2025 - 2026
M1
Option Operator Theory

Exercises sheet 3 - Corrections

Exercise 1:

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1. Let B be the unit ball of $C([a,b],\mathbb{C})$. We want to apply Ascoli's theorem

- Since T is bounded (its norm is bounded by $(\|K\|_\infty)$), one has:

$$\forall x \in [a,b], \forall u \in B, \ |Tu(x)| \leq (x-a)\, \|K\|_\infty \|u\|_\infty \leq (b-a)\|K\|_\infty$$

since $\|u\|_\infty \leq 1$ since $u \in B$.

Hence, for every $x \in [a,b]$, the family $\{Tu(x)\}_{u\in B}$ is bounded.

- Let $x, x' \in [a,b]$ and $u \in B$. One has:

$$|Tu(x) - Tu(x')| = \left| \int_a^x K(x,y)u(y)\,dy - \int_a^{x'} K(x',y)u(y)\,dy \right|$$

$$= \left| \int_a^x (K(x,y) - K(x',y))u(y)\,dy + \int_a^x K(x',y)u(y)\,dy - \int_a^{x'} K(x',y)u(y)\,dy \right|$$

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$$= \left| \int_a^x (K(x,y) - K(x',y))\, u(y)\, dy + \int_{x'}^{x} K(x',y)\, u(y)\, dy \right|$$

$$\leq \underbrace{\|u\|_\infty}_{\leq 1} \sup_{y\in[a,b]} |K(x,y) - K(x',y)|\, (b-a) + |x-x'|\, \|K\|_\infty \underbrace{\|u\|_\infty}_{\leq 1}$$

Since K is continuous on the compact set $[a,b]^2$, it is uniformly continuous on it. Let $\varepsilon > 0$. There exists $\eta > 0$ such that

$$\forall x, x' \in [a,b],\ |x-x'| < \eta \Rightarrow \sup_{y\in[a,b]} |K(x,y) - K(x',y)| \leq \varepsilon.$$

If we moreover assume that $\eta \leq \varepsilon$, one gets that for every $x, x' \in [a,b]$ such that $|x-x'| < \eta$,

$$\forall u \in B,\ |Tu(x) - Tu(x')| \leq \varepsilon (b-a) + \varepsilon \|K\|_\infty$$

Hence, $\{Tu\}_{u\in B}$ is equicontinuous.

- Applying Ascoli's theorem, $T(B)$ is relatively compact: T is a compact operator.

③

2. Since T is a compact operator and $C([a,b],\mathbb{C})$ is of infinite dimension, by Schauder's theorem, $0 \in \sigma(T)$ and $\sigma(T) \setminus \{0\}$ is composed of isolated eigenvalues.

Let $\lambda \neq 0$ such an isolated eigenvalue and let $u \neq 0$ an associated eigenfunction: $Tu = \lambda u$

Then: $\forall x \in [a,b]$, $u(x) = \frac{1}{\lambda} Tu(x) = \frac{1}{\lambda} \int_a^x k(x,y) u(y) \, dy$

Hence: $\forall x \in [a,b]$, $|u(x)| \leq \frac{1}{|\lambda|} \int_a^b |k(x,y)| \, |u(y)| \, dy \leq 0 + \frac{\|k\|_\infty}{|\lambda|} \int_a^x |u(y)| \, dy$

By Gronwall's lemma:

$$\forall x \in [a,b], \; |u(x)| \leq 0 \, e^{\frac{\|k\|_\infty}{|\lambda|}(t-a)} = 0$$

and $u = 0$ in $C([a,b],\mathbb{C})$, hence a contradiction. Therefore, $\sigma(T) \setminus \{0\} = \emptyset$ and $\sigma(T) = \{0\}$.

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Exercise 2:

- T is the multiplication operator by the real-valued and bounded function $\varphi : \begin{array}{l}[0,2] \to \mathbb{R} \\ x \mapsto x\end{array}$ hence it is self-adjoint and bounded with: $\|T\|_{\mathcal{L}(L^2([0,2]))} = \|\varphi\|_\infty = 2$.

- Assume that T is compact. Then, by the Fredholm alternative, either $(I-T)^{-1}$ exists and is bounded or $Tu = u$ admits a non-zero solution u.

But: $Tu = u \iff \forall x \in [0,2]\,,\ x\,u(x) = u(x)$

$$\iff \forall x \in [0,2],\ (x-1)\,u(x) = 0.$$

Hence: $\forall x \in [0,2]\setminus\{1\}$, $u(x) = 0$ and since $\mathrm{Leb}(\{1\}) = 0$, $u = 0$ in $L^2([0,2])$.

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Hence $Tu = u$ does not admit any non-zero solution in $L^2([0,2])$ and $(I-T)^{-1}$ exists and is bounded.

But: $\forall u, v \in L^2([0,2])$, $(I-T)^{-1}v = u \Leftrightarrow v = (I-T)u$

$\Leftrightarrow \forall x \in [0,2]$, $v(x) = u(x) - x\,u(x) = (1-x)\,u(x)$

i.e. $\forall x \in [0,2] \setminus \{1\}$, $u(x) = \frac{1}{1-x} v(x)$

and $(I-T)^{-1} v(x) = \frac{1}{1-x} v(x)$

But $(I-T)^{-1}$ defined this way is not bounded. Indeed, for $v \equiv \frac{1}{\sqrt{2}}$ on $[0,2]$, $\|v\|_{L^2} = 1$ but $\left\|\frac{1}{1-x} v\right\|_{L^2} = \frac{1}{\sqrt{2}} \int_0^2 \frac{dx}{|1-x|^2} = +\infty$

Hence T does not verify the Fredholm alternative, Therefore it is not compact.

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Exercise 3:

1. Let $m, n \in \mathbb{N}$, $m < n$.

By Pythagore : $\forall u \in H$, $\left\| \sum_{j=m}^{n} \lambda_j P_j u \right\|_H^2 = \sum_{j=m}^{n} \| \lambda_j P_j u \|_H^2 \leq \sup_{j \geq m} |\lambda_j|^2 \|u\|_H^2$

Since $\lambda_n \xrightarrow[n \to +\infty]{} 0$, the sequence $\left(\sum_{n=0}^{N} \lambda_n P_n \right)_{N \geq 0}$ is Cauchy in the Banach space $\mathcal{L}(H)$, hence it converges.

Its limit is a limit of finite rank operators, hence it is compact.

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2. Assume that moreover the λ_n are real numbers. The $\lambda_n P_n$ are therefore self-adjoint operators and T is self-adjoint as a strong limit of self-adjoint operators: indeed, since $\lambda_n \in \mathbb{R}$, for every $n \in \mathbb{N}$, $(\lambda_n P_n)^* = \lambda_n P_n^* = \lambda_n P_n$ and:

$$\|(\lambda_n P_n)^* - T^*\|_{\mathcal{L}(H)} = \|(\lambda_n P_n - T)^*\|_{\mathcal{L}(H)} = \|\lambda_n P_n - T\|_{\mathcal{L}(H)} \xrightarrow[n\to\infty]{} 0$$

Hence: $(\lambda_n P_n)^* \xrightarrow[n\to\infty]{} T^*$ and $(\lambda_n P_n)^* = \lambda_n P_n \xrightarrow[n\to\infty]{} T$.

By uniqueness of the limit: $T = T^*$.

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Exercise 4

1. Since $f(x_0) \neq 0$, $Tx_0 \neq 0$ and $T \neq 0$. Moreover:

$\forall x \in E$, $T^2 x = f(Tx)\, x_0 = f(f(x)x_0)\, x_0 = f(x) f(x_0)\, x_0 = f(x_0)\, Tx$.

Hence: $T^2 x = Tx \iff f(x_0) = 1$

$\Leftarrow$) Clear. $\Rightarrow$) we evaluate at $x = x_0$ and use that $Tx_0 \neq 0_E$,

2. T is linear and of finite rank 1 hence it is a compact operator.

By Schauder's theorem: $\sigma(T) = \{0\} \cup \{\text{non-zero eigenvalues}\}$

Let λ a non-zero eigenvalue of T. Then:

$$Tx = \lambda x \iff f(x) x_0 = \lambda x \implies x \in \mathrm{Vect}(x_0) = \mathrm{Im}\, T$$

Hence, if λ is a non-zero eigenvalue of T, its associated eigenspace is necessarily $\mathrm{Vect}(x_0)$.

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But : $Tx_0 = f(x_0)x_0$ and the unique possible eigenvalue is $f(x_0)$.
because $\mathrm{Vect}(x_0) = \mathrm{Ker}(T - f(x_0))$.
Hence $\sigma(T) = \{0, f(x_0)\}$ if E is of infinite dimension.

3. Let $\lambda \notin \{0, f(x_0)\}$. We want to solve in x for any $y \in E$:

$$Tx - \lambda x = y \Leftrightarrow f(x)x_0 - \lambda x = y \quad (*)$$

We apply f to this equality to get: $f(x)f(x_0) - \lambda f(x) = f(y)$

Hence : $f(x) = \dfrac{f(y)}{f(x_0) - \lambda}$ and $\dfrac{f(y)}{f(x_0)-\lambda} x_0 - \lambda x = y$ by $(*)$.

This last expression is equivalent to : $x = \dfrac{1}{\lambda}\left(\dfrac{f(y)}{f(x_0)-\lambda} x_0 - y\right)$

Finally : $x = (T-\lambda)^{-1} y$ with

$$(T-\lambda)^{-1} = -\lambda^{-1} Id + \left(\lambda(f(x_0) - \lambda)\right)^{-1} T$$

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Exercise 5

1. Let $(e_n)_{n\in\mathbb{N}}$ an orthonormal family (which is completed in a Hilbert basis if necessary). Then: $\forall n\in\mathbb{N}$, $\|Te_n\|^2 = (Te_n|Te_n) = (\underbrace{T^*T}_{\geq 0} e_n|e_n)$.

$\Rightarrow$ If $T\in\mathcal{B}_2(H)$, there exists $M>0$,

$$\forall N\geq 0,\ \sum_{n=0}^{N}\|Te_n\|^2 \leq M$$

Hence: $\forall N\geq 0,\ \sum_{n=0}^{N}(T^*Te_n|e_n)\leq M$

Since for every $n\in\mathbb{N}$, $(T^*Te_n|e_n)\geq 0$, the sequence $\left(\sum_{n=0}^{N}(T^*Te_n|e_n)\right)_{N\geq 0}$ is increasing and bounded by M hence it converges.

Then: $$\mathrm{Tr}(T^*T) = \sum_{n=0}^{+\infty}(T^*Te_n|e_n)\leq M<+\infty$$

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($\Leftarrow$) If $\mathrm{Tr}(T^*T) < +\infty$ then by positivity:

$$\forall N \geq 0, \quad \sum_{n=0}^{N} (T^*Te_n | e_n) \leq \mathrm{Tr}(T^*T) := M$$

$$\Leftrightarrow \forall N \geq 0, \quad \sum_{n=0}^{N} \|Te_n\|_H^2 \leq \mathrm{Tr}(T^*T)$$

2. Soient $T \in \mathcal{B}_2(H)$, $\varepsilon > 0$ and $\{e_0, \dots, e_n\}$ an orthonormal family such that:

$$\sum_{n=0}^{N} \|Te_n\|^2 \geq \|T\|_{HS}^2 - \varepsilon^2$$

Let $V = \mathrm{Vect}(e_0, \dots, e_N)$. Then $H = V \overset{\perp}{\oplus} V^\perp$ since V is finite dimensional hence closed.

Let $u \in H$, $\|u\|_H = 1$: $\exists ! (u_V, u_{V^\perp}) \in V \times V^\perp$, $u = u_V + u_{V^\perp}$

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and using Pythagore, $\|u\|_{\mathcal{H}}^2 = \|u_V\|_{\mathcal{H}}^2 + \|u_{V^\perp}\|_{\mathcal{H}}^2 = 1$

In particular, $\|u_V\|_{\mathcal{H}} \leq 1$ and $\|u_{V^\perp}\|_{\mathcal{H}} \leq 1$.

One has:

$$(T - TP_N)u = (T-TP_N)u_V + (T-TP_N)u_{V^\perp}$$

$$= T(I-P_N)u_V + T(I-P_N)u_{V^\perp}$$

But $I-P_N$ is the orthogonal projection on $V^\perp$ and since $\operatorname{Im} P_N = V$,

$$(I-P_N)u_V = 0 \quad \text{and} \quad (I-P_N)u_{V^\perp} = u_{V^\perp}$$

Therefore: $\|(T-TP_N)u\|_H = \|Tu_{V^\perp}\|_H$

Moreover, $\left(e_0, \dots, e_N, \frac{u_{V^\perp}}{\|u_{V^\perp}\|}\right)$ is an orthonormal family

hence by definition of $\|\ \|_{HS}$:

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$$\|T\|_{HS}^2 - \varepsilon^2 + \frac{1}{\|u_{V^\perp}\|^2}\|Tu_{V^\perp}\|^2 \leq \sum_{n=0}^{N}\|Te_n\|^2 + \left\|T\left(\frac{u_{V^\perp}}{\|u_{V^\perp}\|}\right)\right\|^2 \leq \|T\|_{HS}^2$$

In particular: $\cancel{\|T\|_{HS}^2} - \varepsilon^2 + \frac{1}{\|u_{V^\perp}\|^2}\|Tu_{V^\perp}\|^2 \leq \cancel{\|T\|_{HS}^2}$

$$\Rightarrow \quad \|Tu_{V^\perp}\|^2 \leq \varepsilon^2\|u_{V^\perp}\|^2 \leq \varepsilon^2$$

Therefore, $\|(T - TP_N)u\| = \|Tu_{V^\perp}\| \leq \varepsilon$

which implies that $\|T - TP_N\|_{\mathcal{L}(H)} \leq \varepsilon$

3. We deduce that T is the limit of the finite rank operators TP_N, hence it is a compact operator.

4. Let $(e_n)_{n\in\mathbb{N}}$ an orthonormal family of $L^2(X)$ and let $N \geq 1$.

Then: $$\sum_{n=0}^{N} \|T_K e_n\|^2 = \sum_{n=0}^{N} \int_X \Big| \underbrace{\int_X K(x,y)\, e_n(y)\, dy}_{= (e_n \mid \overline{K(x,\cdot)}) = \overline{(K(x,\cdot) \mid e_n)}} \Big|^2 dx$$

$$= \sum_{n=0}^{N} \int_X \left| \left(\overline{K(x,\cdot)} \mid e_n \right) \right|^2 dx$$

$$= \int_X \sum_{n=0}^{N} \left| \left(e_n \mid \overline{K(x,\cdot)} \right) \right|^2 dx$$

But, using Bessel inequality:

$$\sum_{n=0}^{N} \left| \left(e_n \mid \overline{K(x,\cdot)} \right) \right|^2 \leq \sum_{n=0}^{N} \left| \left(e_n \mid \overline{K(x,\cdot)} \right) \right|^2 \leq \|K(x,\cdot)\|^2_{L^2(X)}$$

Hence: $$\sum_{n=0}^{N} \|T_K e_n\|^2 \leq \int_X \|K(x,\cdot)\|^2_{L^2(X)}\, dx \underset{\text{Fubini}}{=} \|K\|^2_{L^2(X\times X)}$$

En particulier, $T_K \in \mathcal{B}_2(H)$.

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Exercise 6:

1. Let $\mathbb{1}_I$ be the characteristic function of the interval I.
Then $\mathbb{1}_I \in L^2([0,1])$ and applying the definition of the positivity of T_k:

$$0 \leq (T_k\mathbb{1}_I \mid \mathbb{1}_I) = \int_0^1 T_k\mathbb{1}_I(x)\,\overline{\mathbb{1}_I(x)}\,dx = \int_I (T_k\mathbb{1}_I)(x)\,dx$$

$$= \int_I \left(\int_0^1 k(x,y)\,\mathbb{1}_I(y)\,dy\right) dx$$

$$= \int_I \int_I k(x,y)\,dx\,dy$$

In particular for $x \in]0,1[$ and $n \geq 1$ such that $I_n = \left[x - \frac{1}{n}, x + \frac{1}{n}\right] \subset [0,1]$

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$\forall n \geq 1, \int_{I_n \times I_n} K(u,v)\,du\,dv \in \mathbb{R}_+$

Using uniform continuity of K, one has as n tends to $+\infty$:

$\forall x \in]0,1[, \int_{I_n \times I_n} K(u,v)\,du\,dv \xrightarrow[n \to +\infty]{} K(x,x) \in \mathbb{R}_+$

By C^0 of K one also has: $\forall x \in [0,1], K(x,x) \in \mathbb{R}_+$.

2. a. Each φ_n satisfies: $\forall x \in [0,1], \int_0^1 K(x,y)\varphi_n(y)\,dy = \lambda_n \varphi_n(x)$

To prove continuity of φ_n it suffices to show that

$x \mapsto \int_0^1 K(x,y)\,\varphi_n(y)\,dy$ is continuous.

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One has, for $x, x' \in [0,1]$,

$$\left| \int_0^1 k(x,y)\varphi_n(y)\,dy - \int_0^1 k(x',y)\varphi_n(y)\,dy \right| \leq \int_0^1 |k(x,y)-k(x',y)|\,|\varphi_n(y)|\,dy$$

Cauchy-Schwarz $\rightarrow$ $$\leq \left(\int_0^1 |k(x,y)-k(x',y)|^2\,dy \right)^{1/2} \underbrace{\|\varphi_n\|_{L^2}}_{=1}$$

But, using uniform C^0 of k on $[0,1]^2$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $|x-x'| < \delta$, $\sup_{y \in [0,1]} |k(x,y)-k(x',y)| \leq \varepsilon$

Hence, for $|x-x'| < \delta$,

$$\left| \int_0^1 k(x,y)\varphi_n(y)\,dy - \int_0^1 k(x',y)\varphi_n(y)\,dy \right| \leq \varepsilon .$$

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Hence $x \mapsto \int_0^1 K(x,y)\,\varphi_n(y)\,dy$ is uniformly continuous hence continuous and so is φ_n since $\lambda_n \neq 0$.

b. Let $N \geq 1$. We set for every $x, y \in [0,1]$,

$$K_N(x,y) = K(x,y) - \sum_{n=0}^{N} \lambda_n \varphi_n(x)\,\overline{\varphi_n(y)}$$

Let us prove that T_{K_N} is a positive operator.

Let $f \in L^2([0,1])$.

$$(T_{K_N} f | f) = \int_0^1 \left(\int_0^1 K_N(x,y) f(y)\,dy \right) \overline{f(x)}\,dx$$

$$= \int_0^1 \left(\int_0^1 K(x,y) f(y)\,dy - \int_0^1 \sum_{n=0}^{N} \lambda_n \varphi_n(x) \overline{\varphi_n(y)}\, f(y)\,dy \right) \overline{f(x)}\,dx$$

$$= (T_K f | f) - \sum_{n=0}^{N} \lambda_n \int_0^1 \varphi_n(x) \Big(\underbrace{\int_0^1 \overline{\varphi_n(y)}\, f(y)\,dy}_{(f|\varphi_n)} \Big) \overline{f(x)}\,dx$$

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$$= (T_K f|f) - \sum_{n=0}^{N} \lambda_n (f|\varphi_n) \int_0^1 \varphi_n(x) \overline{f(x)}\, dx$$

$$= (T_K f|f) - \sum_{n=0}^{N} \lambda_n (f|\varphi_n)(\varphi_n|f)$$

$$= \sum_{n=0}^{+\infty} \lambda_n (f|\varphi_n)(\varphi_n|f) - \sum_{n=0}^{N} \lambda_n (f|\varphi_n)(\varphi_n|f)$$

$$= \sum_{n=N+1}^{+\infty} \lambda_n |(f|\varphi_n)|^2 \geq 0. \text{ since the } \lambda_n \geq 0.$$

Hence T_{K_N} is positive. Using question 1, for every $x \in [0,1]$, $K_N(x,x) \geq 0$.

i.e. $\forall x \in [0,1]$, $K(x,x) \geq \sum_{n=0}^{N} \lambda_n |\varphi_n(x)|^2$

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3\. a. Intregating the inequality obtained at Q2(b) on $[0,1]$:

$$\int_0^1 k(x,x)\,dx \geq \sum_{n=0}^{N} \lambda_n \underbrace{\int_0^1 |\varphi_n(x)|^2 dx}_{\|\varphi_n\|_{L^2}^2 = 1} = \sum_{n=0}^{N} \lambda_n .$$

Since the λ_n are positive, $\left(\sum_{n=0}^{N} \lambda_n\right)_{N\in\mathbb{N}}$ is an increasing sequence which is upper bounded by $\int_0^1 k(x,x)\,dx$, hence $\sum \lambda_n$ cv.

b. One has, for every $x \in [0,1]$,

$$\left| \int_0^1 k(x,y) f(y)\,dy - \sum_{n=0}^{N} \lambda_n (f|\varphi_n)\varphi_n(x) \right| = \left| T_k f(x) - \sum_{n=0}^{N} (f|\varphi_n) T_k\varphi_n(x) \right|$$

$$= \left| T_k f(x) - T_k\left(\sum_{n=0}^{N} (f|\varphi_n)\varphi_n\right)(x) \right| = \left| \int_0^1 k(x,y)\left(f(y) - \sum_{n=0}^{N} (f|\varphi_n)\varphi_n(y)\right) dy \right|$$

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Cauchy-Schwarz

$$\le \|k(x,\cdot)\|_{L^2([0,1])} \left\| f - \sum_{n=0}^{N} (f|\varphi_n)\varphi_n(\cdot) \right\|_{L^2([0,1])}$$

$$\le \|k\|_\infty \left\| f - \sum_{n=0}^{N} (f|\varphi_n)\varphi_n(\cdot) \right\|_{L^2([0,1])} \xrightarrow[N\to+\infty]{} 0$$ independently of x since (φ_n) is an Hilbert basis

Hence the uniform convergence of $\sum \lambda_n (f|\varphi_n)\varphi_n$ to $T_k f$.

c. Let $p<q$ be two integers. One has, for every $x,y\in[0,1]$,

$$\left| \sum_{n=p}^{q} \lambda_n \varphi_n(x)\overline{\varphi_n(y)} \right| \underset{CS}{\le} \left(\sum_{n=p}^{q} \lambda_n |\varphi_n(x)|^2\right)^{1/2} \left(\sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2\right)^{1/2}$$

$$\le \left(\sum_{n=0}^{+\infty} \lambda_n |\varphi_n(x)|^2\right)^{1/2} \left(\sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2\right)^{1/2}$$

But $\left(\sum_{n=p}^{q} \underbrace{\lambda_n |\varphi_n(y)|^2}_{\ge 0}\right)_{q\ge p}$ cv uniformly by Dini's theorem since

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for any y, $\sum \lambda_n |\varphi_n(y)|^2$ cv (it is $\uparrow$ and $\leq K(y,y)$).

Hence: $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$, $\forall q > p \geq N_\varepsilon$, $\sum_{n=p}^{q} \lambda_n |\varphi_n(y)|^2 \leq \varepsilon$ uniformly in y.

Using the uniform Cauchy criterion, it implies that the serie $\sum \lambda_n \varphi_n(x) \overline{\varphi_n(y)}$ cv uniformly in y on $[0,1]$ for any fixed $x \in [0,1]$.

It remains to compute the limit. We apply 3.6. with $f_p = \mathbb{1}_{[y-\frac{1}{p}, y+\frac{1}{p}]}$ for $y \in]0,1[$, $f_p = \mathbb{1}_{[0,\frac{1}{p}]}$ if $y=0$ and $f_p = \mathbb{1}_{[1-\frac{1}{p},1]}$ if $y=1$.

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Then, $(f_p|\varphi_n) = \int_0^1 f_p(t)\overline{\varphi_n(t)}\,dt \xrightarrow[p\to+\infty]{} \overline{\varphi_n(y)}$

Hence: $\forall x,y \in [0,1]$, $\sum_{n\geq 0} \lambda_n (f_p|\varphi_n)\,\varphi_n(x) \xrightarrow[p\to+\infty]{} \sum_{n\geq 0} \lambda_n \overline{\varphi_n(y)}\,\varphi_n(x)$

by uniform convergence.

But $\sum_{n\geq 0} \lambda_n (f_p|\varphi_n)\,\varphi_n(x) = \int_0^1 K(x,t) f_p(t)\,dt \xrightarrow[p\to+\infty]{} K(x,y)$

Hence: $\sum_{n\geq 0} \lambda_n \overline{\varphi_n(y)}\,\varphi_n(x) = K(x,y)$.

d. Since it cv uniformly in y, one can let y tend to x in the previous equality:

$$K(x,x) = \sum_{n=0}^{+\infty} \lambda_n |\varphi_n(x)|^2$$

Then: $$\int_0^1 K(x,x)\,dx = \int_0^1 \sum_{n\geq 0} \lambda_n |\varphi_n(x)|^2\,dx \underset{\text{Beppo-Levi}}{=} \sum_{n\geq 0} \lambda_n \underbrace{\int_0^1 |\varphi_n(x)|^2\,dx}_{=1}$$

$$= \sum_{n\geq 0} \lambda_n$$

Finally: $$\sum_{n\geq 0} \lambda_n = \int_0^1 K(x,x)\,dx$$ (Mercer's formula).

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Exercise 7:

1. Let $f \in H$. We have:

$$\|Tf\|_2^2 = \int_0^1 |Tf(x)|^2 dx = \int_0^1 \left| \int_0^1 K(x,t) f(t) dt \right|^2 dx$$

$$\overset{CS}{\leq} \int_0^1 \left(\int_0^1 |K(x,t)|^2 dt \right) \left(\int_0^1 |f(t)|^2 dt \right) dx$$

$$= \|K\|^2_{L^2([0,1]\times[0,1])} \|f\|^2$$

Hence T is bounded and: $\|T\|_{\mathcal{L}(H)} \leq \|K\|_{L^2([0,1]\times[0,1])}$.

2. We have: $\forall f, g \in H$, $(Tf | g) = \int_0^1 Tf(x) \overline{g(x)} dx$

$$= \int_0^1 \int_0^1 K(x,t) f(t) dt \, \overline{g(x)} \, dx \underset{\text{Fubini}}{=} \int_0^1 f(t) \int_0^1 K(x,t) \overline{g(x)} dx \, dt$$

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since $K(x,t) = K(t,x) = \overline{K(t,x)}$ as K is real-valued

$$= \int_0^1 f(t) \overline{\int_0^1 K(t,x)\, g(x)\, dx}\, dt = \int_0^1 f(t) \overline{Tg(t)}\, dt = (f \mid Tg).$$

Hence T is self-adjoint: $T^* = T$.

3. Let $\varepsilon > 0$. Since K is C^0 on $[0,1] \times [0,1]$ (it is easy check, the only thing to look at is the C^0 on the diagonal $\{(x,t) \mid x = t\}$) which is compact, K is uniformly continuous on $[0,1] \times [0,1]$. Let $\varepsilon > 0$. There exists $\eta > 0$ such that for every $x, y \in [0,1]$ such that $|x - y| < \eta$,

$$\sup_{t \in [0,1]} |K(x,t) - K(y,t)| \leq \varepsilon.$$

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Then we have:

$\forall x,y \in [0,1],\ |x-y|<\eta,\quad \forall f \in H,$

$$|Tf(x)-Tf(y)| = \left|\int_0^1 k(x,t)f(t)\,dt - \int_0^1 k(y,t)f(t)\,dt\right|$$

$$= \left|\int_0^1 (k(x,t)-k(y,t))\,f(t)\,dt\right|$$

$$\overset{CS}{\leq} \left(\int_0^1 |k(x,t)-k(y,t)|^2\,dt\right)^{1/2} \|f\|_2$$

$$\leq \left(\int_0^1 \sup_{t\in[0,1]} |k(x,t)-k(y,t)|^2\,dt\right)^{1/2} \|f\|_2$$

as $|x-y|<\eta$

$$\leq \left(\int_0^1 \varepsilon^2\,dt\right)^{1/2} \|f\|_2 = \varepsilon \|f\|_2$$

Therefore: $\forall x,y \in [0,1],\ |x-y|<\eta \Rightarrow \forall f \in H,\ |Tf(x)-Tf(y)| \leq \varepsilon \|f\|_2.$

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It proves that Tf is continuous on $[0,1]$ for every $f \in H$. We deduce that:

$$\operatorname{Im} T \subset C^0([0,1]).$$

4. Since $\operatorname{Im} T$ is included in the space of continuous functions on $[0,1]$, to prove the compactness of the closure of the image by T of the unit ball of H, one can apply Ascoli's theorem.

But, if $\|f\|_2 \leq 1$, we already show at question 3. that:

$$\forall \varepsilon > 0, \exists \eta > 0, \forall x, y \in [0,1], |x-y| < \eta \Rightarrow \forall f \in \overline{B}_H(0,1), |Tf(x) - Tf(y)| \leq \varepsilon.$$

Hence the equicontinuity of $T(\overline{B}_H(0,1))$.

Moreover: $\forall x \in [0,1]$, $\forall f \in B_H(0,1)$,

$$|Tf(x)| \underset{CS}{\leq} \|K\|_{L^2([0,1]\times[0,1])} \underbrace{\|f\|_2}_{\leq 1} \leq \|K\|_{L^2([0,1]\times[0,1])}$$

Therefore, for every $x \in [0,1]$, $\{Tf(x)\}_{f \in B_H(0,1)}$ is bounded.

One can apply Ascoli's theorem to get that $\overline{T(B_H(0,1))}$ is compact. Hence T is a compact operator.

5. Let $\lambda \in \mathbb{C}\setminus\{0\}$. Let $f \in H$, $Tf = \lambda f$. Then:

$$\forall x \in [0,1],\ f(x) = \frac{1}{\lambda} Tf(x) = \frac{1}{\lambda}\int_0^1 k(x,t) f(t)\,dt$$

$$= \frac{1}{\lambda}\left[\int_0^x t(1-x) f(t)\,dt + \int_x^1 x(1-t) f(t)\,dt\right] \quad (1)$$

But, Tf is C^0 on $[0,1]$ hence $f = \frac{1}{\lambda} Tf$ is also continuous.

We deduce that $x \mapsto (1-x)\int_0^x t f(t)\,dt + x\int_x^1 (1-t) f(t)\,dt$ is C^1 on $[0,1]$ and therefore f is also C^1 on $[0,1]$. By derivation one gets:

$$\forall x \in [0,1],\ f'(x) = \frac{1}{\lambda}\left[-\int_0^x t f(t)\,dt + \cancel{(1-x)x f(x)} + \int_x^1 (1-t) f(t)\,dt \right.$$
$$\left. - \cancel{x(1-x) f(x)}\right]$$

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$$= \frac{1}{\lambda}\left[-\int_0^x t f(t)\,dt + \int_x^1 (1-t) f(t)\,dt\right] \quad (2)$$

Moreover, (1) implies that: $f(0) = f(1) = 0$.

Since f is C^0 and even C^1 on $[0,1]$, we get by (2) that f' is also C^1 on $[0,1]$ and a second derivation leads to:

$$\forall x \in [0,1], \quad f''(x) = \frac{1}{\lambda}\left[-x f(x) - (1-x) f(x)\right]$$

$$= \frac{1}{\lambda}\left[\cancel{-x f(x)} - f(x) + \cancel{x f(x)}\right] = -\frac{1}{\lambda} f(x).$$

Finally:

$$\begin{cases} f'' + \frac{1}{\lambda} f = 0. \\ f(0) = f(1) = 0 \end{cases}$$

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6. We look at the $\lambda \in \mathbb{C} \setminus \{0\}$ for which the equation $Tf = \lambda f$ admits a non-zero solution in H. By 5., it suffices to solve the Dirichlet problem
$$\begin{cases} f'' + \frac{1}{\lambda} f = 0 \\ f(0) = f(1) = 0 \end{cases}$$
and to search for values of $\lambda \neq 0$ for which it admits non-zero solutions.

But since T is self-adjoint we already know that
$$\sigma_p(T) \subset \sigma(T) \subset \mathbb{R}.$$
Hence we look at $\lambda \in \mathbb{R} \setminus \{0\}$.

1st case: $\lambda = -\frac{1}{a^2}$ $a > 0$. Then :
$$\exists A, B \in \mathbb{C}, \ \forall x \in [0,1], \ f(x) = Ae^{ax} + Be^{-ax}$$
Then $f(0) = A + B = 0$ and $f(1) = Ae^{a} + Be^{-a} = 0$
and $A = B = 0$ and $f = 0$ on $[0,1]$.

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In this case, λ is not an eigenvalue of T.

2nd case $\lambda = \frac{1}{a^2}$, $a > 0$. Then :

$\exists C, D \in \mathbb{C}$, $\forall x \in [0,1]$, $f(x) = C \cos(ax) + B \sin(ax)$.

Again, $f(0) = A = 0$ and $f(1) = \underbrace{A}_{=0} \cos a + B \sin a = 0$ hence $B \sin a = 0$.

On can assume $B \neq 0$ (else $f \equiv 0$ and λ is not an eigenvalue) and then $\sin a = 0$

It implies that $a = n\pi$, $n \in \mathbb{N}^*$ as $a > 0$.

Finally: $\lambda = \frac{1}{a^2} = \frac{1}{(n\pi)^2}$, $n \in \mathbb{N}^*$,

And for each of these values of λ, $f_n : \begin{array}{l}[0,1] \to \mathbb{C} \\ x \mapsto \sin(n\pi x)\end{array}$ is an associated eigenfunction. The eigenspace associated to each $\lambda_n = \frac{1}{(n\pi)^2}$ is $\mathrm{Vect}(f_n)$.

We finally have:

$$\sigma_p(T) = \left\{ \frac{1}{(n\pi)^2}, n \in \mathbb{N}^* \right\}$$

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7. Since T is a compact operator on H which is of infinite dimension, Riesz-Schauder's theorem asserts that:

$$\sigma(T) = \{0\} \cup \left\{ \frac{1}{(n\pi)^2},\ n \in \mathbb{N}^* \right\}.$$

8. (a) Since T is self-adjoint and bounded:

$$\|T^2\|_{\mathcal{L}(H)} = \|TT^*\|_{\mathcal{L}(H)} = \|T\|^2_{\mathcal{L}(H)}.$$

By iterating: $\forall n \geq 1$, $\|T^{2^n}\|_{\mathcal{L}(H)} = \|T\|^{2^n}_{\mathcal{L}(H)} \Rightarrow \forall n \geq 1, \|T\|_{\mathcal{L}(H)} = \|T^{2^n}\|^{\frac{1}{2^n}}_{\mathcal{L}(H)}$

Using the spectral radius formula:

$$r(T) = \lim_{p \to +\infty} \|T^p\|^{\frac{1}{p}}_{\mathcal{L}(H)} = \lim_{n \to +\infty} \|T^{2^n}\|^{\frac{1}{2^n}}_{\mathcal{L}(H)} = \lim_{n \to +\infty} \|T\|_{\mathcal{L}(H)} = \|T\|_{\mathcal{L}(H)}$$

(b) But, by definition of the spectral radius

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \underset{7.}{=} \sup_{n \geq 1} \frac{1}{(n\pi)^2} \underset{\text{for } n=1}{=} \frac{1}{\pi^2}$$

Hence: $\|T\|_{\mathcal{L}(H)} = \dfrac{1}{\pi^2}$.

9. We have $P = P^*$ and $P^2 = P$. Hence:

$$\forall f \in H, \ (Pf \mid f) = (P^2 f \mid f) = (PPf \mid f) = (Pf \mid P^* f)$$
$$= (Pf \mid Pf) = \|Pf\|_2^2 \geq 0.$$

and P is positive.

10. We apply to T the spectral theorem for self-adjoint compact operators.

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$$T = \sum_{n\geq 1} \lambda_n P_n$$ where the λ_n are the non-zero eigenvalues of T and P_n is the orthogonal projection on $\mathrm{Ker}(T-\lambda_n)$.

Then : $\forall f \in H$,

$$\underbrace{(Tf|f) =}_{C^0 \text{ of the scalar product}} \sum_{n\geq 1} \lambda_n (P_n f|f) = \sum_{n\geq 1} \underbrace{\frac{1}{(n\pi)^2}}_{>0} \underbrace{\|P_n f\|_2^2}_{\geq 0} \geq 0$$

Hence T is positive.

11. By 6., we have :

$$\sum_{n=1}^{+\infty} \lambda_n = \sum_{n=1}^{+\infty} \frac{1}{(n\pi)^2} = \frac{1}{\pi^2} \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

Using Mercer's theorem : $\displaystyle\sum_{n=1}^{+\infty} \lambda_n = \int_0^1 K(x,x)\,dx$.

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Hence :

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \pi^2 \int_0^1 K(x,x)\,dx = \pi^2 \int_0^1 x(1-x)\,dx$$

$$= \pi^2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \pi^2 \left(\frac{1}{2} - \frac{1}{3} \right) = \pi^2/6 .$$

We find the well-known result!