \lambda)^{-1} (g) \notin F$ although F is stable by $(T_F - \lambda)^{-1}$.
Hence if $\lambda \in \mathbb{C}$, $|\lambda| < 1$ and $\lambda \neq 0$, $\lambda \notin p(T_F)$. i.e. $\lambda \in \sigma(T_F)$.

Finally, : $D(0,1) \setminus \{0\} \subset \sigma(T_F)$

6. Moreover, since $\|T_F\| = \|T\| = 1$, $\sigma(T_F) \subset \overline{D(0,1)}$.

By 5., $D(0,1) \setminus \{0\} \subset \sigma(T_F)$. Since $\sigma(T_F)$ is a closed set of $\mathbb{C}$: $\sigma(T_F) = \overline{D(0,1)}$.

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Exercise 7:

1. Let $\lambda \in [0, 1]$. Assume that: $\forall v \in E, \exists u \in E, (T - \lambda)u = v$

Then: $v = (T - \lambda)u \Leftrightarrow \forall x \in [0, 1], v(x) = xu(x) + (Lu)g(x) - \lambda u(x)$
 $\Leftrightarrow \forall x \in [0, 1], v(x) = (x - \lambda)u(x) + (Lu)g(x).$

- If $g(1) = 0$ then $v(1) = 0$. In this case, $v(1) = 0$ is a necessary condition for v to be in $\text{Im}(T - \lambda)$.
But, if we take $v \equiv 1$ on $[0, 1]$, $v \in E$ and $v(1) \neq 0$.
Hence $v \notin \text{Im}(T - \lambda)$.

Conclusion: if $g(1) = 0$, $T - \lambda$ is not surjective.

- Assume that $g(1) \neq 0$. Assume, by the absurd, that
 $h: \begin{matrix} [0, 1] \rightarrow \mathbb{R} \\ x \mapsto \sqrt{x - \lambda} \end{matrix}$ is in $\text{Im}(T - \lambda)$.

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By definition of the range:

$$\exists u \in E, \forall x \in [0, 1], \sqrt{x-1} = (x-1)u(x) + (Lu) \frac{g(x)}{\sqrt{x-1}}$$

This implies that: $\forall x \in [0, 1] \setminus \{1\}$, $1 = \sqrt{x-1}u(x) + (Lu) \frac{g(x)}{\sqrt{x-1}}$

If x tends to 1 : $1 = 0 + \infty$! since $g(1) \neq 0$ and g is bounded (because it is continuous on $[0, 1]$).

We got a contradiction from which we deduce that $1 \notin \text{Im}(T-1)$. Hence $T-1$ is not surjective.

Conclusion: If $\lambda \in [0, 1]$, $T-\lambda$ is not surjective hence not bijective i.e. $\lambda \in \sigma(T)$.

Hence: $\boxed{[0, 1] \subset \sigma(T)}$

(27)

2. Let $\lambda \in \mathbb{C} \setminus [0, 1]$. In particular: $\forall x \in [0, 1], x \neq 1$. Hence, if $h \in E$:

$$(T-\lambda)u = h \Leftrightarrow \forall x \in [0, 1], (x-\lambda)u(x) + (Lu) \frac{g(x)}{x-\lambda} = h(x)$$

$$\Leftrightarrow \forall x \in [0, 1], u(x) = \frac{h(x)}{x-\lambda} + (Lu) \frac{g(x)}{x-\lambda}$$

$$\Leftrightarrow \begin{cases} \forall x \in [0, 1], u(x) = \frac{h(x)}{x-\lambda} - \mu \frac{g(x)}{x-\lambda} \\ Lu = \mu \in \mathbb{C} \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu = L \left(x \mapsto \frac{h(x)}{x-\lambda} - \mu \frac{g(x)}{x-\lambda} \right) \\ \forall x \in [0, 1], u(x) = \frac{h(x)}{x-\lambda} - \mu \frac{g(x)}{x-\lambda} \end{cases}$$

$$\Leftrightarrow \begin{cases} \forall x \in [0, 1], u(x) = \frac{h(x)}{x-\lambda} - \mu \frac{g(x)}{x-\lambda} \\ \mu = \int_0^1 \frac{g(y)h(y)}{y-\lambda} dy - \mu \int_0^1 \frac{g(y)g(y)}{y-\lambda} dy \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu \left(1 + \int_0^1 \frac{g(y)g(y)}{y-\lambda} dy \right) = \int_0^1 \frac{g(y)h(y)}{y-\lambda} dy \\ \forall x \in [0, 1], u(x) = \frac{h(x)}{x-\lambda} - \mu \frac{g(x)}{x-\lambda} \end{cases}$$

(28)

If $1 + \int_0^1 \frac{f(y)g(y)}{y-\lambda} dy \neq 0$, the first equation admits a unique solution p_0 and the second equation gives us u in a unique way. Hence there exists a unique $u \in E$, such that $(T-\lambda)u = h$.

Hence, if $1 + \int_0^1 \frac{f(y)g(y)}{y-\lambda} dy \neq 0$ and if $\lambda \notin [0,1]$, $T-\lambda$ is bijective and $\lambda \in \rho(T)$.

We deduce that : $\lambda \in \sigma(T) \setminus [0,1] \Rightarrow 1 + \int_0^1 \frac{f(y)g(y)}{y-\lambda} dy = 0$

Fix $\lambda \in \mathbb{C} \setminus [0,1]$ and set:

$$F(\lambda) := 1 + \int_0^1 \frac{f(y)g(y)}{y-\lambda} dy = 0$$

Then F is holomorphic on $\mathbb{C} \setminus [0,1]$ and since $F(\lambda) \xrightarrow[\lambda \rightarrow \infty]{} 1$ (by dominated cv)

F does not vanish identically and its zeroes are isolated.

i.e. $\sigma(T) \setminus [0,1]$ is a set of isolated points.

(29)

Moreover, if $F(\lambda) = 0$, the equation $(T - \lambda)u = 0$ is equivalent to the system:

$$\begin{cases} \int_0^1 \mu \times 0 = 0 \left(= \int_0^1 \frac{f(y)\lambda \times 0}{y-\lambda} dy \right) \\ \forall x \in [0,1], u(x) = -\mu \frac{g(x)}{x-\lambda} \quad (1) \end{cases}$$

This system has an infinite number of solutions $u \in E$, hence $T - \lambda$ is not injective and λ is an eigenvalue of T . By (1), $\ker(T - \lambda) = \text{Vect}\left(x \mapsto \frac{g(x)}{x-\lambda}\right)$

3. We have: $F(\lambda) = 1 + \alpha \int_0^1 \frac{dy}{y-\lambda}$ and for $\lambda \in \mathbb{R} \setminus [0,1]$,

$$\int_0^1 \frac{dy}{y-\lambda} = \ln\left(\frac{|1-\lambda|}{|\lambda|}\right) = \ln\left(\frac{\lambda-1}{\lambda}\right) \text{ since } \lambda \notin [0,1].$$

We also deduce that:
 $\lambda \in \sigma(T) \setminus [0,1] \Leftrightarrow F(\lambda) = 0$,
 and $\sigma(T) \setminus [0,1] \subset \sigma_p(T)$

Hence: $\lambda \in \sigma(T) \setminus [0,1] \Leftrightarrow F(\lambda) = 0 \Leftrightarrow \int_0^1 \frac{dy}{y-\lambda} = -\frac{1}{\alpha}$.

(30)

In particular, $\int_0^1 \frac{dy}{y-1} \in \mathbb{R}$ and therefore:

$$\int_0^1 \frac{dy}{y-1} = -\frac{1}{\alpha} \Leftrightarrow e^{\int_0^1 \frac{dy}{y-1}} = e^{-\frac{1}{\alpha}}$$

But, $\lambda \mapsto \frac{\lambda-1}{\lambda}$ et $\lambda \mapsto e^{\int_0^1 \frac{dy}{y-1}}$ are holomorphic on $\mathbb{C} \setminus [0, 1]$ which is a connected set. Moreover they are equal on $\mathbb{R} \setminus [0, 1] \subset \mathbb{C} \setminus [0, 1]$ which contains non-isolated points. These two functions are therefore equal on $\mathbb{C} \setminus [0, 1]$ by analytic continuation.

Hence: $\forall \lambda \in \sigma(\pi) \setminus [0, 1], \frac{\lambda-1}{\lambda} = e^{-\frac{1}{\alpha}}$

and finally: $\sigma(\pi) = [0, 1] \cup \left\{ \frac{1}{1-e^{-\frac{1}{\alpha}}} \right\}$.

Exercise 8

1. a. Since A is self-adjoint, for any $\phi \in \mathcal{H}$, $(\phi | A\phi) \in \mathbb{R}$.
 Hence, for any $\alpha < 0$,

$$\|(A - \alpha)\phi\|^2 = \underbrace{\|A\phi\|^2}_{\geq 0} + \alpha^2 \|\phi\|^2 - \underbrace{2\alpha(\phi | A\phi)}_{\geq 0} \geq \alpha^2 \|\phi\|^2$$

b. If ϕ is such that $(A - \alpha)\phi = 0$ then $\alpha^2 \|\phi\|^2 = 0$ using 1. a.

But $\alpha < 0$ hence different from zero and $\|\phi\| = 0 \Rightarrow \phi = 0$

Hence $A - \alpha$ is injective.

c. Let $\psi \in (\text{Im}(A - \alpha))^\perp$. Then: $\forall \phi \in \mathcal{H}, ((A - \alpha)\phi | \psi) = 0$

Since A is self-adjoint, so is $A - \alpha$ and:

$$\forall \phi \in \mathcal{H}, (\phi | (A - \alpha)\psi) = 0 \Rightarrow (A - \alpha)\psi = 0.$$

(32)

Then: $(A\psi|\psi) = (x\psi|\psi) = x\|\psi\|^2$.

But $(A\psi|\psi) \geq 0$ and $x < 0$. It implies that $\|\psi\|^2 = 0$ and $\psi = 0$. Hence $(\text{Im}(A-x))^\perp = \{0\}$ and $\overline{\text{Im}(A-x)} = ((\text{Im}(A-x))^\perp)^\perp = \{0\}^\perp = \mathcal{H}$.

which proves that $\text{Im}(A-x)$ is dense in $\mathcal{H}$.

a. We prove that $\text{Im}(A-x)$ is closed.

Let $(v_n)_{n \in \mathbb{N}}$ a sequence in $\text{Im}(A-x)$ which converges to $v \in \mathcal{H}$. Then: $\forall n \in \mathbb{N}, \exists u_n \in \mathcal{H}, v_n = (A-x)u_n$ and using question 1, a:

$$\begin{aligned} \forall n, m \in \mathbb{N}, \quad \|v_n - v_m\| &= \|(A-x)u_n - (A-x)u_m\| \\ &= \|(A-x)(u_n - u_m)\| \geq x \|u_n - u_m\|. \end{aligned}$$

(33)

Since (w_n) cv it is a Cauchy sequence which implies that (u_n) is a Cauchy sequence using the previous inequality. Hence (u_n) cv to some $u \in H$ and by c^o of $A-\lambda$ (or the fact that since it is self-adjoint, it is a closed operator),

$$\forall n \in \mathbb{N} \quad u_n = (A-\lambda) u_n \xrightarrow{\text{notes}} u = (A-\lambda) u \in \text{Im}(A-\lambda)$$

and $\text{Im}(A-\lambda)$ is closed. Since it is dense in H ,

$\text{Im}(A-\lambda) = H$ and $A-\lambda$ is surjective. Since by 1.(b) it is injective, it is bijective.

e. Let $\psi \in H$ and $\phi = R_A(\lambda) \psi \in H$. Then $(A-\lambda)\phi = \psi$ and by 1. a.:

$$\lambda^2 \|R_A(\lambda) \psi\|^2 \leq \|\psi\|^2 \Rightarrow \|R_A(\lambda) \psi\| \leq \frac{1}{|\lambda|} \|\psi\|$$

(34)

Hence $R_A(h)$ is bounded and: $\|R_A(h)\|_{\mathcal{L}(H)} \leq \frac{1}{|h|}$

(f) Since A is self-adjoint, $\sigma(A) \subset \mathbb{R}$.

But, question 2(e) shows that any $x < 0$ is in $\rho(A)$
hence $\sigma(A) \subset \mathbb{R}_+$.

2. We prove the converse. Assume that $\sigma(A) \subset \mathbb{R}_+$.

Since A is self-adjoint, by the spectral theorem, there exists a unique spectral family, E such that:

$$A = \int_{\sigma(A)} t dE(t)$$

But since $\sigma(A) \subset [0, +\infty[$, $A = \int_0^{+\infty} t dE(t)$

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Let $B = \int_0^{+\infty} \sqrt{t} dE(t) = \sqrt{A}$. Then $B^2 = A$ and B is self-adjoint hence:
 $\forall \phi \in \mathcal{H}, (\phi | A \phi) = (\phi | B^2 \phi) = (\phi | \underbrace{B^*}_{=B} B \phi) = (B \phi | B \phi) = \|B \phi\|^2 \geq 0$

Hence A is positive.

(36)

Exercise 9

1. Let $\phi \in \ell^2(\mathbb{N})$. Then if $(\Delta_{\text{disc}} \phi)_n = \phi_{n+1} + \phi_{n-1}$

$$\|H\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq \|\Delta_{\text{disc}}\|_{\mathcal{L}(\ell^2(\mathbb{N}))} + \|V\|_{\mathcal{L}(\ell^2(\mathbb{N}))}$$

But we proved that $\|\Delta_{\text{disc}}\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq 2$ and we

$$\text{have } \|V\|_{\mathcal{L}(\ell^2(\mathbb{N}))} = \|v\|_{\infty}$$

Hence: $\|H\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq 2 + \|v\|_{\infty}$ and H is bounded.

To show that it is self-adjoint it remains to show that it is symmetric.

34) Let $\phi, \psi \in \ell^2(\mathbb{N})$.

$$(H\phi | \psi) = \sum_{n=0}^{+\infty} (H\phi)_n \overline{\psi_n}$$

$$= (-\phi_1 + \sqrt{0}\phi_0) \overline{\psi_0} + \sum_{n=1}^{+\infty} (-\phi_{n+1} - \phi_{n-1} + \sqrt{n}\phi_n) \overline{\psi_n}$$

$$= (-\phi_1 + \sqrt{0}\phi_0) \overline{\psi_0} - \sum_{n=1}^{+\infty} \phi_{n+1} \overline{\psi_n} - \sum_{n=1}^{+\infty} \phi_{n-1} \overline{\psi_n} + \sum_{n=1}^{+\infty} \sqrt{n} \phi_n \overline{\psi_n}$$

$$= (-\phi_1 + \sqrt{0}\phi_0) \overline{\psi_0} - \sum_{n=2}^{+\infty} \phi_n \overline{\psi_{n-1}} - \sum_{n=0}^{+\infty} \phi_n \overline{\psi_{n+1}} + \sum_{n=1}^{+\infty} \phi_n (\sqrt{n} \overline{\psi_n})$$

$$= -\cancel{\phi_1} \overline{\psi_0} + \phi_0 (\sqrt{0} \overline{\psi_0}) + \cancel{\phi_1} \overline{\psi_0} - \sum_{n=1}^{+\infty} \phi_n \overline{\psi_{n-1}} - \phi_0 \overline{\psi_1} - \sum_{n=1}^{+\infty} \phi_n \overline{\psi_{n+1}} + \sum_{n=1}^{+\infty} \phi_n (\sqrt{n} \overline{\psi_n})$$

$$= \phi_0 (\underbrace{-\psi_1 + \sqrt{0}\psi_0}_{(H\psi)_0}) + \sum_{n=1}^{+\infty} \phi_n (\underbrace{-\psi_{n-1} - \psi_{n+1} + \sqrt{n}\psi_n}_{(H\psi)_n})$$

$$= (\phi | H\psi). \quad \text{Hence } H \text{ is self-adjoint.}$$

(38)

2. Since H is self-adjoint, $\sigma(H) \subset \mathbb{R}$ and $\rho(H) \supset \mathbb{C} \setminus \mathbb{R}$.

If $z \in \mathbb{C} \setminus \mathbb{R}$, $H - z$ is invertible and $G(z)$ is well-defined.

One has:

$$G(z) - G(\bar{z}) = (z - \bar{z}) G(z) G(\bar{z}) = 2i \operatorname{Im} z G(z) G(\bar{z})$$

$$\text{But } G(\bar{z}) = (H - \bar{z})^{-1} \stackrel{H=H^*}{=} ((H - z)^*)^{-1} = ((H - z)^{-1})^* = (G(z))^*$$

$$\text{hence } \|G(z)\| = \|G(\bar{z})\|.$$

Then:

$$2 |\operatorname{Im} z| \|G(z) G(\bar{z})\| = \|G(z) - G(\bar{z})\| \leq \|G(z)\| + \|G(\bar{z})\| \leq 2 \|G(z)\|$$

$$\text{hence } |\operatorname{Im} z| \|G(z) G(\bar{z})\| \leq \|G(z)\|$$

$$\text{But } \|G(z) G(\bar{z})\| = \|(G(z))(G(z))^*\| = \|G(z)\|^2$$

$$\text{hence } |\operatorname{Im} z| \|G(z)\|^2 \leq \|G(z)\|.$$

(39)

Since $G(z)$ is invertible, $G(z) \neq 0$ and $\|G(z)\| \neq 0$.

Hence: $|\operatorname{Im} z| \|G(z)\| \leq 1$ and $\|G(z)\| \leq \frac{1}{|\operatorname{Im} z|}$

3. One has, using spectral theorem for H self-adjoint,

$$G(z) = (H - z)^{-1} = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE(\lambda)$$

$$\begin{aligned} \text{Hence: } \langle \delta_m, m, (\delta_m | G(z) \delta_m) \rangle &= \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\delta_m | E(\lambda) \delta_m) \\ &= \int_{\mathbb{R}} \frac{1}{\lambda - z} \psi_m(\lambda) \psi_m(\lambda) d\rho(\lambda) \end{aligned}$$

$$\text{Finally: } G_{m,m}(z) = \int_{\mathbb{R}} \frac{\psi_m(\lambda) \psi_m(\lambda)}{\lambda - z} d\rho(\lambda)$$

(40)

4. One has: $f_m > 0$, $\operatorname{Im} G_{m,m}(i) = \operatorname{Im} \int_{\mathbb{R}} \frac{\psi_m(t) \psi_m(t)}{\lambda - i} dp(t)$

But: $\operatorname{Im} \frac{1}{\lambda - i} = \operatorname{Im} \frac{\lambda + i}{\lambda^2 + 1} = \frac{1}{1 + \lambda^2}$

Therefore: $\operatorname{Im} G_{m,m}(i) = \int_{\mathbb{R}} \frac{(\psi_m(t))^2}{\lambda^2 + 1} dp(t)$

But: $\operatorname{Im} G_{m,m}(i) \leq |G_{m,m}(i)| = |(f_m, G(i) f_m)|$
 $\leq \|f_m\|_{\ell^2} \|G(i) f_m\|_{\ell^2}$
 $\leq \underbrace{\|f_m\|_{\ell^2}}_{=1} \underbrace{\|f_m\|_{\ell^2}}_{=1} \|G(i)\|_{\mathcal{L}(\ell^2(\mathbb{N}))}$
 $= \|G(i)\|_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq \frac{1}{\sqrt{2}} = 1$

Hence: $\int_{\mathbb{R}} \frac{(\psi_m(t))^2}{\lambda^2 + 1} dp(t) \leq 1.$

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5. Let $\varepsilon > 0$.

$$\int_{\mathbb{R}} \theta(\lambda) d\rho(\lambda) = \int_{\mathbb{R}} \sum_{n=0}^{+\infty} \frac{(\psi_n(\lambda))^2}{(\lambda^2+1)(1+n^{\frac{1}{2}+\varepsilon})^2} d\rho(\lambda)$$

$$\text{Beppo-Levi} \rightarrow = \sum_{n=0}^{+\infty} \frac{1}{(1+n^{\frac{1}{2}+\varepsilon})^2} \int_{\mathbb{R}} \frac{(\psi_n(\lambda))^2}{\lambda^2+1} d\rho(\lambda)$$

But, using 4.,

$$\left| \frac{1}{(1+n^{\frac{1}{2}+\varepsilon})^2} \int_{\mathbb{R}} \frac{(\psi_n(\lambda))^2}{\lambda^2+1} d\rho(\lambda) \right| \leq \frac{1}{(1+n^{\frac{1}{2}+\varepsilon})^2} \times 1 \leq \frac{1}{n^{1+2\varepsilon}}$$

$$\text{and } \sum_n \frac{1}{n^{1+2\varepsilon}} < \infty$$

Hence $\int_{\mathbb{R}} \theta(\lambda) d\rho(\lambda) < +\infty$ and $\theta(\lambda) < +\infty$ for ρ -a.e. λ .

6. Since for ρ -a.e. λ , $\theta(\lambda) < +\infty$, the serie which defines $\theta(\lambda)$ is cv for ρ -a.e. λ and for such a λ :

(42)

$$\frac{(\psi_m(b))^2}{(1+b^2)(1+m^{\frac{1}{2}+\varepsilon})^2} \xrightarrow{n \rightarrow \infty} 0$$

Then: $\exists N_0 \in \mathbb{N}$, $\forall m \geq N_0$, $|\psi_m(b)|^2 \leq (1+b^2)(1+m^{\frac{1}{2}+\varepsilon})^2$

and: $\forall m \geq N_0$, $|\psi_m(b)| \leq \sqrt{1+b^2} (1+m^{\frac{1}{2}+\varepsilon})$

and $\psi_m(b)$ is polynomially bounded in m .

(more precisely: $\forall m \geq 0$, $|\psi_m(b)| \leq \max\left(\frac{|\psi_0(b)|}{1}, \frac{|\psi_1(b)|}{1+m^{\frac{1}{2}+\varepsilon}}, \dots, \frac{|\psi_{N_0-1}(b)|}{1+(N_0-1)^{\frac{1}{2}+\varepsilon}} \sqrt{1+b^2}\right) (1+m^{\frac{1}{2}+\varepsilon})$)