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Duality
Bounded and compact operators.
Spectral theorem.

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Contents

Chapter 1

Duality and weak convergence

To obtain compactness in infinite dimension (where closed and bounded sets are not necessarily compact), one can either strengthen the assumption, as in the Ascoli theorem, or try to weaken the conclusion.

It is this second approach that we shall consider in this chapter, by studying the notions of weak convergence and weak compactness. One of the goals of this chapter is to establish the following result: under suitable assumptions on the normed vector space E, the closed unit ball of E is weakly sequentially compact.

Throughout the following, $|\cdot|$ denotes either the absolute value on $\mathbb{R}$ or the modulus on $\mathbb{C}$.

1.1 Topological dual of a normed vector space

Definition 1.1.1. *Let $(E, \|.\|)$ be a normed vector space. The* topological dual *of E, denoted by E', is the set of all continuous linear forms on E.*

Recall that the kernel of a nonzero continuous linear form is a closed hyperplane of E.

Definition 1.1.2. *Let $(E, \|.\|)$ be a normed vector space. For $u \in E'$, the norm of u is defined as the positive real number:*

$$||u||_{E'} = \sup_{x \neq 0} \frac{|u(x)|}{||x||} = \sup_{||x||=1} |u(x)|.$$

Theorem 1.1.3. *Let $(E, \|\|)$ be a normed vector space. Then $(E', ||\ ||_{E'})$ is a Banach space.*

Proof : It is clear that this is a normed vector space. It remains to show that it is complete. Let $(u_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in E'.

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $m, n \geq N$, $||u_m - u_n||_{E'} \leq \varepsilon$. Then, for all $x \neq 0$ in E and for all $m, n \geq N$, we have

$$|u_m(x) - u_n(x)| = |(u_m - u_n)(x)| \leq ||u_m - u_n||_{E'}\ ||x|| \leq \varepsilon ||x||.$$

Hence, the sequence $(u_m(x))_{m\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$), which is complete, so it converges. For every $x \in E$, we set

$$u(x) = \lim_{m\to+\infty} u_m(x).$$

Then $u \in E'$ since each u_m belongs to E' (and by passing to the limit in the linearity equalities and in the continuity inequality). Let $x \in E$ with $||x|| = 1$. Then, for all $m, n \geq N$, and fixing m and letting $n \to \infty$, we obtain

$$\forall m \geq N,\ |u_m(x) - u(x)| \leq \varepsilon.$$

Since ε does not depend on x, we have $||u_m - u||_{E'} \leq \varepsilon$, and thus $(u_m)_{m\in\mathbb{N}}$ converges to u in $(E', || \,||_{E'})$.

□

Example 1.1.4. *If E is a finite-dimensional space, then $E' = E^*$, the algebraic dual of E.*

Example 1.1.5. *If H is a Hilbert space, its topological dual H' can be identified with H by the Riesz representation theorem for linear forms.*

1.2 Hahn-Banach Theorem

1.2.1 Analytic form of the Hahn-Banach theorem

Definition 1.2.1 (Sublinear functional)**.** *Let E be a real vector space. A* sublinear functional *on E is a function $q : E \to \mathbb{R}$ such that*

1. *$q(x+y) \leq q(x) + q(y)$ for all $x, y \in E$;*
2. *$q(\alpha x) = \alpha q(x)$ for all $x \in E$ and all $\alpha \geq 0$.*

Example 1.2.2. *A seminorm $\|.\|$ on E is a sublinear functional. A linear form on E is a sublinear functional. The gauges of convex subsets of E are sublinear functionals (see Exercise Sheet 1).*

In the definition above, note that q may take negative values and that the condition $q(\alpha x) = \alpha q(x)$ is required only for $\alpha \geq 0$.

Theorem 1.2.3 (Hahn-Banach Theorem)**.** *Let E be a real vector space and let q be a sublinear functional on E. Let F be a vector subspace of E, and let $u : F \to \mathbb{R}$ be a linear form such that $u(x) \leq q(x)$ for all $x \in F$. Then there exists a linear form $v : E \to \mathbb{R}$ such that $v|_F = u$ and $v(x) \leq q(x)$ for all $x \in E$.*

Note that such a v is, in particular, an extension of u. Such an extension can always be obtained by completing a basis of F to a basis of E, but the goal here is to show that there exists an extension v satisfying $v \leq q$ on the whole of E. Unlike the extension theorems encountered in analysis, we do not need any assumptions on E here (not even that E be normed). Note also that the Hahn-Banach theorem applies to *real* vector spaces, if only so that the inequality $u \leq q$ makes sense. However, the consequences we shall give in the next section remain valid for both real and complex vector spaces. To prove the Hahn-Banach theorem, we shall need the following lemma, which establishes the theorem in the case where F has codimension 1 in E.

Lemma 1.2.4. *Let $F \subset E$ be a vector subspace of codimension 1 in a real vector space E. Assume that E is endowed with a sublinear functional q and that we are given $u : F \to \mathbb{R}$ such that $u(x) \leq q(x)$ for all $x \in F$. Then there exists a linear form $v : E \to \mathbb{R}$ such that $v|_F = u$ and $v(x) \leq q(x)$ for all $x \in E$.*

Proof : Let $x_0 \in E \backslash F$, so that $E = \mathbb{R}x_0 \oplus F$. Suppose that the desired linear form v exists and set $\alpha_0 = v(x_0)$. Then, for all $y_1 \in F$, $v(x_0) + v(y_1) = v(x_0 + y_1) \leq q(x_0 + y_1)$, hence for all $y_1 \in F$, $\alpha_0 \leq -v(y_1) + q(x_0 + y_1)$. Moreover, for all $y_2 \in F$, $v(-x_0) + v(y_2) = v(-x_0 + y_2) \leq q(-x_0 + y_2)$, hence for all $y_2 \in F$, $\alpha_0 \geq v(y_2) - q(-x_0 + y_2)$. We must therefore have, for all $y_1, y_2 \in F$,

$$v(y_2) - q(-x_0 + y_2) \leq \alpha_0 \leq -v(y_1) + q(x_0 + y_1). \tag{1.1}$$

In particular, $v(y_2) - q(-x_0 + y_2) \le -v(y_1) + q(x_0 + y_1)$. Let us show that this necessary condition is indeed satisfied. It is equivalent to $v(y_1 + y_2) \le q(x_0 + y_1) + q(-x_0 + y_2)$ for all $y_1, y_2 \in F$. Now, for all $y_1, y_2 \in F$, we have

$$v(y_1 + y_2) \le q(y_1 + y_2) = q(y_1 + x_0 - x_0 + y_2) \le q(y_1 + x_0) + q(-x_0 + y_2).$$

Hence, for all $y_1, y_2 \in F$, $v(y_2) - q(-x_0 + y_2) \le -v(y_1) + q(x_0 + y_1)$. Therefore,

$$a := \sup_{y_2 \in F} \{v(y_2) - q(-x_0 + y_2)\} \le \inf_{y_1 \in F} \{-v(y_1) + q(x_0 + y_1)\} =: b.$$

Note that since $y_1, y_2 \in F$ in the above infimum and supremum, we have $v(y_2) = u(y_2)$ and $-v(y_1) = -u(y_1)$, so a and b depend only on u, not on v.

Now choose any $\alpha_0 \in [a, b]$ and *define* $v(tx_0 + y) := t\alpha_0 + u(y)$ for all $t \in \mathbb{R}$ and $y \in F$. Then v is a linear form on $E = \mathbb{R}x_0 \oplus F$, which coincides with u on F. The real number $\alpha_0 = v(x_0)$ thus defined satisfies inequality (1.1), and for all $t \ge 0$ and $y \in F$, writing (1.1) for $y_1 = \frac{y}{t}$ and $y_2 = \frac{y}{t}$ gives

$$v\left(\frac{y}{t}\right) - q\left(-x_0 + \frac{y}{t}\right) \le v(x_0) \le -v\left(\frac{y}{t}\right) + q\left(x_0 + \frac{y}{t}\right).$$

Therefore, on the one hand, $v(tx_0 + y) \le q(tx_0 + y)$ for all $t \ge 0$ and $y \in F$, and on the other hand, $v(-tx_0 + y) \le q(-tx_0 + y)$ for all $t \ge 0$ and $y \in F$. Hence $v(tx_0 + y) \le q(tx_0 + y)$ for all $t \in \mathbb{R}$ and $y \in F$, i.e. $v \le q$ on $E = \mathbb{R}x_0 \oplus F$. □

Observe that this reasoning allows, by a simple induction, to prove the Hahn-Banach theorem when F has finite codimension in E, in particular when E is finite-dimensional. We can then use the lemma to prove the Hahn-Banach theorem in full generality.

Proof : (Proof of the Hahn-Banach theorem). Let Δ be the set of pairs (G, v) where G is a vector subspace of E containing F and v is a linear form on G such that $v|F = u$ and $v \le q$ on G. Let $\le$ be the order relation on Δ defined by $(G, v) \le (G', v')$ if $G \subset G'$ and $v'|_G = v$. We show that this order is inductive. If $(G_i, v_i)_{i \in I}$ is a totally ordered family of elements of Δ, set $G = \cup_{i \in I} G_i$. Since $(G_i)_{i \in I}$ is totally ordered by inclusion, $\cup_{i \in I} G_i$ is a vector subspace of E. Define $v : G \to \mathbb{R}$ by $v(x) = v_i(x)$ if $x \in G_i$. Since $(G_i)_{i \in I}$ is totally ordered by inclusion and since, if $i \le j$, then $v_j|_{G_i} = v_i$, the map v is well defined. Moreover, for all $x \in G$, we have $v(x) \le q(x)$ since $v_i \le q$ for all $i \in I$. Also, by definition of v, we have $v|F = u$. Hence $(G, v) \in \Delta$ and it is an upper bound for the totally ordered family $(G_i, v_i)_{i \in I}$ (that is, $(G_i, v_i) \le (G, v)$ for all $i \in I$). Therefore, by Zorns lemma, Δ possesses a maximal element, denoted (H, w). Suppose $H \subsetneq E$. Then there exists $x \in E \backslash H$, and by Lemma 1.2.4, w can be extended to $H \oplus \mathbb{R}x$ while preserving the domination condition by q, contradicting the maximality of (H, w). Thus, $H = E$ and w is the desired extension of u. □

We recall Zorns lemma.

Lemma 1.2.5 (Zorns Lemma.). *If every totally ordered subset of a (partially) ordered set $(E, \le)$ has an upper bound in E, then $(E, \le)$ has a maximal element.*

Recall that a totally ordered subset $(A, \le)$ is a subset of E in which any two elements are comparable: $\forall a, b \in A,\ a \le b$ or $b \le a$.

An upper bound of A is an element $m \in E$ such that for all $a \in A$, $a \le m$.

A maximal element of E is an element a of E such that for all $x \in E$, $a \le x \;\Rightarrow\; x = a$. If the order is total, the notion of maximal element coincides with that of greatest element.

1.2.2 Extension of linear forms defined on a seminormed space

Throughout this section, we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The following statements are all corollaries of the Hahn-Banach theorem.

Proposition 1.2.6 (A general extension theorem.). *Let E be a $\mathbb{K}$-vector space endowed with a seminorm P. Let F be a vector subspace of E and let $u : F \to \mathbb{K}$ be a linear form such that $|u(x)| \leq P(x)$ for all $x \in F$. Then, there exists a linear form $v : E \to \mathbb{K}$ such that $v|_F = u$ and $|v(x)| \leq P(x)$ for all $x \in E$.*

Proof : *First case*: $\mathbb{K} = \mathbb{R}$. For all $x \in F$, we have $u(x) \leq |u(x)| \leq P(x)$. Since a seminorm is a sublinear functional, the Hahn-Banach theorem implies that there exists $v : E \to \mathbb{R}$ such that $v|_F = u$ and $v(x) \leq P(x)$ for all $x \in E$. Hence, $-v(x) = v(-x) \leq P(-x) = P(x)$ since P is a seminorm. Therefore, $|v(x)| \leq P(x)$ for all $x \in E$.

Second case: $\mathbb{K} = \mathbb{C}$. Set $u_1 = \mathrm{Re}, u$. Then $|u_1| \leq |u| \leq P$ on F. Thus, by the case $\mathbb{K} = \mathbb{R}$, there exists a real-linear form $v_1 : E \to \mathbb{R}$ such that $v_1|_F = u_1$ and $|v_1| \leq P$ on E. Define $v := \widetilde{v_1} : x \mapsto v_1(x) - iv_1(ix)$. Then v is a complex-linear form that satisfies $|v| \leq P$ on E and $v|_F = \widetilde{u_1} = u$. □

Corollary 1.2.7 (Extension of continuous linear forms.). *Let $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space, and let $u : F \to \mathbb{K}$ be a continuous linear form defined on a subspace F of E. Then, there exists a continuous linear form $v : E \to \mathbb{K}$ such that $v|F = u$ and $||v||_{E'} = ||u||_{F'}$.*

Proof : We apply Proposition 1.2.6 with $P(x) = ||u||_{F'}\|x\|$: thus, u admits an extension v satisfying $|v(x)| \leq ||u||_{F'}\|x\|$ for all x in E, so that $||v||_{E'} \leq ||u||_{F'}$. But since v is an extension of u, we also have $||u||_{F'} \leq ||v||_{E'}$, hence equality holds. □

Corollary 1.2.8. *Let $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space. Let $(e_i)_{1 \leq i \leq n}$ be a linearly independent family of vectors in E and let $\lambda_1, ..., \lambda_n \in \mathbb{K}$ be arbitrary. Then there exists a continuous linear form v such that $v(e_i) = \lambda_i$ for all $i \in \{1, ..., n\}$.*

The key point in this statement is that we want to obtain a linear form v that is *continuous* on E.

Proof : Let $F := vect(e_1, ..., e_n) \subset E$; define $u(\sum_{j=1}^{n} \alpha_j e_j) = \sum_{j=1}^{n} \alpha_j \lambda_j$. Since $(e_1, ..., e_n)$ is a basis of F, u is a well-defined linear map and u is continuous *because F is finite-dimensional*. By Corollary 1.2.7, there exists a continuous linear form $v : E \to \mathbb{K}$ that extends u and thus satisfies in particular $v(e_i) = u(e_i) = \lambda_i$ for all $i \in 1, ..., n$. □

1.2.3 Some geometric consequences of the Hahn-Banach theorem in normed vector spaces

Recall that if E is a normed vector space, the notation E' denotes the vector space of *continuous* linear forms on E. The space E' is called the topological dual of E.

Theorem 1.2.9. *Let $(E, \| . \|)$ be a normed $\mathbb{K}$-vector space. Then, for all $x \in E$,*

$$\begin{aligned} \|x\| &= \sup\left\{|u(x)| \,:\, u \in E' \text{ et } ||u||_{E'} \leq 1\right\} \\ &= \sup\left\{|u(x)| \,:\, u \in E' \text{ et } ||u||_{E'} = 1\right\}. \end{aligned}$$

Moreover, this supremum is attained.

Proof : For every $u \in E'$ satisfying $||u||_{E'} \leq 1$, we have $|u(x)| \leq ||u||_{E'} \|x\| \leq \|x\|$, hence

$$\sup\left\{|u(x)| \,:\, u \in E' \text{ et } ||u||_{E'} = 1\right\} \leq \sup\left\{|u(x)| \,:\, u \in E' \text{ et } ||u||_{E'} \leq 1\right\} \leq \|x\|.$$

Now set $F = \mathbb{K}x$ and define $v : F \to \mathbb{K}$ by $v(\lambda x) = \lambda\|x\|$. Then v is a linear form on F satisfying $|v(\lambda x)| = |\lambda|\|x\|$ for all $(\lambda x) \in F$, so $||v||_{F'} = 1$. By Corollary 1.2.7, there exists $w \in E'$ such that $||w||_{E'} = 1$ and $|w(x)| = |v(x)| = \|x\|$, hence

$$\|x\| = \sup\left\{|u(x)| \,:\, u \in E' \text{ et } ||u||_{E'} = 1\right\}.$$

All the inequalities above are therefore equalities, and we have also shown that this supremum is actually attained. □

Corollary 1.2.10. *Let $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space and let F be a* closed *subspace of E. Let $x_0 \in E\backslash F$; denote $d = dist(x_0, F) > 0$. Then, there exists a linear and continuous map $u : E \to \mathbb{K}$ such that $u(x_0) = 1$, $u(x) = 0$ for all $x \in F$, and $\|u\| = \frac{1}{d}$.*

Proof : Let $\pi : E \to E/F$ be the canonical projection. Recall that the quotient vector space E/F is endowed with the norm $\|y + F\| = dist(y, F)$ for all $y \in E$: the topology associated with this norm is the quotient topology of E/F, and the projection π satisfies $\|\pi(x)\| \leq \|x\|$ since $\|x + F\| = \inf_{y \in F} \|x - y\| \leq \|x\|$ because $0 \in F$. By Proposition 1.2.9, there exists a linear and continuous $v : E/F \to \mathbb{K}$ such that $\|v\| = 1$ and $v(x_0 + F) = \|x_0 + F\| = d$. Set $u = \frac{1}{d}(v \circ \pi) : E \to \mathbb{K}$. Then u is linear and continuous and satisfies $u(x_0) = \frac{1}{d}v(x_0 + F) = 1$ and $u(x) = 0$ for all $x \in F$. Moreover, $|u(x)| = \frac{1}{d}|v(x + F)| \leq \frac{1}{d}\|v\|\|x + F\| \leq \frac{1}{d}\|x\|$, hence $\|u\| \leq \frac{1}{d}$. Finally, since $\|v\| = 1$, there exists a sequence $(y_n)n \in \mathbb{N}$ of elements of E such that $\|y_n + F\| < 1$ for all $n \in \mathbb{N}$ and $\lim n \to +\infty|v(y_n + F)| = 1$. For each $n \in \mathbb{N}$, since $dist(y_n, F) < 1$, there exists $x_n \in F$ such that $\|y_n - x_n\| \leq 1$. Then $|u(y_n - x_n)| = \frac{1}{d}|v(y_n + F)| \underset{n \to +\infty}{\longrightarrow} \frac{1}{d}$ with $\|y_n - x_n\| \leq 1$, so $\|u\| = \frac{1}{d}$. □

We then deduce the following characterization of the closure of a subspace of a normed vector space. In particular, when dealing with a closed subspace, we obtain a description of this subspace as the intersection of kernels of continuous linear formsa result already known in finite dimension.

Theorem 1.2.11. *Let $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space and let $F \subset E$ be a vector subspace of E. Then,*

$$\overline{F} = \bigcap_{u \in E', F \subset \operatorname{Ker} u} \operatorname{Ker} u.$$

Proof : Set

$$G = \bigcap_{u \in E', F \subset \operatorname{Ker} u} \operatorname{Ker} u.$$

This intersection is nonempty since the zero linear form satisfies the required conditions. For every continuous linear form $u : E \to \mathbb{K}$, $\operatorname{Ker} u$ is closed in E, so G is closed in E as an intersection of closed sets. Moreover, $G \supset F$, hence $G = \overline{G} \supset \overline{F}$. Conversely, let $x \in G$. Suppose that $x \notin \overline{F}$. Then, by Corollary 1.2.10, there exists a linear and continuous map $u : E \to \mathbb{K}$ such that $u(x) = 1$ and $\overline{F} \subset \operatorname{Ker} u$. But since $x \in G$ and $\operatorname{Ker} u \supset \overline{F} \supset F$, we have $u(x) = 0$, which contradicts $u(x) = 1$. Hence $x \in \overline{F}$, and thus $\overline{F} = G$. □

Corollary 1.2.12. *Let $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space and $F \subset E$ a vector subspace of E. Then F is dense in E if and only if every continuous linear form on E that vanishes on F is identically zero on E.*

Proof : A continuous linear form on E that vanishes on F also vanishes on $\overline{F}$, so if $\overline{F} = E$, then every continuous linear form vanishing on F is zero on E. Conversely, if every continuous linear form on E that vanishes on F is zero on E, then the kernel of such a form is $\operatorname{Ker} u = E$. By Theorem 1.2.11, we have

$$\overline{F} = \bigcap_{u \in E', F \subset \operatorname{Ker} u} \operatorname{Ker} u = \bigcap_{u \in E', F \subset \operatorname{Ker} u} E = E.$$

□

1.3 Weak convergence in Banach spaces

Definition 1.3.1. *Let $(E, \|.\|)$ be a normed vector space and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of E. We say that $(x_n)_{n\in\mathbb{N}}$* converges weakly *to $x \in E$ if, for every $u \in E'$,*

$$\lim_{n\to+\infty} u(x_n) = u(x).$$

We then write $x_n \rightharpoonup x$.

To avoid confusion with the usual notion of convergence in E, we say that $(x_n)_{n\in\mathbb{N}}$ *converges strongly* to x when

$$\lim_{n\to+\infty} ||x_n - x|| = 0$$

and we write $x_n \to x$.

Proposition 1.3.2. *Let $(E, \|.\|)$ be a normed vector space and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of E.*

1. *If $(x_n)_{n\in\mathbb{N}}$ converges strongly to $x \in E$, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to x.*

2. *The weak limit of a weakly convergent sequence is unique.*

3. *If $x_n \rightharpoonup x$, then $(x_n)_{n\in\mathbb{N}}$ is strongly bounded.*

4. *If $x_n \rightharpoonup x$, then*

$$||x|| \leq \liminf ||x_n||.$$

Proof :

1. Let $u \in E'$. Then, for every $n \in \mathbb{N}$,

$$|u(x_n) - u(x)| = |u(x_n - x)| \leq ||u||_{E'} \, ||x_n - x||$$

and since (x_n) converges strongly to x, $||x_n - x|| \to 0$. Hence the weak convergence of (x_n) to x.

2. Suppose that $x_n \rightharpoonup x_1$ and $x_n \rightharpoonup x_2$. Then, for every $u \in E'$,

$$u(x_1) = \lim_{n\to+\infty} u(x_n) = u(x_2).$$

By Theorem 1.2.9, there exists $v \in E'$ such that $||x_1 - x_2|| = v(x_1 - x_2)$. Hence, by linearity of v and the above equality applied to $u = v$, $||x_1 - x_2|| = v(x_1) - v(x_2) = 0$. Thus $x_1 = x_2$.

3. Let $u \in E'$. Since $(u(x_n))_{n\in\mathbb{N}}$ converges, it is bounded by some constant $M_u \geq 0$. For every $n \in \mathbb{N}$, consider the linear form
$$l_n : \begin{array}{ccc} E' & \to & \mathbb{K} \\ u & \mapsto & u(x_n) \end{array}$$
We have: $\forall n \in \mathbb{N}$, $\forall u \in E'$, $|l_n(u)| \leq M_u$. By the Banach-Steinhaus theorem (since E' and $\mathbb{K}$ are Banach spaces), there exists $M \geq 0$ such that
$$\forall u \in E',\ ||u||_{E'} = 1,\ \sup_{n\in\mathbb{N}} |l_n(u)| \leq M.$$
By Theorem 1.2.9, for each $n \in \mathbb{N}$, there exists $u_n \in E'$ such that $|l_n(u_n)| = ||x_n||$. It follows that
$$\forall n \in \mathbb{N},\ ||x_n|| = |l_n(u_n)| \leq M.$$
4. By Theorem 1.2.9, there exists $u \in E'$ such that $|u(x)| = ||x||$ and $||u||_{E'} = 1$. For this u and every $n \in \mathbb{N}$, we have $|u(x_n)| \leq ||u||_{E'}||x_n|| = ||x_n||$.
Since $|u(x_n)| \to |u(x)| = ||x||$, taking the $\liminf$ in the previous inequality yields $||x|| \leq \liminf ||x_n||$. □

Note that the converse of the first point is not true in general. For example, any countable orthonormal family in a Hilbert space converges weakly to 0, but of course does not converge strongly to 0.

However, in finite dimension, weak and strong convergence are equivalent.

Proposition 1.3.3. *Let $(E, \|.\|)$ be a finite-dimensional normed vector space and $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of E. Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $x \in E$ if and only if it converges weakly to x.*

Proof : We have already seen the direct implication, which also holds in infinite dimension. For the converse, since all linear forms are continuous in finite dimension, we have $E' = E^*$, the algebraic dual of E. Applying the definition of $x_n \rightharpoonup x$ with the coordinate linear forms on E, we obtain that for every $i \in \{1, \ldots, d\}$, $x_n^i \to x^i$ (coordinatewise convergence). To show strong convergence in E, it suffices to show it for a convenient norm, since all norms are equivalent in finite dimension. For instance, coordinatewise convergence implies convergence in the sup norm or in the Euclidean norm (identifying E with $\mathbb{R}^d$). Hence the desired result. □

Proposition 1.3.4. *Let $(E, \|.\|)$ be a Banach space, $(x_n)_{n\in\mathbb{N}}$ a sequence in E, and $(u_n)_{n\in\mathbb{N}}$ a sequence in E'. Suppose that $x_n \rightharpoonup x$ and that $(u_n)_{n\in\mathbb{N}}$ converges to u in E'. Then $(u_n(x_n))_{n\in\mathbb{N}}$ converges to $u(x)$ in $\mathbb{R}$ (or $\mathbb{C}$).*

Proof : Let $\varepsilon > 0$. There exists $N_0 \in \mathbb{N}$ such that, for all $n \geq N_0$, $||u_n - u||_{E'} \leq \varepsilon$. Also, there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $|u(x_n) - u(x)| \leq \varepsilon$. Moreover, since $x_n \rightharpoonup x$, the sequence $(||x_n||)_{n\in\mathbb{N}}$ is bounded, say by $M \geq 0$. Then,
$$\begin{aligned} \forall n \geq \max(N_0, N_1),\ |u_n(x_n) - u(x)| &\leq |u_n(x_n) - u(x_n)| + |u(x_n) - u(x)| \\ &\leq ||u_n - u||_{E'}||x_n|| + |u(x_n) - u(x)| \\ &\leq M\varepsilon + \varepsilon. \end{aligned}$$
Hence the result. □

Proposition 1.3.5. *Let $A \subset E$ be a sequentially weakly closed subset of E. Then A is also strongly closed.*

Proof : Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of A such that $x_n \to x$. We show that $x \in A$. Since strong convergence implies weak convergence, $x_n \rightharpoonup x$, and since A is weakly closed, $x \in A$. Hence A is strongly closed. □

1.4 Reflexive spaces

Let $(E, \|.\|)$ be a normed $\mathbb{K}$-vector space and let $x \in E$. Then the map

$$J_x : \begin{array}{ccc} E' & \to & \mathbb{K} \\ u & \mapsto & u(x) \end{array}$$

is a continuous linear form on E' with norm $||x||$, according to Theorem 1.2.9. Hence, it is an element of E''. Thus, the map

$$J : \begin{array}{ccc} E & \to & E'' \\ x & \mapsto & J_x \end{array}$$

is well defined and is an isometry from E into E''. In particular, J is injective.

Definition 1.4.1. *The space E is said to be* reflexive *when the map J is also surjective. In this case, one can identify E isometrically with its topological bidual E''.*

Proposition 1.4.2 (Reflexivity of Hilbert spaces)**.** *Every Hilbert space $\mathcal{H}$ is reflexive. More precisely, the canonical map*

$$J : \begin{array}{ccc} \mathcal{H} & \to & \mathcal{H}'' \\ x & \mapsto & \left(\begin{array}{ccc} \mathcal{H}' & \to & \mathbb{K} \\ u & \mapsto & u(x) \end{array} \right) \end{array}$$

is a surjective isometric isomorphism.

Proof : By the Riesz representation theorem, for every $u \in \mathcal{H}'$ there exists a unique $y \in \mathcal{H}$ such that

$$u(x) = (x|y) \quad \text{and} \quad \|u\| = \|y\|.$$

Hence, the map

$$\Phi : \mathcal{H} \to \mathcal{H}', \quad \forall y \in \mathcal{H},\ \Phi(y) : x \mapsto (x|y)$$

is a linear isometric isomorphism. For $v \in \mathcal{H}''$, define $\tilde{v} = v \circ \Phi \in \mathcal{H}'$. By the Riesz theorem, there exists a unique $z \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H},\ \tilde{v}(x) = (x|z).$$

We thus obtain a linear isometric isomorphism $\Psi : \mathcal{H}' \to \mathcal{H}, \Psi(v) = z$. Finally, for $x \in \mathcal{H}$ and $u_y = \Phi(y)$, we have $(J(x))(u_y) = u_y(x) = (x|y)$, hence $\Psi(J(x)) = x$. Therefore, the map J is bijective and isometric. □

Example 1.4.3. *The space $(C = C([-1,1], \mathbb{R}), ||\ ||_\infty)$ is not reflexive. If it were, we would have $C = C''$, and by Theorem 1.2.9, for every linear form $u \in C'$, there would exist $f \in C'' = C$ such that*

$$||u||_{C'} = u(f) \quad \textit{with} \quad ||f||_\infty = 1.$$

We then define the linear form u by

$$\forall g \in C,\ u(g) = \int_{-1}^{0} g(t)\mathrm{d}t - \int_{0}^{1} g(t)\mathrm{d}t.$$

Then, for all $g \in C$, $|u(g)| < 2||g||_\infty$. But for every $\varepsilon > 0$, one can choose g so that

$$|u(g)| > (2-\varepsilon)||g||_\infty.$$

This shows that $||u||_{C'} = 2$. For $g = f$ as above, we then have $2 < 2 \times 1$, hence a contradiction.

Proposition 1.4.4. *Every reflexive space is a Banach space.*

Proof : Indeed, such a space is in an isometric bijection with its bidual. Since the bidual is the dual of a normed vector space, it is a Banach space, and therefore the original space is Banach as well. □

Proposition 1.4.5. *Let $(E, \|.\|)$ be a normed vector space. If $(E', || \, ||_{E'})$ is separable, then $(E, \|.\|)$ is separable.*

Proof : Since E' is separable, there exists a countable family $\{u_n\}_{n\in\mathbb{N}}$ dense in E'. By the definition of the norm $|| \, ||_{E'}$, for each $n \in \mathbb{N}$ there exists $x_n \in E$ such that $||x_n|| = 1$ and $|u_n(x_n)| > \frac{1}{2}||u_n||_{E'}$. Let us show that the space generated by the x_n is dense in E. To do so, it suffices to check that every linear form vanishing on this space vanishes on all of E. Assume by contradiction that there exists $u \in E'$ such that

$$\forall n \in \mathbb{N},\ u(x_n) = 0 \quad \text{and} \quad ||u||_{E'} = 1.$$

By density of $\{u_n\}_{n\in\mathbb{N}}$ in E', there exists $n \in \mathbb{N}$ such that $||u_n - u|| < \frac{1}{3}$. Since $||u||_{E'} = 1$, we get $||u_n||_{E'} > \frac{2}{3}$. But, since $u(x_n) = 0$, we have

$$\frac{1}{3} > |(u - u_n)(x_n)| = |u_n(x_n)| > \frac{1}{2}||u_n||_{E'}$$

and $||u_n||_{E'} < \frac{2}{3}$, which contradicts the previous inequality. Therefore, such a linear form u cannot exist, and the subspace generated by the x_n is dense in E. Then, finite linear combinations with rational coefficients are also dense in E. Since these form a countable family of vectors in E, we have shown that E is separable. □

Proposition 1.4.6. *Every closed subspace of a reflexive space is reflexive.*

Proof : Let E be a reflexive Banach space and let F be a closed subspace of E. For $u \in E'$, let $u_0 = u|_F \in F'$. By the HahnBanach theorem, every continuous linear form on F can be extended to a continuous linear form on E, so the map

$$\begin{array}{ccc} E' & \to & F' \\ u & \mapsto & u_0 \end{array}$$

is surjective. This surjection induces the following map from F'' into E'': for all $\eta \in F''$, define $\zeta \in E''$ by setting, for every $u \in E'$,

$$\zeta(u) = \eta(u_0).$$

Since E is reflexive, ζ can be identified with some $x \in E$: $\zeta(u) = u(x)$. Then $u(x) = \eta(u_0)$. Let us show that $x \in F$. If u vanishes on F, then $u_0 = 0$ and $u(x) = \eta(0) = 0$. Thus $x \in \text{Ker}(u)$, and by Theorem 1.2.11, $x \in \overline{F}$. Since F is closed, $x \in F$. Hence, $u(x) = u_0(x)$ and we have $u_0(x) = \eta(u_0)$. Since every linear form in F' is the restriction of a linear form in E', this proves that every $\eta \in F''$ can be represented by a vector $x \in F$, hence F is reflexive. □

These two propositions allow us to prove the main result of this chapter: the weak compactness of the unit ball in a reflexive space.

Theorem 1.4.7. *Let $(E, \|.\|)$ be a reflexive space. Then from every sequence of elements in the unit ball of E, one can extract a weakly convergent subsequence whose limit lies in the unit ball of E.*

Proof : Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of vectors in the unit ball of E. Let F be the closure of the subspace generated by the x_n. It is a closed subspace of E. Since E is reflexive, F is reflexive as well by Proposition 1.4.6. Moreover, F'' is also separable because, by density of $\mathbb{Q}$ in $\mathbb{R}$, F is separable and $F'' = F$.

By Proposition 1.4.5, we deduce that F' is separable. Let $\{u_m\}_{m\in\mathbb{N}}$ be a countable dense subset of F'. For each fixed m, by Bolzano-Weierstrass in $\mathbb{R}$, there exists a strictly increasing extraction $\varphi_m : \mathbb{N} \to \mathbb{N}$ of $(x_n)_{n\in\mathbb{N}}$ such that $(u_m(x_{\varphi_m(n)}))_{n\in\mathbb{N}}$ converges. By a diagonal process (setting for all n, $\varphi(n) = \varphi_1 \circ \cdots \circ \varphi_n(n)$), we obtain a strictly increasing extraction $\varphi : \mathbb{N} \to \mathbb{N}$ of $(x_n)_{n\in\mathbb{N}}$ such that, for all $m \in \mathbb{N}$, the sequence $(u_m(x_{\varphi(n)}))_{n\in\mathbb{N}}$ converges.

Now, the sequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ is uniformly bounded (by 1), and since the family $\{u_m\}_{m\in\mathbb{N}}$ is dense in F', we deduce that for every $u \in F'$, the sequence $(u(x_{\varphi(n)}))_{n\in\mathbb{N}}$ converges in $\mathbb{R}$. This limit is linear in u, and we set

$$\forall u \in F', \lim_{n\to+\infty} u(x_{\varphi(n)}) = \tilde{x}(u).$$

Since, for all $u \in F'$ and all $n \in \mathbb{N}$,

$$|u(x_{\varphi(n)})| \leq ||u||_{F'} \, ||x_{\varphi(n)}|| \leq ||u||_{F'},$$

$\tilde{x}$ is a continuous linear form with norm less than or equal to 1. Hence, it is an element of F''. As F is reflexive, there exists $x \in F$ such that for all $u \in F'$, $\tilde{x}(u) = u(x)$, with $\|x\| \leq 1$.

Thus, for every $u \in F'$, the sequence $(u(x_{\varphi(n)}))_{n\in\mathbb{N}}$ converges to $u(x)$. Since the restriction to F of every element of E' is an element of F', we have shown that $(x_{\varphi(n)})_{n\in\mathbb{N}}$ converges weakly to x, and x lies in the unit ball of E. This gives the desired result. □

Furthermore, one can show that if K is a closed convex subset of a normed vector space E and if $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in K that converges weakly to x, then x belongs to K. From this, we deduce the following theorem.

Theorem 1.4.8. *In a reflexive space, every convex, closed, and bounded set is weakly sequentially compact.*

1.5 Duality and reflexivity in L^p spaces

Let μ be any measure on $\mathbb{R}^d$.

Two nonzero real numbers p and q are said to be conjugate when

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 1.5.1. *Let (p, q) be a pair of conjugate exponents, with $p, q \in]1, +\infty[$. Let $g \in L^q(\mu)$. The map φ_g, which associates to each $f \in L^p(\mu)$ the value*

$$\varphi_g(f) = \int_X f\, g \, \mathrm{d}\mu,$$

is a continuous linear form on $L^p(\mu)$, with norm $\|g\|_q$.

Proof : The map φ_g is well-defined by Hölder's inequality, and its linearity follows from that of the integral. Hölder's inequality implies

$$|\varphi_g(f)| \leq \|f\|_p \, \|g\|_q \, ,$$

which shows that φ_g is continuous and that $\|\varphi_g\| \leq \|g\|_q$. If g is the (class of the) zero function, then φ_g is zero and the equality of norms is obvious.

If g is nonzero, choose a representative, still denoted g, such that $\mu(g \neq 0) \neq 0$. Define a fonction f by $f(x) = 0$ if $g(x) = 0$ and

$$f(x) = \frac{\overline{g(x)}}{|g(x)|^{2-q}},$$

if $g(x) \neq 0$. Then f is measurable and satisfies

$$f(x)\, g(x) = |f(x)|^p = |g(x)|^q.$$

In particular, $f \in \mathcal{L}^p(\mu)$ since $g \in \mathcal{L}^q(\mu)$. Therefore,

$$\varphi_g(f) = \int_X |g|^q \, \mathrm{d}\mu = \|g\|_q \left(\int_X |g|^q \, \mathrm{d}\mu \right)^{1-1/q} = \|g\|_q \, \|f\|_p \, .$$

As $f \neq 0$ and $f \in L^p(\mu)$, it follows that $\|\varphi_g\| \geq \|g\|_q$, which proves the equality of norms. □

Theorem 1.5.2. *Let (p, q) be a pair of conjugate exponents, with $p, q \in]1, +\infty[$. The map $\Phi : L^q(\mu) \to (L^p(\mu))'$ defined by $\Phi(g) = \varphi_g$ is a linear isometric isomorphism which is bijective. Thus,*

$$(L^p(\mu))' \simeq L^q(\mu).$$

Proof : By linearity of the integral, Φ is linear. It is an isometry by Proposition 1.5.1, hence injective.

Assuming the reflexivity of $L^p(\mu)$ spaces (which can be obtained via a general convexity argument: every uniformly convex Banach space is reflexive), let us prove that Φ is surjective.

If Φ were not surjective, since $L^q(\mu)$ is closed, by Theorem 1.2.11 there would exist a nonzero $u \in (L^p(\mu))''$ such that $u(v) = 0$ for all $v \in L^q(\mu)$. As $L^p(\mu)$ is reflexive, $u \in L^p(\mu)$, and hence for all $v \in L^q(\mu)$, $\int uv = 0$, which implies $|u|_p = 0$ and thus $u = 0$, a contradiction.

Let us show the result directly in the case $p < 2$ and $\mu(X) = 1$ (for simplicity), without invoking uniform convexity.

Let $f \in L^2(\mu)$. Apply Hölder's inequality to f and 1 for $p' = \frac{2}{p}$ and $q' = \frac{2}{2-p}$:

$$||f||_p^p = \int_X |f(x)|^p \mathrm{d}\mu(x) \leq ||1||_{\frac{2}{2-p}} ||f^p||_{\frac{2}{p}} = \mu(X) ||f||_2^p = ||f||_2^p.$$

Thus, for all $f \in L^2(\mu)$, $||f||_p \leq ||f||_2$.

Let u be a linear form defined on $L^2(\mu)$ which is bounded in the L^p norm:

$$\exists C \geq 0, \; \forall f \in L^2(\mu), \; |u(f)| \leq C \cdot ||f||_p.$$

Then u is also bounded for the $|| \, ||_2$ norm. By the Riesz representation theorem, there exists $g \in L^2(\mu)$ such that

$$\forall f \in L^2(\mu),\ u(f) = \int_X f(x)g(x)\mathrm{d}\mu(x).$$

We now show that $g \in L^q(\mu)$. For this purpose, we take $k \in \mathbb{N}$ and apply the previous inequality with $f = f_k$ defined by:

$$\forall x \in X,\ f_k(x) = |g_k|^{q-1}(x)\mathrm{sign}(g(x)),$$

where $|g_k|(x) = \min\{|g(x)|, k\}$. Then,

$$u(f_k) = \int_X f_k(x)g(x)\mathrm{d}\mu(x) = \int_X |g_k|^{q-1}(x)|g(x)|\mathrm{d}\mu(x) \geq \int_X |g_k|^q(x)\mathrm{d}\mu(x).$$

Moreover,

$$||f_k||_p^p = \int_X |g_k|^{(q-1)p}(x)\mathrm{d}\mu(x) = \int_X |g_k|^q(x)\mathrm{d}\mu(x).$$

Since by hypothesis $|u(f_k)| \leq C||f_k||_p$, we obtain

$$\int_X |g_k|^q(x)\mathrm{d}\mu(x) \leq C\left(\int_X |g_k|^q(x)\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$

hence,

$$||g_k||_q \leq C.$$

Letting $k \to \infty$ and using the monotone convergence theorem (or Fatous lemma), we deduce that $g \in L^q(\mu)$. This proves the desired duality result for $p < 2$ and $\mu(X) = 1$. □

Corollary 1.5.3. *For all $p \in]1, +\infty[$, the space $L^p(\mu)$ is reflexive.*

The previous results extend only partially to the case $p = +\infty$.

Proposition 1.5.4. *Let $g \in L^\infty(\mu)$. The map $\varphi_g : L^1(\mu) \to \mathbb{C}$ defined by*

$$\varphi_g(f) = \int_X f\, g\, \mathrm{d}\mu = \int_B |g|\, \mathrm{d}\mu \geq \alpha\mu(B) = \alpha\, \|f\|_1\,,$$

is a continuous linear form on $L^1(\mu)$ satisfying $\|\varphi_g\| \leq \|g\|_\infty$. If the measure space $(X, \mathcal{M}, \mu)$ is σ-finite, then $\|\varphi_g\| = \|g\|_\infty$.

Proof : By the limiting case of Hölder's inequality, φ_g is well-defined and satisfies $\|\varphi_g\| \leq \|g\|_\infty$. It is also linear by linearity of the integral. Assume $(X, \mathcal{M}, \mu)$ is σ-finite. The reverse inequality is trivial if $g = 0$, so suppose $g \neq 0$. Let $\alpha \in,]0, \|g\|_\infty[$. Let g be a representative of its class, and define $A = \{|g| > \alpha\}$, which has positive measure. Let $(X_n)_{n\in\mathbb{N}}$ be measurable subsets of finite measure whose union is X. For some n, $B = X_n \cap A$ has positive finite measure. Define f by $f(x) = 0$ if $x \notin B$ and $f(x) = g(x)/|g(x)|$ if $x \in B$. Then f is measurable and integrable, with $\|f\|1 = \mu(B)$. We get

$$\varphi_g(f) = \int_X f\, g\, \mathrm{d}\mu = \int_B |g|\, \mathrm{d}\mu \geq \alpha\mu(B) = \alpha\, \|f\|_1\,,$$

so $\|\varphi_g\| \geq \alpha$. Since α is arbitrary, we conclude that $\|\varphi_g\| \geq \|g\|_\infty$, proving equality of norms. □

Finally, let us consider the case $p = 1$.

Proposition 1.5.5. *Let $g \in L^1(\mu)$. The map $\varphi_g : L^\infty(\mu) \to \mathbb{C}$ defined by*

$$\varphi_g(f) = \int_X f\, g \,\mathrm{d}\mu$$

is a continuous linear form on $L^\infty(\mu)$, with norm $\|g\|_1$.

Proof: As in the previous propositions, φ_g is linear and continuous, and $\|\varphi_g\| \le \|g\|_1$. The reverse inequality is trivial if $g = 0$ in $L^1(\mu)$. Otherwise, fixing a representative g, the set $A = g \neq 0$ has positive measure. Define $f = \mathbf{1}_A \frac{\bar{g}}{|g|}$. Then $f \in L^\infty(\mu)$ and $\|f\|_\infty = 1$ (since A is not negligible). Moreover,

$$\varphi_g(f) = \int_X |g| \,\mathrm{d}\mu = \|g\|_1 = \|g\|_1 \, \|f\|_\infty ,$$

which shows that $\|\varphi_g\| \ge \|g\|_1$ and completes the proof. □

One can show that the topological dual of $L^1(\mu)$ identifies with $L^\infty(\mu)$. However, unlike the previous case, in general the topological dual of $L^\infty(\mu)$ cannot be identified with $L^1(\mu)$; we only have the inclusion $L^1(\mu) \subset (L^\infty(\mu))'$ (see Exercise Sheet 1).

Chapter 2

Bounded linear operators on a Hilbert space

$(\mathcal{H}, (\cdot|\cdot))$ denotes a Hilbert space on $\mathbb{R}$ or on $\mathbb{C}$. By convention, when $(\cdot|\cdot)$ is a Hermitian product, it will be semilinear on the right.

2.1 Bounded operators

We begin by defining the space of bounded operators between normed vector spaces and then define several topologies on this space.

Definition 2.1.1. *If $(E, \| \|_E)$ and $(F, \| \|_F)$ are normed vector spaces, a bounded operator from E to F is a continuous linear mapping $T : E \to F$,* i.e, *such that*

$$\exists C > 0,\ \forall u \in E,\ \|Tu\|_F \leq C \|u\|_E .$$

Notation. We denote by $\mathcal{L}(E, F)$ the set of bounded operators from E to F. When $E = F$, we write $\mathcal{L}(E) = \mathcal{L}(E, E)$.

$\mathcal{L}(E, F)$ is a vector space on which we introduce the norm,

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{u \in E \setminus \{0\}} \frac{\|Tu\|_F}{\|u\|_E} = \sup_{\|u\|_E = 1} \|Tu\|_F .$$

The topology induced by this norm on $\mathcal{L}(E, F)$ is called the *uniform operator topology*. If $(F, \| \|_F)$ is a Banach space, then $(\mathcal{L}(E, F), \| \|_{\mathcal{L}(E,F)})$ is also a Banach space. Furthermore, the norm $\| \|_{\mathcal{L}(E,E)}$ is an algebra norm on $(\mathcal{L}(E), +, ., \circ)$ and, more generally, if $(E, \| \|_E)$, $(F, \| \|_F)$, and $(G, \| \|_G)$ are normed vector spaces and $T_1 \in \mathcal{L}(E, F)$ and $T_2 \in \mathcal{L}(F, G)$, then $T_2 \circ T_1 \in \mathcal{L}(E, G)$ and

$$\|T_2 \circ T_1\|_{\mathcal{L}(E,G)} \leq \|T_2\|_{\mathcal{L}(F,G)} \ \|T_1\|_{\mathcal{L}(E,F)} .$$

Notation. Throughout, we denote $T_2 T_1$ as the composition $T_2 \circ T_1$ of two operators $T_1 \in \mathcal{L}(E, F)$ and $T_2 \in \mathcal{L}(F, G)$.

We now introduce two weaker topologies on $\mathcal{L}(E, F)$. Firstly, the *strong operator topology*. It is the smallest topology making the maps $\mathrm{ev}_u : \mathcal{L}(E, F) \to F$, $\mathrm{ev}_u(T) = Tu$ continuous. For this topology, a sequence of bounded operators $(T_n)_{n \in \mathbb{N}}$ converges to a bounded operator T if and only if, for every $u \in E$, $\|T_n u - Tu\|_F \xrightarrow[n \to +\infty]{} 0$. We write $T_n \to T$ in this case.

The second topology is the *weak operator topology*. We could define it for arbitrary Banach spaces E and F, but in what follows we restrict ourselves to bounded operators on $(\mathcal{H}, (\cdot|\cdot))$, so we shall assume $E = F = \mathcal{H}$. Then the weak topology on $\mathcal{L}(\mathcal{H})$ is the smallest topology making the maps $\mathrm{ev}u, v : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$, $\mathrm{ev}u, v(T) = (Tu|v)$ continuous. For this topology, a sequence of bounded operators $(T_n)_{n\in\mathbb{N}}$ converges to a bounded operator T if and only if, for all $u, v \in \mathcal{H}$, $(T_n u|v) \xrightarrow[n\to+\infty]{} (Tu|v)$ in $\mathbb{C}$. We then write $T_n \rightharpoonup T$.
The weak operator topology is weaker than the strong operator topology, which in turn is weaker than the uniform operator topology. The following examples illustrate the differences between these topologies on $\mathcal{L}(\ell^2(\mathbb{N}))$.
The following examples illustrate the differences between these topologies on $\mathcal{L}(\ell^2(\mathbb{N}))$.

Example 2.1.2. *Let $T_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $T_n(x_0, x_1, \ldots) = (\frac{1}{n}x_0, \frac{1}{n}x_1, \ldots)$. Then $(T_n)_{n\in\mathbb{N}}$ uniformly converges to 0, the zero operator.*

Example 2.1.3. *Let $S_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $S_n(x_0, x_1, \ldots) = (0, \ldots, 0, x_n, x_{n+1}, \ldots)$. Then $(S_n)_{n\in\mathbb{N}}$ strongly converges to 0 but not uniformly.*

Indeed, for any $x \in \ell^2(\mathbb{N})$,

$$||S_n x||^2_{\ell^2} = \sum_{k=n}^{+\infty} |x_k|^2 \xrightarrow[n\to+\infty]{} 0.$$

Then, for any $x \in \ell^2(\mathbb{N})$, $||S_n x||_{\ell^2} \leq ||x||_{\ell^2}$ hence $||S_n||_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq 1$. Furthermore, for all $n \in \mathbb{N}$, $||S_n e_n||_{\ell^2} = 1$ where e_n is the sequence being 0 for all $k \neq n$ and 1 at the n-th term. Therefore, for all n, $||S_n||_{\mathcal{L}(\ell^2(\mathbb{N}))} = 1$, and (S_n) does not uniformly converge to 0.

Example 2.1.4. *Let $W_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by $W_n(x_0, x_1, \ldots) = (0, \ldots, 0, x_0, x_1, \ldots)$ with n zeros at the beginning of the sequence. Then $(W_n)_{n\in\mathbb{N}}$ converges weakly to 0, but neither strongly nor uniformly.*

Throughout, we will often consider bounded operators between Hilbert spaces. In this Hilbert framework, we provide a characterization of the operator norm.

Proposition 2.1.5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. Then,*

$$\|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} = \sup\{|(Tu|v)_{\mathcal{H}_2}| \mid \|u\|_{\mathcal{H}_1} \leq 1 \text{ and } \|v\|_{\mathcal{H}_2} \leq 1\}.$$

Proof : Let S be the right-hand side of the equality. By the Cauchy-Schwarz inequality,

$$|(Tu|v)| \leq \|Tu\|_{\mathcal{H}_2} \|v\|_{\mathcal{H}_2} \leq \|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \|u\|_{\mathcal{H}_1} \|v\|_{\mathcal{H}_2} \leq \|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$$

when $\|u\|_{\mathcal{H}_1} \leq 1$ and $\|v\|_{\mathcal{H}_2} \leq 1$. Hence, $S \leq \|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$. Conversely, let M be a positive real number; suppose $S \leq M$. Then, for any $u \in \mathcal{H}_1$, $\|Tu\|_{\mathcal{H}_2} \leq M\|u\|_{\mathcal{H}_1}$. Indeed, if $u = 0$ or $Tu = 0$, the inequality holds. Otherwise, $u' = u/\|u\|_{\mathcal{H}_1}$ and $v' = Tu/\|Tu\|_{\mathcal{H}_2}$ have norm 1, and as $S \leq M$, $|(Tu'|v')| \leq M$. Now, $|(Tu'|v')| = \|Tu\|_{\mathcal{H}_2}/\|u\|_{\mathcal{H}_1}$, hence $\|Tu\|_{\mathcal{H}_2} \leq M\|u\|_{\mathcal{H}_1}$. By the definition of $\|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$, we get $\|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq S$. □

2.2 Adjoint of a Bounded Operator

We will now define the adjoint of a bounded operator, which generalizes to any dimension the transpose of a real matrix or the conjugate transpose of a complex matrix.

Proposition 2.2.1. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. There exists a unique operator $T^* \in \mathcal{L}(\mathcal{H})$ such that*

$$\forall u \in \mathcal{H},\ \forall v \in \mathcal{H},\ (Tu|v) = (u|T^*v). \tag{2.1}$$

Proof : Let $v \in \mathcal{H}$. Then, $\ell_v : u \mapsto (Tu|v)$ is a continuous linear form on $\mathcal{H}$. By the Riesz representation theorem for continuous linear forms, there exists a unique vector $w \in \mathcal{H}$ such that, for all $u \in \mathcal{H}$, $\ell_v(u) = (u|w)$. Let $T^* : \mathcal{H} \to \mathcal{H}$, $T^*v = w$.

T^* is linear. Indeed, for $v_1, v_2 \in \mathcal{H}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, let $v = \lambda_1 v_1 + \lambda_2 v_2$ and $w_1 = T^*(v_1)$, $w_2 = T^*(v_2)$, $T^*(v) = w$. Then,

$$\begin{aligned} \forall u \in \mathcal{H},\ (u|w) &= (Tu|v) = (Tu|\lambda_1 v_1 + \lambda_2 v_2) = \overline{\lambda}_1 (Tu|v_1) + \overline{\lambda}_2 (Tu|v_2) \\ &= \overline{\lambda}_1 (u|w_1) + \overline{\lambda}_2 (u|w_2) = (u|\lambda_1 w_1 + \lambda_2 w_2). \end{aligned}$$

So, $w - \lambda_1 w_1 - \lambda_2 w_2 \in \mathcal{H}^\perp = \{0\}$ and $w = \lambda_1 w_1 + \lambda_2 w_2$, proving linearity of T^*.

T^* is bounded. Indeed, for $u, v \in \mathcal{H}$, $\|u\|_{\mathcal{H}} \le 1$ and $\|v\|_{\mathcal{H}} \le 1$, then

$$|(u|T^*v)| = |(Tu|v)| \le \|Tu\|_{\mathcal{H}} \ \|v\|_{\mathcal{H}} \le \|T\|_{\mathcal{L}(\mathcal{H})}.$$

Thus, taking $u = \frac{T^*v}{\|T^*v\|_{\mathcal{H}}}$, for all $v \in \mathcal{H}$, $\|v\|_{\mathcal{H}} \le 1$ and $T^*v \neq 0$, $\|T^*v\|_{\mathcal{H}} \le \|T\|_{\mathcal{L}(\mathcal{H})}$. If $v \in \mathcal{H}$ is such that $T^*v = 0$, the inequality is still valid. This gives $\|T^*\|_{\mathcal{L}(\mathcal{H})} \le \|T\|_{\mathcal{L}(\mathcal{H})}$ and T^* is bounded.

Finally, for uniqueness, if T_1^* and T_2^* satisfy (2.1), then for all $u, v \in \mathcal{H}$, $(u|(T_1^* - T_2^*)v) = 0$, thus $T_1^* - T_2^* = 0$. □

Definition 2.2.2 (Adjoint). *The bounded operator $T^* \in \mathcal{L}(\mathcal{H})$ is called the adjoint of the operator T.*

Example 2.2.3. *For any Hilbert space $\mathcal{H}$, $\mathrm{Id}_{\mathcal{H}}^* = \mathrm{Id}_{\mathcal{H}}$.*

We state the first properties verified by the adjoint of a bounded operator.

Proposition 2.2.4 (Algebraic Properties of the Adjoint). *Let $T, T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then,*

1. $(T_1 + T_2)^* = T_1^* + T_2^*$;
2. $(\lambda T)^* = \overline{\lambda} T^*$;
3. $(T_1 T_2)^* = T_2^* T_1^*$;
4. $(T^*)^* = T$;
5. *if T has a bounded inverse T^{-1}, T^* also has a bounded inverse, and $(T^*)^{-1} = (T^{-1})^*$.*

Proof : The first two points come from the right semilinearity of the inner product. For the third point, for all $u, v \in \mathcal{H}$, write $(T_1 T_2 u|v) = (T_2 u|T_1^* v) = (u|T_2^* T_1^* v)$. The fourth point is obtained by noticing that in (2.1), vectors u and v play the same role, and $(Tu|v) = (u|T^*v)$ for all $u, v \in \mathcal{H}$ if and only if $(T^*u|v) = (u|Tv)$ for all $u, v \in \mathcal{H}$ by taking conjugates. Finally, for the last point, from $TT^{-1} = I = T^{-1}T$, we deduce by taking the adjoint that $T^*(T^{-1})^* = I^* = I = I^* = (T^{-1})^* T^*$. □

Proposition 2.2.5 (Metric Properties of the Adjoint)**.** *Let $T \in \mathcal{L}(\mathcal{H})$. Then,*

1. $\|T^*\|_{\mathcal{L}(\mathcal{H})} = \|T\|_{\mathcal{L}(\mathcal{H})}$;
2. $\|T^*T\|_{\mathcal{L}(\mathcal{H})} = \|T\|^2_{\mathcal{L}(\mathcal{H})}$.

Proof : From the proof of proposition 2.2.1, $\|T^*\|_{\mathcal{L}(\mathcal{H})} \leq \|T\|_{\mathcal{L}(\mathcal{H})}$. Then, applying this inequality to the bounded operator T^* and using the fact that $(T^*)^* = T$, we obtain $\|T\|_{\mathcal{L}(\mathcal{H})} \leq \|T^*\|_{\mathcal{L}(\mathcal{H})}$, proving the first point. For the second point, we initially have $\|T^*T\|_{\mathcal{L}(\mathcal{H})} \leq \|T^*\|_{\mathcal{L}(\mathcal{H})}\|T\|_{\mathcal{L}(\mathcal{H})} = \|T\|^2_{\mathcal{L}(\mathcal{H})}$. Conversely, if $u \in \mathcal{H}$, $\|u\|_{\mathcal{H}} = 1$,

$$\|Tu\|^2_{\mathcal{H}} = (Tu|Tu) = (T^*Tu|u) \leq \|T^*T\|_{\mathcal{L}(\mathcal{H})},$$

hence $\|T\|^2_{\mathcal{L}(\mathcal{H})} \leq \|T^*T\|_{\mathcal{L}(\mathcal{H})}$. □

Proposition 2.2.6 (Geometric Properties of the Adjoint)**.** *Let $T \in \mathcal{L}(\mathcal{H})$. Then,*

1. $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$ *and* $(\operatorname{Ker} T^*)^\perp = \overline{\operatorname{Im} T}$;
2. *if $F \subset \mathcal{H}$ is a subspace stable under T, then $F^\perp$ is stable under T^*.*

Proof : u belongs to $(\operatorname{Im} T)^\perp$ if and only if, for all $v \in \mathcal{H}$, $(u|Tv) = 0$, which is equivalent to saying that, for all $v \in \mathcal{H}$, $(T^*u|v) = 0$. This is equivalent to $T^*u = 0$, or equivalently $u \in \operatorname{Ker} T^*$. The second property arises from properties of orthogonal spaces in Hilbert spaces.

For the second point, let $v \in F^\perp$ and $u \in F$. Then $Tu \in F$, so $(T^*v|u) = (v|Tu) = 0$. Therefore, $T^*v \in F^\perp$. □

Definition 2.2.7. *An operator $T \in \mathcal{L}(\mathcal{H})$ is called self-adjoint when $T = T^*$.*

Self-adjoint operators are a generalization of symmetric matrices to infinite dimensions. They play a major role in functional analysis and mathematical physics. A structural theorem for these operators asserts that every self-adjoint operator is diagonalizable, in a sense to be specified in infinite dimensions. A primary example of a self-adjoint operator is that of an orthogonal projector.

Definition 2.2.8. *An operator $P \in \mathcal{L}(\mathcal{H})$ is called a projector when $P^2 = P$. Moreover, if $P^* = P$, P is called an orthogonal projector.*

It is noted that the image of a projector is a closed subspace on which P acts as the identity. Furthermore, if P is orthogonal, P acts as the null operator on $(\operatorname{Im} T)^\perp$. Then, the projection theorem for closed subspaces in Hilbert spaces assures that there is a bijection between orthogonal projectors in a Hilbert space $\mathcal{H}$ and closed subspaces of $\mathcal{H}$.

Example 2.2.9. ***(Multiplication Operator).*** *Let (X, μ) be a σ-finite measure space and let $\mathcal{H} = L^2(\mu)$. If $\varphi \in L^\infty(\mu)$, we define the multiplication operator by φ, $M_\varphi : L^2(\mu) \to L^2(\mu)$ such that, for all $u \in \mathcal{H}$, $M_\varphi u = \varphi u$.*

Then, M_φ is in $\mathcal{L}(L^2(\mu))$ and $\|M_\varphi\| = \|\varphi\|_\infty$. Here, $\|\varphi\|_\infty$ denotes the essential supremum, $\|\varphi\|_\infty = \inf\{c > 0 \mid \mu(\{x \in X \mid |\varphi(x)| > c\}) = 0\}$. Therefore, by changing the representative within the class of φ, we can assume that φ is a bounded function.

Moreover, since $\|\varphi u\|_2 \leq \|\varphi\|_\infty \|u\|_2$, M_φ is a bounded operator and $\|M_\varphi\| \leq \|\varphi\|_\infty$. For any $\varepsilon > 0$, as μ is σ-finite, there exists a measurable set A with $0 < \mu(A) < +\infty$ such that $|\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon$ for all $x \in A$. Taking $u = \mu(A)^{-\frac{1}{2}}\mathbf{1}_A$, then $u \in L^2(\mu)$ and $\|u\|_2 = 1$. Thus, $\|M_\varphi\|^2 \geq \|\varphi u\|_2^2 = \mu(A)^{-1}\int_A |\varphi|^2 \mathrm{d}\mu \geq (\|\varphi\|_\infty - \varepsilon)^2$. As ε tends to 0, it holds that $\|M_\varphi\| \geq \|\varphi\|_\infty$.

It is observed that, for any $\varphi \in L^\infty(\mu)$, $M_\varphi^* = M_{\overline{\varphi}}$, where $\overline{\varphi}(x) = \overline{\varphi(x)}$ for all x in X. In particular, if φ takes real values, $M_\varphi^* = M_\varphi$ and M_φ is self-adjoint.

For bounded self-adjoint operators, proposition 2.1.5 can be refined.

Proposition 2.2.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Then, for all $u \in \mathcal{H}$, $(Tu|u) \in \mathbb{R}$, and*

$$\|T\|_{\mathcal{L}(\mathcal{H})} = \sup\{|(Tu|u)| \mid \|u\|_{\mathcal{H}} = 1\}.$$

Proof : Let S be the right-hand side of the equality. According to Proposition 2.1.5, $S \le \|T\|_{\mathcal{L}(\mathcal{H})}$. To prove the other inequality, let's begin by showing that, for all $u \in \mathcal{H}$, $(Tu|u) \in \mathbb{R}$. Indeed, as $T = T^*$, $(Tu|u) = (u|Tu) = \overline{(Tu|u)}$ and $(Tu|u) \in \mathbb{R}$. Then, using the polarization identity, we have

$$\forall u, v \in \mathcal{H},\ \text{Re}\ (Tu|v) = \frac{1}{4}\left((T(u+v)|u+v) - (T(u-v)|u-v)\right).$$

Now, for all $u \in \mathcal{H}$, $|(Tu|u)| \le S\|u\|^2$. Therefore, for all $u, v \in \mathcal{H}$,

$$|\text{Re}\ (Tu|v)| \le \frac{S}{4}\left(\|u+v\|^2 + \|u-v\|^2\right).$$

Then, by the parallelogram identity, for all $u, v \in \mathcal{H}$, $|\text{Re}\ (Tu|v)| \le \frac{S}{2}\left(\|u\|^2 + \|v\|^2\right)$. Hence, if we assume $\|u\| \le 1$ and $\|v\| \le 1$, we obtain $|\text{Re}\ (Tu|v)| \le S$. By replacing v with $e^{-i\theta}v$, where $e^{i\theta}(Tu|v) = |(Tu|v)|$, we then get that, for all $u, v \in \mathcal{H}$, $|(Tu|v)| = (Tu|e^{-i\theta}v) = |\text{Re}\ (Tu|e^{-i\theta}v)| \le S$. Therefore, by Proposition 2.1.5, $\|T\|_{\mathcal{L}(\mathcal{H})} \le S$, which concludes the proof. □

We conclude this section with a result that paves the way for the formalism of unbounded operators.

Theorem 2.2.11 (Hellinger-Toeplitz)**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be an operator such that, for all $u, v \in \mathcal{H}$, $(u|Tv) = (Tu|v)$. Then $T \in \mathcal{L}(\mathcal{H})$.*

Proof : By the closed graph theorem, it suffices to demonstrate that $\Gamma(T)$, the graph of T, is closed. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{H}$ converging to $u \in \mathcal{H}$, such that $(Tu_n)_{n\in\mathbb{N}}$ converges to $v \in \mathcal{H}$. We only need to show that $v = Tu$. For any $w \in \mathcal{H}$,

$$(w|v) = \lim_{n\to\infty}(w|Tu_n) = \lim_{n\to\infty}(Tw|u_n) = (Tw|u) = (w|Tu),$$

thus $v = Tu$. □

This result asserts that there cannot be an unbounded operator defined on the entire space $\mathcal{H}$ that is self-adjoint (or symmetric in general). This poses a problem in quantum mechanics where one wishes to define operators like energy (involving a derivative) that are unbounded while being symmetric in the sense of $(u|Tv) = (Tu|v)$.

Chapter 3

Spectrum of bounded operators

An eigenvalue λ of a matrix A is a scalar such that there exists a non-zero vector x such that $Ax = \lambda x$. This translates to the non-injectivity of the matrix $A - \lambda I$. However, in finite dimensions, a linear map between two spaces of the same dimension is injective if and only if it is bijective. Thus, eigenvalues of a matrix can also be characterized as scalars λ for which $A - \lambda I$ is not invertible. We aim to retain this characterization to define the notion of spectrum for a bounded operator on a Banach space. An issue arises in infinite dimensions : there exist linear maps that are injective but not surjective, for example, the map that associates a bounded sequence $(x_n)_{n\in\mathbb{N}}$ with the bounded sequence $(0, x_0, \ldots)$. There are also linear maps that are surjective but not injective, for example, the map that associates a bounded sequence $(x_n)_{n\in\mathbb{N}}$ with the bounded sequence $(x_1, \ldots)$. This leads us to distinguish between the spectrum of an operator and the set of its eigenvalues. Some scalars in the spectrum are not eigenvalues.

3.1 Spectrum

We begin by providing the definition of the spectrum of a bounded operator. Throughout, $(E, \| \, \|_E)$ denotes a complex Banach space. When there is no ambiguity in notation, for $T \in \mathcal{L}(E)$, we denote $\|T\| := \|T\|_{\mathcal{L}(E)}$.

Notation. For $T \in \mathcal{L}(E)$ and $\lambda \in \mathbb{C}$, we denote $T - \lambda := T - \lambda \mathrm{Id}_E$ where Id_E is the identity linear map of $\mathcal{L}(E)$.

Definition 3.1.1. *Let $T \in \mathcal{L}(E)$. The spectrum of T is the subset of $\mathbb{C}$ defined by*

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not invertible in } \mathcal{L}(E)\}.$$

The elements of $\sigma(T)$ are called spectral values.

It is notable that, according to the isomorphism theorem, T is invertible in $\mathcal{L}(E)$ if and only if T is bijective. Indeed, if T is bounded and bijective, its inverse map is automatically continuous. We derive the following characterization of the spectrum of a bounded operator.

Proposition 3.1.2. *Let $T \in \mathcal{L}(E)$. Then $\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not bijective}\}$.*

Definition 3.1.3. *The set of eigenvalues of $T \in \mathcal{L}(E)$ is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not injective. The set of eigenvalues of T is called the point spectrum of T and is denoted by $\sigma_p(T)$. A non-zero vector $u \in E$ such that $Tu = \lambda u$ is called an eigenvector of T associated with the eigenvalue λ. Finally, the multiplicity of the eigenvalue λ is the (finite or infinite) dimension of* $\mathrm{Ker}\,(T - \lambda)$.

We have $\sigma_p(T) \subset \sigma(T)$. Every eigenvalue is a spectral value, but these two sets are generally not equal, as illustrated by the first example in the introduction.
Before proving the initial properties of the spectrum of a bounded operator, we demonstrate the following lemma, known as the Neumann series lemma.

Lemma 3.1.4 (Neumann Series Lemma). *Let $S \in \mathcal{L}(E)$ such that $\|S\| < 1$. Then $\mathrm{Id}_E - S$ is invertible in $\mathcal{L}(E)$ and $(\mathrm{Id}_E - S)^{-1} = \sum_{n=0}^{+\infty} S^n$. Thus, the group of invertible elements of $\mathcal{L}(E)$, denoted $\mathrm{GL}(E)$, is an open set in $\mathcal{L}(E)$.*

Proof : As $\|S\| < 1$ and for any $n \in \mathbb{N}$, $\|S^n\| \leq \|S\|^n$, the series $\sum \|S^n\|$ converges in $\mathbb{R}$. Therefore, as $(\mathcal{L}(E), || \, ||_{\mathcal{L}(E)})$ is a complete space, the series $\sum S^n$ converges in $\mathcal{L}(E)$. Let

$$U = \sum_{n=0}^{+\infty} S^n = \lim_{N \to +\infty} \sum_{n=0}^{N} S^n.$$

Then, for any $N \geq 1$,

$$(\mathrm{Id}_E - S)\left(\sum_{n=0}^{N} S^n\right) = \left(\sum_{n=0}^{N} S^n\right)(\mathrm{Id}_E - S) = \mathrm{Id}_E - S^{N+1}$$

and $\mathrm{Id}_E - S^{N+1}$ converges in $\mathcal{L}(E)$ to Id_E. Thus $(\mathrm{Id}_E - S)U = U(\mathrm{Id}_E - S) = \mathrm{Id}_E$.

Now, let T_0 be invertible in $\mathcal{L}(E)$. Then, for $S \in \mathcal{L}(E)$, $T_0 + S = T_0(\mathrm{Id}_E + T_0^{-1}S)$, hence $T_0 + S$ is invertible if and only if $\mathrm{Id}_E + T_0^{-1}S$ is. This is the case for S such that $\|T_0^{-1}S\| < 1$, thus $B(T_0, \|T_0^{-1}\|^{-1})$ is contained in $\mathrm{GL}(E)$. This set is a neighborhood of each of its points, hence it is open. □

Proposition 3.1.5. *Let $T \in \mathcal{L}(E)$. Then $\sigma(T)$ is a compact subset of $\mathbb{C}$.*

Proof : The set $\sigma(T)^c$ is the inverse image of the open set $\mathrm{GL}(E)$ under the continuous map $\mathbb{C} \to \mathcal{L}(E)$, $\lambda \mapsto T - \lambda$. Therefore, it is an open set in $\mathbb{C}$ and $\sigma(T)$ is thus closed in $\mathbb{C}$. Furthermore, let $\lambda \in \mathbb{C}$ such that $|\lambda| > \|T\|$. Then $T - \lambda = -\lambda\left(\mathrm{Id}_E - \frac{1}{\lambda}T\right)$ and $\|\frac{1}{\lambda}T\| < 1$. Therefore $\mathrm{Id}_E - \frac{1}{\lambda}T \in \mathrm{GL}(E)$ and $T - \lambda$ is also invertible. Hence $\lambda \notin \sigma(T)$. Therefore $\sigma(T) \subset \overline{D}(0, \|T\|)$ and $\sigma(T)$ is bounded. Therefore, the spectrum of T is a closed bounded subset of $\mathbb{C}$, and hence a compact set in $\mathbb{C}$. □

3.2 Resolvent

We now introduce a key application in the study of the spectrum of an operator, the resolvent.

Notation. The set $\sigma(T)^c$ is called the *resolvent set* of T and is denoted by $\rho(T)$. It is an unbounded open set in $\mathbb{C}$.

Definition 3.2.1. *Let $T \in \mathcal{L}(E)$ and $z \in \mathbb{C}$. The application $R(T) : \rho(T) \to \mathcal{L}(E)$ defined by $R(T)(z) = (T - z)^{-1}$ is called the resolvent of the operator T. For $z \in \rho(T)$, the linear map $R_z(T) := R(T)(z)$ is called the resolvent of T at the point z.*

Proposition 3.2.2. *Let $T \in \mathcal{L}(E)$. The resolvent of T, $z \mapsto R_z(T)$, is holomorphic over the open set $\rho(T)$. Furthermore, $\lim_{|z| \to +\infty} \|R_z(T)\|_{\mathcal{L}(E)} = 0$.*

Proof : Let $z_0 \in \rho(T)$. Then for any $z \in \rho(T)$, $(T-z)^{-1} = (T-z_0-(z-z_0))^{-1} = (T-z_0)^{-1}(\mathrm{Id}_E-(z-z_0)(T-z_0)^{-1})^{-1}$. Now, $\|(z-z_0)(T-z_0)^{-1}\| = |z-z_0|\|(T-z_0)^{-1}\|$ and $\|(T-z_0)^{-1}\| > 0$. Thus, if we suppose that $z \in \rho(T)$ is such that $|z-z_0| < \|(T-z_0)^{-1}\|^{-1}$, then

$$(T-z)^{-1} = (T-z_0)^{-1}\sum_{n=0}^{+\infty}(z-z_0)^n(T-z_0)^{-n} = \sum_{n=0}^{+\infty}(z-z_0)^n(T-z_0)^{-(n+1)}.$$

So $z \mapsto R_z(T)$ is holomorphic at the point z_0. Thus, it is holomorphic over $\rho(T)$.

Let $z \in \mathbb{C}$. Assume $|z| > \|T\|$. Then $z \in \rho(T)$ and

$$(T-z)^{-1} = \left(-z\left(\mathrm{Id}_E - \frac{1}{z}T\right)\right)^{-1} = -\frac{1}{z}\sum_{n=0}^{+\infty}\frac{1}{z^n}T^n,$$

giving

$$\left\|(T-z)^{-1}\right\| \leq \frac{1}{|z|}\sum_{n=0}^{+\infty}\frac{1}{|z|^n}\|T\|^n \leq \frac{1}{|z|-\|T\|} \xrightarrow[|z|\to+\infty]{} 0,$$

proving the second statement of the proposition.

□

Corollary 3.2.3. *If $E \neq \{0\}$ and $T \in \mathcal{L}(E)$, $\sigma(T) \neq \emptyset$.*

Proof : If $\sigma(T) = \emptyset$, then $\rho(T) = \mathbb{C}$ and $R(T) : \mathbb{C} \to \mathcal{L}(E)$ is holomorphic and, by Proposition 3.2.2, $\lim_{|z|\to+\infty}\|R_z(T)\|_{\mathcal{L}(E)} = 0$. Thus, $R(T)$ is an entire and bounded function over $\mathbb{C}$, and by Liouville's theorem, it is constant over $\mathbb{C}$. Since its limit at infinity is zero, this constant can only be 0. Hence, for any $z \in \mathbb{C}$, $(T-z)^{-1} = 0$. But the zero map is bijective only when $E = \{0\}$. Therefore, if $E \neq \{0\}$, $\sigma(T) \neq \emptyset$.

□

The proof of the non-emptiness of the spectrum relies on a result from complex analysis, Liouville's theorem. This is also the case for the proof of the non-emptiness of the spectrum in finite dimension, which is a consequence of d'Alembert-Gauss' theorem, another proof of which relies on Liouville's theorem.

Proposition 3.2.4 (Resolvent Identity). *Let $T \in \mathcal{L}(E)$, z and z' in $\rho(T)$. Then, $R_z(T) - R_{z'}(T) = (z-z')R_z(T)R_{z'}(T)$ and $R_z(T)$ and $R_{z'}(T)$ commute.*

Proof : For all $z, z' \in \rho(T)$, we have

$$\begin{aligned}
R_z(T) - R_{z'}(T) &= (T-z)^{-1} - (T-z')^{-1} \\
&= (T-z)^{-1}(T-z')(T-z')^{-1} - (T-z)^{-1}(T-z)(T-z')^{-1} \\
&= (T-z)^{-1}(T-z'-T+z)(T-z')^{-1} \\
&= (T-z)^{-1}(z-z')(T-z')^{-1} = (z-z')R_z(T)R_{z'}(T),
\end{aligned}$$

yielding the resolvent identity. Interchanging z and z' then shows that $R_z(T)$ and $R_{z'}(T)$ commute.

□

3.3 Spectral Radius

As for $T \in \mathcal{L}(E)$, $\sigma(T)$ is a compact set included in $\overline{D}(0, \|T\|)$, the supremum in the following definition is well defined.

Definition 3.3.1 (Spectral Radius). *Let $T \in \mathcal{L}(E)$. The spectral radius of T is the positive real number $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$.*

Proposition 3.3.2 (Spectral Radius Formula). *Let $T \in \mathcal{L}(E)$. Then*

$$r(T) = \lim_{n \to +\infty} \|T^n\|^{\frac{1}{n}}.$$

Proof : Firstly, since $\sigma(T) \subset \overline{D}(0, \|T\|)$, $r(T) \leq \|T\|$. Let $n \geq 1$. Let's prove that $\sigma(T^n) = \{\lambda^n \mid \lambda \in \sigma(T)\}$. For this, we use the relation $T^n - \lambda^n = (T - \lambda)(T^{n-1} + \ldots + \lambda^{n-1})$. Let $Q_n = T^{n-1} + \ldots + \lambda^{n-1}$. Q_n commutes with $T - \lambda$. Suppose $\lambda^n \notin \sigma(T^n)$. Then there exists $S_n \in \mathcal{L}(E)$ such that $(T^n - \lambda^n)S_n = S_n(T^n - \lambda^n) = \mathrm{Id}_E$. In particular, Q_n and S_n commute. Indeed, $Q_n S_n = \mathrm{Id}_E Q_n S_n = S_n(T^n - \lambda^n) Q_n S_n = S_n Q_n (T^n - \lambda^n) S_n = S_n Q_n \mathrm{Id}_E = S_n Q_n$ since Q_n commutes with T and with $\lambda \mathrm{Id}_E$, hence with $T^n - \lambda^n$. Thus, $(T - \lambda) Q_n S_n = S_n Q_n (T - \lambda) = Q_n S_n (T - \lambda) = \mathrm{Id}_E$. Hence, $T - \lambda$ is invertible and $\lambda \notin \sigma(T)$. Thus, by contraposition, if $\lambda \in \sigma(T)$, $\lambda^n \in \sigma(T^n)$ and $\{\lambda^n \mid \lambda \in \sigma(T)\} \subset \sigma(T^n)$.

Conversely, let $\mu \in \sigma(T^n)$. Then $T^n - \mu = (T - \lambda_1) \ldots (T - \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are the n-th roots of μ. If, for every $i \in \{1, \ldots, N\}$, $T - \lambda_i$ is invertible, then $T^n - \mu$ is also, thus $\mu \notin \sigma(T^n)$. Hence, there exists $i \in \{1, \ldots, N\}$ such that $\lambda_i \in \sigma(T)$. But, $\lambda_i^n = \mu$ by definition. Therefore, for every $\mu \in \sigma(T^n)$, there exists $\lambda \in \sigma(T)$ such that $\mu = \lambda^n$. Thus $\sigma(T^n) \subset \{\lambda^n \mid \lambda \in \sigma(T)\}$.

Therefore, for every $n \geq 1$, $r(T)^n = r(T^n) \leq \|T^n\|$, hence $r(T) \leq \|T^n\|^{\frac{1}{n}}$. So

$$r(T) \leq \liminf_{n \to +\infty} \|T^n\|^{\frac{1}{n}}.$$

For $\xi \in \mathbb{C}$, $0 < |\xi| < \frac{1}{r(T)}$, let $F(\xi) = R_{\xi^{-1}}(T)$. Then, F is holomorphic on the open set $\{\xi \mid 0 < |\xi| < \frac{1}{r(T)}\}$ by Proposition 3.2.2. Moreover, on this set, from the calculations made in the proof of Proposition 3.2.2, $F(\xi) = -\sum_{n=0}^{+\infty} \xi^{n+1} T^n$. So F extends to a holomorphic function on $D(0, r(T)^{-1})$. By the Cauchy inequalities,

$$\forall r < \frac{1}{r(T)}, \ \|T^n\| = \left\| -\frac{F^{(n+1)}(0)}{(n+1)!} \right\| \leq \frac{1}{r^{n+1}} \max_{|\xi| \leq r} \|F(\xi)\|.$$

Thus, for every $n \geq 1$, $\|T^n\|^{\frac{1}{n}} \leq M(r)^{\frac{1}{n}} r^{-1-\frac{1}{n}}$ where $M(r) = \max_{|\xi| \leq r} \|F(\xi)\|$, so

$$\forall r < \frac{1}{r(T)}, \ \limsup_{n \to +\infty} \|T^n\|^{\frac{1}{n}} \leq \frac{1}{r},$$

therefore

$$\limsup_{n \to +\infty} \|T^n\|^{\frac{1}{n}} \leq r(T) \leq \liminf_{n \to +\infty} \|T^n\|^{\frac{1}{n}}.$$

Thus, the sequence $(\|T^n\|^{\frac{1}{n}})_{n \geq 1}$ converges, and its limit is $r(T)$.

□

3.4 The spectrum and the adjoint

We present with two results linking the spectrum of a bounded operator to its adjoint in the case where E is a Hilbert space.

Proposition 3.4.1. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$. Then $\sigma(T^*) = \overline{\sigma(T)} = \{\overline{\lambda} : \lambda \in \sigma(T)\}$. Moreover, for any $z \in \rho(T)$, $R_{\bar{z}}(T^*) = R_z(T)^*$.*

Proof : Indeed, according to point 5 of Proposition 2.2.4, $T - \lambda$ is invertible if and only if $(T - \lambda)^*$ is. Now $(T-\lambda)^* = T^* - \overline{\lambda}$. Therefore,

$$\lambda \in \sigma(T) \Leftrightarrow T - \lambda \text{ non-invertible} \Leftrightarrow (T-\lambda)^* \text{ non-invertible} \Leftrightarrow T^* - \overline{\lambda} \text{ non-invertible} \Leftrightarrow \overline{\lambda} \in \sigma(T^*),$$

which confirms the first assertion. Also, if $z \in \rho(T)$,

$$R_{\bar{z}}(T^*) = (T^* - \bar{z})^{-1} = ((T-z)^*)^{-1} = ((T-z)^{-1})^* = R_z(T)^*,$$

as per point 5 of Proposition 2.2.4. □

The second result concerns self-adjoint operators.

Proposition 3.4.2. *Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be self-adjoint. Then,*

1. *$\sigma(T) \subset \mathbb{R}$;*
2. *Eigenvectors associated with distinct eigenvalues of T are orthogonal.*

Proof : Let λ and μ be two real numbers. Then, for any $u \in H$,

$$\begin{aligned}\|(T-(\lambda+\mathrm{i}\mu))u\|^2 &= ((T-(\lambda+\mathrm{i}\mu))u|(T-(\lambda+\mathrm{i}\mu))u)\\ &= ((T-\lambda)u|(T-\lambda)u) + \mathrm{i}\mu\left[((T-\lambda)u|u) - (u|(T-\lambda)u)\right] - \mathrm{i}^2\mu^2||u||^2\\ &= \|(T-\lambda)u\|^2 + \mu^2\|u\|^2\end{aligned}$$

since T is self-adjoint and λ is a real number, hence $((T-\lambda)u|u) = (u|(T-\lambda)u)$.

Therefore, for any $u \in H$, $\|T-(\lambda+\mathrm{i}\mu))u\|^2 \geq \mu^2\|u\|^2$. If $\mu \neq 0$, $T - (\lambda + \mathrm{i}\mu)$ is injective. Let's assume by contradiction that $T - (\lambda+\mathrm{i}\mu)$ is non-bijective, hence non-surjective, or in other words, Im $(T - (\lambda + \mathrm{i}\mu)) \neq H$. Then,

$$\mathrm{Ker}(T^* - (\lambda - \mathrm{i}\mu)) = \mathrm{Ker}((T-(\lambda+\mathrm{i}\mu))^*) = (\mathrm{Im}(T-(\lambda+\mathrm{i}\mu)))^\perp \neq \{0\}$$

and $\lambda - \mathrm{i}\mu \in \sigma_p(T^*)$. However, $\sigma_p(T^*) = \sigma_p(T)$ because $T = T^*$.

Moreover, we also have $\|T-(\lambda-\mathrm{i}\mu))u\|^2 \geq \mu^2\|u\|^2$, and $\lambda - \mathrm{i}\mu \notin \sigma_p(T)$ if $\mu \neq 0$ (since $T - (\lambda - \mathrm{i}\mu)$ is injective). This leads to a contradiction. Hence, $T - (\lambda + \mathrm{i}\mu)$ is bijective whenever $\mu \neq 0$, and if $\mu \neq 0$, $\lambda + \mathrm{i}\mu \in \rho(T)$, proving the first point.

To prove the second point, let's consider λ_1 and λ_2 as two distinct eigenvalues of T. Let u_1 be an eigenvector associated with λ_1 and u_2 an eigenvector associated with λ_2. We have

$$\lambda_1(u_1|u_2) = (\lambda_1 u_1|u_2) = (Tu_1|u_2) = (u_1|Tu_2) = (u_1|\lambda_2 u_2) = \overline{\lambda}_2(u_1|u_2) = \lambda_2(u_1|u_2),$$

and, since $\lambda_1 \neq \lambda_2$, we must have $(u_1|u_2) = 0$. □

3.5 The Discrete Laplacian in dimension one

We introduce the discrete Laplacian in dimension one. It is the operator Δ defined on the Hilbert space $\ell^2(\mathbb{Z})$ as

$$\forall u \in \ell^2(\mathbb{Z}),\ \forall n \in \mathbb{Z},\ (\Delta u)_n = -(u_{n-1} + u_{n+1}).$$

The operator Δ is the discrete analogue of the second derivative.

Firstly, Δ is bounded. Indeed, if $\|u\|_{\ell^2(\mathbb{Z})} \leq 1$, then $\|\Delta u\|_{\ell^2(\mathbb{Z})} \leq 2\|u\|_{\ell^2(\mathbb{Z})} \leq 2$, so $\|\Delta\| \leq 2$. Moreover, Δ is self-adjoint. Let $u, v \in \ell^2(\mathbb{Z})$. Then

$$\begin{aligned} (\Delta u|v) &= \sum_{n\in\mathbb{Z}} (\Delta u)_n \overline{v_n} = -\sum_{n\in\mathbb{Z}} u_{n-1}\overline{v_n} - \sum_{n\in\mathbb{Z}} u_{n+1}\overline{v_n} = -\sum_{n\in\mathbb{Z}} u_n \overline{v_{n+1}} - \sum_{n\in\mathbb{Z}} u_n \overline{v_{n-1}} \\ &= \sum_{n\in\mathbb{Z}} u_n \overline{(-v_{n-1} - v_{n+1})} = \sum_{n\in\mathbb{Z}} u_n \overline{(\Delta v)_n} = (u|\Delta v). \end{aligned}$$

Therefore, Δ *is a bounded self-adjoint operator*.

Now, let's compute the spectrum of Δ. For this, we introduce the Fourier operator $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2([0, 2\pi])$ defined for any $u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{Z})$ and any $x \in [0, 2\pi]$ as $(\mathcal{F}u)(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$. We then define $S = \mathcal{F} \circ \Delta \circ \mathcal{F}^{-1}$. Let's calculate S. Consider $f \in L^2([0, 2\pi])$. Suppose that, for all $x \in [0, 2\pi]$, $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$. For all $n \in \mathbb{Z}$, $(\mathcal{F}^{-1}f)_n = \hat{f}(n)$. Then, for all $x \in [0, 2\pi]$,

$$\begin{aligned} (Sf)(x) &= -\sum_{n\in\mathbb{Z}} (\hat{f}(n-1) + \hat{f}(n+1))e^{inx} = -\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{\mathrm{i}(n+1)x} - \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{\mathrm{i}(n-1)x} \\ &= -(e^{\mathrm{i}x} + e^{-\mathrm{i}x}) \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx} = (-2\cos(x))f(x). \end{aligned}$$

Therefore, for $x \in [0, 2\pi]$, if we let $\varphi(x) = -2\cos(x)$, $S = M_\varphi$ where M_φ is the multiplication operator by φ. As $\mathcal{F}$ is a unitary transformation, we have $\sigma(\Delta) = \sigma(M_\varphi)$ and $\sigma_p(\Delta) = \sigma_p(M_\varphi)$. Hence, $\sigma(M_\varphi) = \varphi([0, 2\pi]) = [-2, 2]$. Thus,

$$\sigma(\Delta) = [-2, 2].$$

Additionally, since φ is not constant on any subinterval of $[0, 2\pi]$, M_φ has no eigenvalues. Indeed, if $u \in L^2([0, 2\pi])$, the equation $\varphi(x)u(x) = \lambda u(x)$ for all $x \in [0, 2\pi]$ leads to $u = 0$. Therefore,

$$\sigma_p(\Delta) = \emptyset.$$

Remark 3.5.1. *The use of the Fourier transform $\mathcal{F}$ in this example is very common for calculating operator spectra, especially for differential operators. This is because, as $\mathcal{F}$ is unitary, conjugation by $\mathcal{F}$ preserve the spectrum and the point spectrum. Moreover, $\mathcal{F}$ has the particularity of transforming a (here, discrete) derivative into a multiplication. Therefore, conjugating the studied operator by $\mathcal{F}$ helps in computing its spectrum by reducing it to the spectrum calculation of a multiplication operator.*

Chapter 4

Compact operators

b The objective of this chapter is to establish a framework in which many properties of linear applications in finite dimensions can be found. For this purpose, we introduce compact operators, forming a family of operators whose properties are very close to those of linear applications in finite dimensions. In particular, solving linear equations in infinite dimensions, represented by a compact operator, will be analogous to solving linear equations in finite dimensions. This is the subject of Fredholm's alternative for compact operators, which we will demonstrate in this chapter.

4.1 Compact Operators

In this section, we will return to the more general context of operators between Banach spaces. Then, we will focus on operators between Hilbert spaces.

Definition 4.1.1. *Let E and F be two Banach spaces, and $T : E \to F$ be an operator. Let B_E denote the unit ball of E. We say that T is compact if $\overline{T(B_E)}$ is a compact subset of F.*

Notation. Let $\mathcal{B}_\infty(E, F)$ denote the set of compact operators from E to F.

It is observed that $\mathcal{B}_\infty(E, F) \subset \mathcal{L}(E, F)$. Indeed, if $T \in \mathcal{B}_\infty(E, F)$, as $T(B_E)$ is relatively compact (meaning it has compact closure), it is a bounded subset of F. Thus, T is a bounded operator.

Recall that, by the Riesz theorem, a normed vector space is of finite dimension if and only if its unit ball is compact. The topological property of compactness is thus directly related to the algebraic property of finite dimension. This explains why assuming that the image of the unit ball in the departure space by T has compact closure will give T properties close to a linear application in finite dimensions.

Example 4.1.2. *If $E = F$ and is of infinite dimension, the identity $Id : E \to E$ is not compact. Indeed, by the Riesz theorem, B_E is not compact. However, Id is continuous. Therefore, not all bounded operators are compact.*

Example 4.1.3. *Let X be a compact metric space, and let $E = F = C(X, \mathbb{R})$ be the space of continuous functions on X with real values. Let μ be a finite positive measure on $(X, \mathcal{B}(X))$, and $K \in C(X \times X, \mathbb{R})$. For $u \in E$, define*

$$Tu(x) = \int_X K(x, y)u(y)\mathrm{d}\mu(y).$$

Such an operator is an example of an integral kernel operator. Let's demonstrate that the operator T thus defined is a compact operator. Let $u \in B_E$. For $x, x' \in X$,

$$\begin{aligned}|Tu(x) - Tu(x')| &\leq \int_X |(K(x,y) - K(x',y))u(y)| d\mu(y) \\ &\leq \|u\|_\infty \mu(X) \max_{y \in X} |K(x,y) - K(x',y)|.\end{aligned}$$

Now, K is uniformly continuous on the compact set $X \times X$ since it is continuous. For any $\epsilon > 0$, there exists $\delta > 0$ such that, for all $x, x' \in X$,

$$(d(x,x') \leq \delta) \Rightarrow (\forall u \in B_E, \ |Tu(x) - Tu(x')| \leq \mu(X)\epsilon).$$

Therefore, $T(B_E)$ is an equicontinuous subset of $C(X)$. Additionally, $\|Tu\|_\infty \leq \|K\|_\infty \mu(X)\|u\|_\infty$, hence $T(B_E)$ is pointwise bounded. By the Ascoli theorem, $T(B_E)$ is relatively compact in $C(X)$, thus T is compact.

Definition 4.1.4. *An operator $T \in \mathcal{L}(E,F)$ is said to be of finite rank if* $\operatorname{Im} T$ *has finite dimension.*

Example 4.1.5. *An operator $T \in \mathcal{L}(E,F)$ of finite rank is compact. Indeed, by the continuity of T, $T(B_E)$ is bounded in* $\operatorname{Im} T$*. Thus, $\overline{T(B_E)}$ is a closed bounded subset of* $\operatorname{Im} T$*, which is of finite dimension; hence, it is compact in F.*

We will see that this latter example is essential in the context of operators between Hilbert spaces. We will show that any compact operator between Hilbert spaces is, for the operator norm topology, the limit of a sequence of operators of finite rank. Before limiting ourselves to Hilbert spaces, we present two more properties of compact operators applicable to operators between Banach spaces.

Proposition 4.1.6. *Let $(T_n)_{n \in \mathbb{N}}$ be a sequence of compact operators from E to F converging to T in $\mathcal{L}(E,F)$. Then T is compact. Hence, $\mathcal{B}_\infty(E,F)$ is closed in $\mathcal{L}(E,F)$.*

Proof : Firstly, as in a complete space, compacts are precompact sets, T is compact if and only if $T(B_E)$ is a precompact subset of F. Therefore, let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $\|T - T_n\|_{\mathcal{L}(E,F)} \leq \frac{\varepsilon}{2}$. As $T_n(B_E)$ is precompact, there exist vectors $v_j \in E$ such that

$$T_n(B_E) \subset \bigcup_{j=1}^{p} B(v_j, \frac{\varepsilon}{2}).$$

Now, for $u \in B_E$, $\|Tu - T_n u\|_F \leq \frac{\varepsilon}{2}$. Moreover, there exists j_0 such that $T_n u \in B(v_{j_0}, \frac{\varepsilon}{2})$, implying $Tu \in B(v_{j_0}, \varepsilon)$, hence $T(B_E) \subset \cup_{j=1}^{p} B(v_j, \varepsilon)$. Therefore, $T(B_E)$ is precompact. □

Proposition 4.1.7. *Let E, F, and G be three Banach spaces, $T \in \mathcal{L}(E,F)$, and $S \in \mathcal{L}(F,G)$. If T or S is compact, then ST is compact. In particular, $\mathcal{B}_\infty(E)$ is an ideal of $\mathcal{L}(E)$.*

Proof : Suppose T is compact. Then $ST(B_E) = S(T(B_E))$, $T(B_E)$ is relatively compact, and S is continuous. Therefore, $ST(B_E)$ is relatively compact. If S is assumed compact, and $T(B_E)$ is bounded, there exists a real number R such that $T(B_E) \subset RB_F$. So $ST(B_E) \subset RS(B_F)$. Now, $S(B_F)$ is relatively compact, hence $\overline{ST(B_E)}$ is closed in the compact set $\overline{S(B_F)}$; therefore, it is a compact set. □

Now, leaving the general framework of operators between Banach spaces, we focus on operators between Hilbert spaces. We can prove that any compact operator from a separable Hilbert space is the limit of operators of finite rank. Let's start by proving a useful property of compact operators.

Proposition 4.1.8. *Let H be a Hilbert space, and $T \in \mathcal{B}_\infty(H)$. If $(u_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in H, then $(Tu_n)_{n\in\mathbb{N}}$ is a convergent sequence in H for the norm topology on H.*

Proof : Suppose $(u_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in H towards u. Recall that this means that for every $w \in H$, $(u_n|w) \to (u|w)$. By the Banach-Steinhaus theorem, the sequence $(\|u_n\|)_{n\in\mathbb{N}}$ is bounded. Let $v_n = Tu_n$ and $v = Tu$. For any $w \in H$, $(v_n - v|w) = (u_n - u|T^*w)$, therefore $(v_n)_{n\in\mathbb{N}}$ also converges weakly in H to $v = Tu$. Assume by contradiction that $(v_n)_{n\in\mathbb{N}}$ does not norm-converge to v. Then, there exists $\eta > 0$ and a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $\|v_{n_k} - v\| \geq \eta$. However, since $(u_{n_k})_{k\in\mathbb{N}}$ is bounded in the norm of H and T is compact, a subsequence $(u_{n_{k_l}})_{l\in\mathbb{N}}$ can be extracted from $(u_{n_k})_{k\in\mathbb{N}}$ such that $(Tu_{n_{k_l}})_{l\in\mathbb{N}}$ converges to a limit $\tilde{v}$ in the norm of H. This subsequence $(v_{n_{k_l}})_{l\in\mathbb{N}}$ also weakly converges to $\tilde{v}$ and by uniqueness of the weak limit, $\tilde{v} = v$. This contradicts the fact that for all $l \in \mathbb{N}$, $\|v_{n_{k_l}} - v\| \geq \eta$. Therefore, $(Tu_n)_{n\in\mathbb{N}}$ converges in norm to v. □

From Proposition 4.1.6, it follows that any limit of operators of finite rank is a compact operator. Now, we prove that in separable Hilbert spaces, the converse is also true.

Proposition 4.1.9. *Let H be a separable Hilbert space. Every compact operator on H is, for the uniform topology of operators, the limit of a sequence of finite-rank operators.*

Proof : Let $(u_j)_{j\geq 1}$ be an orthonormal basis in H. Let T be a compact operator on H. For $n \geq 1$,

$$\lambda_n = \sup\{\|Tv\| \mid \|v\| = 1 \text{ and } v \in \operatorname{span}(u_1, \ldots, u_n)^\perp\}.$$

The sequence $(\lambda_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non-negative reals, hence it converges to $\lambda \geq 0$. Let's show this limit is zero. Choose a sequence $(v_n)_{n\in\mathbb{N}}$ of elements in $\operatorname{span}(u_1, \ldots, u_n)^\perp$ such that $\|v_n\| = 1$ and $\|Tv_n\| \geq \frac{\lambda}{2}$. As the family $(u_j)_{j\geq 1}$ is total, (v_n) converges weakly to 0 in H. Indeed, if $w \in H$, there exists a family $(\alpha_j)_{j\geq 1}$ such that

$$w = \lim_{N\to+\infty} \sum_{j=1}^{N} \alpha_j u_j.$$

But, from the definition of v_n,

$$\forall n \geq N, \left(v_n \mid \sum_{j=1}^{N} \alpha_j u_j\right) = 0.$$

This implies that

$$\forall n \geq N, (v_n|w) = (v_n|w - \sum_{j=1}^{N} \alpha_j u_j) \leq ||v_n||\, ||w - \sum_{j=1}^{N} \alpha_j u_j||.$$

Let $\varepsilon > 0$. There exists $N_\varepsilon \geq 1$ such that $||w - \sum_{j=1}^{N_\varepsilon} \alpha_j u_j|| \leq \varepsilon$. Therefore,

$$\forall n \geq N_\varepsilon, |(v_n|w)| \leq \epsilon$$

and $(v_n|w) \to 0$ as n tends to infinity.

By Proposition 4.1.8, the sequence $(Tv_n)_{n\in\mathbb{N}}$ converges in norm to 0. Therefore, $\lambda = 0$. By the projection theorem in Hilbert spaces,

$$\lambda_n = \sup_{\|u\|=1} \Big\| Tu - \sum_{j=1}^{n} (u|u_j) Tu_j \Big\|.$$

Hence, as $(\lambda_n)_{n\in\mathbb{N}}$ tends to 0,

$$\Big\| T - \sum_{j=1}^{n} (.|u_j) Tu_j \Big\|_{\mathcal{L}(H)} \to 0. \qquad \square$$

Hence, an operator on a Hilbert space is compact if and only if it is the limit of operators of finite rank. Before using this characterization of compact operators to study linear equations, we conclude with a property sometimes useful for proving an operator's compactness.

Proposition 4.1.10. *Let $T \in \mathcal{L}(H)$. Then T is compact if and only if T^* is compact.*

Proof : Suppose T is compact. Using the notations from the proof of Proposition 4.1.9 and denoting P_n as the projector onto the subspace $\text{span}(u_1, \ldots, u_n)$, we can write $TP_n = \sum_{j=1}^{n} (.|u_j) Tu_j$. As proved in Proposition 4.1.9, $\|TP_n - T\|_{\mathcal{L}(H)} \to 0$ as n tends to infinity. Consequently, as $(TP_n - T)^* = (P_n - I)^* T^* = (P_n - I) T^*$, we obtain that

$$\|TP_n - T\|_{\mathcal{L}(H)} = \|(TP_n - T)^*\|_{\mathcal{L}(H)} = \|P_n T^* - T^*\|_{\mathcal{L}(H)} \to 0.$$

As $P_n T^*$ has finite rank, T^* is a limit of a sequence of finite-rank operators; hence, it is compact. Assuming T^* is compact, then $T = (T^*)^*$ is also compact, yielding the equivalence. $\square$

4.2 Fredholm's Alternative

We have presented several properties of compact operators so far without yet providing a result highlighting the significance of their introduction. We will now present an essential result for solving linear equations in infinite dimensions, known as Fredholm's alternative. This asserts that if T is a compact operator, then either $Tu = u$ has a nontrivial solution or $(I - T)^{-1}$ exists. This is a similar alternative to the case of linear systems in finite dimensions and is equivalent to the result on endomorphisms in finite dimensions, asserting that their injectivity implies bijectivity. In practice, it greatly simplifies the demonstration of the existence of solutions to linear equations. Fredholm's alternative tells us that if for any $v \in H$, there exists at most one solution $u \in H$ to the linear equation $Tu + v = u$, then there exists exactly one solution. If there exists at most one solution, $I - T$ is injective, hence $Tu = u$ has no nontrivial solution. Thus, $(I - T)^{-1}$ exists, and for any $v \in H$, the unique solution to $Tu + v = u$ is given by $u = (I - T)^{-1} v$. The compactness of the operator and the *a priori* uniqueness of the solution imply the existence of the solution.

Fredholm's alternative is not generally satisfied by all bounded operators. For instance, the multiplication operator defined on $L^2([0,2])$ by $Tu(x) = xu(x)$ for all $x \in [0,2]$ does not satisfy it. Although $Tu = u$ has no nontrivial solution, $(I - T)^{-1}$ is not a bounded operator on $L^2([0,2])$.

Since Fredholm's alternative holds for operators on a finite-dimensional space, the idea to prove it for compact operators acting on a Hilbert space is to use the fact that they are limits of sequences of finite-rank operators. Thus, such a compact operator can be written as $T = P + R$, where P is a finite-rank operator, and R is an operator of small norm, a perturbation.

Theorem 4.2.1 (Analytical Fredholm Theorem)**.** *Let H be a Hilbert space, and D be a connected open subset of $\mathbb{C}$. Let $f : D \to \mathcal{L}(H)$ be an analytic function such that for every $z \in D$, $f(z)$ is a compact operator. Then only one of the following occurs:*

1. *$(I - f(z))^{-1}$ does not exist for any $z \in D$;*
2. *$(I - f(z))^{-1}$ exists for all $z \in D \setminus \mathcal{S}$, where $\mathcal{S}$ is a discrete subset of D. In this case, $(I - f(z))^{-1}$ is meromorphic on D, analytic in $D \setminus \mathcal{S}$, and the residues at the poles are operators of finite rank. Moreover, if $z \in \mathcal{S}$, then the equation $f(z)u = u$ has a nontrivial solution in H.*

Proof : Firstly, as D is connected, by analytic continuation, it suffices to prove the theorem in the neighborhood of any point in D. Let $z_0 \in D$. Due to the continuity of f at z_0, there exists $r > 0$ such that for $z \in D$ with $|z - z_0| < r$, we have $\|f(z) - f(z_0)\|_{\mathcal{L}(H)} < \frac{1}{2}$. As the operator $f(z_0)$ is compact, there exists P, a finite-rank operator, such that $\|f(z_0) - P\|_{\mathcal{L}(H)} < \frac{1}{2}$. Therefore, for $z \in D(z_0, r)$, $\|f(z) - P\|_{\mathcal{L}(H)} < 1$. We can then use Neumann's series lemma to prove the existence of $(I - f(z) + P)^{-1} \in \mathcal{L}(H)$ and that $z \mapsto (I - f(z) + P)^{-1}$ is analytic on $D(z_0, r)$.

Now, as P is of finite rank, there exists a linearly independent family $(u_1, \ldots, u_N)$ of N vectors and vectors $v_1, \ldots, v_N$ such that for any $u \in H$, $Pu = \sum_{i=1}^{N} (u|v_i)u_i$. For $z \in D(z_0, r)$, we set $v_i(z) = ((I - f(z) + P)^{-1})^* v_i$ and define the operator $g(z)$ as

$$\forall w \in H,\ g(z)w = P(I - f(z) + P)^{-1} w = \sum_{i=1}^{N} (w|v_i(z))u_i.$$

Observe that for any $z \in D(z_0, r)$, $(I - g(z))(I - f(z) + P) = I - f(z)$. Therefore, for $z \in D(z_0, r)$, $I - f(z)$ is invertible in $\mathcal{L}(H)$ if and only if $I - g(z)$ is. Similarly, the equation $f(z)u = u$ has a nontrivial solution if and only if $g(z)w = w$ has one.

Suppose there exists $w \in H$ such that $g(z)w = w$. We can decompose w as $w = \sum_{n=1}^{N} \alpha_n u_n$, and the coefficients α_n satisfy, due to the freedom of the family $(u_1, \ldots, u_N)$,

$$\forall n \in \{1, \ldots, N\},\ \alpha_n = \sum_{m=1}^{N} (u_m|v_n(z))\alpha_m. \tag{4.1}$$

Conversely, if for a fixed z the system (4.1) has a solution $(\alpha_1, \ldots, \alpha_N)$, then the vector $w = \sum_{n=1}^{N} \alpha_n u_n$ is a solution of $g(z)w = w$. Hence, we've reduced it to studying a finite-dimensional linear system, and the equation $g(z)w = w$ has a nontrivial solution if and only if the determinant $d(z) = \det(I - [(u_m|v_n(z))]_{m,n}) = 0$. As $(u_m|v_n(z))$ is analytic on $D(z_0, r)$, $d(z)$ is also analytic, so the set $\mathcal{S}_r = \{z \in D(z_0, r) \mid d(z) = 0\}$ of zeros of $d(z)$ is either discrete in $D(z_0, r)$ or equal to $D(z_0, r)$. In the latter case, $(I - f(z))^{-1}$ does not exist for any $z \in D(z_0, r)$, and we fall into case 1 of Fredholm's alternative.

Now, suppose that $\mathcal{S}_r \neq D(z_0, r)$, which corresponds to case 2 of Fredholm's alternative. If $z \in \mathcal{S}_r$, the equation $f(z)u = u$ has a nontrivial solution in H, proving the last assertion of the theorem.

Lastly, if $z \notin \mathcal{S}_r$, then $d(z) \neq 0$. Given $u \in H$, we can solve the equation $(I - g(z))w = u$ by setting $w = u + \sum_{n=1}^{N} \beta_n u_n$ if and only if the β_n satisfy the system

$$\forall n \in \{1, \ldots, N\},\ \beta_n = (u|v_n(z)) + \sum_{n=1}^{N} (u_m|v_n(z))\beta_m. \tag{4.2}$$

Since we assumed $d(z) \neq 0$, the system (4.2) has a unique solution. Thus, $(I - g(z))^{-1}$ exists in $\mathcal{L}(H)$. Furthermore, we can explicitly solve the linear system (4.2) using Cramer's formulas, enabling us to express $(I - g(z))^{-1}$, and consequently $(I - f(z))^{-1}$, as a meromorphic function whose residues at the poles are polynomials in P, and therefore operators of finite rank. Thus, case 2 of Fredholm's alternative is proven. □

Corollary 4.2.2 (Fredholm's Alternative). *Let H be a Hilbert space, and T be a compact operator on H. Then either $(I - T)^{-1}$ exists and is bounded, or $Tu = u$ has a nontrivial solution.*

Proof : We apply Theorem 4.2.1 to the analytic function $f(z) = zT$ at the point $z = 1$. We know that for $z = 0$, $I - f(z) = I$ is invertible, so we are necessarily in case 2 of the analytic Fredholm alternative. Then, if $I - T$ is not invertible, we have $z = 1 \in \mathcal{S}$, hence $f(1)u = u$ has a unique nontrivial solution. This gives the result. □

4.3 Dirichlet Problem in $\mathbb{R}^3$

We conclude this chapter by providing an example of a linear equation that can be addressed using Fredholm's alternative. We examine the Dirichlet problem in $\mathbb{R}^3$.
Let D be a connected bounded open set in $\mathbb{R}^3$, with the boundary ∂D being a C^∞ surface in $\mathbb{R}^3$. The Dirichlet problem for the Laplace equation is as follows: given a continuous function f on ∂D, find a function u, which is of class C^2 in D and continuous on $\overline{D}$, satisfying

$$\forall x \in D,\ \Delta u(x) = 0 \text{ and } \forall x \in \partial D,\ u(x) = f(x).$$

To solve this problem, we introduce the kernel $K(x, y) = (x - y|n_y)/|x - y|^3$ defined on $D \times \partial D$, where n_y is the outward normal to ∂D at the point $y \in \partial D$. The function $x \mapsto K(x, y)$ is harmonic, $\Delta_x K(x, y) = 0$ for every $x \in D$ and for all $y \in \partial D$. This leads us to seek a solution u in the form of a superposition,

$$u(x) = \int_{\partial D} K(x, y)\varphi(y)\mathrm{d}S(y),$$

where φ is a continuous function on ∂D and $\mathrm{d}S$ is the surface measure on ∂D. Indeed, for $x \in D$, the integral is well-defined, and $\Delta u(x) = 0$. Now, let's see how to extend u to ∂D. Suppose $x_0 \in \partial D$ and $x \to x_0$, $x \in D$, then we can demonstrate that

$$u(x) \to -\varphi(x_0) + \int_{\partial D} K(x_0, y)\varphi(y)\mathrm{d}S(y). \tag{4.3}$$

Furthermore, if $x \to x_0$, $x \in \overline{D}^c$, it can also be shown that

$$u(x) \to \varphi(x_0) + \int_{\partial D} K(x_0, y)\varphi(y)\mathrm{d}S(y). \tag{4.4}$$

Thus, $\int_{\partial D} K(x_0, y)\varphi(y)\mathrm{d}S(y)$ exists and is a continuous function of x_0 on ∂D. As ∂D is a C^∞ surface, for all $x, y \in \partial D$, $(x - y|n_y) = c|x - y|^2 + o(|x - y|^2)$ as $x \to y$.
We want to verify the boundary condition $u(x) = f(x)$ for every $x \in \partial D$. Hence, we need to demonstrate the existence of a function φ continuous on ∂D such that $\forall x \in \partial D$, $f(x) = -\varphi(x) + \int_{\partial D} K(x, y)\varphi(y)\mathrm{d}S(y)$. For this purpose, we introduce the operator $T : C(\partial D) \to C(\partial D)$ defined by

$$\forall \varphi \in C(\partial D),\ \forall x \in \partial D,\ T\varphi = \int_{\partial D} K(x, y)\varphi(y)\mathrm{d}S(y).$$

Then T is a compact operator. For $\delta > 0$, let $K_\delta(x,y) = (x-y|n_y)/(|x-y|^3+\delta)$. Then K_δ is continuous and, from example 4.1.3, the associated operator T_δ is compact. Additionally, we have the estimate

$$|(T_\delta u)(x) - (Tu)(x)| \leq \|u\|_\infty \int_{\partial D} |K_\delta(x,y) - K(x,y)| \mathrm{d}S(y). \tag{4.5}$$

Now, if we fix $\varepsilon > 0$, we can split the integral into two parts:

$$\int_{\partial D} |K_\delta(x,y) - K(x,y)| \mathrm{d}S(y) = \int_{|x-y|\geq\varepsilon} |K_\delta(x,y) - K(x,y)| \mathrm{d}S(y) + \int_{|x-y|<\varepsilon} |K_\delta(x,y) - K(x,y)| \mathrm{d}S(y).$$

In the first integral, $K_\delta(x,y)$ converges uniformly to $K(x,y)$ as δ tends to 0. The integrability of K allows us to make the second integral arbitrarily small, uniformly in x, by choosing ε small enough. Hence, we've just demonstrated that $T_\delta u$ uniformly converges to Tu as δ tends to 0. Then, from (4.5), we obtain $\|T_\delta - T\|_{\mathcal{L}(C(\partial D))} \to 0$ as δ tends to 0. Therefore, the operator T is compact as the limit of compact operators.

As T is compact, we can apply Fredholm's alternative to it. Either there exists $\psi \in C(\partial D)$, not identically zero, such that $T\psi = \psi$, or for every $f \in C(\partial D)$, the equation $-f = (I-T)\varphi$ has a unique solution. Let's assume we're in the first alternative. Define, for every $x \in D \cup \partial D$, $u(x) = \int_{\partial D} K(x,y)\psi(y)\mathrm{d}S(y)$. Then, for every $x \in \partial D$, $u(x) = T\psi(x) = \psi(x)$. Therefore, for every $x \in D \cup \partial D$, $u(x) = \int_{\partial D} K(x,y)u(y)\mathrm{d}S(y)$. However, by the maximum principle (remember that u is harmonic in D), it follows that $u = 0$ in D. Moreover, $\frac{\partial u}{\partial n}$ is continuous on ∂D and hence is also equal to 0 on ∂D. By integration by parts, this implies that u is also identically zero on ∂D. From (4.3) and (4.4), it follows that $2\psi(x) = 0$ for every $x \in \partial D$, and ψ is identically zero. Hence, the first alternative does not hold. Therefore, for every $f \in C(\partial D)$, the equation $-f = (I-T)\varphi$ has a unique solution, establishing the existence and uniqueness of the solution to the Dirichlet problem for the Laplace equation in $\mathbb{R}^3$.

Chapter 5

Spectrum of compact operators

Compact operators possess properties similar to operators in finite dimensions. This was evident when resolving linear systems by studying Fredholm's alternative. We will now explore the spectral properties of compact operators. In particular, we'll see that both the spectrum of compact operators and the diagonalization properties of self-adjoint compact operators are the limiting cases of corresponding results in finite dimensions. Furthermore, we obtain a classification of self-adjoint compact operators up to unitary equivalence.

5.1 Spectrum of Compact Operators

Let's start by providing a general result on the structure of the spectrum of compact operators.

Theorem 5.1.1 (Riesz-Schauder)**.** *Let H be a Hilbert space, and $T \in \mathcal{B}_\infty(H)$. Then, $\sigma(T) \setminus \{0\}$ is a discrete set in $\mathbb{C}$ consisting of finite multiplicity eigenvalues of T. Additionally, if H is of infinite dimension, $0 \in \sigma(T)$.*

Note that when $0 \in \sigma(T)$, 0 might not be an eigenvalue of T. Moreover, 0 could be an accumulation point of $\sigma(T)$, as we'll see shortly.

Proof : Consider, for all $z \in \mathbb{C}$, the function $f(z) = zT$. Then f is a holomorphic map from $\mathbb{C}$ to $\mathcal{B}_\infty(H)$. Let $\mathcal{S} = \{z \neq 0 \mid zTu = u \text{ has a non-zero solution } u\}$. If $z \in \mathcal{S}$, $\frac{1}{z}$ is an eigenvalue of T. Since $z = 0 \notin \mathcal{S}$, by Theorem 4.2.1, $\mathcal{S}$ is a discrete set. If $\frac{1}{z} \notin \mathcal{S}$, then

$$(T - z)^{-1} = \frac{1}{z}\left(\frac{1}{z}T - \mathrm{Id}_E\right)^{-1}$$

exists, again by Theorem 4.2.1. Therefore, $\sigma(T) \setminus \{0\} = \{\frac{1}{z} \mid z \in \mathcal{S}\}$, and $\sigma(T) \setminus \{0\}$ is a discrete set of eigenvalues of T according to the definition of $\mathcal{S}$.

If $\lambda \in \sigma_p(T)$, $\lambda \neq 0$, let $F = \mathrm{Ker}\,(T - \lambda)$. Then, if B_F represents the unit ball in F and B_H in H, we have

$$B_F = \frac{1}{\lambda}\lambda B_F = \frac{1}{\lambda}T(B_F) \subset \frac{1}{\lambda}T(B_H).$$

Since T is compact, $T(B_H)$ is relatively compact, and so is B_F. By a Riesz theorem, F is of finite dimension. Therefore, each non-zero eigenvalue of T has finite multiplicity.

Suppose H is of infinite dimension. If $0 \notin \sigma(T)$, then T is bijective, and T^{-1} is continuous. Thus, $B_H = T^{-1}(T(B_H))$ is relatively compact as $T(B_H)$ is compact due to T's compactness. Hence, again by the same Riesz theorem, H is of finite dimension. This contradicts our initial assumption, so $0 \in \sigma(T)$. □

The proof of the Riesz-Schauder theorem relies on the analytical Fredholm alternative. We confined ourselves to the case of Hilbert spaces since we didn't prove the analytical Fredholm alternative (see Theorem 4.2.1) in full generality but only for Hilbert spaces. Nevertheless, the Riesz-Schauder theorem is still valid for compact operators on any Banach space. The analytical Fredholm alternative remains true in the framework of Banach spaces, but its proof is more challenging.

5.2 Diagonalization of Self-adjoint Compact Operators

In this section, we present a generalization for self-adjoint compact operators of the result asserting that any real symmetric matrix is diagonalizable in an orthonormal basis. Throughout, H will be a complex Hilbert space.

Lemma 5.2.1. *Let $T \in \mathcal{L}(H)$. If T is compact and self-adjoint, then either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

Proof : If $T = 0$, then 0 is an eigenvalue of T and $\|T\| = 0$. Suppose $T \neq 0$. By Proposition 2.2.10, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of unit vectors such that $|(Tu_n|u_n)| \to \|T\|$ as n approaches infinity. By extracting a subsequence (due to the compactness of the set $\overline{D(0,\|T\|)}$ because $|(Tu_n|u_n)| \leq \|T\|$), we may assume $(Tu_n|u_n) \to \lambda$ as n tends to infinity, where $|\lambda| = \|T\|$. Then,

$$0 \leq \|(T-\lambda)u_n\|_H^2 = \|Tu_n\|_H^2 - 2\lambda(Tu_n|u_n) + \lambda^2 \leq 2\lambda^2 - 2\lambda(Tu_n|u_n) \to 0$$

as n tends to infinity. Thus, $\|(T-\lambda)u_n\|_H \to 0$ as n approaches infinity. Hence, due to the compactness of T, there exists $u \in H$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\|Tu_{n_k} - u\|_H \to 0$ as k tends to infinity. Considering $u_{n_k} = \frac{1}{\lambda}(\lambda - T)u_{n_k} + \frac{1}{\lambda}Tu_{n_k}$ converges to $\frac{1}{\lambda}u$. Thus, $1 = \|\lambda^{-1}u\|_H = |\lambda|^{-1}\|u\|_H$ and $u \neq 0$. Moreover, $Tu_{n_k} \to \frac{1}{\lambda}Tu$ due to the continuity of T. Therefore, by the uniqueness of the limit, $u = \lambda^{-1}Tu$ and $Tu = \lambda u$, $u \neq 0$. Thus, $\lambda \in \sigma_p(T)$. □

Proposition 5.2.2. *Let $T \in \mathcal{L}(H)$ be a self-adjoint compact operator. Then,*

$$H = \mathrm{Ker}\, T \oplus \widehat{\bigoplus}_{\lambda\in\sigma(T)\setminus\{0\}} \mathrm{Ker}\,(T-\lambda).$$

Proof : Recall that for $\lambda \neq \mu$ in $\sigma_p(T)$, $\mathrm{Ker}\,(T-\lambda) \perp \mathrm{Ker}\,(T-\mu)$. Let $F = \widehat{\oplus}_{\lambda\in\sigma(T)\setminus\{0\}}\mathrm{Ker}\,(T-\lambda)$. Then F is closed and stable under T. If $u = \sum_{\lambda\in\sigma(T)\setminus\{0\}} u_\lambda$ with $\sum \|u_\lambda\|_H^2$ convergent, then $Tu = \sum_{\lambda\in\sigma(T)\setminus\{0\}} \lambda u_\lambda \in F$. Additionally, since T is self-adjoint, $F^\perp$ is also stable under T (see Proposition 2.2.6). Let $T_0 : F^\perp \to F^\perp$ be the restriction of T to $F^\perp$. Then T_0 is self-adjoint and compact. We have $r(T_0) = \|T_0\|$. Moreover, if $r(T_0) > 0$, T_0 has a non-zero eigenvalue λ_0 because, according to the Riesz-Schauder theorem, every non-zero element in $\sigma(T_0)$ is an eigenvalue as T_0 is compact. But, since $\mathrm{Ker}\,(T_0 - \lambda_0) \subset \mathrm{Ker}\,(T - \lambda_0)$, we would have $\mathrm{Ker}\,(T-\lambda_0) \cap F^\perp \neq \{0\}$, which is absurd because for every $\lambda \neq 0$, $F^\perp \perp \mathrm{Ker}\,(T-\lambda)$. Therefore, $r(T_0) = 0$, $\|T_0\| = 0$, and T_0 is the null operator. Hence, $F^\perp \subset \mathrm{Ker}\, T$. Additionally, $\mathrm{Ker}\, T \subset (\mathrm{Ker}\,(T-\lambda))^\perp$ for every $\lambda \neq 0$ and $\mathrm{Ker}\, T \subset F^\perp$. Therefore, $\mathrm{Ker}\, T = F^\perp$. Since F is closed, $H = F \oplus F^\perp$, and indeed $H = \mathrm{Ker}\, T \oplus \widehat{\oplus}_{\lambda\in\sigma(T)\setminus\{0\}}\mathrm{Ker}\,(T-\lambda)$. □

We can now prove the theorem of diagonalization of self-adjoint compact operators, also known as the spectral theorem for compact operators.

Theorem 5.2.3 (Spectral Theorem for Self-adjoint Compact Operators). *Let $T \in \mathcal{L}(H)$ be a self-adjoint compact operator. Denote the non-zero eigenvalues of T by $\{\lambda_1, \lambda_2, \ldots\}$ and P_n as the projection of H onto* $\operatorname{Ker}(T - \lambda_n)$. *Then, for $n \neq m$, $P_n P_m = P_m P_n = 0$, and P_n is of finite rank. Moreover, if T has finite rank, the set of eigenvalues of T is finite, and if T does not have finite rank, $\lambda_n \to 0$ as n tends to infinity. Finally,*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

where the series converges in the operator norm (or is a finite sum in the case where T has finite rank).

Proof : According to Lemma 5.2.1, there exists a real number $\lambda_1 \in \sigma_p(T)$ such that $|\lambda_1| = \|T\|$. Let $F_1 = \operatorname{Ker}(T - \lambda_1)$ and P_1 be the projection of H onto F_1. Define $H_2 = F_1^{\perp}$. Since T leaves F_1 invariant and is self-adjoint, it also leaves H_2 invariant. Let $T_2 = T|_{H_2}$ be the restriction of T to H_2. Then T_2 is a self-adjoint compact operator. Therefore, by Lemma 5.2.1, a real number $\lambda_2 \in \sigma_p(T_2)$ with $|\lambda_2| = \|T_2\|$ exists. Let $F_2 = \operatorname{Ker}(T_2 - \lambda_2)$. Thus $F_2 = \operatorname{Ker}(T - \lambda_2)$ (if $u \in \operatorname{Ker}(T - \lambda_2)$, decompose u into $u_{F_1} + u_{H_2}$, apply T and use the fact that $F_2 = F_1^{\perp}$ to get that $u_{F_1} = 0$) and since $F_2 \subset F_1^{\perp}$, $\lambda_1 \neq \lambda_2$. Consider P_2 as the projection of H onto F_2 and define $H_3 = (F_1 \oplus F_2)^{\perp}$. As $\|T_2\| \leq \|T\|$, it follows that $|\lambda_2| \leq |\lambda_1|$.

By recurrence, we build a sequence of eigenvalues of T such that $|\lambda_1| \geq |\lambda_2| \geq \ldots$. If T has finite rank, this construction stops after a finite number of steps. If T does not have finite rank, an infinite sequence is constructed. Moreover, for all $n \geq 1$, let $F_n = \operatorname{Ker}(T - \lambda_n)$, then $|\lambda_{n+1}| = \|T|_{(F_1 \oplus \cdots \oplus F_n)^{\perp}}\|$. For all $n \geq 1$, let P_n be the projection of H onto F_n. The relation $P_n P_m = P_m P_n = 0$ for $n \neq m$ is due to the pairwise orthogonality of the F_n. Finally, by Theorem 5.1.1, the spectrum of T is at most countable, and the construction here demonstrates that $\{\lambda_1, \ldots\} = \sigma(T) \setminus \{0\}$.

In the rest of the proof, assume that T does not have finite rank. Let's prove that the sequence $(\lambda_n)_{n \geq 1}$ defined in this way converges to 0. Firstly, since $|\lambda_1| \geq |\lambda_2| \geq \ldots$, the sequence $(|\lambda_n|)_{n \geq 1}$ is convergent, say to α. Then, for all $n \geq 1$, choose $u_n \in F_n$ with $\|u_n\|_H = 1$. As T is compact, there exists $u \in H$ and a subsequence $(u_{n_k})_{k \geq 1}$ such that $\|Tu_{n_k} - u\|_H \to 0$ as k approaches infinity. Now, for $n \neq m$, $u_n \perp u_m$ and for all $k \geq 1$, $Tu_{n_k} = \lambda_{n_k} u_{n_k}$. Hence, for $k, l \geq 1$, it follows that $\|Tu_{n_k} - Tu_{n_l}\|_H^2 = \lambda_{n_k}^2 + \lambda_{n_l}^2 \geq 2\alpha^2$. But as $(Tu_{n_k})_{k \geq 1}$ is a Cauchy sequence, we must have $\alpha = 0$.

Let $k \in \{1, \ldots, n\}$ and $u \in F_k$. Then $(T - \sum_{j=1}^{n} \lambda_j P_j)u = Tu - \lambda_k u = 0$. So, $F_1 \oplus \cdots \oplus F_n \subset \operatorname{Ker}(T - \sum_{j=1}^{n} \lambda_j P_j)$. Now, if $u \in (F_1 \oplus \cdots \oplus F_n)^{\perp}$, then $P_j u = 0$ for all $j \in \{1, \ldots, n\}$ and $(T - \sum_{j=1}^{n} \lambda_j P_j)u = Tu$. As T also leaves $(F_1 \oplus \cdots \oplus F_n)^{\perp}$ invariant, we obtain

$$\left\| T - \sum_{j=1}^{n} \lambda_j P_j \right\| = \left\| T|_{(F_1 \oplus \cdots \oplus F_n)^{\perp}} \right\| = |\lambda_{n+1}| \to 0$$

as n approaches infinity. Therefore, the series $\sum \lambda_n P_n$ converges in operator norm to T. □

From this theorem, the following corollary can be deduced, which demonstrates the existence of a Hilbert basis for the compact self-adjoint operator T.

Corollary 5.2.4. *Let $T \in \mathcal{L}(H)$ be a compact self-adjoint operator. There exists a Hilbert basis $(\phi_n)_{n \in \mathbb{N}}$ of H such that, for all $n \in \mathbb{N}$, there exists a real number λ_n such that $T\phi_n = \lambda_n \phi_n$ and $\lambda_n \to 0$ as n tends to infinity.*

Proof: By Proposition 5.2.2, a Hilbert basis $(\phi_n)_{n\in\mathbb{N}}$ of H can be constructed by joining the bases of Ker $(T-\lambda)$ for $\lambda \in \sigma(T)$. Hence, by renumbering the eigenvalues of T, for all $n \in \mathbb{N}$, $T\phi_n = \lambda_n\phi_n$ where the λ_n are given by Theorem 5.2.3. Furthermore, by Theorem 5.2.3, $\lambda_n \to 0$ as n tends to infinity. □

5.3 Reduction of Self-Adjoint Compact Operators

The spectral theorem for self-adjoint compact operators states that every self-adjoint compact operator can be diagonalized in a Hilbert space. Thus, in a sense that we will now define, every self-adjoint compact operator is *unitarily equivalent* to an infinite diagonal matrix. In finite dimension, two diagonalizable matrices are equivalent if and only if they have the same eigenvalues with the same multiplicities. This result will be generalized for self-adjoint compact operators.

Definition 5.3.1. *Let H and K be two Hilbert spaces. Let $S \in \mathcal{L}(H)$ and $T \in \mathcal{L}(K)$. S and T are said to be unitarily equivalent if there exists a Hilbert space isomorphism $U : H \to K$ such that $USU^{-1} = T$.*

Definition 5.3.2. *Let $T \in \mathcal{B}_\infty(H)$. The multiplicity function of T is the function $m_T : \mathbb{C} \to \mathbb{N} \cup \{+\infty\}$ defined by $m_T(\lambda) = \dim \text{Ker}\,(T-\lambda)$.*

Then, $m_T(\lambda) > 0$ if and only if λ is an eigenvalue of T. Additionally, if $\lambda \neq 0$, by the Riesz-Schauder theorem, $m_T(\lambda) < +\infty$.

Proposition 5.3.3. *If T and S are two unitarily equivalent compact operators, and $U : H \to K$ is an isomorphism such that $USU^{-1} = T$, then $\text{Ker}\,(T-\lambda) = U\text{Ker}\,(S-\lambda)$ for all $\lambda \in \mathbb{C}$. In particular, $m_T = m_S$.*

Proof: Indeed, if $v \neq 0$ such that $Sv = \lambda v$, then $TUv = USv = \lambda Uv$, hence $Uv \in \text{Ker}\,(T-\lambda)$. Therefore, $U\text{Ker}\,(S-\lambda) \subset \text{Ker}\,(T-\lambda)$. Conversely, if $w \in \text{Ker}\,(T-\lambda)$ and $v = U^{-1}w$, then $Sv = SU^{-1}w = U^{-1}Tw = \lambda v$. Hence $\text{Ker}\,(T-\lambda) \subset U\text{Ker}\,(S-\lambda)$. As U is an isomorphism of vector spaces, we obtain $m_T = m_S$. □

It follows from this proposition that equality of multiplicity functions is a necessary condition for two compact operators to be unitarily equivalent. We will now demonstrate that for self-adjoint compact operators, it is also a sufficient condition.

Theorem 5.3.4. *Two self-adjoint compact operators are unitarily equivalent if and only if they have the same multiplicity function.*

Proof: Let $S \in \mathcal{L}(H)$ and $T \in \mathcal{L}(K)$ be two compact self-adjoint operators. If S and T are unitarily equivalent, as we have just shown in Proposition 5.3.3, $m_T = m_S$. Now suppose that $m_T = m_S$ and construct an isomorphism $U : H \to K$ such that $UTU^{-1} = S$.

By the spectral theorem for compact operators, we can write $T = \sum_{n=1}^{\infty} \lambda_n P_n$ and $S = \sum_{n=1}^{\infty} \mu_n Q_n$ where for $m \neq n$, $\lambda_n \neq \lambda_m$ and $\mu_n \neq \mu_m$, and the projectors P_n and Q_n have finite rank. Let P_0 be the projector of H onto Ker T and Q_0 the projector of K onto Ker S. Also, let $\lambda_0 = \mu_0 = 0$. As $m_T = m_S$, for all $n \in \mathbb{N}$, $m_S(\lambda_n) = m_T(\lambda_n) > 0$. Hence, the λ_n are also eigenvalues of S. Therefore, for all $n \in \mathbb{N}$, there exists a unique μ_j such that $\mu_j = \lambda_n$. Define $\pi : \mathbb{N} \to \mathbb{N}$ by $\mu_{\pi(n)} = \lambda_n$ and set $\pi(0) = 0$. Additionally, as $m_T(\mu_n) = m_S(\mu_n) > 0$, all the μ_n are also eigenvalues of T, and for all $n \in \mathbb{N}$, there exists $j \in \mathbb{N}$, $\pi(j) = n$. Thus, π is a bijection.

For all $n \in \mathbb{N}$, $\dim \operatorname{Im} P_n = m_T(\lambda_n) = m_S(\mu_{\pi(n)}) = \dim \operatorname{Im} Q_{\pi(n)}$ (equality of Hilbert space dimensions), there exists an isomorphism of Hilbert spaces $U_n : P_nH \to Q_{\pi(n)}K$. Define $U : H \to K$ by setting $U = U_n$ on P_nH and extending by linearity. Then, U is indeed an isomorphism since $\widehat{\bigoplus}_{n\in\mathbb{N}} \operatorname{Im} P_n = H$. Moreover, if $v \in P_nH$, then $UTv = \lambda_n Uv = \mu_{\pi(n)}Uv = SUv$. Thus, we indeed have $UTU^{-1} = S$. □

In general, the multiplicity function is not sufficient to characterize the unitary equivalence of two arbitrary compact operators. For example, if V is the Volterra operator, $m_V = 0$, yet V and the zero operator are not unitarily equivalent. No known necessary and sufficient conditions exist for two compact operators to be unitarily equivalent. In fact, even in finite dimensions, there are no known necessary and sufficient conditions for two operators to be unitarily equivalent.

Chapter 6

Spectral theorem

We will generalize the classical result asserting that any real symmetric matrix is diagonalizable in an orthonormal basis to the framework of bounded operators on a Hilbert space.
A good way to state this theorem for matrices is to write that for any real symmetric matrix $A \in M_n(\mathbb{R})$, there exist real numbers $\lambda_1, \ldots, \lambda_n$ and orthogonal projectors $P_1, \ldots, P_n$ such that:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

It is this formulation that we will generalize to infinite dimension by transforming the sum into an integral against measures with projector values.

6.1 Spectral Families

Definition 6.1.1. *A spectral family (or identity resolution) on $\mathcal{H}$ is a function $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ such that:*

1. *For all $t \in \mathbb{R}$, $E(t)$ is an orthogonal projection, i.e., $E(t)^2 = E(t)$ and $E(t)^* = E(t)$.*
2. *Monotonicity: $\forall s \leq t$, $E(s) \leq E(t)$, i.e., $\forall u \in \mathcal{H}, (E(s)u|u) \leq (E(t)u|u)$.*
3. *Right-continuous: $\forall u \in \mathcal{H},\ E(t+\varepsilon)u \xrightarrow[\varepsilon \to 0^+]{} E(t)u$.*
4. *Normalization at infinity: $\forall u \in \mathcal{H},\ E(t)u \xrightarrow[t \to -\infty]{} 0$ and $E(t)u \xrightarrow[t \to +\infty]{} u$.*

In particular, points 1 and 2 imply that $E(t)E(s) = E(s)E(t)$ for all s, t and if $s \leq t$, $E(s)E(t) = E(s)$.

Also: $\forall u \in \mathcal{H}, \forall t \in \mathbb{R},\ (E(t)u|u) = ||E(t)u||^2 \geq 0$ (or with 2 and letting s tend to $-\infty$ for a fixed t).

Remark 6.1.2. *The concept of a spectral family is analogous to the cumulative distribution function of a random variable in probabilities.*

Example 6.1.3. *Let $M \subset \mathbb{R}^d$ be measurable and $g : M \to \mathbb{R}$ be measurable. We define $M(t) = \{x \in M \mid g(x) \leq t\}$. Then $M(t)$ increases towards M in terms of inclusion. We then define for $u \in L^2(M)$ and $t \in \mathbb{R}$, $E(t)u = \chi_{M(t)}u$. Then, $E : t \mapsto E(t)$ is a spectral family.*

Example 6.1.4. *If T is a self-adjoint operator, with a discrete spectrum and such that for all $u \in \mathcal{H}$, $(Tu|u) \geq C||u||^2$, then there exists a sequence λ_i of real numbers increasing towards infinity and an orthonormal basis $\{u_i\}_{i \in \mathbb{N}}$ of $\mathcal{H}$ such that*

$$\forall u \in \mathcal{H},\ Tu = \sum_{i=0}^{+\infty} \lambda_i (u|u_i) u_i.$$

This resembles the spectral theorem for self-adjoint compact operators. We then define for all $t \in \mathbb{R}$, $E(t)$ as the orthogonal projector onto Vect$\{u_0, \dots u_j | \lambda_j \leq t\}$. Then $t \mapsto E(t)$ is a spectral family.

6.2 Spectral Theorem

Let $u, v \in \mathcal{H}$. By the polarization identity, the function $F_{u,v}(\lambda) = (E(\lambda)u|v)$ is a complex linear combination of four right-continuous, non-decreasing functions at every point:

$$F_{u,v}(\lambda) = \frac{1}{4}\left(\|E(\lambda)(u+v)\|^2 - \|E(\lambda)(u-v)\|^2 + \mathrm{i}\,\|E(\lambda)(u+\mathrm{i}v)\|^2 - \mathrm{i}\,\|E(\lambda)(u-\mathrm{i}v)\|^2\right),$$

and we note this expression as $F_{u,v}(\lambda) = \alpha_1 F_1(\lambda) + \cdots + \alpha_4 F_4(\lambda)$. According to the Stieljes integration theory, there exist four Borel measures $\mu_1, \dots, \mu_4$ corresponding to F_i such that for any function ϕ in $\mathcal{L}^1(\mathbb{R}, \mu_1 + \cdots + \mu_4)$,

$$\int_{\mathbb{R}} \phi(\lambda) \mathrm{d}F_{u,v}(\lambda) = \alpha_1 \int_{\mathbb{R}} \phi(\lambda) \mathrm{d}\mu_1 + \cdots + \alpha_4 \int_{\mathbb{R}} \phi(\lambda) \mathrm{d}\mu_4.$$

The measures μ_i depend on u and v, and, by the normalization property of spectral families, each μ_i is a finite measure. Indeed, we have $\mu_1(\mathbb{R}) \leq \|u+v\|^2, \dots, \mu_4(\mathbb{R}) \leq \|u - \mathrm{i}v\|^2$.

Example 6.2.1. *Let's revisit the second example from the previous section. If $u \in \mathcal{H}$, then $F_{u,u}(\lambda) = (E(\lambda)u|u)$. If $u = u_0$, then for $\lambda < \lambda_0$, $F_{u_0,u_0}(\lambda) = 0$ and for $\lambda \geq \lambda_0$, $F_{u_0,u_0}(\lambda) = ||u_0||^2 = 1$. Therefore, $\mathrm{d}F_{u_0,u_0} = \delta_{\lambda_0}$. If $u = au_0 + bu_1$, then $\mathrm{d}F_{u,u} = |a|^2\delta_{\lambda_0} + |b|^2\delta_{\lambda_1}$. More generally, if $u = \sum_{i=0}^{+\infty} a_i u_i$ with $\sum |a_i|^2 < +\infty$, then $\mathrm{d}F_{u,u} = \sum_{i=0}^{+\infty} |a_i|^2 \delta_{\lambda_i}$.*

We can now state the spectral theorem for self-adjoint operators.

Theorem 6.2.2 (Spectral Theorem for Bounded Operators)**.** *Let T be a self-adjoint operator. There exists a unique spectral family $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ such that*

$$T = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda) = \int_{\sigma(T)} \lambda \, \mathrm{d}E(\lambda)$$

where, for all $u, v \in \mathcal{H}$,

$$(Tu|v) = \int_{\sigma(T)} \lambda \, \mathrm{d}F_{u,v}(\lambda).$$

Proof : We outline the main steps of the construction.

We start by defining for $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, and $u \in \mathcal{H}$, $F(z) = (R_z(T)u|u)$. Then F is holomorphic in the upper complex half-plane, and we verify that $\operatorname{Im} F(z) > 0$. It is therefore a Herglotz function that satisfies the inequality

$$|F(z)| \leq \frac{C}{|\operatorname{Im} z|}.$$

We can thus associate it with a positive Borel measure of finite mass, with a distribution function w_u such that

$$F(z) = \int_{-\infty}^{+\infty} \frac{1}{z - \lambda} dw_u(\lambda).$$

Through polarization, we then obtain a complex Borel measure $\mathrm{d}w_{u,v}$ that similarly represents $(R_z(T)u|v)$ for all $u, v \in \mathcal{H}$. Moreover, by harmonic analysis results,

$$w_{u,v}(\lambda) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{-\infty}^{\lambda+\delta} ((R_{s-i\varepsilon}(T) - R_{s+i\varepsilon})u, v) \mathrm{d}s.$$

Now, the map $(u, v) \mapsto w_{u,v}(\lambda)$ is a continuous sesquilinear form, so for any $\lambda \in \mathbb{R}$, there exists a unique operator $E(\lambda) \in \mathcal{L}(\mathcal{H})$ such that

$$w_{u,v}(\lambda) = (E(\lambda)u|v).$$

We then demonstrate that $\lambda \mapsto E(\lambda)$ is a spectral family that satisfies the desired representation formula for T. □

6.3 Functional Calculus

If $\phi : \mathbb{R} \to \mathbb{C}$ is a locally bounded Borel map on $\mathbb{R}$ and T is a self-adjoint operator, we can define the operator $\phi(T)$ as follows:

$$\forall u, v \in \mathcal{H},\ (\phi(T)u|v) = \int_{\sigma(T)} \phi(\lambda) \mathrm{d}F_{u,v}(\lambda),$$

where $F_{u,v}$ comes from the spectral family associated with T via the spectral theorem. This allows the development of a functional calculus on self-adjoint operators.
Note that if ϕ takes real values, then $\phi(T)$ is also self-adjoint. Then we have the following property.

Proposition 6.3.1. *Let f and g be two bounded Borel functions and T a self-adjoint operator. For all $u, v \in \mathcal{H}$,*

$$(f(T)u|g(T)v) = \int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)} \mathrm{d}F_{u,v}(\lambda),$$

where $F_{u,v}(\lambda) = (E(\lambda)u|v)$ with E being the spectral family associated with T.

Proof : This is demonstrated by considering f and g as indicator functions of Borel sets, then by linear combinations of such functions (step functions), and eventually passing to the limit. □

An initial application of functional calculus is the following formula for the resolvent of a self-adjoint operator.

Proposition 6.3.2. *Let T be a self-adjoint operator. Let $z \in \mathbb{C}$, $z \notin \sigma(T)$. Then*

$$R_z(T) = (z - T)^{-1} = \int_{\mathbb{R}} \frac{1}{z - \lambda} \mathrm{d}E(\lambda)$$

where E is the spectral family associated with T. Furthermore,

$$||(z - T)^{-1}|| \leq \frac{1}{\mathrm{dist}(z, \sigma(T))}.$$

Proof : The first point follows immediately from the definition of functional calculus. Then, for $u \in \mathcal{H}$,

$$\begin{aligned} ||(z - T)^{-1}u||^2 &= ((z - T)^{-1}u|(z - T)^{-1}u) \\ &= \int_{\sigma(T)} (z - \lambda)^{-1}\overline{(z - \lambda)^{-1}} \mathrm{d}(E(\lambda)u|u) \\ &= \int_{\sigma(T)} |(z - \lambda)|^{-2} \mathrm{d}(E(\lambda)u|u) \\ &\leq \sup_{\lambda \in \sigma(T)} |z - \lambda|^{-2} \int_{\mathbb{R}} \mathrm{d}(E(\lambda)u|u) = \frac{1}{(\mathrm{dist}(z, \sigma(T)))^2} ||u||^2. \end{aligned}$$

□

The spectral theorem also allows us to define the notion of a spectral projector on a Borel set B in $\mathbb{R}$ using the formula:

$$E_B = \mathbb{1}_B(T).$$

In particular, if B is an interval and if E is the spectral family associated with T, let's denote

$$E_{(a,b)} = E(b^-) - E(a^+) \quad \text{and} \quad E_{[a,b]} = E(b^+) - E(a^-).$$

Proposition 6.3.3 (Stone's formula.)**.** *Let T be a self-adjoint operator. For all $a < b$,*

$$s - \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_a^b (R_{s-i\varepsilon}(T) - R_{s+i\varepsilon}(T))\mathrm{d}s = \frac{1}{2}\left(E_{[a,b]} + E_{(a,b)}\right).$$

Proof : For a complete and detailed proof, see [2], Theorem 2.13, page 37. □

Using the spectral theorem and the functional calculus it induces, we can define for $t \in \mathbb{R}$ and T a self-adjoint operator, the unitary operator $U(t) = \mathrm{e}^{\mathrm{i}tT}$. Let's summarize the properties of this operator.

Proposition 6.3.4.

1. *For any $t \in \mathbb{R}$, $U(t)$ is unitary, and if $s, t \in \mathbb{R}$, $U(t+s) = U(t)U(s)$.*
2. *If $\psi \in \mathcal{H}$ then $U(t)\psi \underset{t \to t_0}{\longrightarrow} U(t_0)\psi$.*
3. *If $\psi \in \mathcal{H}$, then $\frac{U(t)\psi - \psi}{t} \underset{t \to 0}{\longrightarrow} \mathrm{i}T\psi$.*

Proof : See Theorem VIII.7 in [6]. □

The unitary operator $U(t)$ allows to solve the Schrödinger equation:

$$\begin{cases} \partial_t \psi &= \mathrm{i}T\psi \\ \psi|_{t=0} &= \psi_0 \end{cases} \quad \text{with } \psi_0 \in D(T).$$

Indeed, for any $t \geq 0$, $\psi(t) = U(t)\psi_0$.