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Random operators theory and application to Anderson localization

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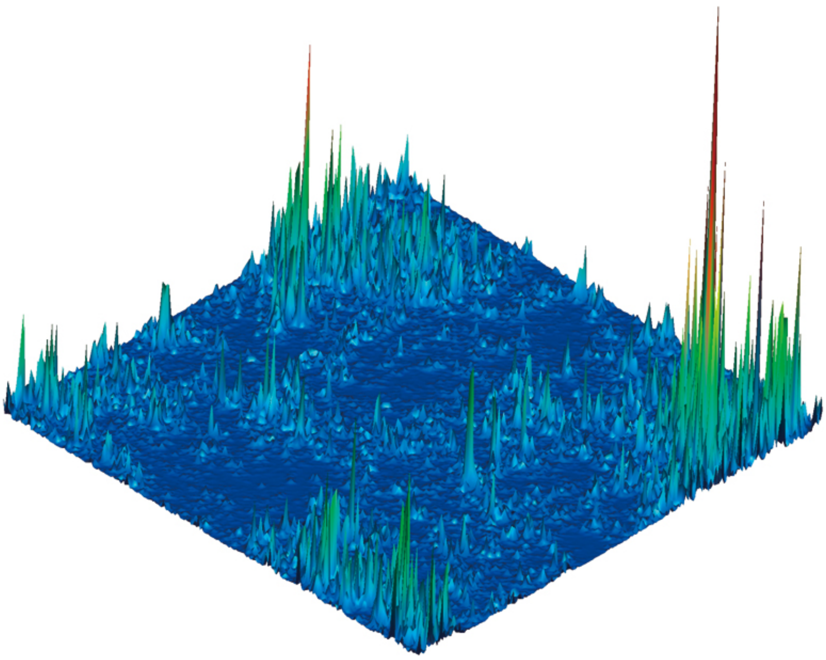

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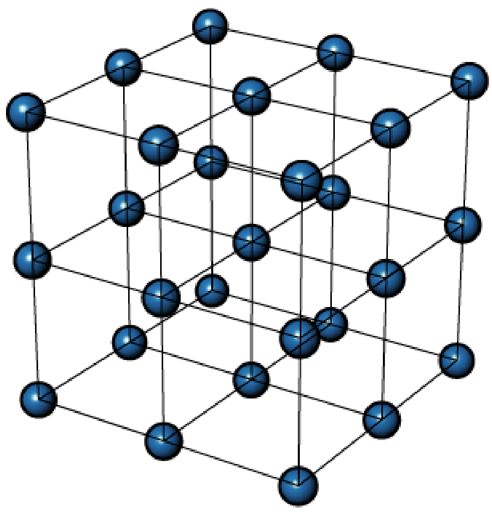
Contents

Chapter 1

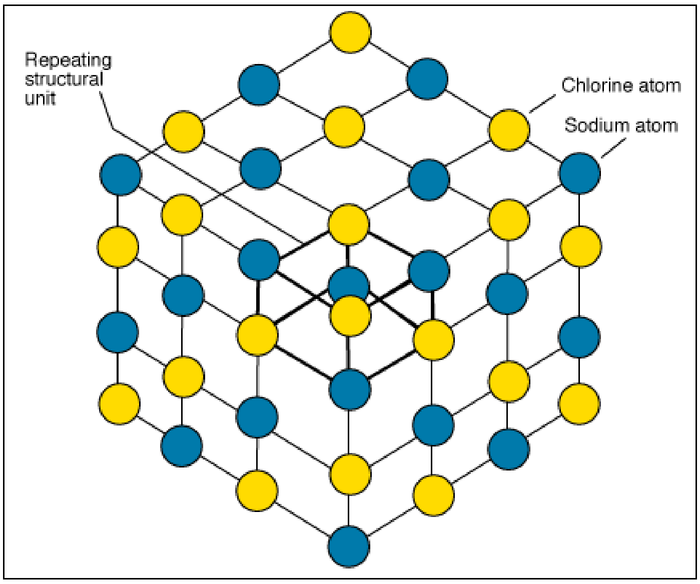
Anderson model

1.1 The physical phenomenon

In an ideal crystal, made up of atoms arranged on a periodic lattice, an electron can move without encountering any obstacle: if the crystal has an energy taking certain values above a minimal energy, it then behaves as an electrical conductor. These energies correspond to what is called the spectrum of the crystal and take values in given intervals, somewhat like the lines of the hydrogen atom spectrum, with thickness added.

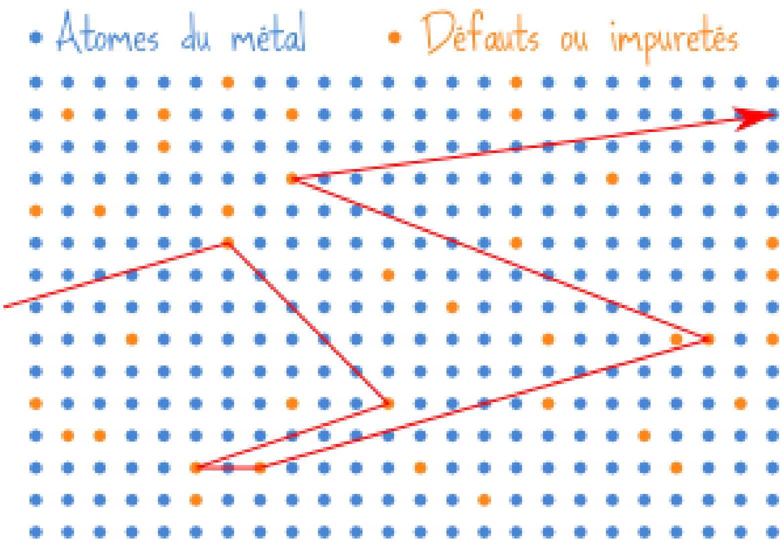


However, in nature, ideal crystals do not exist; they always contain defects. These defects or impurities can be of different kinds. For example, one can observe the presence of ionized atoms in the lattice forming the crystal or, in the case where the crystal is not made up of identical atoms but results from an alloy of several materials, it may happen that the lattice is no longer perfectly periodic, but that here and there an atom is not located at the correct position. Finally, some atoms are sometimes slightly displaced with respect to their ideal position on the periodic lattice. In all these cases, the physical properties of the crystal are modified.

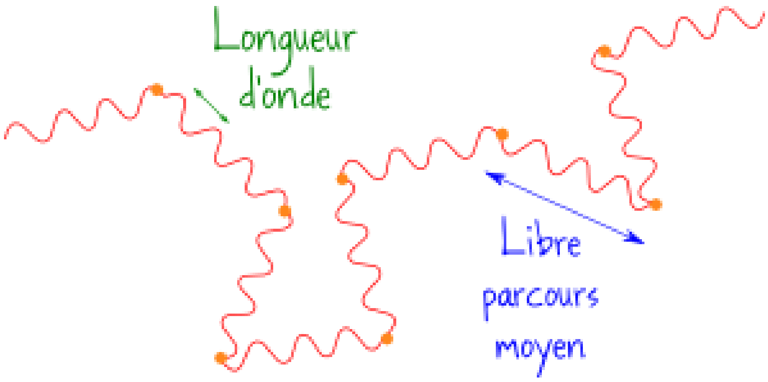


How can one then model these impurities in a crystal and their impact on electronic transport? The first to propose a model explaining the effects of disorder on the quantum behavior of electrons in an imperfect crystal was the American physicist Philip Warren Anderson in a seminal 1958 paper. By introducing random terms into the Schrödinger equation (the equation governing the quantum behavior of electrons in the crystal), two new phenomena were highlighted: Anderson localization and the Anderson transition for three-dimensional crystals. These discoveries earned him the Nobel Prize in Physics in 1977, jointly with Nevill Mott and John van Vleck.

The phenomenon of Anderson localization can be stated as follows: at a fixed energy of the spectrum, beyond a certain amount of disorder in the crystal, the electron ceases to move freely through it and remains confined in a localized region. The crystal ceases to be a conductor and becomes an insulator. To explain this phenomenon, one must recall that in quantum mechanics, an electron can just as well be viewed as a particle endowed with mass as an associated wave: this is the wave–particle duality discovered by Louis de Broglie in 1924. Thus, at each collision of the electron with an impurity of the crystal, its associated wave is scattered.



The "mean free path" is defined as the average distance traveled by the electron between two collisions. One might expect that as disorder increases, the mean free path decreases continuously. But this is not what happens. After a certain critical amount of impurities, the diffusion of the electron stops abruptly. This sudden halt occurs when the mean free path becomes shorter than the wavelength of the electron: if the wave is scattered before even completing a first period, it can no longer truly be regarded as a wave...



The phenomenon of localization goes beyond the framework of quantum mechanics. It can be observed in other situations where a wave propagates in a disordered medium. This may be the case for light waves, microwaves, or acoustic waves.

From a mathematical point of view, Anderson localization is now a relatively well-understood phenomenon. The first mathematically rigorous proofs date back to the 1970s with the work of Goldsheid, Molchanov, and Pastur, followed by that of Kunz and Souillard. These first rigorous results all concerned one-dimensional crystal models. From there, the study of random operators became a very active research area. As early as 1983, Fröhlich and Spencer, by introducing multiscale analysis, succeeded in obtaining a first proof of Anderson localization for crystals of arbitrary dimension, when their energy is close to the minimal energy. This mathematical technique was subsequently widely used to obtain various results related to the Anderson model and other models in condensed matter physics. In 2001, Damanik, Sims, and Stolz proved that Anderson localization occurs at all energies of the spectrum for a weakly disordered one-dimensional crystal governed by the Anderson model, thus completing the study of this model in dimension 1. It remains today to prove an important and very difficult conjecture asserting that Anderson localization also occurs at all energies for a disordered two-dimensional crystal. In dimension 2 and higher, we only know that Anderson localization manifests itself for energies close to the minimal energy; we know nothing for the other energies.

The second phenomenon discovered after the introduction of the Anderson model is the existence of a transition between an insulating state and a conducting state for a three-dimensional crystal at a certain critical energy, regardless of the amount of impurities in the crystal. For one-dimensional and two-dimensional crystals, this transition does not exist: at any energy, Anderson localization appears for sufficiently large disorder. Let us specify that this is well proved in dimension 1 but remains a conjecture in dimension 2. Mathematical research on the proof of this Anderson transition is very active, but has not yet led to rigorous results for the Anderson model. Initial results by Germinet, Klein, and Schencker or by Aizenmann and Warzel have highlighted this transition for other random models, but the techniques employed do not yet seem sufficient to obtain a rigorous proof of the existence of the Anderson transition in dimensions higher than 3.

It should be specified here that the Anderson mathematical model does not take into account the interactions between the electrons moving in the crystal and those associated with the atoms forming the lattice. A priori, the electrons circulating in the crystal interact with the electrons of the lattice, and the latter are themselves modified by the presence of the former. Taking these interactions into account could lead to results very different from those cited above, such as, for example, the appearance of the insulator–conductor transition in dimensions 1 and 2 and not only in dimensions higher than 3.

1.2 Mathematical modeling

1.2.1 Reminders of quantum mechanics

Let us first recall that in quantum mechanics, the state of a system of N particles is described by its wave function

$$\psi \; : \; \begin{array}{ccc} \mathbb{R}^{3N} \times \mathbb{R} & \to & \mathbb{C} \\ (x,t) & \mapsto & \psi(x,t) \end{array},$$

with $\psi \in L^2(\mathbb{R}^{3N} \times \mathbb{R})$. The probability density of presence of the system at time t is then given by $x \mapsto |\psi(x,t)|^2$, *i.e.*, if A is a Borel subset of $\mathbb{R}^{3N}$,

$$\mathsf{P}((x_1, \ldots, x_N) \in A) = \int_A |\psi(x_1, \ldots, x_N, t)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N.$$

Moreover,

$$\forall t \in \mathbb{R}, \; \int_{\mathbb{R}^{3N}} |\psi(x_1, \ldots, x_N, t)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N = 1.$$

The evolution of the system is then described through the Schrödinger equation as we have already seen in the previous chapter. If the classical energy of the system is given by the Hamiltonian $H(x_1, \ldots, x_N, p_1, \ldots, p_N)$, one defines by (Weyl) quantization the self-adjoint operator

$$\hat{H} = H(x_1, \ldots, x_N, -\mathrm{i}\hbar\partial_{x_1}, \ldots, -\mathrm{i}\hbar\partial_{x_N}).$$

For example, if $H(x,p) = \frac{|p|^2}{2m} + V(x_1, \ldots, x_N)$, then $\hat{H} = -\frac{\hbar^2}{2m}\Delta + V$. The Laplacian thus corresponds to the kinetic energy of the system and the operator of multiplication by V to the potential energy of the system, that is, its interaction with its environment.
Moreover, any quantity associated with the system that can be measured is called an observable. For example, energy is one, but so is the position x of a particle, even if one prefers to it the probability of being in a Borel set A defined above.

The postulates of quantum mechanics can be summarized as follows:

1. The configuration space of a quantum system is a Hilbert space $(\mathcal{H}, (\cdot|\cdot))$.
2. The possible states of the system are the elements of $\mathcal{H}$ of norm 1.
3. An observable $\mathfrak{h}$ of a quantum system is represented by a linear self-adjoint operator H on $\mathcal{H}$.
4. If the system is in the state ψ, the value of the measurement $\mathfrak{h}$ at time t is the real number
$$\mathbb{E}_\psi = (\psi|H\psi).$$
5. Since this expectation is not always finite, in general H is only defined on a dense subspace $D(H)$ of $\mathcal{H}$.

1.2.2 The Anderson model

We wish to study the evolution of an electron in a crystal. If the crystal is perfect, the atoms are distributed on a periodic lattice, for example $\mathbb{Z}^d$. At the point $x \in \mathbb{R}^d$, the electron experiences a potential of the form $qf(x-i)$ due to an atom of charge q located at $i \in \mathbb{Z}^d$. The total potential experienced by the electron in the crystal is therefore

$$V(x) = \sum_{i \in \mathbb{Z}^d} qf(x-i).$$

There are several ways to model disorder in a crystal. We begin by presenting Anderson's idea. To model imperfections in the crystal, one can consider the charge of the atoms in the crystal as a random variable. This leads to considering a potential experienced at the point x of the form

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega) f(x-i),$$

where the q_i are *i.i.d.* random variables on a probability space $(\Omega, \mathcal{A}, \mathsf{P})$. A priori, the q_i take only finitely many values, but we will see later that in order to carry out the mathematical study of the Schrödinger operator associated with this potential, it may be simpler to consider the case where the q_i have a continuous distribution.

This idea leads to introducing a random family of Schrödinger operators:

$$\forall\omega\in\Omega,\ H_\omega^A = -\Delta_d + \sum_{i\in\mathbb{Z}^d} q_i(\omega) f(x-i),$$

acting on the Sobolev space $H^2(\mathbb{R}^d)$, where f is a function supported in the unit cube $[0,1]^d$. The family of operators $\{H_\omega^A\}_{\omega\in\Omega}$ is called the **continuous Anderson model**.
In particular, in dimension $d=1$, for all $\omega\in\Omega$, for all $u\in H^2(\mathbb{R})$ and for all $x\in\mathbb{R}$,

$$(H_\omega^A u)(x) = -u''(x) + \sum_{i\in\mathbb{Z}} q_i(\omega) f(x-i) u(x).$$

The discrete analogue of this model is given by

$$\forall n\in\mathbb{Z}^d,\ (h_\omega u)_n = -\sum_{|m-n|=1} u_m + q_n(\omega) u_n,$$

acting on $\ell^2(\mathbb{Z}^d)$. In particular, for $d=1$, for all $\omega\in\Omega$, for all $u\in\ell^2(\mathbb{Z})$ and for all $n\in\mathbb{Z}$,

$$(h_\omega u)_n = -(u_{n-1}+u_{n+1}) + q_n(\omega) u_n.$$

There are other ways to model imperfections in a crystal. For example, one can consider the random displacement model:

$$\forall\omega\in\Omega,\ H_\omega^D = -\Delta_d + \sum_{i\in\mathbb{Z}^d} f(x-i-x_i(\omega)),$$

where the $x_i(\omega)\in]-\frac{1}{2},\frac{1}{2}[^d$ are *i.i.d.* random variables. For this type of model, one generally assumes that they follow a Poisson distribution.
Another way to conceive disorder in a crystal is to consider quasi-periodic models. For $\alpha\in\mathbb{R}\setminus\mathbb{Q}$, $\phi\in\mathbb{T}$ and $Q:\mathbb{T}\to\mathbb{R}$ continuous, one sets

$$\forall x\in\mathbb{R},\ q_{\phi,\alpha}(x) = Q(\phi+\alpha x).$$

This potential is called a quasi-periodic potential of frequency α and phase ϕ. One then defines

$$\forall\alpha\in\mathbb{R}\setminus\mathbb{Q},\ \forall\phi\in\mathbb{T},\ H_{\alpha,\phi} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q_{\alpha,\phi}(x),$$

acting on $H^2(\mathbb{R})$. Similarly, in the discrete case: $\forall\alpha\in\mathbb{R}\setminus\mathbb{Q},\ \forall\phi\in\mathbb{T},\ \forall u\in\ell^2(\mathbb{Z}), \forall n\in\mathbb{Z}$,

$$(h_{\alpha,\phi} u)_n = -(u_{n-1}+u_{n+1}) + q_{\alpha,\phi}(n) u_n.$$

Chapter 2

Spectral types and dynamics

2.1 The spectral theorem for self-adjoint operators

We are going to generalize to the setting of unbounded operators on a Hilbert space the classical result asserting that any real symmetric matrix can be diagonalized in an orthonormal basis.
A convenient way to state this theorem for matrices is to write that for any real symmetric matrix $A \in M_n(\mathbb{R})$, there exist real numbers $\lambda_1, \dots, \lambda_n$ and orthogonal projectors $P_1, \dots, P_n$ such that:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

It is this formulation that we are going to generalize to infinite dimension, by transforming the sum into an integral with respect to projector-valued measures.

Definition 2.1.1. *A spectral family (or resolution of the identity) on $\mathcal{H}$ is a mapping $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ such that:*

1. *For all $t \in \mathbb{R}$, $E(t)$ is an orthogonal projection,* i.e. $E(t)^2 = E(t)$ *and* $E(t)^* = E(t)$.
2. $\forall s \leq t$, $E(s) \leq E(t)$, i.e. $\forall u \in \mathcal{H}, (E(s)u|u) \leq (E(t)u|u)$.
3. *Strong right continuity:* $\forall u \in \mathcal{H},\ E(t+\varepsilon)u \xrightarrow[\varepsilon \to 0^+]{} E(t)u.$
4. $\forall u \in \mathcal{H},\ E(t)u \xrightarrow[t \to -\infty]{} 0$ *and* $E(t)u \xrightarrow[t \to +\infty]{} u.$

Let $u, v \in \mathcal{H}$. We consider the function $F_{u,v}(\lambda) = (E(\lambda)u|v)$. It is then a complex linear combination of four right-continuous increasing functions at every point:

$$F_{u,v}(\lambda) = \frac{1}{4}\left(\|E(\lambda)(u+v)\|^2 - \|E(\lambda)(u-v)\|^2 + \mathrm{i}\,\|E(\lambda)(u+\mathrm{i}v)\|^2 - \mathrm{i}\,\|E(\lambda)(u-\mathrm{i}v)\|^2 \right),$$

and we denote this expression by $F_{u,v}(\lambda) = \alpha_1 F_1(\lambda) + \cdots + \alpha_4 F_4(\lambda)$. According to Stieltjes integration theory, there therefore exist four Borel measures $\mu_1, \dots, \mu_4$ corresponding to the F_i such that one can define, for any function ϕ in $\mathcal{L}^1(\mathbb{R}, \mu_1 + \cdots + \mu_4)$,

$$\int_{\mathbb{R}} \phi(\lambda)\mathrm{d}F_{u,v}(\lambda) = \alpha_1 \int_{\mathbb{R}} \phi(\lambda)\mathrm{d}\mu_1 + \cdots + \alpha_4 \int_{\mathbb{R}} \phi(\lambda)\mathrm{d}\mu_4.$$

The measures μ_i depend on u and v and, by the normalization property of spectral families, each μ_i is a finite measure. Indeed, one has $\mu_1(\mathbb{R}) \leq \|u+v\|^2, \dots, \mu_4(\mathbb{R}) \leq \|u - \mathrm{i}v\|^2$.

We can now state the spectral theorem for self-adjoint operators.

Theorem 2.1.2 (Spectral theorem for self-adjoint operators.)**.** *Let T be a self-adjoint operator. There exists a unique spectral family $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ such that*

$$T = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda) = \int_{\sigma(T)} \lambda \, \mathrm{d}E(\lambda)$$

in the sense that, for all $u, v \in \mathcal{H}$,

$$(Tu|v) = \int_{\sigma(T)} \lambda \, \mathrm{d}F_{u,v}(\lambda).$$

If $\phi : \mathbb{R} \to \mathbb{C}$ is a locally bounded Borel function on $\mathbb{R}$ and if T is a self-adjoint operator, one can define the operator $\phi(T)$ by:

$$D(\phi(T)) = \{u \in \mathcal{H} \mid \int_{\mathbb{R}} |\phi(\lambda)|^2 \mathrm{d}F_{u,u}(\lambda) < +\infty\}$$

and

$$\forall u, v \in \mathcal{H}, \; (\phi(T)u|v) = \int_{\sigma(T)} \phi(\lambda) \mathrm{d}F_{u,v}(\lambda),$$

where $F_{u,v}$ comes from the spectral family associated with T by the spectral theorem. This allows one to develop a functional calculus for self-adjoint operators.
Note that if ϕ is real-valued, then $\phi(T)$ is also self-adjoint.

A first application of the functional calculus is the following formula for the resolvent of a self-adjoint operator.

Proposition 2.1.3. *Let T be a self-adjoint operator. Let $z \in \mathbb{C}$, $z \notin \sigma(T)$. Then*

$$R_z(T) = (z - T)^{-1} = \int_{\mathbb{R}} \frac{1}{z - \lambda} \mathrm{d}E(\lambda)$$

where E is the spectral family associated with T. Moreover,

$$||(z - T)^{-1}|| \leq \frac{1}{\mathrm{dist}(z, \sigma(T))}.$$

Proof : The first point follows immediately from the definition of the functional calculus. Then, if $u \in \mathcal{H}$,

$$\begin{aligned}
||(z - T)^{-1}u||^2 &= ((z - T)^{-1}u|(z - T)^{-1}u) \\
&= \int_{\sigma(T)} (z - \lambda)^{-1}\overline{(z - \lambda)^{-1}} \mathrm{d}(E(\lambda)u|u) \\
&= \int_{\sigma(T)} |(z - \lambda)|^{-2} \mathrm{d}(E(\lambda)u|u) \\
&\leq \sup_{\lambda \in \sigma(T)} |z - \lambda|^{-2} \int_{\mathbb{R}} \mathrm{d}(E(\lambda)u|u) = \frac{1}{(\mathrm{dist}(z, \sigma(T)))^2} ||u||^2.
\end{aligned}$$

□

The spectral theorem also makes it possible to define the notion of a spectral projector on a Borel set B of $\mathbb{R}$ via the formula:

$$E_B = \mathbb{1}_B(T).$$

In particular, if B is an interval and if E is the spectral family associated with T, let us note

$$E_{(a,b)} = E(b^-) - E(a^+) \quad \text{and} \quad E_{[a,b]} = E(b^+) - E(a^-).$$

2.2 Spectral types

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Let $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ be the spectral family associated with T. For $u \in \mathcal{H}$, we will denote throughout by μ_u the spectral measure associated with T and u, defined by:

$$\forall t \in \mathbb{R},\ \mu_u((-\infty, t)) = (E(t)u|u).$$

This is the measure associated with the function $F_{u,u}$ defined previously.

2.2.1 Spectrum, point spectrum, and spectral measures

We begin by giving a characterization of the spectrum in terms of spectral measures.

Proposition 2.2.1. *The following are equivalent:*

1. $\lambda \in \sigma(T)$.
2. *For every* $\varepsilon > 0$, $E(\lambda+\varepsilon) - E(\lambda-\varepsilon) = E_{(\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0$.
3. *For every* $\varepsilon > 0$, *there exists* $u = u_\varepsilon \in D(T)$ *such that* $\mu_u((\lambda-\varepsilon, \lambda+\varepsilon]) > 0$.

Proof : Assume 1 and prove 2. Let $\lambda \in \sigma(T)$. Suppose, by contradiction, that there exists $\varepsilon > 0$ such that $E(\lambda+\varepsilon) - E(\lambda-\varepsilon) = 0$. By Weyl's criterion, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(T)$ such that $||u_n|| = 1$ for all n and $(T-\lambda)u_n \to 0$. Then

$$||(T-\lambda)u_n||^2 = \int_{-\infty}^{+\infty} |\lambda - t|^2 \mathrm{d}\mu_{u_n}(t).$$

Now, E is constant on $(\lambda-\varepsilon, \lambda+\varepsilon]$, hence for all $u \in \mathcal{H}$, the function $t \mapsto \mu_{u_n}((-\infty, t])$ vanishes on this interval. Therefore,

$$\int_{-\infty}^{+\infty} |\lambda - t|^2 \mathrm{d}\mu_{u_n}(t) = \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon]} |\lambda - t|^2 \mathrm{d}\mu_{u_n}(t) \geq \varepsilon^2 \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon]} \mathrm{d}\mu_{u_n}(t) = \varepsilon^2 ||u_n||^2.$$

Hence $||(T-\lambda)u_n||^2 \geq \varepsilon^2 ||u_n||^2 = \varepsilon^2$, which contradicts $(T-\lambda)u_n \to 0$. This proves 2.

Assume 2 and prove 1. Let $\lambda \in \mathbb{R}$ such that for every $\varepsilon > 0$, $E(\lambda+\varepsilon) - E(\lambda-\varepsilon) \neq 0$. Then, for every $n \geq 1$, $E(\lambda+\frac{1}{n}) - E(\lambda-\frac{1}{n}) \neq 0$. Let $u_n \in \mathrm{Im}\ \left(E(\lambda+\frac{1}{n}) - E(\lambda-\frac{1}{n})\right)$ with $||u_n|| = 1$. Then $E_{(\lambda-\frac{1}{n},\lambda+\frac{1}{n})]}u_n = u_n$ for all $n \geq 1$, and since $\mu_{u_n}(\mathbb{R} \setminus (\lambda - \frac{1}{n}, \lambda + \frac{1}{n})]) = 0$, we have

$$||(T-\lambda)u_n||^2 = \int_{(\lambda-\frac{1}{n},\lambda+\frac{1}{n})]} |\lambda - t|^2 \mathrm{d}\mu_{u_n}(t) \leq \frac{1}{n^2}.$$

By Weyl's criterion, $\lambda \in \sigma(T)$.

Assume 2. Let $u \in \mathrm{Im}\ (E(\lambda+\varepsilon) - E(\lambda-\varepsilon))$ with $||u|| = 1$. Then $\mu_u(\mathbb{R} \setminus (\lambda-\varepsilon, \lambda+\varepsilon]) = 0$, hence $\mu_u((\lambda-\varepsilon, \lambda+\varepsilon]) > 0$, which gives 3.

If there exists $\varepsilon > 0$ such that $E(\lambda+\varepsilon) - E(\lambda-\varepsilon) = 0$, then for all $u \in \mathcal{H}$, $\mu_u((\lambda-\varepsilon, \lambda+\varepsilon]) = 0$. Thus 3 implies 2 by contraposition.

□

Corollary 2.2.2. *We have:*

$$\sigma(T) = \overline{\bigcup_{u\in\mathcal{H}} \mathrm{supp}\ \mu_u} \neq \emptyset.$$

Recall that the point spectrum of T is the set of its eigenvalues:

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is not injective } \}.$$

Proposition 2.2.3. *The following are equivalent:*

1. $\lambda \in \sigma_p(T)$.
2. $E_{\{\lambda\}} = E(\lambda) - E(\lambda^-) \neq 0$ *(discontinuity).*
3. *There exists* $u \in D(T)$ *such that* $\mu_u(\{\lambda\}) > 0$.

Proof : Assertions 2 and 3 are equivalent since $\mu_u(\{\lambda\}) = (E_{\{\lambda\}}u|u) = ||E_{\{\lambda\}}u||^2$.

Assume 2 and let $u \in \operatorname{Im} E_{\{\lambda\}}$, $u \neq 0$. Then

$$||(T-\lambda)u||^2 = \int_{\mathbb{R}} |\lambda - t|^2 \mathrm{d}\mu_u(t) = 0$$

since supp $\mu_u = \{\lambda\}$. Thus $Tu = \lambda u$ with $u \neq 0$, hence $\lambda \in \sigma_p(T)$.

The converse follows from the same computation, starting this time from $0 = ||(T - \lambda)u||^2$.

□

Remark 2.2.4. *This leads to defining the multiplicity of an eigenvalue* $\lambda \in \sigma_p(T)$ *as the dimension of* $\operatorname{Im} E_{\{\lambda\}}$. *This multiplicity may be infinite.*

Corollary 2.2.5. *If* λ *is an isolated point of* $\sigma(T)$*, then* $\lambda \in \sigma_p(T)$.

Proof : By definition, there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \cap \sigma(T) = \{\lambda\}$. Hence $E_{\{\lambda\}} \neq 0$. Indeed, $E_{(\lambda-\varepsilon,\lambda)} = 0$ since $(\lambda - \varepsilon, \lambda) \cap \sigma(T) = \emptyset$. Similarly, $E(\lambda + \varepsilon) - E(\lambda) = 0$. Therefore $E(\lambda) - E(\lambda^-) \neq 0$ and $\lambda \in \sigma_p(T)$.

□

More precisely, if λ is an isolated point of $\sigma(T)$, let $\varepsilon > 0$ be such that

$$(]\lambda - 2\varepsilon, \lambda[\cup]\lambda, \lambda + 2\varepsilon[) \cap \sigma(T) = \emptyset.$$

One can then define the Riesz projection

$$P_\lambda = \frac{1}{2\mathrm{i}\pi} \int_{C_\varepsilon} (z - T)^{-1} \mathrm{d}z$$

where C_ε is the circle $\{z \in \mathbb{C};\ |z - \lambda| = \varepsilon\}$ oriented counterclockwise. The integral is well defined since $z \mapsto (z - T)^{-1}$ is analytic in the annulus $\{z \in \mathbb{C};\ 0 < |z - \lambda| < 2\varepsilon\}$.

Proposition 2.2.6. P_λ *is the spectral projection onto the eigenspace associated with the eigenvalue* λ.

Proof : See [16], Proposition 3.2, page 42.

□

2.2.2 Essential spectrum and discrete spectrum

The last corollary of the previous section leads us to the following distinction.

Definition 2.2.7. *The essential spectrum of T is the set*

$$\sigma_{\text{ess}}(T) = \{\lambda \in \sigma(T) \mid \lambda \text{ is not isolated or is an eigenvalue of infinite multiplicity}\}.$$

The discrete spectrum of T is the set

$$\sigma_d(T) = \{\lambda \in \sigma(T) \mid \lambda \text{ is isolated and of finite multiplicity}\}.$$

Remark 2.2.8. *A real number λ belongs to the essential spectrum of T if and only if for every $\varepsilon > 0$, the rank of the spectral projection onto the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ is infinite.*

The spectrum of T is the disjoint union of its essential spectrum and its discrete spectrum. Moreover, $\sigma_d(T) \subset \sigma_p(T)$.

Example 2.2.9.

1. *If $T = \text{Id}_{\mathcal{H}}$, then $\sigma(T) = \{1\}$ and $\sigma(T) = \sigma_{\text{ess}}(T)$ if $\dim \mathcal{H} = \infty$, while $\sigma(T) = \sigma_d(T)$ if $\dim \mathcal{H} < \infty$.*
2. *If $\dim \mathcal{H} < \infty$, then $\sigma_{\text{ess}}(T) = \emptyset$.*
3. *If T is compact, then $\sigma(T) \cap (\mathbb{R} \setminus \{0\}) = \sigma_p(T) \cap (\mathbb{R} \setminus \{0\}) = \sigma_d(T) \cap (\mathbb{R} \setminus \{0\})$ and $\sigma_{\text{ess}}(T) = \{0\}$.*

Theorem 2.2.10 (Weyl criterion for the essential spectrum.). *Let T be a self-adjoint operator. Then $\lambda \in \sigma_{\text{ess}}(T)$ if and only if there exists an orthonormal family $\{u_n\}_{n\in\mathbb{N}}$ of elements of $D(T)$ such that $||(T-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0$.*

Proof : We prove the direct implication.

Case 1: if λ is an eigenvalue of infinite multiplicity, we take for $\{u_n\}$ an orthonormal sequence of eigenvectors.

Case 2: if λ is not isolated in $\sigma(T)$, there exists a sequence (λ_n) of pairwise distinct elements of $\sigma(T)$ converging to λ. Moreover, for each n, there exists $\varepsilon_n > 0$ such that the intervals $]\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n[$ are pairwise disjoint. Since $\lambda_n \in \sigma(T)$, we have $E(\lambda_n + \varepsilon_n) - E(\lambda_n - \varepsilon_n) \neq 0$. We can therefore choose $u_n \in \text{Im}\,(E(\lambda_n + \varepsilon_n) - E(\lambda_n - \varepsilon_n))$ such that $||u_n|| = 1$. Moreover,

$$(u_n|u_m) = (E_{]\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n[}u_n|E_{]\lambda_m-\varepsilon_m,\lambda_m+\varepsilon_m[}u_m) = (u_n|E_{]\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n[}E_{]\lambda_m-\varepsilon_m,\lambda_m+\varepsilon_m[}u_m) = 0$$

since $E_{]\lambda_n-\varepsilon_n,\lambda_n+\varepsilon_n[}E_{]\lambda_m-\varepsilon_m,\lambda_m+\varepsilon_m[} = 0$ because the two intervals are disjoint. Thus $\{u_n\}$ is orthonormal. Finally,

$$||(T-\lambda)u_n||^2 = \int_{\mathbb{R}} |t-\lambda|^2 \mathrm{d}\mu_{u_n}(t) \leq 2(|\lambda_n - \lambda|^2 + \varepsilon_n^2) \int_{\mathbb{R}} \mathrm{d}\mu_{u_n}(t)$$

since $\operatorname{supp} \mu_{u_n} \subset [\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n]$. The upper bound tends to 0 because $\int_{\mathbb{R}} \mathrm{d}\mu_{u_n}(t) = ||u_n||^2 = 1$.

For the converse, suppose that there exists an orthonormal family $\{u_n\}_{n\in\mathbb{N}}$ of elements of $D(T)$ such that $||(T-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0$. Then for every $\varepsilon > 0$, $\text{Im}\,(E(\lambda+\varepsilon) - E(\lambda-\varepsilon))$ is infinite-dimensional. Indeed, if this projection had finite rank, it would be compact and would map an orthonormal sequence, which converges weakly to 0, to another sequence converging weakly to 0. In particular, the sequence (μ_{u_n}) would then converge vaguely to 0 on the interval $(\lambda - \varepsilon, \lambda + \varepsilon]$. Thus we may write:

$$\begin{aligned} ||(T-\lambda)u_n||^2 &= \int_{\mathbb{R}} |t-\lambda|^2 \mathrm{d}\mu_{u_n}(t) \\ &= \int_{\mathbb{R}\setminus(\lambda-\varepsilon,\lambda+\varepsilon]} |t-\lambda|^2 \mathrm{d}\mu_{u_n}(t) + \int_{(\lambda-\varepsilon,\lambda+\varepsilon]} |t-\lambda|^2 \mathrm{d}\mu_{u_n}(t) \\ &\geq \varepsilon^2 + \int_{(\lambda-\varepsilon,\lambda+\varepsilon]} |t-\lambda|^2 \mathrm{d}\mu_{u_n}(t). \end{aligned}$$

Letting $n \to \infty$ yields the contradiction $0 \geq \varepsilon^2$.

Therefore, either $\mathrm{Im}\,(E_{\{\lambda\}})$ is infinite-dimensional and λ is an eigenvalue of infinite multiplicity, or this projection has finite rank and $\mathrm{Im}\,(E(\lambda+\varepsilon) - E(\lambda-\varepsilon))$ is still infinite-dimensional. In this case, $(]\lambda-\varepsilon, \lambda[\cup]\lambda, \lambda+\varepsilon[) \cap \sigma(T) \neq \emptyset$, so λ is not isolated. Hence $\lambda \in \sigma_{\mathrm{ess}}(T)$.

□

This Weyl criterion for the essential spectrum allows one to prove a stability result for the essential spectrum.

Theorem 2.2.11. *Let T and S be two self-adjoint operators, with S additionally assumed to be compact. Then $\sigma_{\mathrm{ess}}(T+S) = \sigma_{\mathrm{ess}}(T)$.*

Proof : We prove the first inclusion. Let $\lambda \in \sigma_{\mathrm{ess}}(T+S)$. By Weyl's criterion, there exists an orthonormal sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$||(T+S-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0.$$

Since $(u_n)_{n\in\mathbb{N}}$ is orthonormal, it converges weakly to 0, and since S is compact, $Su_n \to 0$. Thus $||(T-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0$, and by the converse of Weyl's criterion, $\lambda \in \sigma_{\mathrm{ess}}(T)$.

The second inclusion follows by applying the first one to $T+S$, which is self-adjoint by the Kato–Rellich theorem (since S is compact, hence bounded), and to $-S$, which is compact.

□

2.2.3 Pure point, absolutely continuous, and singular continuous spectra

Let μ be a Borel measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) < \infty$. Then $\mu(\{x\}) = 0$ except for at most a countable set of real numbers x.

Definition 2.2.12. *Let M be a Borel subset of $\mathbb{R}$.*

1. *We say that μ is a pure point measure if*

$$\mu(M) = \sum_{x\in M} \mu(\{x\}).$$

In particular, its support is countable.

2. *We say that μ is a continuous measure if $\forall x \in \mathbb{R},\ \mu(\{x\}) = 0$.*

3. *We say that μ is absolutely continuous with respect to Lebesgue measure if*

$$\mathrm{Leb}\,(M) = 0 \ \Rightarrow\ \mu(M) = 0.$$

4. *We say that μ is singular continuous if μ is continuous and there exists a Borel set $S \subset \mathbb{R}$ such that $\mu(S) = 0$ and $\mathrm{Leb}(\mathbb{R} \setminus S) = 0$.*

Example 2.2.13. 1. *Pure point measure: the Dirac measure at a point $a \in \mathbb{R}$ defined by $\delta_a(M) = 1$ if $a \in M$ and 0 otherwise.*

2. *Absolutely continuous measure: let $f \in L^1(\mathbb{R}, \mathrm{Leb})$. Then the measure μ defined by*

$$\forall g \in C_b^0(\mathbb{R}), \int_{\mathbb{R}} g(x)\mathrm{d}\mu(x) = \int_{\mathbb{R}} g(x)f(x)\mathrm{d}x$$

is absolutely continuous with respect to Lebesgue measure. Conversely, by the Radon–Nikodym theorem, all such measures are of this form and $f = \frac{\mathrm{d}\mu}{\mathrm{d}x}$ is the Radon–Nikodym derivative of μ.

3. *Singular continuous measure: the Cantor measure, whose distribution function is the Devil's staircase defined on the triadic Cantor set.*

The interest of these notions is that every finite Borel measure decomposes into a sum of measures having these properties.

Theorem 2.2.14 (Lebesgue–Radon–Nikodym.)**.** *Every finite Borel measure μ decomposes uniquely as*

$$\mu = \mu_{\mathrm{pp}} + \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}}$$

where μ_{pp} is a pure point measure, μ_{ac} is absolutely continuous with respect to Lebesgue measure, and μ_{sc} is singular continuous with respect to Lebesgue measure.

Proof : See [21], Section 6.10.

□

We now apply this decomposition of finite Borel measures to the spectral measures associated with a self-adjoint operator.

Let T be a self-adjoint operator on $\mathcal{H}$ and let $u \in \mathcal{H}$. Let μ_u be the spectral measure associated with T and u. It is a finite Borel measure with values in $\mathbb{C}$ a priori. We define:

1. $\mathcal{H}_{\mathrm{pp}} = \{u \in \mathcal{H} \mid \mu_u \text{ is a pure point measure}\}$.

2. $\mathcal{H}_{\mathrm{ac}} = \{u \in \mathcal{H} \mid \mu_u \text{ is absolutely continuous}\}$.

3. $\mathcal{H}_{\mathrm{sc}} = \{u \in \mathcal{H} \mid \mu_u \text{ is singular continuous}\}$.

Theorem 2.2.15. *$\mathcal{H}_{\mathrm{pp}}$, $\mathcal{H}_{\mathrm{ac}}$, and $\mathcal{H}_{\mathrm{sc}}$ are closed subspaces of $\mathcal{H}$, pairwise orthogonal, and such that*

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}.$$

Moreover, each of these subspaces is invariant under T.

Proof : The first point follows from the fact that if u and v are in $\mathcal{H}_{\mathrm{pp}}$, then there exist countable sets X and Y such that $E_X u = u$ and $E_Y v = v$. Then $E_{X \cup Y} u = E_{X \cup Y} E_X u = E_{(X \cup Y) \cap X} u = E_X u = u$, and similarly $E_{X \cup Y} v = v$. Hence $E_{X \cup Y}(u + v) = u + v$, and $X \cup Y$ is countable. The same holds for scalar multiplication. The same argument shows that the other two sets are vector subspaces. For $\mathcal{H}_{\mathrm{sc}}$ one uses the fact that if X and Y are null sets, then $X \cup Y$ is also a null set.

For the second point, let $u \in \mathcal{H}$ and write $\mu_u = \mu_{u,\mathrm{pp}} + \mu_{u,\mathrm{ac}} + \mu_{u,\mathrm{sc}}$. Let X be a countable set such that $\mu_{u,\mathrm{pp}}(\mathbb{R} \setminus X) = 0$. Choose Y such that $Y \cap X = \emptyset$, $\mathrm{Leb}(Y) = 0$, and $\mu_{u,\mathrm{sc}}(\mathbb{R} \setminus$

$Y) = 0$. Finally, set $Z = \mathbb{R} \setminus (X \cup Y)$. Then $\mathbb{R} = X \cup Y \cup Z$ is a disjoint union, and we have $\mu_{u,\mathrm{ac}} = \mu_u(Z \cap \cdot)$, $\mu_{u,\mathrm{sc}} = \mu_u(Y \cap \cdot)$, and $\mu_{u,\mathrm{pp}} = \mu_u(X \cap \cdot)$. We write:

$$u = E_{\mathbb{R}} u = E_{X \cup Y \cup Z} u = E_X u + E_Y u + E_Z u.$$

Moreover, $\mu_{E_Z u} = \chi_Z \mu_u << \text{Leb}$, hence $E_Z u \in \mathcal{H}_{\mathrm{ac}}$. Similarly, $E_Y u \in \mathcal{H}_{\mathrm{sc}}$ and $E_X u \in \mathcal{H}_{\mathrm{pp}}$. Pairwise orthogonality follows from the fact that the projections are orthogonal.

For invariance under T, let $u \in D(T)$ and associate to it μ_u. We compute μ_{Tu}. We have

$$\int_{-\infty}^{t} \mathrm{d}\mu_{Tu}(s) = (E(t)(Tu)|Tu) = (TE(t)Tu|u) = (E(id)E(\chi_{]-\infty,t]})E(id)u|u) = (E(id\, \chi_{]-\infty,t]}\, id)|u)$$

If $u_t = id\, \chi_{]-\infty,t]}\, id$, then $u_t(s) = s^2 \chi_{]-\infty,t]}(s)$. Hence

$$\int_{-\infty}^{t} \mathrm{d}\mu_{Tu}(s) = \int_{-\infty}^{t} s^2 \mathrm{d}\mu_u(s).$$

Thus $\mu_{Tu} = t^2 \mu_u$. We deduce the invariance of the three subspaces under T.

□

One can then define three spectral types that decompose the spectrum of T.

Definition 2.2.16. *Let T be a self-adjoint operator.*

1. *The pure point spectrum of T is the set $\sigma_{\mathrm{pp}}(T) = \sigma(T|_{\mathcal{H}_{\mathrm{pp}}})$.*
2. *The absolutely continuous spectrum of T is the set $\sigma_{\mathrm{ac}}(T) = \sigma(T|_{\mathcal{H}_{\mathrm{ac}}})$.*
3. *The singular continuous spectrum of T is the set $\sigma_{\mathrm{sc}}(T) = \sigma(T|_{\mathcal{H}_{\mathrm{sc}}})$.*

One then has:

$$\sigma(T) = \sigma_{\mathrm{pp}}(T) \cup \sigma_{\mathrm{ac}}(T) \cup \sigma_{\mathrm{sc}}(T)$$

but this union is not necessarily disjoint (see the examples of embedded eigenvalues in the absolutely continuous spectrum).

We say that the operator T has purely absolutely continuous spectrum (resp. pp, resp. sc) when $\mathcal{H}_{\mathrm{ac}} = \mathcal{H}$ (resp. $\mathcal{H}_{\mathrm{pp}} = \mathcal{H}$, resp. $\mathcal{H}_{\mathrm{sc}} = \mathcal{H}$). Although this does imply that $\sigma(T) = \sigma_{\mathrm{ac}}(T)$, the converse is not true in general.

Moreover, one has:

$$\sigma_{\mathrm{pp}}(T) = \overline{\sigma_p(T)}.$$

Indeed, any countable set is a countable union of singletons.

Finally, with this decomposition, one can also define the notions of continuous spectrum and singular spectrum. Set $\mathcal{H}_{\mathrm{c}} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}$; this is the space of vectors such that μ_u is a continuous measure. The space $\mathcal{H}_{\mathrm{c}}$ is again invariant under T, and we define:

$$\sigma_{\mathrm{c}}(T) = \sigma(T|_{\mathcal{H}_{\mathrm{c}}}).$$

One can also define $\mathcal{H}_{\mathrm{s}} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_{\mathrm{sc}}$ and $\sigma_{\mathrm{s}}(T) = \sigma(T|_{\mathcal{H}_{\mathrm{s}}})$, the singular spectrum of T.
We conclude this section with a stability result for the absolutely continuous spectrum. Beforehand, we introduce a definition.

Definition 2.2.17. *Let $\mathcal{H}$ be a Hilbert space and $\{u_n\}_{n\in\mathbb{N}}$ a Hilbert basis of $\mathcal{H}$. A self-adjoint, bounded, and positive operator T is said to be of* trace class *when:*

$$\mathrm{Tr}(T) = \sum_{n\in\mathbb{N}} (u_n|Tu_n) < \infty.$$

The number $\mathrm{Tr}(T)$ does not depend on the choice of the Hilbert basis, and more generally a bounded self-adjoint operator T is said to be of trace class when $\mathrm{Tr}(T) < \infty$. A trace class operator is compact.

Theorem 2.2.18 (Kato–Rosenblum)**.** *Let T and S be two self-adjoint operators with S of trace class. Then*

$$\sigma_{\mathrm{ac}}(T+S) = \sigma_{\mathrm{ac}}(T).$$

Proof : For a more complete statement giving a result on wave operators as well as for the proof, see [14], Theorem 4.4, page 542.

□

2.3 Examples of spectra

2.3.1 The discrete Laplacian in dimension 1

We introduce the discrete Laplacian in one dimension. It is the operator Δ defined on the Hilbert space $\ell^2(\mathbb{Z})$ by

$$\forall u \in \ell^2(\mathbb{Z}),\ \forall n \in \mathbb{Z},\ (\Delta u)_n = u_{n-1} + u_{n+1}.$$

The operator Δ is the discrete analogue of the second derivative.

First of all, Δ is bounded. Indeed, if $\|u\|_{\ell^2(\mathbb{Z})} \leq 1$, then $\|\Delta u\|_{\ell^2(\mathbb{Z})} \leq 2\|u\|_{\ell^2(\mathbb{Z})} \leq 2$, hence $\|\Delta\| \leq 2$. Next, let us show that Δ is self-adjoint. Let $u, v \in \ell^2(\mathbb{Z})$. Then

$$\begin{aligned}(\Delta u|v) &= \sum_{n\in\mathbb{Z}} (\Delta u)_n \overline{v_n} = \sum_{n\in\mathbb{Z}} u_{n-1}\overline{v_n} + \sum_{n\in\mathbb{Z}} u_{n+1}\overline{v_n} = \sum_{n\in\mathbb{Z}} u_n \overline{v_{n+1}} + \sum_{n\in\mathbb{Z}} u_n \overline{v_{n-1}} \\ &= \sum_{n\in\mathbb{Z}} u_n \overline{(v_{n-1}+v_{n+1})} = \sum_{n\in\mathbb{Z}} u_n \overline{(\Delta v)_n} = (u|\Delta v).\end{aligned}$$

Thus, Δ *is a bounded self-adjoint operator*.

We now compute the spectrum of Δ. To this end, we introduce the Fourier operator $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2([0,2\pi])$ defined for all $u = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{C})$ and all $x \in [0,2\pi]$ by $(\mathcal{F}u)(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx}$. We then set $S = \mathcal{F} \circ \Delta \circ \mathcal{F}^{-1}$. Let us compute S. Let $f \in L^2([0,2\pi])$. Suppose that, for all $x \in [0,2\pi]$, $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$. For all $n \in \mathbb{Z}$, $(\mathcal{F}^{-1}f)_n = \hat{f}(n)$. Then, for all $x \in [0,2\pi]$,

$$\begin{aligned}(Sf)(x) &= \sum_{n\in\mathbb{Z}} (\hat{f}(n-1)+\hat{f}(n+1))e^{inx} = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{\mathrm{i}(n+1)x} + \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{\mathrm{i}(n-1)x} \\ &= (e^{\mathrm{i}x}+e^{-\mathrm{i}x}) \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx} = (2\cos(x))f(x).\end{aligned}$$

Thus, setting for all $x \in [0,2\pi]$, $\varphi(x) = 2\cos(x)$, we have $S = M_\varphi$, where M_φ is the operator of multiplication by φ. Since $\mathcal{F}$ is a unitary transformation, we have $\sigma(\Delta) = \sigma(M_\varphi)$ and $\sigma_p(\Delta) = \sigma_p(M_\varphi)$. Hence, $\sigma(M_\varphi) = \varphi(\mathbb{R}) = [-2,2]$. Therefore,

$$\sigma(\Delta) = [-2,2].$$

Moreover, since φ is not constant on any interval of $\mathbb{R}$, M_φ has no eigenvalues. Indeed, if $u \in L^2([0,\pi])$, the equation $\varphi(x)u(x) = \lambda u(x)$ for all $x \in [0, 2\pi]$ forces $u = 0$. Thus,

$$\sigma_p(\Delta) = \emptyset.$$

For all $\lambda \in \mathbb{R}$, let $M(\lambda) = \{x \in M \mid \varphi(x) \leq \lambda\}$. For all $\lambda \in \mathbb{R}$ and all $u \in L^2([0, 2\pi])$, define $E(\lambda)u = \mathbf{1}_{M(\lambda)}u$. Then $E : \mathbb{R} \to \mathcal{L}(L^2([0, 2\pi]))$ is a spectral family. Moreover, $M_\varphi = \int_{-2}^{2} \lambda \mathrm{d}E(\lambda)$, so E is the spectral family associated with M_φ by the spectral theorem. Since $\mathcal{F}$ is unitary, the map $\tilde{E} = \mathcal{F}^{-1} \circ E \circ \mathcal{F}$ is again a spectral family, and one can check, as for E, that

$$\Delta = \int_{-2}^{2} \lambda \, \mathrm{d}\tilde{E}(\lambda).$$

This also reflects the fact that the spectrum of Δ is purely absolutely continuous:

$$\sigma(\Delta) = \sigma_{\mathrm{ac}}(\Delta).$$

Remark 2.3.1. *All the results obtained in this example generalize directly to the case of arbitrary dimension $d \geq 1$.*

Remark 2.3.2. ***Fourier transform and spectrum.*** *The use of the Fourier transform $\mathcal{F}$ in this example is very common in the computation of spectra of operators, especially for differential operators. This is due to the fact that, since $\mathcal{F}$ is unitary, conjugation by $\mathcal{F}$ leaves the spectrum and the point spectrum invariant. Moreover, $\mathcal{F}$ has the property of transforming a (here discrete) differentiation into a multiplication. Thus, conjugating the operator under study by $\mathcal{F}$ allows one to reduce the computation of its spectrum to the computation of the spectrum of a multiplication operator.*

2.3.2 A Sturm–Liouville operator

Let $q \in C([0,1], \mathbb{R})$. Let $(T, D(T))$ be the operator defined on $L^2([0,1])$ by

$$D(T) = \{f \in C^2([0,1]) \mid f(0) = f(1) = 0\} \text{ and } \forall f \in D(T),\ Tf = -f'' + qf.$$

First of all, for all $f, g \in D(T)$, one proves by integration by parts that $(Tf|g) = (f|Tg)$. Thus, the operator $(T, D(T))$ is symmetric. One can then show that it is essentially self-adjoint, and in the sequel we consider its closure $(\overline{T}, D(\overline{T}))$, which is self-adjoint. The operator $\overline{T}$ is an example of a Sturm–Liouville operator.

One then proves, using the variation of constants of order 2, that if $z \notin \sigma_p(T)$, then there exists $K_z \in L^2([0,1]^2)$ (and even $K_z \in C([0,1]^2)$) such that

$$\forall g \in C^2([0,1]),\ (T - z)f = g \Leftrightarrow f(t) = \int_0^1 K_z(t,s)g(s)\mathrm{d}s.$$

This means that, for all $z \notin \sigma_p(T)$, the resolvent $R_z(T)$ is the restriction of a Hilbert–Schmidt operator and is therefore a compact operator. In particular, for all $z \notin \sigma_p(T)$, $R_z(T)$ is the restriction of a bounded operator. This remains valid for $\overline{T}$, hence, if $z \notin \sigma_p(T)$, then $z \notin \sigma(\overline{T})$, from which one deduces that $\sigma(\overline{T}) = \sigma_p(\overline{T}) = \sigma_p(T)$.

Moreover, if $\lambda \in \sigma_p(T)$ and $f \in \mathrm{Ker}\,(T - \lambda)$, then, by integration by parts, $\lambda\|f\|_2^2 = (Tf|f) = \|f'\|_2^2 + \int_0^1 q(t)\bar{f}(t)f(t)\mathrm{d}t \geq (\inf_{[0,1]} q)\|f\|_2^2$, hence $\lambda \geq \inf_{[0,1]} q$. Therefore $\sigma(\overline{T}) \subset [\inf_{[0,1]} q, +\infty[$.

If $\mu < \inf_{[0,1]} q$, then $\mu \notin \sigma(\overline{T})$ and $R_\mu(\overline{T})$ is compact and self-adjoint since $\overline{T}$ is. Hence the spectrum of $R_\mu(\overline{T})$ consists of 0 and eigenvalues of finite multiplicity. In fact,

$$\sigma(R_\mu(\overline{T})) = \{\frac{1}{\mu - \lambda_n} \mid \lambda_n \in \sigma_p(\overline{T})\}.$$

Moreover, there exists a Hilbert basis $(f_n)_{n\in\mathbb{N}}$ of $L^2([0,1])$ such that

$$\forall g \in L^2([0,1]),\ R_\mu(\overline{T})g = \sum_{n=0}^{\infty} \frac{1}{\mu - \lambda_n}(g|f_n)f_n,$$

where the f_n are eigenfunctions of $R_\mu(\overline{T})$ associated with the eigenvalues $\frac{1}{\mu-\lambda_n}$. Hence the f_n are also eigenfunctions of $\overline{T}$ associated with the eigenvalues λ_n. From all this one deduces that

$$\overline{T} = \sum_{n\geq 0} \lambda_n(.|f_n)f_n \text{ and } D(\overline{T}) = \{f \in L^2([0,1]) \mid \sum_{n\geq 0} |\lambda_n(f|f_n)|^2 < +\infty\}.$$

Thus, although $\overline{T}$ is not bounded, the fact that its resolvent is compact still allows us to diagonalize it in a Hilbert basis, as for a compact operator.

2.3.3 Periodic Schrödinger operators

We consider the example of a periodic Schrödinger operator acting on the Sobolev space $\mathsf{H}^2(\mathbb{R})$,

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V, \tag{2.1}$$

where V is a piecewise continuous function that is periodic on $\mathbb{R}$, with period $2L_0$ to fix the notation.

Then the spectrum of the operator H is purely absolutely continuous, H has no eigenvalues, and its spectrum is a union of bands called spectral bands:

$$\sigma(H) = \bigcup_{p\geq 0} \left[E^p_{\mathrm{min}}, E^p_{\mathrm{max}}\right].$$

The real numbers E^p_{min} and E^p_{max} are called the band edges, and the intervals $(E^p_{\mathrm{max}}, E^{p+1}_{\mathrm{min}})$ are the spectral gaps.

The band edges are solutions of equations with quasi-periodic boundary conditions. Let $\omega \in [-L_0, L_0]$. We consider the restriction $H(\omega)$ of H to $\mathsf{H}^2([-L_0, L_0])$, the Sobolev space of functions $\psi \in \mathsf{H}^2(\mathbb{R})$ such that

$$\forall x \in \mathbb{R},\ \psi(x + 2L_0) = \mathrm{e}^{\mathrm{i}(\frac{\pi}{L_0}\omega+\pi)}\psi(x). \tag{2.2}$$

Since $\mathsf{H}^2([-L_0, L_0]) \subset C^1([-L_0, L_0])$, this condition is equivalent to the boundary conditions:

$$\psi(L_0) = \mathrm{e}^{\mathrm{i}(\frac{\pi}{L_0}\omega+\pi)}\psi(-L_0) \quad \text{and} \quad \psi'(L_0) = \mathrm{e}^{\mathrm{i}(\frac{\pi}{L_0}\omega+\pi)}\psi'(-L_0). \tag{2.3}$$

The operator $H(\omega)$ is self-adjoint. Its resolvent is even Hilbert–Schmidt and therefore compact. Hence its spectrum is purely point, and the band edges are then the eigenvalues of $H(-L_0)$ and of $H(0)$. Indeed, one also has that $H(2L_0 - \omega)$ and $H(\omega)$ are unitarily equivalent for all $\omega \in [-L_0, L_0]$. Moreover, the functions $\omega \mapsto E^p(\omega)$ are strictly monotone on $[-L_0, 0]$, increasing for even p and decreasing for odd p. One then has:

$$\sigma(H(-L_0)) := \{E^0_{\mathrm{min}}, E^1_{\mathrm{max}}, E^2_{\mathrm{min}}, E^3_{\mathrm{max}}, \ldots\}$$

and

$$\sigma(H(0)) := \{E^0_{\mathrm{max}}, E^1_{\mathrm{min}}, E^2_{\mathrm{max}}, E^3_{\mathrm{min}}, \ldots\}.$$

The link between H and $H(\omega)$ is given by the notion of direct integral of operators. First, if $\mathcal{H}'$ is a Hilbert space and if $(X, \mathrm{d}\mu)$ is a measure space, one considers the "continuous sum" of copies of $\mathcal{H}'$,

$$\mathcal{H} = \int_X^{\oplus} \mathcal{H}'\mathrm{d}\mu := L^2(X, \mathrm{d}\mu, \mathcal{H}').$$

Then, if $A'(\cdot)$ is a measurable function from X into the set of self-adjoint operators on $\mathcal{H}'$, one defines an operator A on $\mathcal{H}$ by:

$$D(A) = \left\{\psi \in \mathcal{H},\ \psi(x) \in D(A'(x)) \text{ for a.e. } x \in X \text{ and } \int_X ||A'(x)\psi(x)||^2_{\mathcal{H}'} \mathrm{d}\mu(x) < +\infty\right\}$$

and $(A\psi)(x) = A'(x)\psi(x)$. One then writes

$$A = \int_X^{\oplus} A'(x)\mathrm{d}\mu(x).$$

With this formalism, one then shows that H is the direct integral of the operators $H(\omega)$:

$$H = \int_{[-L_0,L_0]}^{\oplus} H(\omega)\mathrm{d}\omega.$$

The result on the band spectrum of H then follows from the fact that one can show that the eigenvalues of $H(\omega)$ are analytic in ω and non-constant (they are even strictly monotone, increasing for even bands and decreasing for odd ones). A general result on direct integrals of operators then ensures that the spectrum is purely absolutely continuous (the Radon–Nikodym derivative on each band is given by the inverse of the derivative of $\omega \mapsto E_p(\omega)$).

2.3.4 Quasi-periodic Schrödinger operators in dimension 1

We are interested in a discrete Schrödinger operator:

$$h_{\alpha,\omega,f} = -\Delta_d + V \text{ acting on } \ell^2(\mathbb{Z}).$$

where V is a quasi-periodic potential, *i.e.* of the form

$$\forall n \in \mathbb{Z},\ V(n) = f(\omega + n\alpha),\ \alpha, \omega \in \mathbb{T}^d,\ f \in C(\mathbb{T}^d, \mathbb{R}).$$

The vector $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{T}^d$ has rationally independent coordinates (after adding 1 to the coordinates):

$$k_j \in \mathbb{Z},\ 1 \le j \le d,\ k_1\alpha_1 + \dots + k_d\alpha_d = 0 \in \mathbb{T} \ \Rightarrow\ k_j = 0,\ 1 \le j \le d.$$

Generically, such an operator has purely singular continuous spectrum.

Theorem 2.3.3. *There exists a residual set $\mathcal{F}_{sc} \subset C(\mathbb{T}^d, \mathbb{R})$ such that for all $f \in \mathcal{F}_{sc}$ and for Lebesgue-almost every $\omega \in \mathbb{T}^d$, the spectrum of $h_{\alpha,\omega,f}$ is purely singular continuous.*

Moreover, this spectrum is generically a Cantor set (a closed subset, with no isolated points and containing no interval).

Theorem 2.3.4. *There exists a residual set $\mathcal{F}_{cantor} \subset C(\mathbb{T}^d, \mathbb{R})$ such that for all $f \in \mathcal{F}_{cantor}$ and for all $\omega \in \mathbb{T}^d$, the spectrum of $h_{\alpha,\omega,f}$ is a Cantor set.*

It turns out that the case of quasi-periodic Schrödinger operators is more subtle than this. Indeed, by introducing a coupling parameter in front of the potential, other spectral types may appear. To illustrate this phenomenon, we consider the only quasi-periodic model that can be regarded as completely understood, the almost Mathieu operator. Let $H_\omega^{\alpha,\lambda}$ act on $\ell^2(\mathbb{Z})$ by:

$$\forall n \in \mathbb{Z}, (H_\omega^{\alpha,\lambda}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda\cos(2\pi(\omega + n\alpha))\psi(n).$$

We then set

$$\Sigma^{\alpha,\lambda} = \bigcup_{\omega \in \mathbb{T}} \sigma(H_\omega^{\alpha,\lambda}).$$

We then have the following theorems, where α is always assumed to be irrational.

Theorem 2.3.5 (Measure of the spectrum.). *For all $\lambda > 0$ and for all irrational $\alpha \in \mathbb{T}$, one has*

$$\mathrm{Leb}(\Sigma^{\alpha,\lambda}) = 4|1-\lambda|.$$

Theorem 2.3.6 (Metal–insulator transition.).

1. *If $\lambda < 1$, then for all α and all ω, $\sigma(H_\omega^{\alpha,\lambda})$ is purely absolutely continuous.*
2. *If $\lambda = 1$, for all α and for all ω except a countable set, $\sigma(H_\omega^{\alpha,\lambda})$ is purely singular continuous.*
3. *If $\lambda > 1$, for almost every α and almost every ω, $\sigma(H_\omega^{\alpha,\lambda})$ is purely point and the associated eigenfunctions decay exponentially.*
4. *If $\lambda > 1$, for generic α and for all ω, $\sigma(H_\omega^{\alpha,\lambda})$ is purely singular continuous.*
5. *If $\lambda > 1$, for all α and for generic ω, $\sigma(H_\omega^{\alpha,\lambda})$ is purely singular continuous.*

Theorem 2.3.7 (Ten Martini Problem.). *For all $\lambda > 0$ and all irrational $\alpha \in \mathbb{T}$, $\Sigma^{\alpha,\lambda}$ is a Cantor set.*

2.4 Spectral types and dynamics

We consider a quantum system such as, for instance, an electron propagating in a crystal. Given an initial state $\psi_0 \in \mathcal{H}$, there must exist a unique state $\psi(\cdot,t) : x \mapsto \psi(x,t)$ representing the state of the system at time t. We write:

$$\forall t \in \mathbb{R},\ \psi(\cdot,t) = U(t)\psi_0.$$

For each t, $U(t)$ is a linear operator on $\mathcal{H}$ which is unitary ($\|U(t)\psi\| = \|\psi\|$) and satisfies:

$$U(0) = \mathrm{Id},\ \text{ and } \ \forall s,t \in \mathbb{R},\ U(s+t) = U(s)U(t).$$

We further assume that for all $\psi \in \mathcal{H}$ and all $t_0 \in \mathbb{R}$,

$$\lim_{t\to t_0} U(t)\psi = U(t_0)\psi.$$

We say that U is a strongly continuous one-parameter unitary group.
We then define:

$$D(H) = \left\{ \psi \in \mathcal{H} \ \middle|\ \lim_{t\to 0} \frac{1}{t}(U(t)\psi - \psi) \text{ exists} \right\}$$

and, for all $\psi \in D(H)$,

$$H\psi = \lim_{t\to 0} \frac{\mathrm{i}}{t}(U(t)\psi - \psi) \ \longleftrightarrow\ U(t) = \mathrm{e}^{\mathrm{i}tH}.$$

The operator $(H, D(H))$ is called the Hamiltonian of the quantum system and corresponds to the energy of this system. If $\psi_0 \in D(H)$, then ψ is a solution of the Schrödinger equation:

$$\partial_t \psi = \mathrm{i}H\psi \ \text{ and } \ \psi(\cdot,0) = \psi_0.$$

Using functional calculus, one can therefore write:

$$\psi(\cdot,t) = \mathrm{e}^{\mathrm{i}tH}\psi_0.$$

Studying the dynamics of a system means studying the time evolution of its states. We will see in the following that the evolution of $\psi(\cdot,t)$ strongly depends on the space in which the initial state ψ_0 lies in the decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{pp}} \oplus \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}}.$$

Conversely, these spaces are characterized by the dynamics of the states they induce.

2.4.1 Reminders on the Fourier transform of a measure

Let μ be a Borel measure on $\mathbb{R}$. The Fourier transform of μ is defined by:

$$\forall t \in \mathbb{R},\ \hat{\mu}(t) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}xt} \mathrm{d}\mu(x).$$

We then have the following three results:

1. Riemann–Lebesgue lemma: if μ is absolutely continuous with respect to the Lebesgue measure, then $\hat{\mu}(t) \xrightarrow[t \to \pm\infty]{} 0$.

2. Plancherel theorem: $\hat{\mu} \in L^2(\mathbb{R})$ if and only if μ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}x} \in L^2(\mathbb{R})$. In this case:

$$\int_{\mathbb{R}} |\hat{\mu}(t)|^2 \mathrm{d}t = 2\pi \int_{\mathbb{R}} \left| \frac{\mathrm{d}\mu}{\mathrm{d}x}(x) \right|^2 \mathrm{d}x.$$

3. Wiener theorem:

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t |\hat{\mu}(s)|^2 \mathrm{d}s = \sum_{x \in \mathbb{R}} |\mu(\{x\})|^2.$$

2.4.2 The R.A.G.E. theorem

Theorem 2.4.1 (R.A.G.E.). *Let H be a self-adjoint operator acting on $\mathcal{H} = L^2(\mathbb{R}^d)$ (or on $\ell^2(\mathbb{Z}^d)$ with adapted notation).*

1. *$\psi \in \mathcal{H}_{\mathrm{pp}}$ if and only if*

$$\lim_{R \to +\infty} \sup_{t \geq 0} \left(\int_{\mathbb{R}^d \setminus B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH} \psi(x)|^2 \mathrm{d}x \right) = 0.$$

Such a ψ is called a bounded state.

2. *$\psi \in \mathcal{H}_{\mathrm{c}}$ if and only if*

$$\forall R > 0,\ \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left(\int_{B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH} \psi(x)|^2 \mathrm{d}x \right) \mathrm{d}t = 0.$$

Such a ψ is called a diffusive state.

3. *If $\psi \in \mathcal{H}_{\mathrm{ac}}$ then*

$$\forall R > 0,\ \lim_{t \to +\infty} \int_{B(0,R)} |\mathrm{e}^{-\mathrm{i}tH} \psi(x)|^2 \mathrm{d}x = 0,$$

but the converse is not true.

One may replace $B(0,R)$ by any compact subset of $\mathbb{R}^d$; what matters is the applicability of the dominated convergence theorem.
Let $\psi \in \mathcal{H}$, $\|\psi\| = 1$. Then,

$$T_R(\psi) = \int_0^{+\infty} \int_{B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH} \psi(x)|^2 \mathrm{d}x$$

is the sojourn time of the particle in the state ψ inside $B(0,R)$.
Let $\psi \in \mathcal{H}$, $\|\psi\| = 1$. Then,

$$W_R(\psi) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \left(\int_{B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH} \psi(x)|^2 \mathrm{d}x \right) \mathrm{d}t$$

is the probability of presence of the particle in the state ψ inside $B(0,R)$.

Corollary 2.4.2. *1. If, for all $R > 0$ and all $\psi \in \mathcal{H}$ with norm 1, one has $T_R(\psi) = \infty$ and $W_R(\psi) > 0$, then $\mathcal{H} = \mathcal{H}_{\rm pp}$ and H has purely point spectrum.*

2. *If, for all $R > 0$ and all $\psi \in \mathcal{H}$ with norm 1, one has $T_R(\psi) = \infty$ and $W_R(\psi) = 0$, then $\mathcal{H} = \mathcal{H}_{\rm sc}$ and H has purely singular continuous spectrum.*

3. *If, for all $R > 0$ and all $\psi \in \mathcal{H}$ with norm 1, one has $T_R(\psi) < \infty$, then $\mathcal{H} = \mathcal{H}_{\rm ac}$ and H has purely absolutely continuous spectrum.*

Proof : [Proof of the R.A.G.E. theorem]

1. Let us begin by proving the direct implication in point 1. Let $\psi \in \mathcal{H}_{\rm pp}$. Up to considering linear combinations (and using Pythagore's theorem since the eigenspaces are pairwise orthogonal) and passing to the limit using the dominated convergence theorem, we may assume that ψ is an eigenfunction associated with an eigenvalue $\lambda \in \mathbb{R}$ of H (since H is self-adjoint). Then:

$$\mathrm{e}^{\pm \mathrm{i}tH}\psi = \mathrm{e}^{\pm \mathrm{i}\lambda t}\psi$$

and since $\lambda \in \mathbb{R}$ and $\psi \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d \setminus B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH}\psi(x)|^2 \mathrm{d}x = \int_{\mathbb{R}^d \setminus B(0,R)} |\psi(x)|^2 \mathrm{d}x \xrightarrow[R\to+\infty]{} 0.$$

2. Let us prove the direct implication of point 2. Let $R > 0$. Let $\varphi, \psi \in \mathcal{H}$ and let $E(\cdot)$ denote the spectral family associated with H. Then:

$$(\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\lambda t} \mathrm{d}\mu_{\varphi,\psi}(\lambda) = \hat{\mu}_{\varphi,\psi}(t),$$

where $\mu_{\varphi,\psi}(\lambda) = (\varphi|E(\lambda)\psi)$. We apply Wieners theorem to $\mu_{\varphi,\psi}$:

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T |\hat{\mu}_{\varphi,\psi}(t)|^2 \mathrm{d}t = \sum_{\lambda \in \mathbb{R}} |\mu_{\varphi,\psi}(\{\lambda\})|^2.$$

Hence,

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T |(\varphi|\mathrm{e}^{\pm \mathrm{i}tH}\psi)|^2 \mathrm{d}t = \sum_{\lambda \in \mathbb{R}} |(\varphi|E(\{\lambda\})\psi)|^2.$$

If $\psi \in \mathcal{H}_{\rm c}$, then for all $\lambda \in \mathbb{R}$, $E(\{\lambda\})\psi = 0$. Therefore, for all $\varphi \in \mathcal{H}$,

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T |(\varphi|\mathrm{e}^{\pm \mathrm{i}tH}\psi)|^2 \mathrm{d}t = 0.$$

We obtain the desired result by choosing $\varphi = \mathbf{1}_{B(0,R)}\mathrm{e}^{\pm \mathrm{i}tH}\psi$.

1-2 bis. We prove simultaneously the converses of points 1 and 2. Let:

$$\mathcal{H}_a = \{\psi \in \mathcal{H} \mid \lim_{R\to+\infty} \sup_{t\geq 0} \left(\int_{\mathbb{R}^d \setminus B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH}\psi(x)|^2 \mathrm{d}x \right) = 0\}$$

and

$$\mathcal{H}_b = \{\psi \in \mathcal{H} \mid \forall R > 0, \lim_{T\to+\infty} \frac{1}{T} \int_0^T \left(\int_{B(0,R)} |\mathrm{e}^{\pm \mathrm{i}tH}\psi(x)|^2 \mathrm{d}x \right) \mathrm{d}t = 0\}.$$

We have just shown that: $\mathcal{H}_{\text{pp}} \subset \mathcal{H}_a$ and $\mathcal{H}_{\text{c}} \subset \mathcal{H}_b$. Let $\varphi \in \mathcal{H}_a$ and $\psi \in \mathcal{H}_b$. Then, for all $T > 0$,

$$\begin{aligned}(\varphi|\psi) &= \frac{1}{T}\int_0^T (\varphi|\psi)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T (\mathrm{e}^{-\mathrm{i}tH}\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T ((1-\mathbf{1}_{B(0,R)})\mathrm{e}^{-\mathrm{i}tH}\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi)\mathrm{d}t + \frac{1}{T}\int_0^T (\mathbf{1}_{B(0,R)}\mathrm{e}^{-\mathrm{i}tH}\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi)\mathrm{d}t.\end{aligned}$$

The first term can be made arbitrarily small for R large enough since $\varphi \in \mathcal{H}_a$, and one can bound the integrand by the uniform supremum over $t \geq 0$ after applying the Cauchy–Schwarz inequality. The second term tends to 0 as T goes to infinity for any fixed $R > 0$ since $\psi \in \mathcal{H}_b$, again using Cauchy–Schwarz. We can therefore let T tend to infinity and then R tend to infinity to obtain $(\varphi|\psi) = 0$. Hence $\mathcal{H}_a \perp \mathcal{H}_b$. Using the inclusions already obtained: $\mathcal{H}_a \perp \mathcal{H}_{\text{c}}$ and $\mathcal{H}_b \perp \mathcal{H}_{\text{pp}}$. Thus, since $\mathcal{H} = \mathcal{H}_{\text{pp}} \oplus \mathcal{H}_{\text{c}}$, we have $\mathcal{H}_a \subset \mathcal{H}_{\text{c}}^{\perp} = \mathcal{H}_{\text{pp}}$ and $\mathcal{H}_b \subset \mathcal{H}_{\text{pp}}^{\perp} = \mathcal{H}_{\text{c}}$. This yields the desired equalities.

3. By a previous computation, for all $\varphi, \psi \in \mathcal{H}$, $(\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi) = \hat{\mu}_{\varphi,\psi}(t)$. Moreover, for any Borel set M:

$$|\mu_{\varphi,\psi}(M)| = |(E(M)\varphi|E(M)\psi| \leq ||E(M)\varphi|| \cdot ||E(M)\psi|| = |\mu_\varphi(M)|^{\frac{1}{2}}|\mu_\psi(M)|^{\frac{1}{2}}.$$

Assume now that $\psi \in \mathcal{H}_{\text{ac}}$. Then μ_ψ is absolutely continuous with respect to Lebesgue measure, and the same is therefore true for $\mu_{\varphi,\psi}$ for any $\varphi \in \mathcal{H}$. By the Riemann–Lebesgue lemma, it follows that

$$(\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi) = \hat{\mu}_{\varphi,\psi}(t) \xrightarrow[t\to\infty]{} 0.$$

Let $\varphi = \mathbf{1}_{B(0,R)}\mathrm{e}^{\pm\mathrm{i}tH}\psi$. Then $\varphi \in \mathcal{H}$ and

$$(\varphi|\mathrm{e}^{-\mathrm{i}tH}\psi) = \int_{B(0,R)} |\mathrm{e}^{-\mathrm{i}tH}\psi(x)|^2\mathrm{d}x,$$

which yields the desired result.

□

The second point of the R.A.G.E. theorem admits the following generalization.

Proposition 2.4.3. *Let H be a self-adjoint operator. Let K be an operator, either compact, or such that $K(H+\mathrm{i})^{-1}$ is compact. Then, for all $\psi \in \mathcal{H}_{\text{c}}$,*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \|K\mathrm{e}^{-\mathrm{i}tH}\psi\|^2\mathrm{d}t = 0.$$

Proof : If K is compact, it is the limit of finite-rank operators. It therefore suffices to prove the result for finite-rank operators. Such an operator is a linear combination of operators of the form $K = (\phi|\cdot)\varphi$, and by the triangle inequality we are reduced to point 2 of the R.A.G.E. theorem.

Suppose now that $K(H+\mathrm{i})^{-1}$ is compact. Since $D(H) \cap \mathcal{H}_{\text{c}}$ is dense in $\mathcal{H}_{\text{c}}$, it suffices to prove the result for $\psi \in D(H) \cap \mathcal{H}_{\text{c}}$. We then write:

$$\|K\mathrm{e}^{-\mathrm{i}tH}\psi\| = \|K(H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}tH}(H+\mathrm{i})\psi\|,$$

which reduces the problem to the previous case.

□

Chapter 3

Random operators

Throughout this chapter, $(\Omega, \mathcal{A}, \mathrm{P})$ denotes a complete probability space.

3.1 Measurable families of operators

We begin by defining the notion of measurability for a family of bounded random operators.

Definition 3.1.1. *Let $\mathcal{H}$ be a Hilbert space. Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of bounded operators on $\mathcal{H}$. We say that $\{H_\omega\}_{\omega\in\Omega}$ is measurable if for all $\varphi, \psi \in \mathcal{H}$, the map $\omega \mapsto (\varphi|H_\omega\psi)$ is measurable.*

The discrete Anderson model acting on $\ell^2(\mathbb{Z}^d)$ is a measurable family of bounded operators. In order to pass to the case of unbounded operators, we restrict ourselves to self-adjoint operators so as to be able to use the functional calculus.

Definition 3.1.2. *Let $\mathcal{H}$ be a Hilbert space. Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of self-adjoint operators on $\mathcal{H}$. We say that $\{H_\omega\}_{\omega\in\Omega}$ is measurable if for every bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, the family $\{f(H_\omega)\}_{\omega\in\Omega}$ is measurable in the sense of bounded operator families.*

One can then prove the following characterizations of measurability.

Proposition 3.1.3. *Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of self-adjoint operators on $\mathcal{H}$. For each $\omega \in \Omega$, let E_ω denote the spectral family associated with H_ω. Then:*

1. *$\{H_\omega\}_{\omega\in\Omega}$ is measurable if and only if, for every Borel set B of $\mathbb{R}$, the family $\{E_\omega(B)\}_{\omega\in\Omega}$ is measurable.*

2. *$\{H_\omega\}_{\omega\in\Omega}$ is measurable if and only if there exists $z \in \mathbb{C} \setminus \mathbb{R}$ such that $\{(H_\omega + z)^{-1}\}_{\omega\in\Omega}$ is measurable.*

3. *$\{H_\omega\}_{\omega\in\Omega}$ is measurable if and only if for all $t \in \mathbb{R}$, the family $\{\mathrm{e}^{\mathrm{i}tH_\omega}\}_{\omega\in\Omega}$ is measurable.*

Proof : For details, see [8], Proposition V.1.2, page 243.

□

The first point of the previous proposition yields a measurability result for the projections onto the different spectral types. Let us first recall that the spectral theorem allows one to define, for every Borel set B of $\mathbb{R}$, a spectral projection by the formula $E(B) = \chi_B(H)$. Since E decomposes as a sum $E = E_{\mathrm{pp}} + E_{\mathrm{ac}} + E_{\mathrm{sc}}$ according to the Lebesgue decomposition of the spectral measure of H, one can define, for every Borel set B of $\mathbb{R}$, the projections $E_{\mathrm{pp}}(B)$, $E_{\mathrm{ac}}(B)$ and $E_{\mathrm{sc}}(B)$.

Proposition 3.1.4. *Let $\{H_\omega\}_{\omega\in\Omega}$ be a family of self-adjoint operators on $\mathcal{H}$ such that for each $\omega \in \Omega$, E_ω denotes the spectral family associated with H_ω. Suppose that $\{H_\omega\}_{\omega\in\Omega}$ is measurable. Then for every Borel set B of $\mathbb{R}$, the families $\{E_\omega(B)\}_{\omega\in\Omega}$, $\{E_{\omega,\mathrm{pp}}(B)\}_{\omega\in\Omega}$, $\{E_{\omega,\mathrm{ac}}(B)\}_{\omega\in\Omega}$ and $\{E_{\omega,\mathrm{sc}}(B)\}_{\omega\in\Omega}$ are measurable.*

Proof : See [8], Proposition V.1.7, page 246.

□

We now verify that the Schrödinger operators we wish to study in the sequel are indeed measurable. We have the following general result.

Proposition 3.1.5. *Let $H = -\Delta + V$ act on $L^2(\mathbb{R}^d)$ and be essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. If $\{V_\omega\}_{\omega\in\Omega}$ is a stochastic process measurable in x and ω such that for every ω, $H_\omega = H + V_\omega$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, then $\{H_\omega\}_{\omega\in\Omega}$ is measurable.*

Proof : See [8], Proposition V.3.1, page 257.

□

Example 3.1.6.
1. *The Anderson model is measurable.*
2. *The quasi-periodic model is measurable.*
3. *The Poisson random displacement model is measurable.*

3.2 Ergodic families of operators

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a complete probability space and let $\{T_i\}_{i\in\mathbb{Z}^d}$ be a group of measurable transformations preserving the measure P. A set $A \in \mathcal{A}$ is said to be invariant under the action of $\{T_i\}$ if $T_i^{-1}A = A$ for all $i \in \mathbb{Z}^d$.
Then $\{T_i\}$ is said to be ergodic if every invariant set has measure zero or equal to 1.
The family $\{T_i\}$ is ergodic if and only if for every measurable function $f : \Omega \to \mathbb{R}$,

$$(\forall i,\ f(T_i\omega) = f(\omega) \text{ for almost every } \omega) \ \Rightarrow\ (f \text{ is almost surely constant}).$$

Finally, if $\{T_i\}$ is ergodic, a potential $x \mapsto V_\omega(x)$ for $x \in \mathbb{R}^d$ and $\omega \in \Omega$ is said to be $\mathbb{Z}^d$-transitive (with respect to $\{T_i\}_{i\in\mathbb{Z}^d}$) when:

$$\forall i \in \mathbb{Z}^d,\ \forall x \in \mathbb{R}^d,\ V_{T_i\omega}(x) = V_\omega(x - i).$$

Similarly, if $\{T_y\}_{y\in\mathbb{R}^d}$ is a group of measurable transformations preserving the measure P, the potential $V_\omega(x)$ is said to be $\mathbb{R}^d$-transitive (with respect to $\{T_y\}_{y\in\mathbb{R}^d}$) when:

$$\forall y \in \mathbb{R}^d,\ \forall x \in \mathbb{R}^d,\ V_{T_y\omega}(x) = V_\omega(x - y).$$

Definition 3.2.1. *A measurable family of self-adjoint operators $\{H_\omega\}_{\omega\in\Omega}$ is said to be $\mathbb{Z}^d$-ergodic (resp. $\mathbb{R}^d$-ergodic) if there exists a family of unitary operators U_y defined by $U_y\phi(x) = \phi(x - y)$ such that:*

$$\forall i \in \mathbb{Z}^d,\ \forall \omega \in \Omega,\ H_{T_i\omega} = U_i H_\omega U_i^*.$$

The interest of this definition is that, for all $\omega \in \Omega$ and all $i \in \mathbb{Z}^d$, $\sigma(H_{T_i\omega}) = \sigma(H_\omega)$, which leads to the existence of a deterministic set equal to the spectrum of H_ω P-almost surely.

We also have the following property.

Proposition 3.2.2. *If the family $\{H_\omega\}_{\omega\in\Omega}$ is ergodic, then for every measurable and bounded function f, the family $\{f(H_\omega)\}_{\omega\in\Omega}$ is ergodic.*

3.3 The almost-sure spectrum

We begin with a first result that guides us toward the notion of the almost-sure spectrum.

Lemma 3.3.1. *If $\{P_\omega\}_{\omega\in\Omega}$ is an ergodic family of orthogonal projectors, then the rank of P_ω is* P*-almost surely constant.*

Proof : Let $\{\varphi_k\}_{k\geq 1}$ be a Hilbert basis of $\mathcal{H}$. For every $\omega \in \Omega$ we define

$$r(\omega) := \text{Tr } P_\omega = \sum_{k\geq 1} (P_\omega \varphi_k | \varphi_k).$$

Then r is a positive random variable and for all i and all ω,

$$r(T_i\omega) = \sum_{k\geq 1} (P_{T_i\omega}\varphi_k|\varphi_k) = \sum_{k\geq 1} (U_i^* P_\omega U_i \varphi_k|\varphi_k) = \sum_{k\geq 1} (P_\omega U_i\varphi_k|U_i\varphi_k) = \text{Tr } P_\omega = r(\omega).$$

By ergodicity, we deduce that r is almost surely constant.

□

We then have the theorems of Pastur and of Kunz–Souillard / Kirsch–Martinelli, which ensure that for an ergodic family of self-adjoint operators, almost surely their spectrum is independent of ω.

Theorem 3.3.2. *Let $\{H_\omega\}_{\omega\in\Omega}$ be an ergodic family of self-adjoint operators. Then there exists a closed set $\Sigma \subset \mathbb{R}$ such that $\Sigma = \sigma(H_\omega)$* P*-almost surely.*

The set Σ is called the almost-sure spectrum of the family $\{H_\omega\}_{\omega\in\Omega}$.

Proof : Let us first note that since H_ω is self-adjoint, it is closed, and therefore its spectrum $\sigma(H_\omega)$ is a closed subset of $\mathbb{R}$. Using the structure of open sets of $\mathbb{R}$ as countable unions of pairwise disjoint open intervals with rational endpoints, we may write:

$$\sigma(H_\omega) = \mathbb{R} \setminus \left(\bigcup_{a<b,\ a,b\in\mathbb{Q},\ \sigma(H_\omega)\cap]a,b[=\emptyset}]a,b[\right).$$

Let $a < b$ be two real numbers. We define:

$$\Omega_{a,b} = \{\omega \in \Omega \mid \sigma(H_\omega)\cap]a,b[= \emptyset\}.$$

Note that

$$(\sigma(H_\omega)\cap]a,b[= \emptyset) \Leftrightarrow (E_\omega(]a,b[) = 0).$$

Since $\{H_\omega\}_{\omega\in\Omega}$ is measurable, the family $\{E_\omega\}_{\omega\in\Omega}$ is also measurable, and since $]a,b[$ is a Borel set of $\mathbb{R}$, $\Omega_{a,b}$ is measurable. Moreover, the spectrum is invariant under the unitary transformation U_i. Hence:

$$\begin{aligned} \forall i \in \mathbb{Z}^d,\ T_i^{-1}(\Omega_{a,b}) &= \{\omega \in \Omega \mid \sigma(H_{T_i\omega})\cap]a,b[= \emptyset\} \\ &= \{\omega \in \Omega \mid \sigma(U_i H_\omega U_i^*)\cap]a,b[= \emptyset\} \\ &= \{\omega \in \Omega \mid \sigma(H_\omega)\cap]a,b[= \emptyset\} \\ &= \Omega_{a,b}. \end{aligned}$$

Since the family $\{T_i\}_{i\in\mathbb{Z}^d}$ is ergodic, we deduce that

$$\mathsf{P}(\Omega_{a,b}) = 0 \quad \text{or} \quad \mathsf{P}(\Omega_{a,b}) = 1.$$

We introduce the set:

$$\Omega' = \left(\bigcap_{a<b,\ a,b\in\mathbb{Q},\ \mathsf{P}(\Omega_{a,b})=1} \Omega_{a,b} \right) \setminus \left(\bigcup_{c<d,\ c,d\in\mathbb{Q},\ \mathsf{P}(\Omega_{c,d})=0} \Omega_{c,d} \right).$$

By countability of $\mathbb{Q}$, $\mathsf{P}(\Omega') = 1$.

Let $\omega \in \Omega'$. Then, for all $a, b \in \mathbb{Q}$, $a < b$, such that $\mathsf{P}(\Omega_{a,b}) = 1$, we have $\sigma(H_\omega) \cap]a, b[= \emptyset$. Moreover, for all $c, d \in \mathbb{Q}$, $c < d$, such that $\mathsf{P}(\Omega_{c,d}) = 0$, we have $\omega \notin \Omega_{c,d}$, *i.e.*, $\sigma(H_\omega) \cap]c, d[\neq \emptyset$. Hence,

$$\forall \omega \in \Omega', \quad \bigcup_{a<b,\ a,b\in\mathbb{Q},\ \sigma(H_\omega)\cap]a,b[=\emptyset}]a, b[= \bigcup_{a<b,\ a,b\in\mathbb{Q},\ \mathsf{P}(\Omega_{a,b})=1}]a, b[.$$

We observe that the left-hand side depends on ω, while the right-hand side is independent of ω. We therefore define

$$\Sigma = \mathbb{R} \setminus \left(\bigcup_{a<b,\ a,b\in\mathbb{Q},\ \mathsf{P}(\Omega_{a,b})=1}]a, b[\right).$$

The set Σ is a closed subset of $\mathbb{R}$ and we have:

$$\forall \omega \in \Omega',\ \sigma(H_\omega) = \Sigma.$$

This yields the desired result since $\mathsf{P}(\Omega') = 1$.

□

The proof given above extends directly to the spectral types by replacing the projector E_ω with the projectors $E_{\omega,\mathrm{pp}}$, $E_{\omega,\mathrm{ac}}$ and $E_{\omega,\mathrm{sc}}$.

Theorem 3.3.3. *Let $\{H_\omega\}_{\omega\in\Omega}$ be an ergodic family of self-adjoint operators. Then there exist closed sets Σ_{pp}, Σ_{sc} and Σ_{ac} in $\mathbb{R}$ such that P-almost surely,*

$$\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(H_\omega), \quad \Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(H_\omega) \quad \text{and} \ \Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(H_\omega).$$

Finally, we have a result concerning the essential and discrete spectra.

Theorem 3.3.4. *Let $\{H_\omega\}_{\omega\in\Omega}$ be an ergodic family of self-adjoint operators. Then the essential spectrum and the discrete spectrum of H_ω are P-almost surely constant.*

Proof : Once again, the proof of the Ishii–Pastur theorem adapts directly to these two types of spectrum by considering the associated spectral projectors.

□

3.4 Examples of almost-sure spectra

3.4.1 The discrete Anderson model

Let $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathsf{P}})$ be a complete probability space and set

$$(\Omega, \mathcal{A}, \mathsf{P}) = \left(\bigotimes_{n\in\mathbb{Z}^d} \tilde{\Omega}, \bigotimes_{n\in\mathbb{Z}^d} \tilde{\mathcal{A}}, \bigotimes_{n\in\mathbb{Z}^d} \tilde{\mathsf{P}} \right).$$

We aim to compute the almost-sure spectrum of the ergodic family of operators $\{h_\omega\}_{\omega\in\Omega}$ acting on $\ell^2(\mathbb{Z}^d)$ defined by:

$$(h_\omega u)_n = -\sum_{|m-n|=1} u_m + \omega_n u_n$$

where, for each $\omega \in \Omega$ and $n \in \mathbb{Z}^d$, ω_n is a random variable on $\tilde{\Omega}$. We assume this family of random variables is *i.i.d.*.
Let us briefly justify the ergodicity of this random family. First, for each $i \in \mathbb{Z}^d$ and $\omega \in \Omega$, we define $T_i(\omega) = (\omega_{n-i})_{n\in\mathbb{Z}^d}$. Then the transformations T_i preserve the product measure P since P is the product of copies of the same measure. Moreover, the family $\{T_i\}_{i\in\mathbb{Z}^d}$ defined in this way is ergodic. Indeed, one shows that for all $A, B \in \mathcal{A}$,

$$\mathsf{P}(T_i^{-1}A \cap B) \xrightarrow[||i||_\infty\to\infty]{} \mathsf{P}(A)\mathsf{P}(B). \tag{3.1}$$

Recall that, by definition of the product σ-algebra, $\mathcal{A}$ is generated by cylinders of the form

$$\{\omega \in \Omega \mid \omega_{i_1} \in \tilde{\mathcal{A}}, \ldots, \omega_{i_k} \in \tilde{\mathcal{A}}\}.$$

If A and B are such cylinders, then by the definition of the product measure and since each T_i preserves the measure, (3.1) holds. Since the set of A and B for which (3.1) holds is a σ-algebra and cylinders generate $\mathcal{A}$, (3.1) is true for all of $\mathcal{A}$. Now, let M be a set invariant under the action of $\{T_i\}_{i\in\mathbb{Z}^d}$. Applying (3.1) with $A = B = M$ gives

$$\mathsf{P}(M) = \mathsf{P}(M\cap M) = \mathsf{P}(T_i^{-1}M \cap M) \xrightarrow[||i||_\infty\to\infty]{} \mathsf{P}(M)\mathsf{P}(M) = \mathsf{P}(M)^2.$$

Hence $\mathsf{P}(M)$ is either 0 or 1. Finally, the family $\{T_i\}_{i\in\mathbb{Z}^d}$ is ergodic.
We can then define the unitary operators U_i as translations:

$$\forall i \in \mathbb{Z}^d,\ \forall u \in \ell^2(\mathbb{Z}^d), \forall n \in Z^d,\ (U_i u)_n = u_{n-i}.$$

Then we have the relation:

$$\forall i \in \mathbb{Z}^d,\ \forall \omega \in \Omega,\ h_{T_i\omega} = U_i h_\omega U_i^*.$$

Indeed, for all $i \in \mathbb{Z}^d$, $u \in \ell^2(\mathbb{Z}^d)$ and $n \in \mathbb{Z}^d$,

$$\begin{aligned}
(U_i h_\omega U_i^* u)_n &= (U_i h_\omega u_{\cdot+i})_n \\
&= U_i\left(-\sum_{|m-n|=1} u_{m+i} + \omega_n u_{n+i}\right) \\
&= -\sum_{|m-n|=1} u_m + \omega_{n-i} u_n \\
&= (h_{T_i\omega} u)_n.
\end{aligned}$$

Since the ω_n are *i.i.d.*, let ν denote their common law. The support of ν is by definition

$$\operatorname{supp}\nu = \{x \in \mathbb{R} \mid \ \forall\varepsilon > 0,\ \nu((x-\varepsilon, x+\varepsilon)) > 0\}.$$

Furthermore, define

$$\ell_0^2(\mathbb{Z}^d) = \{u \in \ell^2(\mathbb{Z}^d) \mid u_n = 0 \text{ for all but finitely many } n\}.$$

Since the discrete Laplacian is bounded, by the Kato–Rellich theorem, h_ω is essentially self-adjoint on $\ell_0^2(\mathbb{Z}^d)$.
We begin by proving a lemma on the random potential.

Lemma 3.4.1. *There exists a set $\Omega_0 \subset \Omega$ of probability 1 such that: for every $\omega \in \Omega_0$, for every finite set $\Lambda \subset \mathbb{Z}^d$, for every sequence $(q_i)_{i\in\Lambda}$ of elements $q_i \in \operatorname{supp}\nu$, and for every $\varepsilon > 0$, there exists a sequence $(j_n)_{n\in\mathbb{N}}$ of elements in $\mathbb{Z}^d$ such that $||j_n||_\infty \xrightarrow[n\to+\infty]{} +\infty$ and*

$$\sup_{i\in\Lambda} |q_i - \omega_{i+j_n}| < \varepsilon.$$

Proof : Fix a finite set $\Lambda \subset \mathbb{Z}^d$, a sequence $(q_i)_{i\in\Lambda}$ of elements in $\operatorname{supp}\nu$, and $\varepsilon > 0$. By the definition of the support and the *i.i.d.* property of the ω_i, we have $\mathsf{P}(A) > 0$ where A is the event $A = \{\omega \in \Omega \mid \sup_{i\in\Lambda} |q_i - \omega_i| < \varepsilon\}$.

Let $(l_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}^d$ such that the distance between any two terms l_n and l_m ($n \neq m$) is greater than twice the diameter of Λ. Then, by translation invariance in law and independence, the events

$$A_n = \{\omega \in \Omega \mid \sup_{i\in\Lambda} |q_i - \omega_{i+l_n}| < \varepsilon\}.$$

are independent and satisfy for all n, $\mathsf{P}(A_n) = \mathsf{P}(A) > 0$. By the Borel–Cantelli lemma, the event

$$\Omega_{\Lambda,\{q_i\},\varepsilon} = \{\omega \in \Omega \mid \omega \in A_n \text{ infinitely often}\}$$

has probability 1. Moreover, since $\operatorname{supp}\nu \subset \mathbb{R}$, it contains a countable dense subset denoted S. Also, the set F_d of finite subsets of $\mathbb{Z}^d$ is countable, so:

$$\Omega_0 = \bigcap_{\Lambda\in F_d,\ (q_i)\in S^\Lambda,\ n\in\mathbb{N}^*} \Omega_{\Lambda,(q_i),\frac{1}{n}}$$

is a countable intersection of sets of probability 1, hence $\mathsf{P}(\Omega_0) = 1$. Therefore, Ω_0 is suitable.

□

We can now compute the almost-sure spectrum of $\{h_\omega\}_{\omega\in\Omega}$.

Theorem 3.4.2. *For P-almost every $\omega \in \Omega$, $\sigma(h_\omega) = [-2d, 2d] + \operatorname{supp}\nu$.*

Proof : First, recall that the spectrum of a multiplication operator is given by the closure of the image of the function defining it. Thus, almost surely, $\sigma(V_\omega) = \operatorname{supp}\nu$ where V_ω is the multiplication operator by the sequence $(\omega_n)_{n\in\mathbb{Z}^d}$. Moreover, the spectrum of the discrete Laplacian is deterministic and given by $[-2d, 2d]$. Since V_ω is bounded and symmetric and $-\Delta_{\text{disc}}$ is also bounded, we have by [14, Theorem 4.10, Chapter V], for almost every ω,

$$\sigma(-\Delta_{\text{disc}} + V_\omega) \subset \sigma(-\Delta_{\text{disc}}) + \sigma(V_\omega) = [-2d, 2d] + \operatorname{supp}\nu.$$

Conversely, let $\lambda \in [-2d, 2d] + \operatorname{supp}\nu$. Write $\lambda = \lambda_0 + \lambda_1$, with $\lambda_0 \in [-2d, 2d]$ and $\lambda_1 \in \operatorname{supp}\nu$. Since h_ω is essentially self-adjoint on $\ell^2_0(\mathbb{Z}^d)$, by Weyl's criterion, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\ell^2_0(\mathbb{Z}^d)$ of norm 1 such that

$$||(-\Delta_{\text{disc}} - \lambda_0)u_n|| \xrightarrow[n\to+\infty]{} 0.$$

Furthermore, since $\lambda_1 \in \operatorname{supp}\nu$, by Lemma 3.4.1, there exists a sequence $(j_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^d$ such that $||j_n||_\infty$ tends to infinity as $n \to \infty$ and for all $n \geq 1$

$$\sup_{i\in\operatorname{supp} u_n} |\omega_{i+j_n} - \lambda_1| < \frac{1}{n}.$$

Finally, set $v_n = u_{n-j_n}$. Then $||v_n||_2 = 1$ for all n and, for almost every ω,

$$\begin{aligned} ||(h_\omega - \lambda)v_n|| &\leq ||(-\Delta_{\text{disc}} - \lambda_0)v_n|| + ||(V_\omega - \lambda_1)v_n|| \\ &\leq ||(-\Delta_{\text{disc}} - \lambda_0)u_n|| + ||(V_\omega - \lambda_1)v_n|| \\ &\leq ||(-\Delta_{\text{disc}} - \lambda_0)u_n|| + \frac{1}{n}. \end{aligned}$$

Hence, for almost every ω, $||(h_\omega - \lambda)v_n||$ tends to 0, and by Weyl's criterion, $\lambda \in \sigma(h_\omega)$. This gives almost surely the desired equality.

□

3.4.2 A quasi one-dimensional continuous Anderson model

Consider the random family of matrix-valued one-dimensional Anderson-Bernoulli operators

$$H_{l,\omega} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N + V + \sum_{n \in \mathbb{Z}} \begin{pmatrix} c_1 \omega_1^{(n)} \mathbf{1}_{[0,l]}(x - ln) & & 0 \\ & \ddots & \\ 0 & & c_N \omega_N^{(n)} \mathbf{1}_{[0,l]}(x - ln) \end{pmatrix} \tag{3.2}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^N$, where $N \geq 1$ is an integer, I_N is the identity matrix of order N and $l > 0$ is a real number. The matrix V is a real $N \times N$ symmetric matrix, the space of these matrices being denoted by $\mathrm{S_N}(\mathbb{R})$. The constants $c_1, \dots, c_N$ are non-zero real numbers.

For every $i \in \{1, \dots, N\}$, $(\omega_i^{(n)})_{n \in \mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d., for short) random variables on a complete probability space $(\widetilde{\Omega}_i, \widetilde{\mathcal{A}}_i, \widetilde{\mathsf{P}}_i)$, of common law ν_i such that $\{0,1\} \subset \operatorname{supp} \nu_i$ and $\operatorname{supp} \nu_i$ is bounded. In particular, the $\omega_i^{(n)}$'s can be Bernoulli random variables. The family $\{H_{l,\omega}\}_{\omega \in \Omega}$ is a family of random operators indexed by the product space

$$(\Omega, \mathcal{A}, \mathsf{P}) = \left(\bigotimes_{n \in \mathbb{Z}} (\widetilde{\Omega}_1 \otimes \cdots \otimes \widetilde{\Omega}_N), \bigotimes_{n \in \mathbb{Z}} (\widetilde{\mathcal{A}}_1 \otimes \cdots \otimes \widetilde{\mathcal{A}}_N), \bigotimes_{n \in \mathbb{Z}} (\widetilde{\mathsf{P}}_1 \otimes \cdots \otimes \widetilde{\mathsf{P}}_N) \right).$$

We also set, for every $n \in \mathbb{Z}$, $\omega^{(n)} = (\omega_1^{(n)}, \dots, \omega_N^{(n)})$, which is a random variable on $(\widetilde{\Omega}_1 \otimes \cdots \otimes \widetilde{\Omega}_N, \widetilde{\mathcal{A}}_1 \otimes \cdots \otimes \widetilde{\mathcal{A}}_N, \widetilde{\mathsf{P}}_1 \otimes \cdots \otimes \widetilde{\mathsf{P}}_N)$ of law $\nu_1 \otimes \cdots \otimes \nu_N$. The expectation value with respect to P will be denoted by $\mathbb{E}(\cdot)$.

As a bounded perturbation of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N$, the operator $H_{l,\omega}$ is self-adjoint on the Sobolev space $H^2(\mathbb{R}) \otimes \mathbb{C}^N$ and thus, for every $\omega \in \Omega$, the spectrum of $H_{l,\omega}$, denoted by $\sigma(H_{l,\omega})$, is included in $\mathbb{R}$. Moreover, because of the periodicity in law of the random potential of $H_{l,\omega}$, the family $\{H_{l,\omega}\}_{\omega \in \Omega}$ is $l\mathbb{Z}$-ergodic. Thus, there exists $\Sigma \subset \mathbb{R}$ such that, for P-almost every $\omega \in \Omega$, $\Sigma = \sigma(H_{l,\omega})$. There also exist Σ_{pp}, Σ_{ac} and Σ_{sc}, subsets of $\mathbb{R}$, such that, for P-almost every $\omega \in \Omega$, $\Sigma_{\text{pp}} = \sigma_{\text{pp}}(H_{l,\omega})$, $\Sigma_{\text{ac}} = \sigma_{\text{ac}}(H_{l,\omega})$ and $\Sigma_{\text{sc}} = \sigma_{\text{sc}}(H_{l,\omega})$, respectively the pure point, absolutely continuous and singular continuous spectrum of $H_{l,\omega}$.

We can give an explicit description of the almost-sure spectrum Σ of $\{H_{l,\omega}\}_{\omega \in \Omega}$. For $\omega^{(0)} = (\omega_1^{(0)}, \dots, \omega_N^{(0)}) \in \operatorname{supp}(\nu_1 \otimes \cdots \otimes \nu_N)$, we denote by $E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}$ the real eigenvalues of the real symmetric matrix $V + \operatorname{diag}(c_1 \omega_1^{(0)}, \dots, c_N \omega_N^{(0)})$. Then, we have

$$\Sigma = [0, +\infty) + \bigcup_{\omega^{(0)} \in \operatorname{supp}(\nu_1 \otimes \cdots \otimes \nu_N)} \{E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}\}. \tag{3.3}$$

In particular, Σ does not depend on the parameter l. In the proof, we will use the specific form of the potential, in particular the fact that V is constant and, in the random part, the fact that the single site potential is of the form $\mathbf{1}_{[0,l]}$ instead of a generic single site potential $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ supported on $[0,l]$.

We fix $l > 0$. Let $\omega \in \Omega$ and $n \in \mathbb{Z}$. If V is a matrix in $\mathrm{S}_{\mathrm{N}}(\mathbb{R})$, we set

$$V_{\omega^{(n)}} = V + \mathrm{diag}(c_1\omega_1^{(n)}, \dots, c_N\omega_N^{(n)}) \in \mathrm{S}_{\mathrm{N}}(\mathbb{R})\,. \tag{3.4}$$

Let $x \in \mathbb{R}$. We set

$$V_\omega(x) = \sum_{n\in\mathbb{Z}} \mathbf{1}_{[0,l]}(x - ln) \otimes V_{\omega^{(n)}}. \tag{3.5}$$

We denote by V_ω the maximal multiplication operator by the function $x \mapsto V_\omega(x)$. Since $(\omega^{(n)})_{n\in\mathbb{Z}}$ is a sequence of i.i.d. random variables and since for every $n \in \mathbb{Z}$ the function $x \mapsto \mathbf{1}_{[0,l]}(x - ln) \otimes V_{\omega^{(n)}}$ is constant on $[ln, l(n+1)]$, the almost-sure spectrum of the $l\mathbb{Z}$-ergodic family $\{V_\omega\}_{\omega\in\Omega}$ is

$$\Sigma(V_\omega) = \bigcup_{\omega^{(0)}\in \mathrm{supp}(\nu_1\otimes\cdots\otimes\nu_N)} \{E_1^{\omega^{(0)}}, \dots, E_N^{\omega^{(0)}}\}. \tag{3.6}$$

We recall that if we consider the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L^2(\mathbb{R}) \otimes \mathbb{C}$ of domain $H^2(\mathbb{R}) \otimes \mathbb{C}$, we have

$$\sigma\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) = [0, +\infty). \tag{3.7}$$

Now, since the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N$ is deterministic, its almost-sure spectrum is

$$\Sigma\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N\right) = [0, +\infty). \tag{3.8}$$

For any $\omega \in \Omega$, V_ω is a bounded self-adjoint operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^N$ and $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}$ is self-adjoint on $H^2(\mathbb{R}) \otimes \mathbb{C}^N$. Then, adapting the proof of [14, Theorem 4.10, Chapter V], we get that every $\lambda \in \Sigma$ satisfies $\lambda \in \Sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N) + \Sigma(V_\omega)$ and thus, $\Sigma \subset \Sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N) + \Sigma(V_\omega)$. Indeed, if $\lambda < \min(0, \inf \Sigma(V_\omega))$ then it is almost-surely in the resolvent set of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N + V_\omega$. At least, a direct application of the result of Kato leads to

$$\sigma \subset [-\max(|\inf \Sigma(V_\omega)|, |\sup \Sigma(V_\omega)|), +\infty).$$

Conversely, let $\alpha \in \Sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}})$ and let $\beta \in \Sigma(V_\omega)$. Then, $\alpha \in [0, +\infty)$ and in particular, $\alpha \in \sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2})$. One can find a sequence $(g_p)_{p\in\mathbb{N}}$ of elements of $H^2(\mathbb{R}) \otimes \mathbb{C}$ such that

(i) $||g_p||_{L^2(\mathbb{R})\otimes\mathbb{C}} = 1$ for every $p \in \mathbb{N}$,

(ii) $|| - g_p'' - \alpha g_p||_{L^2(\mathbb{R})\otimes\mathbb{C}}$ tends to 0 as p tends to infinity,

(iii) $\mathrm{supp}\, g_p \subset [-(p+1), (p+1)]$ for every $p \in \mathbb{N}$.

To construct such a sequence $(g_p)_{p\in\mathbb{N}}$, we can consider a solution of the ordinary differential equation $-u'' = \alpha u$, for example the function $u : x \mapsto \mathrm{e}^{\mathrm{i}\sqrt{\alpha}x}$. Then, we multiply this solution by a sequence of functions $(\chi_p)_{p\in\mathbb{N}}$ which are compactly supported in the interval $[-(p+1), (p+1)]$, constant on the interval $[-p, p]$ and such that $||\chi_p||_{L^2(\mathbb{R})\otimes\mathbb{C}} = 1$ for every $p \in \mathbb{N}$.

Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. We set

$$\Omega_n^{(m)} = \{\omega \in \Omega\,;\ \beta \in \sigma(V_{\omega^{(n)}}) \text{ and } \omega^{(n)} = \omega^{(n+1)} = \cdots = \omega^{(n+m)}\}, \tag{3.9}$$

where $\sigma(V_{\omega^{(n)}}) = \{E_1^{\omega^{(n)}}, \ldots, E_N^{\omega^{(n)}}\}$. We remark that

$$\Sigma(V_\omega) = \bigcup_{\omega^{(0)} \in \mathrm{supp}(\nu_1 \otimes \cdots \otimes \nu_N)} \sigma(V_{\omega^{(0)}}) = \bigcup_{\omega^{(n)} \in \mathrm{supp}(\nu_1 \otimes \cdots \otimes \nu_N)} \sigma(V_{\omega^{(n)}}), \text{ for any } n \in \mathbb{Z}. \tag{3.10}$$

We also set

$$\Omega^{(m)} = \{\omega \in \Omega \,;\, \beta \in \sigma(V_{\omega^{(n)}}) \text{ and } \omega^{(n)} = \cdots = \omega^{(n+m)} \text{ for infinitely many } n \in \mathbb{Z}\}. \tag{3.11}$$

Since $(\omega^{(n)})_{n \in \mathbb{Z}}$ is a sequence of i.i.d. random variables, we have

$$\mathsf{P}(\Omega_i^{(m)}) = \mathsf{P}(\Omega_j^{(m)}) \text{ for any } (i,j) \in \mathbb{Z}^2 \text{ such that } i \neq j. \tag{3.12}$$

Moreover, as $\beta \in \Sigma(V_\omega)$, by (3.10) and the fact that the random variables $\omega^{(n)}$ are i.i.d., we have

$$\mathsf{P}(\Omega_n^{(m)}) > 0 \text{ for any } n \in \mathbb{Z}. \tag{3.13}$$

Finally, the events $(\Omega_{(m+1)n}^{(m)})_{n \in \mathbb{Z}}$ are independent and we can apply Borel-Cantelli's lemma to obtain

$$\mathsf{P}(\Omega^{(m)}) = 1 \text{ for any } m \in \mathbb{N}. \tag{3.14}$$

We set

$$\Omega_1 = \{\omega \in \Omega \,; \forall m \in \mathbb{N},\ \beta \in \sigma(V_{\omega^{(n)}}) \text{ and } \omega^{(n)} = \cdots = \omega^{(n+m)} \text{ for infinitely many } n \in \mathbb{Z}\}.$$

Then, Ω_1 is the countable intersection of the $\Omega^{(m)}$ and, by (3.14), $\mathsf{P}(\Omega_1) = 1$. We set

$$\Omega_2 = \{\omega \in \Omega \,;\, \sigma(H_{l,\omega}) = \Sigma\}.$$

By definition of Σ, we have $\mathsf{P}(\Omega_2) = 1$. Thus, if we set $\Omega_0 = \Omega_1 \cap \Omega_2$, we have $\mathsf{P}(\Omega_0) = 1$. In particular, $\Omega_0 \neq \emptyset$. We fix $\omega \in \Omega_0$.

Let $p \in \mathbb{N}$ and let $m \in \mathbb{N}$ such that $m > 2p + 2$. Since $\omega \in \Omega_0$, there exists $n \in \mathbb{Z}$ such that $\beta \in \sigma(V_{\omega^{(n)}})$ and $\omega^{(n+i)} = \omega^{(n)}$ for every $i \in \{1, \ldots, m\}$. Since properties (i), (ii) and (iii) of $(g_p)_{p \in \mathbb{N}}$ are invariant by translation, one may assume that $\mathrm{supp}\, g_p \subset [n, n+m]$. Moreover, since $\omega^{(n+i)} = \omega^{(n)}$ for every $i \in \{1, \ldots, m\}$, $V_{\omega^{(n+i)}} = V_{\omega^{(n)}}$ for every $i \in \{1, \ldots, m\}$. Since $\beta \in \sigma(V_{\omega^{(n)}})$, one can find an eigenvector $f_n \in \mathbb{C}^N$ of the matrix $V_{\omega^{(n)}}$ associated to β which is also an eigenvector of $V_{\omega^{(n+i)}}$ associated to β for every $i \in \{1, \ldots, m\}$. We can assume that $||f_n||_{\mathbb{C}^N} = 1$, where $|| \cdot ||_{\mathbb{C}^N}$ is any norm on $\mathbb{C}^N$.

Then, we set $f \in L^2(\mathbb{R}) \otimes \mathbb{C}^N$ which is equal to f_n on $[n, n+m]$ and equal to 0 on $\mathbb{R} \setminus [n, n+m]$. We have

$$(V_\omega - \beta) \cdot f = \sum_{j=n}^{n+m} (V_{\omega^{(j)}} - \beta) \cdot f_n + \sum_{j \in \mathbb{Z} \setminus \{n, \ldots, n+m\}} (V_{\omega^{(j)}} - \beta) \cdot 0 = 0, \tag{3.15}$$

since f_n is a common eigenvector to the matrices $V_{\omega^{(j)}}$ for $j \in \{n, \ldots, n+m\}$.

Now we can define $h_p = g_p f$ for every $p \in \mathbb{N}$. Then $h_p \in H^2(\mathbb{R}) \otimes \mathbb{C}^N$ because $g_p \in H^2(\mathbb{R}) \otimes \mathbb{C}$, $\mathrm{supp}\, g_p \subset [n, n+m]$ and $f \in L^2(\mathbb{R}) \otimes \mathbb{C}^N$ is constant on $[n, n+m]$. We have

$$\begin{aligned} ||h_p||^2_{L^2(\mathbb{R}) \otimes \mathbb{C}^N} &= \int_{\mathbb{R}} ||g_p(x) f(x)||^2_{\mathbb{C}^N} \mathrm{d}x \\ &= \int_{[n,n+m]} |g_p(x)|^2 \, ||f_n||^2_{\mathbb{C}^N} \mathrm{d}x \\ &= \int_{[n,n+m]} |g_p(x)|^2 \mathrm{d}x = ||g_p||_{L^2(\mathbb{R}) \otimes \mathbb{C}^N} = 1, \end{aligned} \tag{3.16}$$

since supp $g_p \subset [n, n+m]$. We also have, using (3.15),

$$\begin{aligned}
||(H_{l,\omega} - (\alpha+\beta)) \cdot h_p||_{L^2(\mathbb{R})\otimes\mathbb{C}^N} &= \left|\left|\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N - \alpha\right) \cdot h_p + (V_\omega - \beta) \cdot h_p\right|\right|_{L^2(\mathbb{R})\otimes\mathbb{C}^N} \\
&= \left|\left|(-g_p'' - \alpha g_p) f + g_p (V_\omega - \beta) \cdot f\right|\right|_{L^2(\mathbb{R})\otimes\mathbb{C}^N} \\
&= \left|\left|(-g_p'' - \alpha g_p) f\right|\right|_{L^2(\mathbb{R})\otimes\mathbb{C}^N} \\
&\leq \left|\left|-g_p'' - \alpha g_p\right|\right|_{L^2(\mathbb{R})\otimes\mathbb{C}} \times \left(\sup_{x\in\mathbb{R}} ||f(x)||^2_{\mathbb{C}^N}\right) \\
&= \left|\left|-g_p'' - \alpha g_p\right|\right|_{L^2(\mathbb{R})\otimes\mathbb{C}},
\end{aligned}$$

by $\sup_{x\in\mathbb{R}} ||f(x)||^2_{\mathbb{C}^N} = ||f_n||^2_{\mathbb{C}^N} = 1$. By (ii), we obtain that $||(H_{l,\omega} - (\alpha+\beta)) \cdot h_p||_{L^2(\mathbb{R})\otimes\mathbb{C}^N}$ tends to 0 when p tends to infinity. Combining this with (3.16) and applying Weyl's criterion with the sequence $(h_p)_{p\in\mathbb{N}}$, we obtain $\alpha+\beta \in \sigma(H_{l,\omega})$. Since $\omega \in \Omega_2$, we have $\alpha+\beta \in \Sigma$. Thus, $\Sigma(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}) + \Sigma(V_\omega) \subset \Sigma$ and finally

$$\Sigma = \Sigma\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}\right) + \Sigma(V_\omega) = [0, +\infty) + \bigcup_{\omega^{(0)} \in \mathrm{supp}(\nu_1 \otimes \cdots \otimes \nu_N)} \{E_1^{\omega^{(0)}}, \ldots, E_N^{\omega^{(0)}}\}. \tag{3.17}$$

Chapter 4

Lyapunov exponents

The aim of this chapter is to study the convergence of sequences of products of *i.i.d.* random matrices. In particular, we seek to understand how the law of large numbers generalizes to this setting.

Throughout the sequel, $N \geq 1$ denotes an integer.

4.1 Upper Lyapunov exponent and the Furstenberg–Kesten theorem

Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a complete probability space. We begin by introducing the upper Lyapunov exponent of a sequence of *i.i.d.* random matrices in $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$, denoted by $(A_n^\omega)_{n\in\mathbb{N}}$.

Definition 4.1.1. *Assume that the expectation* $\mathbb{E}(\log^+ ||A_0^\omega||)$ *is finite. Then the following limit belongs to* $\mathbb{R} \cup \{-\infty\}$*:*

$$\gamma = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log ||A_{n-1}^\omega \cdots A_0^\omega||)$$

We call it the upper Lyapunov exponent associated with the sequence $(A_n^\omega)_{n\in\mathbb{N}}$.

By equivalence of norms in finite dimension, γ does not depend on the choice of the norm on $\mathcal{M}_{\mathrm{N}}(\mathbb{R})$.

Proof : Let $S_n = A_{n-1}^\omega \cdots A_0^\omega$. Choosing a sub-multiplicative norm on $\mathcal{M}_{\mathrm{N}}(\mathbb{R})$, we have

$$||S_n|| \leq ||A_{n-1}^\omega|| \cdots ||A_0^\omega||.$$

Then, under the assumption that the expectation $\mathbb{E}(\log^+ ||A_0^\omega||)$ is finite, we also have integrability of $\log^+ ||S_n||$ by independence and multiplicativity of expectation in the independent case. We may then write, for all $p, n \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}(\log^+ ||S_{n+p}||) &\leq \mathbb{E}(\log ||A_{n+p}^\omega \cdots A_n^\omega||) + \mathbb{E}(\log ||A_{n-1}^\omega \cdots A_0^\omega||) \\ &= \mathbb{E}(\log^+ ||S_p||) + \mathbb{E}(\log^+ ||S_n||),\end{aligned}$$

using the *i.i.d.* property in the first term. Thus, the sequence $(\mathbb{E}(\log^+ ||S_n||))_{n\in\mathbb{N}}$ is sub-additive and $\frac{1}{n}\mathbb{E}(\log^+ ||S_n||)$ converges to $\inf_{m\geq 1} \frac{1}{m}\mathbb{E}(\log^+ ||S_m||)$ in $\mathbb{R} \cup \{-\infty\}$. □

Example 4.1.2. *If we assume that each matrix* A_n^ω *is diagonal with diagonal entries* $a_{n,i}^\omega$ *for* $i \in \{1, \ldots, N\}$*, then* $\gamma = \sup_i \mathbb{E}(\log |a_{0,i}^\omega|)$ *and, by the usual strong law of large numbers,*

$$\gamma = \lim_{n\to+\infty} \frac{1}{n} \log(||S_n||)$$

for almost every ω.

Thus, convergence in expectation is easily obtained. This convergence also holds almost surely, which is a less obvious result. It can be proved directly using cocycle theory, as done in [2]. Almost sure convergence is also a consequence of Kingman's sub-additive ergodic theorem.

Let T be a set that can be either $\mathbb{N}$ or $\mathbb{Z}$, or $\mathbb{R}$ or $[0, +\infty)$. We then consider a semigroup $\Theta = \{\theta_t \mid t \in T\}$ of transformations of Ω that preserve P. The quadruple $(\Omega, \mathcal{A}, \Theta, \mathsf{P})$ is called a dynamical system.

A random variable Z on Ω is said to be invariant with respect to Θ if for all $t \in T$, $Z \circ \theta_t = Z$. Finally, a dynamical system $(\Omega, \mathcal{A}, \Theta, \mathsf{P})$ is said to be ergodic if every random variable invariant under Θ is P-almost surely constant.

Definition 4.1.3. *A real-valued stochastic process $\{X(t) \mid t \in T\}$ on $(\Omega, \mathcal{A}, \Theta, \mathsf{P})$ is said to be sub-additive if*

$$X(0) = 0 \text{ and } \forall s, t \in T,\ X(t+s) \leq X(t) + X(s) \circ \theta_t.$$

In the case of equality, the process is called additive.

Theorem 4.1.4 (Kingman)**.** *Let $\{X(n) \mid n \in \mathbb{N}\}$ be a sub-additive process such that*

1. *for all $n \in \mathbb{N}$, $X(n)$ is integrable,*
2. *the sequence $(\frac{1}{n}\mathbb{E}(X(n)))_{n\geq 1}$ is bounded from below.*

Then there exists a random variable Z, invariant with respect to Θ, such that $(\frac{1}{n}X(n))_{n\geq 1}$ converges P-almost surely and in expectation to Z. Moreover,

$$\mathbb{E}(Z) = \inf_{n\geq 1} \tfrac{1}{n}\mathbb{E}(X(n)).$$

Using this theorem, one can prove the Furstenberg–Kesten theorem.

Theorem 4.1.5 (Furstenberg–Kesten)**.** *Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of* i.i.d. *random matrices in* $\mathrm{GL}_N(\mathbb{R})$. *Assume that the expectation $\mathbb{E}(\log^+ ||A_0^\omega||)$ is finite. Then, P-almost surely, the following limit exists and*

$$\lim_{n\to+\infty} \frac{1}{n} \log ||A_{n-1}^\omega \dots A_0^\omega|| = \gamma,$$

the upper Lyapunov exponent.

Proof : **Case 1.** Assume $\gamma \in \mathbb{R}$. Then the process $(\log(||S_n||))_{n\in\mathbb{N}}$ is sub-additive for $\Theta = \{\mathrm{Id}\}$, and the assumptions of Kingman's theorem are satisfied by the proof of Definition 4.1.1. Hence the result in this case.

Case 2. Assume $\gamma = -\infty$. In this case, Kingman's theorem can no longer be applied directly. Fix $m \in \mathbb{N}^*$ and, for every integer $n \geq 1$, perform the Euclidean division of n by m, $n = qm + r$ with $q \geq 0$ and $0 \leq r < m$. We have

$$\begin{aligned}
\frac{1}{n}\log(||S_n||) &\leq \frac{1}{n}\log(||A_{n-1}^\omega \cdots A_{qm+1}^\omega||) + \frac{1}{n}\log(||A_{qm}^\omega \cdots A_0^\omega||) \\
&\leq \frac{1}{n}\sum_{i=qm+1}^{n-1} \log(||A_i^\omega||) + \frac{1}{n}\sum_{j=0}^{q-1}\log(||A_{(j+1)m}^\omega \cdots A_{jm+1}^\omega||) \\
&\leq \frac{1}{n}\sum_{i=qm+1}^{n-1} \log(||A_i^\omega||) + \frac{1}{m}\left(\frac{1}{q}\sum_{j=0}^{q-1}\log(||A_{(j+1)m}^\omega \cdots A_{jm+1}^\omega||)\right).
\end{aligned}$$

The first sum has a fixed finite number of terms bounded by m, independent of n. Hence the first term of the upper bound tends to 0 as n tends to infinity. For the second term, we may apply the usual law of large numbers to obtain

$$\lim_{q\to+\infty} \frac{1}{q}\sum_{j=0}^{q-1} \log(||A^{\omega}_{(j+1)m}\cdots A^{\omega}_{jm+1}||) = \mathbb{E}(\log(||A^{\omega}_{m-1}\cdots A^{\omega}_{0}||)).$$

Since q tends to infinity as n tends to infinity, we deduce that

$$\forall m\in\mathbb{N}^*,\ \limsup_{n\to+\infty}\frac{1}{n}\log(||S_n||) \le \frac{1}{m}\mathbb{E}(\log(||A^{\omega}_{m}\cdots A^{\omega}_{1}||)).$$

Letting m tend to infinity finally yields

$$-\infty \le \liminf_{n\to+\infty}\frac{1}{n}\log(||S_n||) \le \limsup_{n\to+\infty}\frac{1}{n}\log(||S_n||) \le \gamma = -\infty.$$

Thus, the conclusion of the theorem holds with the limit and γ equal to $-\infty$.

□

4.2 Sequences of random matrices in $\mathrm{GL}_2(\mathbb{R})$

Before turning to the general case, let us examine what happens for products of invertible random 2×2 matrices, since the geometric interpretations are simpler in this setting.

4.2.1 Furstenberg's theorem

We seek simple conditions ensuring that, for two nonzero vectors x and y in $\mathbb{R}^N$, the random vectors $S_n x$ and $S_n y$ tend to align in the same direction at an exponential rate.

For a nonzero $x\in\mathbb{R}^N$, we denote by $\bar{x}$ its equivalence class in $P(\mathbb{R}^N)$, and if $M\in\mathrm{GL}_\mathrm{N}(\mathbb{R})$, we set $M\bar{x}=\overline{Mx}$. Then, if x and y are two unit vectors in $\mathbb{R}^N$, we define their distance by

$$\delta(\bar{x},\bar{y}) = (1-(x|y)^2)^{\frac{1}{2}},$$

which is the modulus of the sine of the angle between $\bar{x}$ and $\bar{y}$ when $N=2$. We now assume that $N=2$. In this case, if $x=(x_1,x_2)$ and $y=(y_1,y_2)$, then

$$\delta(\bar{x},\bar{y}) = \frac{|x_1y_2-x_2y_1|}{||x||\,||y||},$$

and this indeed defines a distance on $P(\mathbb{R}^2)$. Possibly replacing each A^{ω}_n by $|\det(A^{\omega}_n)|^{-\frac{1}{2}}A^{\omega}_n$, which does not change $S_n\bar{x}$ (since we are working in projective space), we may assume that $|\det(A^{\omega}_n)|=1$. In this case, a direct computation for

$$S_n = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$$

with $|ad-bc|=1$ yields

$$\delta(S_n\bar{x},S_n\bar{y}) = \frac{||x||\,||y||}{||S_nx||\,||S_ny||}\delta(\bar{x},\bar{y}).$$

Now, if $\gamma>0$, we are guaranteed that there exists a unique direction in the plane along which $||S_nx||$ decays exponentially. This is precisely the content of Oseledets' theorem, which ensures that if $\bar{x}\neq\bar{y}$, then one of the two norms $||S_nx||$ or $||S_ny||$ grows exponentially to $+\infty$. Hence, the distance $\delta(S_n\bar{x},S_n\bar{y})$ always decays exponentially to 0.

Theorem 4.2.1 (Oseledets in $\mathrm{SL}_2(\mathbb{R})$). *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of matrices in $\mathrm{SL}_2(\mathbb{R})$ such that*

(i) $\lim_{n\to+\infty} \frac{1}{n}\ln||A_n|| = 0$,

(ii) $\gamma = \lim_{n\to+\infty} \frac{1}{n}\ln(||A_n\cdots A_1||) > 0$.

Then there exists a one-dimensional subspace $V_- \subset \mathbb{R}^2$ such that

1. $\forall v \in V_-,\ v \neq 0,\ \lim_{n\to+\infty} \frac{1}{n}\ln(||A_n\cdots A_1 v||) = -\gamma$,
2. $\forall v \notin V_-,\ \lim_{n\to+\infty} \frac{1}{n}\ln(||A_n\cdots A_1 v||) = \gamma$.

We therefore seek conditions ensuring that $\gamma > 0$. These conditions are given by Furstenberg's theorem.

Theorem 4.2.2 (Furstenberg). *Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathrm{GL}_2(\mathbb{R})$ with common law μ. Let G_μ be the smallest closed subgroup of $\mathrm{GL}_2(\mathbb{R})$ containing the support of μ, called the Furstenberg subgroup associated with $(A_n^\omega)_{n\in\mathbb{N}}$. Assume that*

(i) for all $M \in G_\mu$, $|\det(M)| = 1$,

(ii) G_μ is not compact,

(iii) there exists no finite union L of lines in $\mathbb{R}^2$ such that $M(L) = L$ for all $M \in G_\mu$.

Then:

1. *for all $\bar{x}, \bar{y} \in P(\mathbb{R}^2)$, P-almost surely,*
$$\lim_{n\to+\infty} \delta(S_n\bar{x}, S_n\bar{y}) = 0,$$
2. *if $\mathbb{E}(\log^+(||A_0^\omega||))$ is finite, there exists $\gamma > 0$ such that for all $x \in \mathbb{R}^2$, $x \neq 0$, P-almost surely,*
$$\lim_{n\to+\infty} \frac{1}{n}\log(||S_n x||) = \lim_{n\to+\infty} \frac{1}{n}\log(||S_n||) = \gamma.$$

Assumption (iii) can be difficult to verify. However, if μ is such that (i) and (ii) hold, then (iii) is equivalent to the following property $(iii)'$:

$$(iii)' \qquad \forall \bar{x} \in P(\mathbb{R}^2),\ \#\{M.\bar{x} \mid M \in G_\mu\} \geq 3.$$

For a proof of this equivalence, see [2][Proposition 4.3], page 31.

4.2.2 Application to the one-dimensional discrete scalar Anderson model

We apply Furstenberg's theorem to a first example of a sequence of *i.i.d.* random matrices in $\mathrm{GL}_2(\mathbb{R})$ arising from the one-dimensional discrete scalar Anderson model.

Let $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathsf{P}})$ be a complete probability space, and let $(\omega_n)_{n\in\mathbb{Z}}$ be a sequence of *i.i.d.* random variables with common law ν, such that there exist $a \neq b$, $\{a,b\} \subset \operatorname{supp}\nu$, and $\operatorname{supp}\nu$ is bounded. Set

$$(\Omega, \mathcal{A}, \mathsf{P}) = \left(\bigotimes_{n\in\mathbb{Z}} \tilde{\Omega}, \bigotimes_{n\in\mathbb{Z}} \tilde{\mathcal{A}}, \bigotimes_{n\in\mathbb{Z}} \tilde{\mathsf{P}}\right).$$

If

$$h_\omega \ :\ \begin{array}{ccc} \ell^2(\mathbb{Z}) & \to & \ell^2(\mathbb{Z}) \\ (u_n)_{n\in\mathbb{Z}} & \mapsto & -(u_{n+1}+u_{n-1}) + \omega_n u_n \end{array}$$

then $\{h_\omega\}_{\omega\in\Omega}$ is an ergodic family of random operators with almost-sure spectrum $[-2,2]+\operatorname{supp}\nu$.

We aim to study the asymptotic behavior of generalized eigenfunctions of h_ω. If E belongs to the almost-sure spectrum of h_ω, we seek to understand the asymptotic behavior of sequences $(u_n)_{n\in\mathbb{Z}}$ such that

$$\forall n\in\mathbb{Z},\ -(u_{n+1}+u_{n-1})+\omega_n u_n = E u_n.$$

This equation can be written as

$$\forall n\in\mathbb{Z},\ \begin{pmatrix} u_{n+1}\\ u_n\end{pmatrix} = \begin{pmatrix} \omega_n - E & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} u_n\\ u_{n-1}\end{pmatrix}. \tag{4.1}$$

We then set

$$T_n^\omega(E) = \begin{pmatrix} \omega_n - E & -1\\ 1 & 0\end{pmatrix}.$$

The matrix $T_n^\omega(E)$ is called the transfer matrix at site n. The sequence $(T_n^\omega(E))_{n\in\mathbb{Z}}$ is a sequence of random matrices in $SL_2(\mathbb{R})$, independent and identically distributed, with common law μ induced by the law ν of the random variables ω_n.

The asymptotic behavior of $(u_n)_{n\in\mathbb{Z}}$ reduces, by iteration of (4.1), to that of the product $T_n^\omega(E)\cdots T_0^\omega(E)$. The upper Lyapunov exponent of this sequence of matrices is well defined, and we will show that it is strictly positive.

Since the transfer matrices are *i.i.d.*, the smallest subgroup of $SL_2(\mathbb{R})$ containing the support of μ is given by

$$G_\mu = \overline{\langle\{T_0^\omega(E)\mid \omega_0\in\operatorname{supp}\nu\}\rangle}.$$

First, all matrices in G_μ indeed have determinant 1. Let us show that G_μ is not compact. To this end, we exhibit an unbounded sequence in G_μ. By the assumption on the law of ω_0, we have

$$\left\langle \begin{pmatrix} a-E & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix} b-E & -1\\ 1 & 0\end{pmatrix}\right\rangle \subset G_\mu.$$

Then,

$$\begin{pmatrix} a-E & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} b-E & -1\\ 1 & 0\end{pmatrix}^{-1} = \begin{pmatrix} 1 & a-b\\ 0 & 1\end{pmatrix}\in G_\mu,$$

and for all $n\in\mathbb{N}$,

$$\begin{pmatrix} 1 & a-b\\ 0 & 1\end{pmatrix}^n = \begin{pmatrix} 1 & n(a-b)\\ 0 & 1\end{pmatrix}\in G_\mu,$$

which therefore contains an unbounded sequence since $a-b\neq 0$.

It remains to show that for all $\bar{x}\in P(\mathbb{R}^2)$, $\#\{M\bar{x}\mid M\in G_\mu\}\geq 3$. Let $x=(x_1,x_2)\in\mathbb{R}^2$, $x\neq 0$. Suppose that $x_2\neq 0$. Then, by solving linear systems explicitly, one shows that for

$$A = \left(\begin{smallmatrix} 1 & a-b\\ 0 & 1\end{smallmatrix}\right),$$

the vectors Ax, A^2x, and A^3x are pairwise non-colinear, which allows one to construct three distinct elements in $\{M\bar{x}\mid M\in G_\mu\}$. If $x_2=0$, then $x_1\neq 0$ and we consider

$$B = \left(\begin{smallmatrix} 1 & 0\\ a-b & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} a-E & -1\\ 1 & 0\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} b-E & -1\\ 1 & 0\end{smallmatrix}\right)\in G_\mu,$$

and then the vectors Bx, B^2x, and B^3x, which are pairwise non-colinear.

Thus, Furstenberg's theorem applies and ensures that the Lyapunov exponent associated with the sequence of transfer matrices is strictly positive. More precisely: $\forall E\in\mathbb{R},\ \gamma(E)>0$.

By Oseledets' theorem, P-almost surely there exist only exponentially growing or exponentially decaying solutions to the equation $h_\omega u = Eu$. An exponentially decaying solution (as $+\infty$) is obtained only for an initial condition $v_{+\infty} \in V_-^+$. Any other initial condition leads to a solution that grows exponentially as $+\infty$.

One may define the Lyapunov exponent at $-\infty$ by the formula

$$\gamma_-(E) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log ||(T^\omega_{-n+1}(E))^{-1} \cdots (T^\omega_0(E))^{-1}||).$$

Since

$$(T^\omega_{-n+1}(E))^{-1} \cdots (T^\omega_0(E))^{-1} = (T^\omega_0(E) \cdots T^\omega_{-n+1}(E))^{-1},$$

we have, by the *i.i.d.* property,

$$\mathbb{E}(\log ||(T^\omega_{-n+1}(E))^{-1} \cdots (T^\omega_0(E))^{-1}||) = \mathbb{E}((S_n(E))^{-1}).$$

Now, if $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have

$$^t(JS_n(E)J^{-1}) = (S_n(E))^{-1},$$

and since transposition, J, and J^{-1} are isometries, we obtain $\gamma_-(E) = \gamma_+(E) = \gamma(E)$. This exponent is also strictly positive for all $E \in \mathbb{R}$ by Furstenberg's theorem. Again by Oseledets' theorem, an exponentially decaying solution as $-\infty$ is obtained only for an initial condition $v_{-\infty} \in V_-^-$. Thus, to obtain an eigenvector in $\ell^2(\mathbb{Z})$, one must have $\mathrm{Vect}(v_{+\infty}) = \mathrm{Vect}(v_{-\infty})$.

It is important to note that the above statements depend on E. If E varies in an uncountable set (for instance an interval of $\mathbb{R}$), the set of $\omega \in \Omega$ for which, for all E, the Lyapunov exponent is not strictly positive could have nonzero measure, or even measure 1. For example, one cannot assert that P-almost surely, for all $E \in \mathbb{R}$, every solution of $h_\omega u = Eu$ is exponentially growing or decaying. This requires a more refined analysis.

On the other hand, we will see in Chapter 6 that this already implies the almost-sure absence of absolutely continuous spectrum for the family $\{h_\omega\}_{\omega\in\Omega}$.

4.3 Sequences of random matrices in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$

4.3.1 Exterior powers

We define the exterior powers of a finite-dimensional vector space E. In order to fix ideas, and since we will in any case work in this framework later on, we assume that this vector space is $\mathbb{R}^N$ for a fixed integer N.

Definition 4.3.1. *For $p \in \{1, \ldots, N\}$, $\wedge^p \mathbb{R}^N$ is the vector space of alternating p-linear forms on the dual $(\mathbb{R}^N)^*$. The vector space $\wedge^p \mathbb{R}^N$ is called the p-th exterior power of the space $\mathbb{R}^N$.*

For $u_1, \ldots, u_p$ in $\mathbb{R}^N$ and $f_1, \ldots, f_p$ in $(\mathbb{R}^N)^*$, we set:

$$(u_1 \wedge \ldots \wedge u_p)(f_1, \ldots, f_p) = \det((f_i(u_j))_{i,j})$$

Any vector of the form $u_1 \wedge \ldots \wedge u_p$ is called a decomposable p-vector. Using these decomposable p-vectors, one can define a basis of the space $\wedge^p \mathbb{R}^N$ in the following way.

Proposition 4.3.2. *Let $p \in \{1, \ldots, N\}$. The following properties hold:*

1. *If $(u_1, \ldots, u_N)$ is a basis of $\mathbb{R}^N$, then $\{u_{i_1} \wedge \ldots \wedge u_{i_p} \mid 1 \le i_1 < \ldots i_p \le N\}$ is a basis of $\wedge^p \mathbb{R}^N$.*

2. *For $u_1, \ldots, u_p$ in $\mathbb{R}^N$, $u_1 \wedge \ldots \wedge u_p$ is nonzero if and only if the vectors $u_1, \ldots, u_p$ are linearly independent.*

3. *$(u_1, \ldots, u_p)$ and $(v_1, \ldots, v_p)$ generate the same subspace if and only if there exists a real number $\lambda \neq 0$ such that:*
$$u_1 \wedge \ldots \wedge u_p = \lambda(v_1 \wedge \ldots \wedge v_p)$$

This proposition justifies the fact that the operations we shall define on $\wedge^p \mathbb{R}^N$ will be defined simply on the set of decomposable p-vectors.

In the following, we will study the asymptotic behavior of the norms of sequences of matrices whose p-th exterior power we wish to consider. For the moment, we have neither defined what the exterior power of a matrix is, nor the norm on an exterior power of a vector space.

First, we define an inner product on $\wedge^p \mathbb{R}^N$. For two p-tuples $(u_1, \ldots, u_p)$ and $(v_1, \ldots, v_p)$ of vectors in $\mathbb{R}^N$, we define the inner product on decomposable p-vectors by:

$$(u_1 \wedge \ldots \wedge u_p, v_1 \wedge \ldots \wedge v_p) = \det((\langle u_i, v_j \rangle)_{i,j})$$

where $\langle ., . \rangle$ denotes the usual inner product on $\mathbb{R}^N$. We denote by $|| \; ||$ the associated norm.

It remains to define how an element of the linear group $GL_N(\mathbb{R})$ acts on $\wedge^p \mathbb{R}^N$. If $M \in GL_N(\mathbb{R})$, we define an automorphism $\wedge^p M$ of $\wedge^p \mathbb{R}^N$ by defining it on decomposable p-vectors by:
$$(\wedge^p M)(u_1 \wedge \ldots \wedge u_p) = Mu_1 \wedge \ldots \wedge Mu_p$$
We then have the multiplicativity property: $\wedge^p(MN) = (\wedge^p M)(\wedge^p N)$.

The norm of the linear map or matrix $\wedge^p M$ is the norm induced by $|| \; ||$ on $\wedge^p \mathbb{R}^N$:

$$|| \wedge^p M|| = \sup\{||(\wedge^p M)v|| \mid v \in \wedge^p \mathbb{R}^N, \; ||v|| = 1\}$$

We also denote it by $|| \; ||$, since in what follows there will be no possible confusion in practice between the norm on $\wedge^p \mathbb{R}^N$ and the operator norm it induces. By the multiplicativity property of the exterior power and the submultiplicativity of the operator norm, we have for M and N in $GL_N(\mathbb{R})$:
$$|| \wedge^p (MN)|| \leq || \wedge^p M|| \; || \wedge^p N||$$

We note that if K is an orthogonal matrix for the usual inner product on $\mathbb{R}^N$, then $\wedge^p K$ is also orthogonal. This remark leads to the following proposition:

Proposition 4.3.3. *Let M be a matrix in $GL_N(\mathbb{R})$ and let $a_1(M) \geq \ldots \geq a_N(M) > 0$ be the positive square roots of the eigenvalues of ${}^tMM$. Then for all $p \in \{1, \ldots, N\}$, we have:*

$$|| \wedge^p M|| = a_1(M) \ldots a_p(M)$$

Proof : We write a polar decomposition of M, $M = KAU$ with K and U orthogonal matrices and $A = \mathrm{diag}(a_1(M), \ldots, a_N(M))$. Since $\wedge^p K$ and $\wedge^p U$ are also orthogonal, we have: $|| \wedge^p M|| = || \wedge^p A||$.

Now, if $(e_1, \ldots, e_N)$ is the canonical basis of $\mathbb{R}^N$:

$$(\wedge^p A)(e_{i_1} \wedge \ldots \wedge e_{i_p}) = (a_{i_1} \ldots a_{i_p})(e_{i_1} \wedge \ldots \wedge e_{i_p})$$

Hence:

$$||(\wedge^p A)|| = \sup\{a_{i_1} \ldots a_{i_p} \mid 1 \leq i_1 < \ldots < i_p \leq N\} = a_1(M) \ldots a_p(M)$$

This completes the proof.

□

Finally, we can also estimate the logarithm of the norm of $\wedge^p M$.

Proposition 4.3.4. *Let M be a matrix in* $\mathrm{GL_N}(\mathbb{R})$ *. We set:*

$$\ell(M) = \max(\log^+ ||M||, \log^+ ||M^{-1}||)$$

Then for all $p \in \{1, \dots, N\}$, and for any unit vector w in $\wedge^p \mathbb{R}^N$, we have:

$$|\log(|| \wedge^p M||)| \leq p\ell(M) \text{ and } |\log(||(\wedge^p M)w||)| \leq p\ell(M)$$

Proof : This follows from the previous proposition as explained in [2], Lemma 5.4, page 62. □

4.3.2 Furstenberg theorem for $\mathrm{GL_N}(\mathbb{R})$

Let us first note that Furstenberg's theorem extends directly to the case of sequences of random matrices in $\mathrm{GL_N}(\mathbb{R})$. It then yields the strict positivity of the upper Lyapunov exponent.

The upper Lyapunov exponent describes the asymptotic exponential behavior of the sequence $(A_n^\omega)_{n\in\mathbb{N}}$ in the whole space $\mathbb{R}^N$, that is, the global asymptotic behavior of this sequence. However, by itself, this exponent does not allow one to determine the asymptotic exponential behavior on the invariant subspaces of the matrix $A_{n-1}^\omega \dots A_0^\omega$ and therefore does not fully capture the associated dynamics of this sequence of random matrices. To overcome this difficulty, one defines other, intermediate Lyapunov exponents, which allow for a more refined analysis of the asymptotic behavior of $(A_n^\omega)_{n\in\mathbb{N}}$. A precise dynamical interpretation of these Lyapunov exponents is given by Oseledets' theorem.

Definition 4.3.5. *Let $(A_n^\omega)_{n\in\mathbb{Z}}$ be a sequence of random matrices,* i.i.d. *in* $\mathrm{GL_N}(\mathbb{R})$ *, such that the expectation $\mathbb{E}(\log^+ ||A_0^\omega||)$ is finite. The Lyapunov exponents $\gamma_1, \dots, \gamma_N$ associated with the sequence $(A_n^\omega)_{n\in\mathbb{N}}$ are defined inductively by $\gamma_1 = \gamma$ (the upper Lyapunov exponent) and for $p \geq 2$,*

$$\sum_{i=1}^{p} \gamma_i = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log || \wedge^p (A_{n-1}^\omega \dots A_0^\omega)||)$$

One can see that the sums $\sum_{i=1}^{p} \gamma_i$ are in fact the upper Lyapunov exponents associated with the sequences $(\wedge^p(A_n^\omega))_{n\in\mathbb{Z}}$ as p varies. Thus, the limits exist and these sums belong to $\mathbb{R} \cup \{-\infty\}$. If moreover $\mathbb{E}(\log^+ ||(A_0^\omega)^{-1}||) < +\infty$ (which will be automatically satisfied in the symplectic case), then Proposition 4.3.4 applies and for all p:

$$\left| \frac{1}{p} \sum_{i=1}^{p} \gamma_i \right| \leq \max(\mathbb{E}(\log^+ ||A_0^\omega||), \mathbb{E}(\log^+ ||(A_0^\omega)^{-1}||)) < +\infty$$

As a consequence, the case where one of the sums would be equal to $-\infty$ being excluded, all Lyapunov exponents are finite.

One can give a characterization of these Lyapunov exponents in terms of the sequence of eigenvalues of the matrices ${}^t(A_{n-1}^\omega \dots A_0^\omega)(A_{n-1}^\omega \dots A_0^\omega)$.

Proposition 4.3.6. *If $a_1(n) \geq \ldots \geq a_N(n) > 0$ are the square roots of the eigenvalues of the symmetric positive definite matrix ${}^t(A_{n-1}^\omega \dots A_0^\omega)(A_{n-1}^\omega \dots A_0^\omega)$, then,* P*-almost surely,*

$$\forall p \in \{1, \dots, n\},\ \gamma_p = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}(\log a_p(n)) = \lim_{n\to\infty} \frac{1}{n} \log a_p(n)$$

Proof : See [2], Proposition 5.6, page 63.

□

In particular, this proposition justifies the numbering of the Lyapunov exponents and the terminology of upper Lyapunov exponent, since it implies that $\gamma_1 \geq \cdots \geq \gamma_N$. We can now give a geometric interpretation of these intermediate Lyapunov exponents.

Theorem 4.3.7 (Oseledets)**.** *Let $(A_n^\omega)_{n\in\mathbb{Z}}$ be a sequence of random matrices,* i.i.d. *in* $\mathrm{GL_N}(\mathbb{R})$ *, such that the expectation $\mathbb{E}(\log^+ ||A_0^\omega||)$ is finite. Let $\gamma_1 \geq \cdots \geq \gamma_N$ be its Lyapunov exponents. Let r denote the number of distinct Lyapunov exponents, with values $\lambda_1, \ldots, \lambda_r$.*

There exists a decreasing sequence of measurable subspaces of $\mathbb{R}^N$, $(\mathcal{V}_i^\omega)_{1\leq i\leq r+1}$ such that

1. $\{0\} = \mathcal{V}_{r+1}^\omega \subset \mathcal{V}_r^\omega \subset \cdots \subset \mathcal{V}_1^\omega \subset \mathbb{R}^N$.
2. $\forall i \in \{1, \ldots, r\}$, $x \in \mathcal{V}_i^\omega \setminus \mathcal{V}_{i+1}^\omega \Leftrightarrow \lim_{n\to+\infty} \frac{1}{n} \log(||S_n x||) = \lambda_i$.
3. $\dim(\mathcal{V}_i^\omega) - \dim(\mathcal{V}_{i+1}^\omega)$ *is equal to the multiplicity of* λ_i.

We now wish to generalize Furstenberg's theorem and obtain conditions under which the Lyapunov exponents are pairwise distinct. Strong irreducibility generalizes as follows:

Definition 4.3.8. *Let $p \in \{1, \ldots, N\}$. A subset S of* $\mathrm{GL_N}(\mathbb{R})$ *is said to be p-strongly irreducible if there does not exist a finite family of proper subspaces of $\wedge^p \mathbb{R}^N$, $V_1, \ldots, V_k$ such that:*

$$(\wedge^p M)(V_1 \cup \ldots \cup V_k) = V_1 \cup \ldots \cup V_k$$

for all $M \in S$.

The non-compactness condition of the Furstenberg subgroup generalizes to the notion of p-contractivity. Before that, let us define the notion of a contracting set.

The index of a subset T of $\mathrm{GL_N}(\mathbb{R})$ is the smallest integer r such that there exists a sequence $(M_n)_{n\in\mathbb{N}}$ of elements of T such that $(||M_n||^{-1}M_n)_{n\in\mathbb{N}}$ converges to a matrix of rank r. A subset of $\mathrm{GL_N}(\mathbb{R})$ is said to be contracting if it has index 1.

Recall that a probability measure P on $P(\mathbb{R}^N)$ is said to be proper if for every hyperplane H in $\mathbb{R}^N$, $\mathsf{P}(\{\bar{x} \in P(\mathbb{R}^N),\ x \in H \setminus \{0\}\}) = 0$.

Then, if $T \subset \mathrm{GL_N}(\mathbb{R})$ is a contracting set, for any proper probability measure P on $P(\mathbb{R}^N)$, there exists a sequence $(M_n)_{n\in\mathbb{N}}$ of elements of T such that $M_n\mathsf{P}$ converges weakly to a Dirac measure.

Indeed, let $(M_n)_{n\in\mathbb{N}}$ be a sequence of elements of T such that $(||M_n||^{-1}M_n)_{n\in\mathbb{N}}$ converges to a matrix M of rank 1. For any $x \in \mathbb{R}^N$ such that $Mx \neq 0$, we have $M_n\bar{x} \xrightarrow[n\to+\infty]{} M\bar{x}$. But since P is proper and since $\{x \in \mathbb{R}^N,\ Mx = 0\}$ is a hyperplane, we have

$$\lim_{n\to+\infty} M_n\bar{x} = M\bar{x} \quad \text{for } \mathsf{P}\text{-almost every } \bar{x}.$$

Thus, if $\operatorname{Im} M = \operatorname{Vect}(z)$, we obtain

$$\lim_{n\to+\infty} M_n\bar{x} = \bar{z} \quad \text{for } \mathsf{P}\text{-almost every } \bar{x}.$$

This means that $M_n\mathsf{P}$ converges weakly to $\delta_{\bar{z}}$.

Definition 4.3.9 (p-contracting set). *Let T be a subset of* $\mathrm{GL_N}(\mathbb{R})$ *and p an integer in* $\{1,\ldots,N-1\}$. *The set T is said to be p-contracting if there exists a sequence $(M_n)_{n\in\mathbb{N}}$ in T such that the following limit exists:*

$$\lim_{n\to\infty}\frac{\wedge^p M_n}{||\wedge^p M_n||}=M$$

and is a matrix of rank 1.

This property can be seen as the analogue of non-compactness in the case $N=2$. Indeed, assuming that all matrices of the Furstenberg group have determinant of modulus 1 (assumption (i) of Furstenberg's theorem), then the non-compactness of G_μ is equivalent to its contracting property.

Assume that G_μ is contracting. Let $(M_n)_{n\in\mathbb{N}}$ be a sequence in G_μ such that $(||M_n||^{-1}M_n)$ converges to a matrix M of rank 1. Then,

$$0=|\det(M)|=\lim_{n\to+\infty}|\det(||M_n||^{-1}M_n)|=\lim_{n\to+\infty}||M_n||^{-2}|\det(M_n)|=\lim_{n\to+\infty}||M_n||^{-2}. \tag{4.2}$$

In particular, $||M_n||$ tends to $+\infty$ and G_μ is unbounded, hence non-compact.

Conversely, if G_μ is non-contracting, since it is a subgroup of $\mathrm{GL_2}(\mathbb{R})$, it must have index 2 since it cannot have index 0 or 1. Let $(M_n)_{n\in\mathbb{N}}$ be any sequence of elements of G_μ. Then the sequence $(||M_n||^{-1}M_n)$ takes values in the unit sphere, which is compact, so one can extract a convergent subsequence, say $(||M_{n_m}||^{-1}M_{n_m})$, converging to a limit M. Since G_μ has index 2, M has rank 2 and is therefore invertible. Then, by (4.2),

$$0\neq|\det(M)|=\lim_{m\to+\infty}||M_{n_m}||^{-2}.$$

In particular, $(||M_{n_m}||)_{m\in\mathbb{N}}$ converges, and hence so does the sequence $(M_{n_m})_{m\in\mathbb{N}}$. Therefore, from any sequence in G_μ one can extract a convergent subsequence, and thus G_μ is compact.

p-contractivity can also be interpreted as the fact that the set T is sufficiently "large" so that one can find within it a sequence converging to a rank-one matrix. We now give a simple way to ensure that a subset of $\mathrm{GL_N}(\mathbb{R})$ is p-contracting for all p.

Proposition 4.3.10. *A subset T of* $\mathrm{GL_N}(\mathbb{R})$ *which contains a matrix having N eigenvalues with distinct moduli is p-contracting for all* $p\in\{1,\ldots,N-1\}$.

Proof. Let M be a matrix in T having N eigenvalues with distinct moduli, $\{\lambda_1,\ldots,\lambda_N\}$. Then $\wedge^p M$ has a simple dominant eigenvalue for all p in $\{1,\ldots,N-1\}$. Indeed, if $u_1,\ldots,u_N$ are the eigenvectors associated with the eigenvalues of M, then:

$$\forall p,\ (\wedge^p M)(u_1\wedge\ldots\wedge u_N)=\lambda_1\ldots\lambda_N(u_1\wedge\ldots\wedge u_N)$$

It then follows, from the Jordan decomposition of $\wedge^p M$, that $\frac{\wedge^p M^{2n}}{||\wedge^p M^{2n}||}$ converges to a projection matrix with image $\ker(M-(\lambda_1\ldots\lambda_N)I)$ of dimension 1. Therefore, the sequence $((\wedge^p M)^n)_n$ is p-contracting for all p in $\{1,\ldots,N-1\}$. □

Of course, this criterion is not applicable in full generality to prove that a subset of $\mathrm{GL_N}(\mathbb{R})$ is p-contracting for all p, but in some simple cases it may be sufficient.

We can now state a criterion for the separation of Lyapunov exponents in $\mathrm{GL_N}(\mathbb{R})$.

Theorem 4.3.11. *Let $(A_n^\omega)_{n\in\mathbb{N}}$ be an* i.i.d. *sequence of invertible random matrices of size N with common distribution μ, and let p be an integer in $\{1,\ldots,N\}$. Assume that the Furstenberg group G_μ associated with the sequence $(A_n^\omega)_{n\in\mathbb{N}}$ is p-contracting and p-strongly irreducible, and that the expectation $\mathbb{E}(\log||A_0^\omega||)$ is finite. Then:*

1. $\gamma_p > \gamma_{p+1}$
2. *For every nonzero element x of $\wedge^p R^N$ and for* P*-almost every ω,*

$$\lim_{n\to\infty} \frac{1}{n} \log || \wedge^p (A^\omega_{n-1} \dots A^\omega_0) x|| = \sum_{i=1}^p \gamma_i$$

4.3.3 Furstenberg Theorem for $\mathrm{Sp_N}(\mathbb{R})$

In the models we are interested in, we are led to study sequences of transfer matrices that naturally belong to the symplectic group due to the Hamiltonian nature of the flow of these models.

Definition 4.3.12. *The group of symplectic matrices of order $2N$ is the subgroup of $\mathrm{GL_{2N}}(\mathbb{R})$ consisting of matrices M satisfying:*

$${}^tMJM = J$$

where J is the $2N \times 2N$ matrix defined by $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. It is denoted by $\mathrm{Sp_N}(\mathbb{R})$. In the definition of J, I is the $N \times N$ identity matrix.

We can then state some first properties that will be useful in what follows.

Proposition 4.3.13. *If M is a matrix in $\mathrm{Sp_N}(\mathbb{R})$.*

1. *${}^tM$ belongs to $\mathrm{Sp_N}(\mathbb{R})$.*
2. *If λ is an eigenvalue of M, then $\frac{1}{\lambda}$ is also an eigenvalue of M.*
3. *There exist two orthogonal matrices K and U in $\mathrm{Sp_N}(\mathbb{R})$ and a diagonal matrix $A = \mathrm{diag}(a_1, \dots, a_N, \frac{1}{a_1}, \dots, \frac{1}{a_N})$ with $a_1 \geq \dots \geq a_N \geq 1$ such that: $M = KAU$.*
4. $||M|| = ||M^{-1}||$.

Proof : For point 1, one checks directly that if ${}^tMJM = J$, then by taking inverses, $M^{-1}J^{-1}({}^tM)^{-1} = J^{-1}$. Since $J^{-1} = -J$, we obtain $M^{-1}J({}^tM)^{-1} = J$, hence $J = MJ{}^tM$ and ${}^tM$ belongs to $\mathrm{Sp_N}(\mathbb{R})$.

For point 2, if λ is an eigenvalue of M associated with the eigenvector v, then:

$${}^tMJv = \frac{1}{\lambda}{}^tMJMv = \frac{1}{\lambda}Jv$$

Thus $\frac{1}{\lambda}$ is an eigenvalue of ${}^tM$ and hence of M.

For point 3, we refer to the proof of Lemma 3.1, page 88 in [2], where one sees that the a_i are precisely the positive square roots of the eigenvalues of the symmetric positive definite matrix ${}^tMM$.

Finally, for point 4, this is an immediate consequence of point 3 since then $||M|| = ||A|| = ||A^{-1}|| = ||M^{-1}||$.

□

The definition of Lyapunov exponents and Oseledets' theorem remain valid in the setting of *i.i.d.* sequences of random matrices in $\mathrm{Sp_N}(\mathbb{R})$. We even have two additional properties in this setting. The first is that if $\mathbb{E}(\log^+ ||(A^\omega_0)||) < +\infty$, then by point 4 of Proposition 4.3.13, we also have $\mathbb{E}(\log^+ ||(A^\omega_0)^{-1}||) < +\infty$, and all Lyapunov exponents are finite. The second is a symmetry property.

Proposition 4.3.14. *If $\gamma_1 \geq \ldots \geq \gamma_{2N}$ are the Lyapunov exponents associated with an* i.i.d. *sequence of random matrices in* $\mathrm{Sp_N}(\mathbb{R})$ *, then for all* $1 \leq i \leq N$*:*

$$\gamma_{2N-i+1} = -\gamma_i.$$

Proof : This is a direct consequence of point 3 of Proposition 4.3.13 applied to the symplectic matrix ${}^t(A^\omega_{n-1} \ldots A^\omega_0)(A^\omega_{n-1} \ldots A^\omega_0)$, of Proposition 4.3.6, and of the fact that $\log \frac{1}{a_i(n)} = -\log(a_i(n))$.

□

This symmetry property shows that the Lyapunov exponents associated with an *i.i.d.* sequence of random symplectic matrices can be grouped into pairs, and we will see that to prove their separation it will suffice to study the first N.

Definition 4.3.15. *The Lyapunov exponents are said to be separated when they are distinct:*

$$\gamma_1 > \gamma_2 > \ldots > \gamma_{2N}$$

We now wish to obtain a criterion for the separation of Lyapunov exponents analogous to the one obtained in $\mathrm{GL_N}(\mathbb{R})$. Unfortunately, the action of $\mathrm{Sp_N}(\mathbb{R})$ on $\wedge^2\mathbb{R}^{2N}$ is not irreducible (and therefore *a fortiori* not p-strongly irreducible). Indeed, the space spanned by $\sum_{i=1}^N e_i \wedge e_{N+i}$ is invariant under every element of $\mathrm{Sp_N}(\mathbb{R})$. The notion of p-irreducibility is therefore not suited to the symplectic setting. We thus refine this definition to adapt it to our needs.

We introduce the p-Lagrangian subvariety of $\mathbb{R}^{2N}$. Let $(e_1, \ldots, e_{2N})$ be the canonical basis of $\mathbb{R}^{2N}$.

Definition 4.3.16. *For every p in $\{1, \ldots, N\}$, let L_p be the subspace of $\wedge^p\mathbb{R}^{2N}$ generated by $\{Me_1 \wedge \ldots \wedge Me_p \mid M \in \mathrm{Sp_N}(\mathbb{R})\}$. It is called the p-Lagrangian subvariety of $\mathbb{R}^{2N}$.*

The projective space $P(L_p)$ is the set of isotropic subspaces of dimension p in $\mathbb{R}^{2N}$ for the bilinear form given by the usual scalar product.

Definition 4.3.17 (L_p-strong irreducibility)**.** *Let T be a subset of $\mathrm{Sp_N}(\mathbb{R})$ and let p be an integer in $\{1, \ldots, N\}$. We say that T is L_p-strongly irreducible if there does not exist a finite union W of strict subspaces of L_p such that $\wedge^p M(W) = W$ for all $M \in T$.*

Here again, by strict subspace we mean a subspace of L_p different from L_p and from $\{0\}$. Restricting to strict subspaces of L_p makes it possible to avoid the pitfall encountered by the first definition. However, this definition is less intuitive from a geometric point of view than the previous one, which is why we did not state it directly.

We can now state the theorem providing the main criterion for the separation of Lyapunov exponents.

Theorem 4.3.18. *Let $(A^\omega_n)_{n\in\mathbb{N}}$ be an* i.i.d. *sequence of random symplectic matrices of order $2N$, with common distribution μ, and let p be an integer in $\{1, \ldots, N\}$. Assume that the Furstenberg subgroup G_μ associated with the sequence $(A^\omega_n)_{n\in\mathbb{N}}$ is p-contracting and L_p-strongly irreducible, and that the expectation $\mathbb{E}(\log ||A^\omega_0||)$ is finite. Then:*

1. $\gamma_p > \gamma_{p+1}$
2. *For every nonzero element x of L_p and for* P*-almost every ω,*

$$\lim_{n\to\infty} \frac{1}{n} \log || \wedge^p (A^\omega_{n-1} \ldots A^\omega_0)x|| = \sum_{i=1}^{p} \gamma_i$$

This result will mainly be of interest to us in the form of the following corollary:

Corollary 4.3.19. *Let $(A_n^\omega)_{n\in\mathbb{Z}}$ be an* i.i.d. *sequence of random symplectic matrices of order $2N$, with common distribution μ. Assume that the Furstenberg subgroup G_μ associated with the sequence $(A_n^\omega)_{n\in\mathbb{Z}}$ is p-contracting and L_p-strongly irreducible for all $p \in \{1,\ldots,N\}$, and that the expectation $\mathbb{E}(\log ||A_0^\omega||)$ is finite.*

Then the Lyapunov exponents associated with the sequence $(A_n^\omega)_{n\in\mathbb{Z}}$ are separated; in particular:

$$\gamma_1 > \gamma_2 > \ldots > \gamma_N > 0$$

Proof: The fact that $\gamma_1 > \gamma_2 > \ldots > \gamma_N$ follows from repeated application of point 1 of Theorem 4.3.18 above, for each p in $\{1,\ldots,N-1\}$. Then γ_N is strictly positive because, by Proposition 4.3.14, $\gamma_N = -\gamma_{N+1}$ and by Theorem 4.3.18, $\gamma_N > \gamma_{N+1}$, and therefore one cannot have $\gamma_N = \gamma_{N+1} = 0$.

□

We thus observe that a link is established between, on the one hand, a dynamical property associated with an *i.i.d.* sequence of random symplectic matrices, namely the separation of its Lyapunov exponents, and, on the other hand, more geometric properties of an algebraic object associated with this sequence, namely the Furstenberg subgroup. This constitutes a very powerful criterion in practice, since it allows one to avoid studying directly limits of matrix products and instead to reduce the problem to finite products of such matrices in the Furstenberg subgroup.

4.3.4 Goldsheid–Margulis criterion

It turns out that the properties of p-contractivity and L_p-strong irreducibility are difficult to handle for the models we are interested in. In this section, we present a powerful algebraic criterion that will allow us to deal with our models.

We begin by recalling the definition of density with respect to the Zariski topology in a linear Lie group (i.e. a Lie subgroup of $\mathrm{GL_{2N}}(\mathbb{R})$). We keep in mind that we wish to apply these definitions and results to the symplectic group $\mathrm{Sp_N}(\mathbb{R})$.

We first define the Zariski topology on $\mathcal{M}_\mathrm{N}(\mathbb{R})$. For this purpose, we identify $\mathcal{M}_\mathrm{N}(\mathbb{R})$ with $\mathbb{R}^{N^2}$. Then, for any subset S of $\mathbb{R}[X_1,\ldots,X_{N^2}]$, we set:

$$V(S) = \{x \in \mathbb{R}^{N^2} \mid \forall P \in S,\ P(x) = 0\}$$

Thus, $V(S)$ consists of the common zeros of the polynomials in S. The set $V(S)$ is called the affine algebraic set defined by S. One can then prove that an arbitrary intersection of affine algebraic sets is again an affine algebraic set, as well as any finite union.

These affine algebraic sets are therefore the closed sets of a topology on $\mathbb{R}^{N^2}$, which is called the Zariski topology. The Zariski topology is the topology whose closed sets are the common zeros of families of polynomials.

Considering the Zariski topology on $\mathcal{M}_\mathrm{N}(\mathbb{R})$ amounts to considering polynomials in the N^2 matrix coefficients. The Zariski topology on $\mathrm{GL_N}(\mathbb{R})$ is the topology induced by the Zariski topology just defined. The same applies to the Zariski topology on any subgroup of $\mathrm{GL_N}(\mathbb{R})$, such as $\mathrm{Sp_N}(\mathbb{R})$ or the Furstenberg subgroup associated with an *i.i.d.* sequence of random matrices.

We can now define the Zariski closure of a subset of $\mathrm{GL_N}(\mathbb{R})$.

Definition 4.3.20. *The Zariski closure of a subset G of $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$ is the smallest Zariski-closed subset of $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$ containing G. It is denoted by $\mathrm{Cl}_{\mathrm{Z}}(G)$. A subset $G' \subset G$ is then said to be Zariski-dense in G if $\mathrm{Cl}_{\mathrm{Z}}(G') = \mathrm{Cl}_{\mathrm{Z}}(G)$.*

In other words, if G is a subset of $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$, $\mathrm{Cl}_{\mathrm{Z}}(G)$ is the set of zeros in $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$ of the polynomials that vanish on G. Thus, G' is Zariski-dense in G if every polynomial that vanishes on G' also vanishes on G.

The classical example of a Zariski-dense subset of another is $\mathbb{Z}$, which is Zariski-dense in $\mathbb{R}$.

We now state an important property of the Zariski closure: it preserves the group structure.

Proposition 4.3.21. *If G is a closed subgroup of $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$, then its Zariski closure $\mathrm{Cl}_{\mathrm{Z}}(G)$ is also a subgroup of $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$.*

Moreover, since the Zariski closure of a subset is also closed for the usual topology, it follows that the Zariski closure of a Lie subgroup of $\mathrm{GL}_{\mathrm{N}}(\mathbb{R})$ is again a Lie group. This will be essential in order to consider the Lie algebra of the Zariski closure of the Furstenberg subgroup.

We now have at our disposal the elementary definitions needed to present the Goldsheid–Margulis criterion.

Theorem 4.3.22 (Goldsheid and Margulis). *Let G be a subgroup of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$. If G is Zariski-dense in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$, then G is p-contracting and L_p-strongly irreducible for all $p \in \{1,\dots,N\}$.*

Proof : According to [10], Lemma 6.2 and Theorem 6.3, page 57, it suffices to prove that the connected component of the identity of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is irreducible in L_p and that $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is p-contracting, for all p. For p-contractivity, by Proposition 4.3.10, it suffices to find an element of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ whose eigenvalues have pairwise distinct moduli. For example, the matrix $\mathrm{diag}(2,3,\dots,N+1,\frac{1}{2},\frac{1}{3},\dots,\frac{1}{N+1})$ has eigenvalues with pairwise distinct moduli. Thus, we have already shown that $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is p-contracting for all p.

To study the irreducibility in L_p of the connected component of the identity in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$, we first recall that $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is connected. Hence, the connected component of the identity in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is the symplectic group $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ itself. We are therefore reduced to proving that $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is irreducible in L_p for all p. We must thus show that there is no strict subspace V of L_p such that $(\wedge^p M)(V) \subset V$ for all M in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$.

Suppose by contradiction that such a V exists. We write $M = KAU$ with K, A, and U as in point 3 of Proposition 4.3.13. Then, if $(e_1,\dots,e_{2N})$ is the canonical basis of $\mathbb{R}^{2N}$, we have:

$$(\wedge^p A)(e_1 \wedge \dots \wedge e_p) = a_1 \dots a_p e_1 \wedge \dots \wedge e_p$$

and thus $(\wedge^p A)(V) \subset V$. But if $e_1 \wedge \dots \wedge e_p$ belongs to V, then $Me_1 \wedge \dots \wedge Me_p$ belongs to V, and therefore $V = L_p$. Thus, V is not a strict subspace of L_p. Hence, $e_1 \wedge \dots \wedge e_p$ belongs to W, the orthogonal complement of V in L_p. Then, if $w \in W$ and $v \in V$:

$$(\wedge^p Mw, v) = (w, \wedge^p M^* v) = 0$$

for all M in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ (since ${}^tM$ also belongs to $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$). Thus, $Me_1 \wedge \dots \wedge Me_p$ belongs to W for all M, and hence $W = L_p$. This again contradicts the fact that V is a strict subspace of L_p. Therefore, $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is indeed L_p-strongly irreducible. □

We thus have at our disposal a powerful criterion, effective in certain cases, which allows one to prove that a subgroup of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is L_p-strongly irreducible and p-contracting for all $p \in \{1,\dots,N\}$. This criterion is sufficient, for example, to study the case of discrete quasi-1d Schrödinger operators, as we shall see in the following chapter.

Chapter 5

Quasi-$1d$ discrete and continuous Anderson models

5.1 Quasi-$1d$ discrete Anderson model

Let $N \geq 1$. We want to study the operator

$$h_\omega^N : \begin{array}{ccc} \ell^2(\mathbb{Z}, \mathbb{C}^L) & \to & \ell^2(\mathbb{Z}, \mathbb{C}^L) \\ (u_n) & \mapsto & (-(u_{n+1} + u_{n-1}) + V_{\omega^{(n)}} u_n) \end{array}$$

where

$$V_{\omega^{(n)}} = \begin{pmatrix} \omega_1^{(n)} & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & \omega_N^{(n)} \end{pmatrix}$$

with $(\omega_1^{(n)})_{n\in\mathbb{Z}}, \ldots, (\omega_N^{(n)})_{n\in\mathbb{Z}}$ being sequences of independent and identically distributed real random variables on a complete probability space $(\Omega, \mathcal{B}, \mathsf{P})$, with common law ν whose support contains at least 2 points (for instance 0 and 1 in the Bernoulli case). We also set $\omega^{(n)} = (\omega_1^{(n)}, \ldots, \omega_N^{(n)})$, which has distribution $\nu \otimes \cdots \otimes \nu$.

To study the generalized eigenfunctions of the operator h_ω^N, we solve the equation

$$-(u_{n+1} + u_{n-1}) + V_{\omega^{(n)}} u_n = E u_n,$$

which leads to the introduction of the transfer matrices

$$T_{\omega^n}(E) = \begin{pmatrix} V_{\omega^{(n)}} - E & -I \\ I & 0 \end{pmatrix}.$$

These transfer matrices form a sequence of *i.i.d.* matrices in $\mathrm{Sp_N}(\mathbb{R})$ with common law μ_E, and one can therefore consider the associated Fürstenberg subgroup

$$G_{\mu_E} = \overline{\langle \operatorname{supp} \mu_E \rangle}.$$

We then have

$$G_{\{0,1\}}(E) = \overline{\langle T_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N \rangle} \subset \overline{\langle T_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \operatorname{supp} \nu^{\otimes N} \rangle}.$$

We thus want to show that the subgroup generated by 2^N matrices, $G_{\{0,1\}}(E)$, is Zariski-dense in $\mathrm{Sp_N}(\mathbb{R})$. We denote by $X(E)$ its Zariski closure in $\mathrm{Sp_N}(\mathbb{R})$.

Recall that the Lie algebra of $\mathrm{Sp_N}(\mathbb{R})$ is given by

$$\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b_1 \\ b_2 & -{}^t a \end{pmatrix},\ a \in \mathcal{M}_{\mathrm{N}}(\mathbb{R}),\ b_1 \text{ and } b_2 \text{ symmetric} \right\}.$$

For $i,j \in \{1,\dots,N\}$, let E_{ij} denote the matrix in $\mathcal{M}_{\mathrm{N}}(\mathbb{R})$ with a coefficient 1 at the intersection of the i-th row and the j-th column, and 0 elsewhere. We also set

$$\forall i,j \in \{1,\dots,N\},\ X_{ij} = \frac{1}{2}\begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix},\ Y_{ij} = {}^t X_{ij},\ Z_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}.$$

Let δ_{ij} be the Kronecker symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } \ i=j \\ 0 & \text{if } \ i \neq j. \end{cases}$$

We observe that the family $\{X_{ij}, Y_{ij}, Z_{ij}\}_{i,j=1..N}$ is a basis of $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$. A direct computation yields, for all $i,j,k,r \in \{1,\dots,N\}$,

(i) $[Z_{ij}, X_{kr}] = \delta_{jk}X_{ir} + \delta_{jr}X_{ik}$

(ii) $[Y_{kr}, Z_{ij}] = \delta_{ik}Y_{rj} + \delta_{ir}Y_{kj}$

(iii) $[X_{ij}, Y_{kr}] = \frac{1}{4}(\delta_{jk}Z_{ir} + \delta_{jr}Z_{ik} + \delta_{ki}Z_{jr} + \delta_{ir}Z_{jk})$

where $[\ ,\ ]$ denotes the usual Lie bracket on the Lie algebras of linear Lie groups. From these relations one deduces that $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$ is generated by

$$\{X_{ij}, Y_{ij} \mid i,j \in \{1,\dots,N\}, |i-j| \leq 1\}.$$

Indeed, let $\mathfrak{g}$ be the Lie algebra generated by this set. Let $i \in \{1,\dots,N\}$. Then $Z_{ii} = 2[X_{ii}, Y_{ii}] \in \mathfrak{g}$ and $Z_{i,i+1} = 2[X_{ii}, Y_{i,i+1}] \in \mathfrak{g}$. Thus, for all $i,j \in \{1,\dots,N\}$ with $|i-j| \leq 1$, we have $Z_{ij} \in \mathfrak{g}$. It follows that $X_{i,i+2} = [Z_{i,i+1}, Y_{i+1,i+2}]$, $Y_{i,i+2} = [Y_{i,i+1}, Z_{i+1,i+2}] \in \mathfrak{g}$ and $Z_{i,i+2} = 2[X_{i,i+1}, Y_{i+1,i+2}] \in \mathfrak{g}$. Hence, for all $i,j \in \{1,\dots,N\}$ with $|i-j| = 2$, X_{ij}, Y_{ij} and Z_{ij} belong to $\mathfrak{g}$. By induction, one proceeds similarly for indices i,j such that $|i-j| = 3$, and more generally for all $i,j \in \{1,\dots,N\}$. One thus proves that $\{X_{ij}, Y_{ij}, Z_{ij}\}_{i,j=1..N}$ is contained in $\mathfrak{g}$, hence $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}) \subset \mathfrak{g}$. Finally, $\mathfrak{g} = \mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$.

Therefore, to prove Lemma 5.2.6, it suffices to show that, for all $E \in \mathbb{R}$, the Lie algebra generated by $\{T_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N\}$ contains all the matrices X_{ij} and Y_{ij} for $i,j \in \{1,\dots,N\}$ with $|i-j| \leq 1$. Let

$$\mathfrak{a}(E) = \mathrm{Lie}\{T_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N\}. \tag{5.1}$$

To show that $\mathfrak{a}(E)$ contains all matrices X_{ij} and Y_{ij} for $i,j \in \{1,\dots,N\}$ with $|i-j| \leq 1$, we proceed in successive steps. Fix $E \in \mathbb{R}$.

Step 1. We prove that matrices of the form

$$\begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \text{ and } \begin{pmatrix} I & 0 \\ D & I \end{pmatrix}$$

where D is diagonal, belong to $X(E)$.

We choose T_1 and T_2 in G_{μ_E} associated with two realizations V_1 and V_2 of $V_{\omega^{(0)}}$. Since $X(E)$ is a group, it is stable under inversion and multiplication. We therefore have

$$B := T_1 T_2^{-1} = \begin{pmatrix} I & V_1 - V_2 \\ 0 & I \end{pmatrix} \in G_{\mu_E}$$

and

$$C := T_1^{-1}T_2 = \begin{pmatrix} I & 0 \\ V_2 - V_1 & I \end{pmatrix} \in G_{\mu_E}.$$

Fix $i \in \{1, \dots, N\}$. We may choose V_1 and V_2 such that

$$B = \begin{pmatrix} I & E_{ii} \\ 0 & I \end{pmatrix} \text{ and } C = \begin{pmatrix} I & 0 \\ E_{ii} & I \end{pmatrix}.$$

Then, for all $n \in \mathbb{Z}$,

$$B^n = \begin{pmatrix} I & 0 \\ nE_{ii} & I \end{pmatrix} \text{ and } C^n = \begin{pmatrix} I & nE_{ii} \\ 0 & I \end{pmatrix}.$$

Let P be a polynomial in $\mathbb{R}[X_{1,1}, \dots, X_{2N,2N}]$ such that $\forall n \in \mathbb{Z},\ P(C^n) = 0$. We fix $X_{j,j} = 1$ for all $j \in \{1, \dots, 2N\}$ and $X_{r,l} = 0$ for $r \neq l$ except for $X_{i,N+i}$, and we consider

$$\widetilde{P} : X_{i,N+i} \mapsto P \begin{pmatrix} I & \begin{pmatrix} 0 & | & 0 \\ - & X_{i,N+i} & - \\ 0 & | & 0 \end{pmatrix} \\ 0 & I \end{pmatrix}.$$

$\widetilde{P}$ is a one-variable polynomial with infinitely many roots, namely the integers $n \in \mathbb{Z}$. Hence $\widetilde{P}$ is the zero polynomial and $\forall \alpha \in \mathbb{R},\ \widetilde{P}(\alpha) = 0$. This means that P vanishes on all matrices $\begin{pmatrix} I & \alpha E_{ii} \\ 0 & I \end{pmatrix}$. Moreover,

$$\forall \alpha \in \mathbb{R}, \begin{pmatrix} I & \alpha E_{ii} \\ 0 & I \end{pmatrix} \in X(E).$$

Since i was arbitrary, we obtain for distinct i and j:

$$\forall \alpha_1, \dots, \alpha_N \in \mathbb{R}, \begin{pmatrix} I & \alpha_1 E_{11} \\ 0 & I \end{pmatrix} \dots \begin{pmatrix} I & \alpha_N E_{NN} \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & \alpha_1 E_{11} + \dots + \alpha_N E_{NN} \\ 0 & I \end{pmatrix} \in X(E).$$

This implies that

$$\begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \in X(E)$$

for all diagonal matrices D. The other part of this first step is treated in the same way using C instead of B.

Step 2: We write

$$\begin{pmatrix} I & D_1 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + 1 \times \begin{pmatrix} 0 & D_1 \\ 0 & 0 \end{pmatrix} \in X(E).$$

By differentiating at the point $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ we obtain

$$\begin{pmatrix} 0 & D_1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{a}(E).$$

We also have

$$\begin{pmatrix} 0 & 0 \\ D_2 & 0 \end{pmatrix} \in \mathfrak{a}(E),$$

and hence the sum

$$\begin{pmatrix} 0 & D_1 \\ D_2 & 0 \end{pmatrix} \in \mathfrak{a}(E)$$

for all diagonal matrices D_1 and D_2. In particular,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathfrak{a}(E).$$

But, for all $t \in \mathbb{R}$,

$$\exp(tJ) = \begin{pmatrix} \cos(t)I & \sin(t)I \\ -\sin(t)I & \cos(t)I \end{pmatrix} \in X(E).$$

For $t = \frac{\pi}{2}$ we obtain $J \in X(E)$.

Step 3: Since we are working in a *linear* Lie group, if $z \in \mathfrak{a}(E)$ and $A \in X(E)$, then $AzA^{-1} \in \mathfrak{a}(E)$.

Step 4: If

$$z = \begin{pmatrix} a & b_1 \\ b_2 & -{}^t a \end{pmatrix} \in \mathfrak{a}(E),$$

since $J \in X(E)$, applying Step 3 yields

$$JzJ^{-1} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} a & b_1 \\ b_2 & -{}^t a \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} -{}^t a & -b_2 \\ -b_1 & a \end{pmatrix} \in \mathfrak{a}(E).$$

In particular, if

$$\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{a}(E),$$

then

$$\begin{pmatrix} 0 & 0 \\ b_1 & 0 \end{pmatrix} \in \mathfrak{a}(E)$$

(since if $z \in \mathfrak{a}(E)$, then $-z \in \mathfrak{a}(E)$). The same holds for

$$\begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix}.$$

We have thus shown that, in our construction, b_1 and b_2 play the same role.

Step 5: Since

$$A = \begin{pmatrix} V_{\omega^{(0)}} & -I \\ I & 0 \end{pmatrix} \in X(E),$$

we have

$$AJ^{-1} = \begin{pmatrix} I & V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \in X(E).$$

Step 6: If

$$z = \begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix} \in \mathfrak{a}(E)$$

and

$$A = \begin{pmatrix} I & V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \in X(E),$$

then

$$\begin{aligned} z_1 &:= AzA^{-1} - z \\ &= \begin{pmatrix} I & V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix} \begin{pmatrix} I & -V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} V_{\omega^{(0)}} b_2 & 0 \\ b_2 & 0 \end{pmatrix} \begin{pmatrix} I & -V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} V_{\omega^{(0)}} b_2 & -V_{\omega^{(0)}} b_2 V_{\omega^{(0)}} \\ 0 & -b_2 V_{\omega^{(0)}} \end{pmatrix} \in \mathfrak{a}(E). \end{aligned}$$

And

$$z_1' = A^{-1} z A - z = \begin{pmatrix} -V_{\omega^{(0)}} b_2 & -V_{\omega^{(0)}} b_2 V_{\omega^{(0)}} \\ 0 & b_2 V_{\omega^{(0)}} \end{pmatrix} \in \mathfrak{a}(E).$$

Finally,

$$\frac{1}{2}(z_1 + z_1') = \begin{pmatrix} V_{\omega^{(0)}} b_2 & 0 \\ 0 & -b_2 V_{\omega^{(0)}} \end{pmatrix} \in \mathfrak{a}(E).$$

If one can construct in $\mathfrak{a}(E)$ a matrix with an off-diagonal block, then one can place this block on the diagonal.

Step 7: Let us show that one can do the reverse, namely that one can move a block from the diagonal off the diagonal. If

$$z = \begin{pmatrix} a & 0 \\ 0 & -{}^t a \end{pmatrix} \in \mathfrak{a}(E)$$

and

$$A = \begin{pmatrix} I & V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \in X(E),$$

then

$$z_2 = \begin{pmatrix} 0 & V^t_{\omega^{(0)}} a + a V_{\omega^{(0)}} \\ 0 & 0 \end{pmatrix} \in \mathfrak{a}(E) \text{ and } \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + t z_2 \in X(E)$$

for every real t. Indeed, $z_2 = z - AzA^{-1}$ and it suffices to take the exponential map from the Lie algebra to the Lie group.

Step 8: We apply Steps 2 and 6 with $b_2 = D$ a diagonal matrix and $V_{\omega^{(0)}} = I$, to obtain

$$z_1 = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in \mathfrak{a}(E).$$

Step 9: We may therefore assume that $D = E_{ii}$. We then have, for every $i \in \{1, \ldots, N\}$,

$$\begin{pmatrix} E_{ii} & \\ 0 & -E_{ii} \end{pmatrix} \in \mathfrak{a}(E).$$

Moreover, by Step 1, if $D_{\omega^{(0)}}$ denotes the diagonal of $V_{\omega^{(0)}}$, then

$$\begin{pmatrix} I & D_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \in X(E).$$

Furthermore, by Step 5,

$$\begin{pmatrix} I & V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \in X(E),$$

and

$$\begin{pmatrix} I & V_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & D_{\omega^{(0)}} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & V_{\omega^{(0)}} - D_{\omega^{(0)}} \\ 0 & I \end{pmatrix} \in X(E).$$

Now,

$$V_{\omega^{(0)}} - D_{\omega^{(0)}} = \begin{pmatrix} 0 & 1 & & & 0 \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix} := V_0.$$

If V_i denotes the matrix containing only the i-th row and the i-th column of V_0, then

$$V_i = E_{ii} V_0 + V_0 E_{ii}$$

(since its only diagonal entry that would carry a factor 2 is zero). Then, for this V_i, by Step 7, we have

$$\begin{pmatrix} I & V_i \\ 0 & I \end{pmatrix} \in \mathfrak{a}(E).$$

Step 10. Recall that $X_{ij} = X_{ji}$ and $Y_{ij} = Y_{ji}$. Let $i \in \{1, \dots, N\}$. By applying Step 7 again with $E_{i-1,i-1}$ or $E_{i+1,i+1}$ and V_i, we obtain that

$$Y_{i,i-1} + Y_{i,i+1} \in \mathfrak{a}(E).$$

For $i = 1$, this means that $Y_{1,2} \in \mathfrak{a}(E)$. Then

$$\frac{1}{2} Z_{1,2} = [X_{1,1}, Y_{1,2}] \in \mathfrak{a}(E),$$

hence $Z_{1,2} \in \mathfrak{a}(E)$. But we also have

$$2X_{1,2} = [Z_{1,2}, X_{2,2}] \in \mathfrak{a}(E),$$

and therefore $X_{1,2} \in \mathfrak{a}(E)$. For $i = 2$, we have

$$Y_{2,1} + Y_{2,3} \in \mathfrak{a}(E).$$

But we have just shown that $Y_{2,1} \in \mathfrak{a}(E)$, hence $Y_{2,3} \in \mathfrak{a}(E)$. By induction, we obtain

$$\forall i \in \{1, \dots, N\},\ Y_{i,i+1} \in \mathfrak{a}(E).$$

We also have, for all $i \in \{1, \dots, N\}$,

$$[X_{ii}, Y_{i,i+1}] = \frac{1}{2} Z_{i,i+1} \in \mathfrak{a}(E) \quad \text{and} \quad [Z_{i,i+1}, X_{i+1,i+1}] = 2X_{i,i+1} \in \mathfrak{a}(E).$$

This shows that all matrices X_{ij} and Y_{ij} for $i, j \in \{1, \dots, N\}$ and $|i - j| \leq 1$ belong to $\mathfrak{a}(E)$. Hence,

$$\mathfrak{a}(E) = \mathfrak{sp}_{\mathrm{N}}(\mathbb{R}).$$

This allows us to deduce that the Furstenberg group associated with the operator $\{h_\omega^N\}_{\omega \in \Omega}$ is Zariski-dense in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$. It follows that the associated Lyapunov exponents are separated and, in particular, that they are all strictly positive, for every energy $E \in \mathbb{R}$.

By applying Oseledets theorem, we deduce that for each fixed energy E, and for P-almost every ω, one can construct N solutions of $h_\omega^N u = Eu$ that decay exponentially to 0 as $+\infty$. One can do the same at $-\infty$, with L solutions a priori distinct from the L obtained at $+\infty$. Indeed, in this case all Lyapunov exponents have multiplicity 1.

As in the scalar case, this leads, by adapting Kotani theory to the setting of matrix-valued Schrödinger operators, to the absence of absolutely continuous spectrum.

5.2 Quasi-1d continuous Anderson model

5.2.1 The model and its almost-sure spectrum

We discuss properties of a random family of matrix-valued one-dimensional Anderson-Bernoulli operators

$$H_{l,\omega} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_N + V + \sum_{n\in\mathbb{Z}} \begin{pmatrix} c_1\omega_1^{(n)}\mathbf{1}_{[0,l]}(x-ln) & & 0 \\ & \ddots & \\ 0 & & c_N\omega_N^{(n)}\mathbf{1}_{[0,l]}(x-ln) \end{pmatrix} \tag{5.2}$$

acting on $L^2(\mathbb{R})\otimes\mathbb{C}^N$, where $N \geq 1$ is an integer, I_N is the identity matrix of order N and $l > 0$ is a real number. The matrix V is a real $N\times N$ symmetric matrix, the space of these matrices being denoted by $\mathrm{S_N}(\mathbb{R})$. The constants $c_1,\dots,c_N$ are non-zero real numbers.

For every $i \in \{1,\dots,N\}$, $(\omega_i^{(n)})_{n\in\mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d., for short) random variables on a complete probability space $(\widetilde{\Omega}_i, \widetilde{\mathcal{A}}_i, \widetilde{\mathsf{P}}_i)$, of common law ν_i such that $\{0,1\} \subset \operatorname{supp}\nu_i$ and $\operatorname{supp}\nu_i$ is bounded. In particular, the $\omega_i^{(n)}$'s can be Bernoulli random variables. The family $\{H_{l,\omega}\}_{\omega\in\Omega}$ is a family of random operators indexed by the product space

$$(\Omega,\mathcal{A},\mathsf{P}) = \left(\bigotimes_{n\in\mathbb{Z}}(\widetilde{\Omega}_1\otimes\cdots\otimes\widetilde{\Omega}_N), \bigotimes_{n\in\mathbb{Z}}(\widetilde{\mathcal{A}}_1\otimes\cdots\otimes\widetilde{\mathcal{A}}_N), \bigotimes_{n\in\mathbb{Z}}(\widetilde{\mathsf{P}}_1\otimes\cdots\otimes\widetilde{\mathsf{P}}_N)\right).$$

We also set, for every $n\in\mathbb{Z}$, $\omega^{(n)} = (\omega_1^{(n)},\dots,\omega_N^{(n)})$, which is a random variable on $(\widetilde{\Omega}_1\otimes\cdots\otimes\widetilde{\Omega}_N, \widetilde{\mathcal{A}}_1\otimes\cdots\otimes\widetilde{\mathcal{A}}_N, \widetilde{\mathsf{P}}_1\otimes\cdots\otimes\widetilde{\mathsf{P}}_N)$ of law $\nu_1\otimes\cdots\otimes\nu_N$. The expectation value with respect to P will be denoted by $\mathbb{E}(\cdot)$.

As a bounded perturbation of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\otimes I_N$, the operator $H_{l,\omega}$ is self-adjoint on the Sobolev space $H^2(\mathbb{R})\otimes\mathbb{C}^N$ and thus, for every $\omega\in\Omega$, the spectrum of $H_{l,\omega}$, denoted by $\sigma(H_{l,\omega})$, is included in $\mathbb{R}$. Moreover, because of the periodicity in law of the random potential of $H_{l,\omega}$, the family $\{H_{l,\omega}\}_{\omega\in\Omega}$ is $l\mathbb{Z}$-ergodic. Thus, there exists $\Sigma\subset\mathbb{R}$ such that, for P-almost every $\omega\in\Omega$, $\Sigma = \sigma(H_{l,\omega})$. There also exist Σ_{pp}, Σ_{ac} and Σ_{sc}, subsets of $\mathbb{R}$, such that, for P-almost every $\omega\in\Omega$, $\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(H_{l,\omega})$, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(H_{l,\omega})$ and $\Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(H_{l,\omega})$, respectively the pure point, absolutely continuous and singular continuous spectrum of $H_{l,\omega}$.

We can give an explicit description of the almost-sure spectrum Σ of $\{H_{l,\omega}\}_{\omega\in\Omega}$. For $\omega^{(0)} = (\omega_1^{(0)},\dots,\omega_N^{(0)}) \in \operatorname{supp}(\nu_1\otimes\cdots\otimes\nu_N)$, we denote by $E_1^{\omega^{(0)}},\dots,E_N^{\omega^{(0)}}$ the real eigenvalues of the real symmetric matrix $V + \operatorname{diag}(c_1\omega_1^{(0)},\dots,c_N\omega_N^{(0)})$. Then, we have

$$\Sigma = [0,+\infty) + \bigcup_{\omega^{(0)}\in\operatorname{supp}(\nu_1\otimes\cdots\otimes\nu_N)} \{E_1^{\omega^{(0)}},\dots,E_N^{\omega^{(0)}}\}. \tag{5.3}$$

In particular, Σ does not depend on the parameter l. It is important that V is constant and, in the random part, that the single site potential is of the form $\mathbf{1}_{[0,l]}$ instead of a generic single site potential $v\in L^1_{\mathrm{loc}}(\mathbb{R})$ supported on $[0,l]$.

If we want to consider the case of a generic single site potential v supported on $[0,l]$, we will face two problems. The first one is that, from our proof of the structure of Σ, we can not recover that Σ is independent of l, which may lead to an empty statement in thge result of localization that we aim at proving. The second problem is that in this case, or if instead of V we choose a matrix-valued function $x\mapsto V(x)$ from $\mathbb{R}$ to $\mathrm{S_N}(\mathbb{R})$ which is not constant, the particular form of the transfer matrices will not be simple anymore. Indeed, our analysis rests on the fact that the transfer matrices are exponentials of matrices. If neither V nor the single site potential $\mathbf{1}_{[0,l]}$ are constant, the transfer matrices become time-ordered exponentials instead of exponentials of matrices. In this case, we can not compute anymore the logarithms of these time-ordered exponentials and all the algebraic approach fails.

5.2.2 Transfer matrices

Let $E \in \mathbb{R}$. We want to understand the exponential asymptotic behaviour of a solution $u : \mathbb{R} \to \mathbb{C}^N$ of the second order differential system

$$H_{l,\omega} u = Eu. \tag{5.4}$$

For this, we transform (5.4) into an Hamiltonian differential system of order 1 and we introduce the transfer matrix $T_{\omega^{(n)}}(E)$ of $H_{l,\omega}$ from ln to $l(n+1)$ which maps a solution (u, u') of the order 1 system at position ln to the solution at position $l(n+1)$. The transfer matrix $T_{\omega^{(n)}}(E)$ is therefore defined by the relation

$$\begin{pmatrix} u(l(n+1)) \\ u'(l(n+1)) \end{pmatrix} = T_{\omega^{(n)}}(E) \begin{pmatrix} u(ln) \\ u'(ln) \end{pmatrix} \tag{5.5}$$

for all $n \in \mathbb{Z}$. Since $T_{\omega^{(n)}}(E)$ is the solution of an Hamiltonian differential system of order 1 at time 1, the transfer matrix $T_{\omega^{(n)}}(E)$ lies in the symplectic group $\mathrm{Sp_N}(\mathbb{R})$. The sequence $(T_{\omega^{(n)}}(E))_{n\in\mathbb{Z}}$ is also a sequence of *i.i.d.* symplectic matrices because of the *i.i.d.* character of the $\omega_i^{(n)}$'s for every i in $\{1, \ldots, N\}$ and the non-overlapping of these random variables. By iterating the relation (9.31) we get the asymptotic behaviour of (u, u').

Definition 5.2.1. *For every $E \in \mathbb{R}$, the* Furstenberg group *of $\{H_{l,\omega}\}_{\omega\in\Omega}$ is defined by*

$$G_{\mu_E} = \overline{< \mathrm{supp}\, \mu_E >},$$

where μ_E is the common distribution of the $T_{\omega^{(n)}}(E)$ and the closure is taken for the usual topology in $\mathrm{Sp_N}(\mathbb{R})$.

As the $T_{\omega^{(n)}}(E)$ are i.i.d., $\mu_E = (T_{\omega^{(0)}}(E))_* (\nu_1 \otimes \cdots \otimes \nu_N)$ and we have the internal description of G_{μ_E} :

$$G_{\mu_E} = \overline{< T_{\omega^{(0)}}(E)\, ;\, \omega^{(0)} \in \mathrm{supp}(\nu_1 \otimes \cdots \otimes \nu_N) >} \text{ for all } E \in \mathbb{R}. \tag{5.6}$$

As, for every $i \in \{1, \ldots, N\}$, $\{0,1\} \subset \mathrm{supp}\, \nu_i$, we also have

$$\overline{< T_{\omega^{(0)}}(E)\, ;\, \omega^{(0)} \in \{0,1\}^N >} \subset G_{\mu_E}. \tag{5.7}$$

We will denote by $G_{\{0,1\}}(E)$ the subgroup $< T_{\omega^{(0)}}(E)\, ;\, \omega^{(0)} \in \{0,1\}^N >$ of G_{μ_E} with 2^N generators.

We will prove that, for almost every $V \in \mathrm{S_N}(\mathbb{R})$ and for all $E \in \mathbb{R}$ except those in a finite set, $G_{\{0,1\}}(E)$ is dense in $\mathrm{Sp_N}(\mathbb{R})$. Actually, we will prove it for V_0 the multiplication operator by the tridiagonal matrix V_0 having a null diagonal and coefficients on the upper and lower diagonals all equal to 1. Then one can apply a genericity argument to obtain the desired results.

We finish this section by giving the explicit form of the transfer matrices $T_{\omega^{(n)}}(E)$. Let $V \in \mathrm{S_N}(\mathbb{R})$, $E \in \mathbb{R}$, $n \in \mathbb{Z}$ and $\omega^{(n)} \in \mathrm{supp}(\nu_1 \otimes \cdots \otimes \nu_N)$. We set

$$M_{\omega^{(n)}}(E, V) = V + \mathrm{diag}(c_1\omega_1^{(n)}, \ldots, c_N\omega_N^{(n)}) - EI_{\mathrm{N}}. \tag{5.8}$$

Then, we set the matrix in the Lie algebra $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}) \subset \mathcal{M}_{2\mathrm{N}}(\mathbb{R})$

$$X_{\omega^{(n)}}(E,V) = \begin{pmatrix} 0 & I_{\mathrm{N}} \\ M_{\omega^{(n)}}(E,V) & 0 \end{pmatrix}. \tag{5.9}$$

By solving the constant coefficients system (5.4) on $[ln, l(n+1)]$, we have

$$T_{\omega^{(n)}}(E) = \exp\left(l X_{\omega^{(n)}}(E,V)\right) \tag{5.10}$$

for every $l > 0$, every $n \in \mathbb{Z}$, every $V \in \mathrm{S}_{\mathrm{N}}(\mathbb{R})$ and every $E \in \mathbb{R}$.

Since the transfer matrices are exponentials of matrices, we cannot directly apply the approach of the matrix-valued discrete case. Indeed, the morphism property of the exponential fails in the non-commutative case. Thus, we need an other approach which is based upon a general result of Breuillard and Gelander.

5.2.3 Generating dense subgroups in a Lie group

Proving that a subgroup of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is Zariski-dense in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ remains a constructive problem that can be quite difficult to implement. Already in the work of Goldsheid and Margulis, the construction made for a matrix-valued discrete Schrödinger operator is relatively intricate. In the case of matrix-valued continuous operators, one faces the fact that the transfer matrices are more complicated than in the discrete case and that the construction of [10] proves to be unstable under perturbation. To study the continuous model, one must therefore find a way to prove that a subgroup of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is Zariski-dense. In the work of Breuillard and Gelander, a stronger criterion is provided that allows one to prove that a subgroup of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is dense for the usual topology in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$. It is this criterion and its consequences that we now present.

We present a criterion that reduces the question of whether a subgroup of a semi-simple Lie group G generated by a finite number of elements is dense to a problem of reconstructing the Lie algebra of G. This result is presented in [7], where some of its consequences are explored.

In the work of Breuillard and Gelander, the result we will state is given in the framework of topologically perfect groups. Let us therefore start by giving a definition and see why the real symplectic group $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ satisfies it.

Definition 5.2.2. *A connected Lie group G is said to be topologically perfect if its derived group* $[G,G]$ *is dense in G.*

In our context, recall that the symplectic group $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is connected and semi-simple. Moreover, a connected semi-simple Lie group G always satisfies $[G,G] = G$, hence it is topologically perfect. Indeed, let us recall the definition of a semisimple group.

Definition 5.2.3. *A connected Lie group G is said to be* topologically perfect *if its derived group* $[G,G]$ *(that is, the subgroup of G generated by the commutators* $[x,y] = x^{-1}y^{-1}xy$*) is dense in G.*

In our framework, we recall that the symplectic group $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ is connected and semisimple. Now, any connected semisimple Lie group G always satisfies $[G,G] = G$, and is therefore topologically perfect. Indeed, recall that by definition a Lie group is said to be semisimple when it has no non-trivial abelian quotient. By construction, the quotient $G/[G,G]$ is abelian. Hence, if G is semisimple, this quotient can only be equal to $\{e\}$. Therefore, $G = [G,G]$.
We may therefore apply the following theorem with $G = \mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$, which is semisimple.

Theorem 5.2.4 (Breuillard and Gelander). *Let G be a connected real Lie group that is topologically perfect, with Lie algebra* $\mathfrak{g}$*. There exists a neighborhood of the identity* $\mathcal{O} \subset G$ *on which* $\log = \exp^{-1}$ *is a well-defined diffeomorphism, such that* $g_1, \dots g_m \in \mathcal{O}$ *generate a dense subgroup of G if and only if* $\log(g_1), \dots, \log(g_m)$ *generate* $\mathfrak{g}$.

This result tells us that in a certain neighborhood of the identity (which by construction depends only on G), a finite subset is topologically generating for G if and only if it is algebraically generating for $\mathfrak{g}$.

5.2.4 Separation of Lyapunov Exponents

Notations

Before stating our main results, we need to introduce some more notations. Let $\mathcal{O}$ be the neighborhood of $I_{2\mathrm{N}}$ in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ given by Theorem 5.2.4 applied to $G = \mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$.

We set

$$d_{\log \mathcal{O}} = \max\{R > 0\,;\, B(0,R) \subset \log \mathcal{O}\}, \tag{5.11}$$

where $B(0,R)$ is the open ball, centered on 0 and of radius $R > 0$, for the metric induced on the Lie algebra $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$ of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ by the matrix norm induced by the euclidean norm on $\mathbb{R}^{2N}$.

For $\omega^{(0)} = (\omega_1^{(0)}, \ldots, \omega_N^{(0)}) \in \{0,1\}^N$, let

$$M_{\omega^{(0)}}(0,V) = V + \mathrm{diag}(c_1\omega_1^{(0)}, \ldots, c_N\omega_N^{(0)}).$$

As $M_{\omega^{(0)}}(0,V) \in \mathrm{S}_{\mathrm{N}}(\mathbb{R})$, it has $\lambda_1^{\omega^{(0)}}, \ldots, \lambda_N^{\omega^{(0)}}$ as real eigenvalues. We set,

$$\lambda_{\min} = \min_{\omega^{(0)} \in \{0,1\}^N} \min_{1 \le i \le N} \lambda_i^{\omega^{(0)}}, \qquad \lambda_{\max} = \max_{\omega^{(0)} \in \{0,1\}^N} \max_{1 \le i \le N} \lambda_i^{\omega^{(0)}} \tag{5.12}$$

and $\delta = (\lambda_{\max} - \lambda_{\min})/2$. We also set

$$l_C := l_C(N,V) = \min\left(1, \frac{d_{\log \mathcal{O}}}{\delta}\right) \tag{5.13}$$

and, for every $l \in (0, l_C)$,

$$I(N,V,l) = \left[\lambda_{\max} - \frac{d_{\log \mathcal{O}}}{l}, \lambda_{\min} + \frac{d_{\log \mathcal{O}}}{l}\right]. \tag{5.14}$$

We remark that, as l tends to 0^+, $I(N,V,l)$ tends to the whole real line. We can now state our main result. For $E \in \mathbb{R}$, let G_{μ_E} be the Fürstenberg group associated to $H_{l,\omega}$ (see Definition 5.2.1).

Results

We can now state the result on the separation of Lyapunov exponents.

Theorem 5.2.5. *For almost every $V \in \mathrm{S}_{\mathrm{N}}(\mathbb{R})$, there exist a finite set $\mathcal{S}_{\mathrm{V}} \subset \mathbb{R}$ and a real number $\ell_C(N,V) > 0$ such that, for all $\ell \in]0, \ell_C(N,V)[$, there exists a compact interval $I(N,V,\ell) \subset \mathbb{R}$ on which the N positive Lyapunov exponents $\gamma_1(E), \ldots, \gamma_N(E)$ of $\{H_{\omega,\ell}^{(N)}\}_{\omega \in \Omega}$ satisfy:*

$$\forall E \in I(N,V,\ell) \setminus \mathcal{S}_{\mathrm{V}}, \quad \gamma_1(E) > \cdots > \gamma_N(E) > 0. \tag{5.15}$$

As we shall see presently, the real number $\ell_C(N,V)$ and the interval $I(N,V,\ell)$ for $\ell < \ell_C(N,V)$ are defined in such a way that one can apply Theorem 5.2.4 to the Furstenberg group of $\{H_{\omega,\ell}^{(N)}\}_{\omega \in \Omega}$.

Theorem 5.2.4 provides the outline of our proof.

1. We construct $\ell_C(N,V)$ and $I(N,V,\ell)$ so that for all $\ell \in]0, \ell_C(N,V)[$ and all $E \in I(N,V,\ell)$, one has $T_{\omega^{(0)}}(E) \in \mathcal{O}$ for all $\omega^{(0)} \in \{0,1\}^N$, where $\mathcal{O}$ is the neighborhood of I_{2N} given by Theorem 5.2.4 applied to $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$.

2. For $\ell < \ell_C(N,V)$, we compute $\log T_{\omega^{(0)}}(E)$.

3. Finally, we prove that the Lie algebra generated by these logarithms,

$$\mathrm{Lie}\{\log T_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N\} = \mathfrak{sp}_{\mathrm{N}}(\mathbb{R}),$$

is the Lie algebra of $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$.

An algebraic lemma

We assume that $V = V_0$, where V_0 denotes the multiplication operator by the tridiagonal matrix whose diagonal coefficients are zero and whose sub- and super-diagonal coefficients are equal to 1.

Lemma 5.2.6. *Let $N \geq 1$ and $E \in \mathbb{R}$. The Lie algebra generated by $\{X_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N\}$ is equal to $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$.*

Proof : First, we recall that :

$$\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b_1 \\ b_2 & -{}^t a \end{pmatrix},\ a \in \mathcal{M}_{\mathrm{N}}(\mathbb{R}),\ b_1 \text{ and } b_2 \text{ symmetric} \right\}.$$

For $i,j \in \{1,\ldots,N\}$, let E_{ij} be the matrix in $\mathcal{M}_{\mathrm{N}}(\mathbb{R})$ with a 1 coefficient at the intersection of the ith row and the jth column, and 0 elsewhere. We also set

$$\forall i,j \in \{1,\ldots,N\},\ X_{ij} = \frac{1}{2}\begin{pmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{pmatrix},\ Y_{ij} = {}^t X_{ij},\ Z_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}.$$

We also denote by δ_{ij} the Kronecker's symbol :

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

We remark that the set $\{X_{ij}, Y_{ij}, Z_{ij}\}_{i,j=1..N}$ is a basis of $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$. By direct computation, we get the relations, for every $i,j,k,r \in \{1,\ldots,N\}$,

$[Z_{ij}, X_{kr}] = \delta_{jk}X_{ir} + \delta_{jr}X_{ik}$

(ii) $[Y_{kr}, Z_{ij}] = \delta_{ik}Y_{rj} + \delta_{ir}Y_{kj}$

(iii) $[X_{ij}, Y_{kr}] = \frac{1}{4}(\delta_{jk}Z_{ir} + \delta_{jr}Z_{ik} + \delta_{ki}Z_{jr} + \delta_{ir}Z_{jk})$

where [,] is the usual bracket on Lie algebra of linear Lie groups. From these relations, we deduce that $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$ is generated by

$$\{X_{ij}, Y_{ij} \mid i,j \in \{1,\ldots,N\}, |i-j| \leq 1\}.$$

Indeed, let $\mathfrak{g}$ be the Lie algebra generated by this set. Let $i \in \{1,\ldots,N\}$. Then, $Z_{ii} = 2[X_{ii}, Y_{ii}] \in \mathfrak{g}$ and $Z_{i,i+1} = 2[X_{ii}, Y_{i,i+1}] \in \mathfrak{g}$. Thus, for every $i,j \in \{1,\ldots,N\}$, $|i-j| \leq 1$, $Z_{ij} \in \mathfrak{g}$. Then we have, $X_{i,i+2} = [Z_{i,i+1}, Y_{i+1,i+2}]$, $Y_{i,i+2} = [Y_{i,i+1}, Z_{i+1,i+2}] \in \mathfrak{g}$ and $Z_{i,i+2} = 2[X_{i,i+1}, Y_{i+1,i+2}] \in \mathfrak{g}$. Thus, for every $i,j \in \{1,\ldots,N\}$, $|i-j| = 2$, X_{ij}, Y_{ij}, $Z_{ij} \in \mathfrak{g}$. By induction, we do the same for indices i,j such that $|i-j| = 3$ and more generally for all indices $i,j \in \{1,\ldots,N\}$. Thus, we proved that $\{X_{ij}, Y_{ij}, Z_{ij}\}_{i,j=1..N}$ is included in $\mathfrak{g}$ and then $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}) \subset \mathfrak{g}$. Finally, $\mathfrak{g} = \mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$.

According to this, to prove Lemma 5.2.6, we only have to prove that, for every $E \in \mathbb{R}$, the Lie algebra generated by $\{X_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N\}$ contains all the matrices X_{ij} and Y_{ij} for $i,j \in \{1,\dots,N\}$, $|i-j| \leq 1$. Let

$$\mathfrak{a}(E) = \mathrm{Lie}\big\{X_{\omega^{(0)}}(E) \;\big|\; \omega^{(0)} \in \{0,1\}^N\big\}. \tag{5.16}$$

To prove that $\mathfrak{a}(E)$ contains the matrices X_{ij} and Y_{ij} for $i,j \in \{1,\dots,N\}$, $|i-j| \leq 1$, we will proceed in several steps. We fix $E \in \mathbb{R}$.

Step 1. We prove that the matrices Z_{ii} for $i \in \{1,\dots,N\}$ are in $\mathfrak{a}(E)$. Let $\omega^{(0)}$ and $\tilde{\omega}^{(0)}$ in $\{0,1\}^N$. We have :

$$\begin{aligned}
&[X_{\omega^{(0)}}(E), X_{\tilde{\omega}^{(0)}}(E)] = X_{\omega^{(0)}}(E)X_{\tilde{\omega}^{(0)}}(E) - X_{\tilde{\omega}^{(0)}}(E)X_{\omega^{(0)}}(E)\\
&= \mathrm{diag}(c_1(\tilde{\omega}_1^{(0)} - \omega_1^{(0)}),\dots,c_N(\tilde{\omega}_N^{(0)} - \omega_N^{(0)}), c_1(\omega_1^{(0)} - \tilde{\omega}_1^{(0)}),\dots,c_N(\omega_N^{(0)} - \tilde{\omega}_N^{(0)})).
\end{aligned}$$

In particular, for $\omega^{(0)} = (0,\dots,0)$ and $\tilde{\omega}^{(0)} = (0,\dots,1,\dots,0)$, with a 1 at the ith place and 0 elsewhere, we get $Z_{ii} = [X_{\omega^{(0)}}(E), X_{\tilde{\omega}^{(0)}}(E)] \in \mathfrak{a}(E)$.

Step 2. With the same choice of $\omega^{(0)}$ and $\tilde{\omega}^{(0)}$, we get

$$X_{\tilde{\omega}^{(0)}}(E) - X_{\omega^{(0)}}(E) = Y_{ii}.$$

Thus, for every $i \in \{1,\dots,N\}$, $Y_{ii} \in \mathfrak{a}(E)$.

Step 3. We fix $\omega^{(0)} \in \{0,1\}^N$ and $i \in \{1,\dots,N\}$. We have :

$$\begin{aligned}
[X_{\omega^{(0)}}(E), Z_{ii}] &= \begin{pmatrix} 0 & -2E_{ii} \\ M_{\omega^{(0)}}(0)E_{ii} + E_{ii}M_{\omega^{(0)}}(0) - 2EE_{ii} & 0 \end{pmatrix}\\
&= -2X_{ii} + 2Y_{i,i-1} + 2Y_{i,i+1} + 2(\omega_i^{(0)} - E)Y_{ii}
\end{aligned}$$

with the convention that Y_{ij} is zero if the index j is not in $\{1,\dots,N\}$. Thus, dividing by 2, one gets,

$$\forall i \in \{1,\dots,N\},\ -X_{ii} + Y_{i,i-1} + Y_{i,i+1} + (\omega_i^{(0)} - E)Y_{ii} \in \mathfrak{a}(E). \tag{5.17}$$

Step 4. We prove that the matrix J is in $\mathfrak{a}(E)$. We fix $\omega^{(0)} = (0,\dots,0)$. By summing (5.17) for $i \in \{1,\dots,N\}$, we stay in $\mathfrak{a}(E)$ and we have :

$$\sum_{i=1}^{N}(-X_{ii} + Y_{i,i-1} + Y_{i,i+1} - EY_{ii}) = \sum_{i=1}^{N}(-X_{ii}) + \begin{pmatrix} 0 & 0 \\ M_{\omega^{(0)}}(E) & 0 \end{pmatrix} \in \mathfrak{a}(E).$$

We can substract $X_{\omega^{(0)}}(E) \in \mathfrak{a}(E)$ from this, to get :

$$\sum_{i=1}^{N}(-X_{ii}) + \begin{pmatrix} 0 & -I_{\mathrm{N}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2I_{\mathrm{N}} \\ 0 & 0 \end{pmatrix} \in \mathfrak{a}(E).$$

Thus, $\left(\begin{smallmatrix} 0 & -I_{\mathrm{N}} \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{a}(E)$. But, by Step 2, all the Y_{ii}'s are in $\mathfrak{a}(E)$, so we also have :

$$\sum_{i=1}^{N} Y_{ii} = \begin{pmatrix} 0 & 0 \\ I_{\mathrm{N}} & 0 \end{pmatrix} \in \mathfrak{a}(E).$$

By adding these two matrices, $J = \left(\begin{smallmatrix} 0 & -I_{\mathrm{N}} \\ I_{\mathrm{N}} & 0 \end{smallmatrix}\right) \in \mathfrak{a}(E)$.

Step 5. For every $i \in \{1,\dots,N\}$, $[J, Z_{ii}] = 2Y_{ii} + 2X_{ii} \in \mathfrak{a}(E)$. But $Y_{ii} \in \mathfrak{a}(E)$, so $2X_{ii} = [J, Z_{ii}] - 2Y_{ii} \in \mathfrak{a}(E)$ and, for every $i \in \{1,\dots,N\}$, $X_{ii} \in \mathfrak{a}(E)$.

Step 6. We recall that $X_{ij} = X_{ji}$ and $Y_{ij} = Y_{ji}$. Let $i \in \{1,\dots,N\}$. Substracting $(\omega^{(0)} - E)Y_{ii} \in \mathfrak{a}(E)$ and adding $X_{ii} \in \mathfrak{a}(E)$ in (5.17) we get $Y_{i,i-1} + Y_{i,i+1} \in \mathfrak{a}(E)$. For $i = 1$, it means that $Y_{1,2} \in \mathfrak{a}(E)$. Then, $\frac{1}{2}Z_{1,2} = [X_{1,1}, Y_{1,2}] \in \mathfrak{a}(E)$ and $Z_{1,2} \in \mathfrak{a}(E)$. But we also have $2X_{1,2} = [Z_{1,2}, X_{2,2}] \in \mathfrak{a}(E)$ and $X_{1,2} \in \mathfrak{a}(E)$. Now, for $i = 2$, we have $Y_{2,1} + Y_{2,3} \in \mathfrak{a}(E)$. But we just proved that $Y_{2,1} \in \mathfrak{a}(E)$, thus $Y_{2,3} \in \mathfrak{a}(E)$. Inductively, we prove that :

$$\forall i \in \{1,\dots,N\},\ Y_{i,i+1} \in \mathfrak{a}(E).$$

Also, for every $i \in \{1,\dots,N\}$,

$$[X_{ii}, Y_{i,i+1}] = \frac{1}{2}Z_{i,i+1} \in \mathfrak{a}(E) \quad \text{and} \quad [Z_{i,i+1}, X_{i+1,i+1}] = 2X_{i,i+1} \in \mathfrak{a}(E).$$

It proves that all the matrices X_{ij} and Y_{ij} for $i,j \in \{1,\dots,N\}$ and $|i-j| \leq 1$ are in $\mathfrak{a}(E)$. Thus, $\mathfrak{a}(E) = \mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$.

□

Continuation of the proof

Let us return to the first step of the proof. First, the eigenvalues of $X_{\omega^{(0)}}(E)^t X_{\omega^{(0)}}(E)$ are 1, $(\lambda_1^{\omega^{(0)}} - E)^2, \dots, (\lambda_N^{\omega^{(0)}} - E)^2$, hence:

$$||X_{\omega^{(0)}}(E)|| = \max\left(1, \max_{1\leq i\leq N} |\lambda_i^{\omega^{(0)}} - E|\right),$$

where $||\ ||$ denotes the matrix norm associated with the Euclidean norm on $\mathbb{R}^{2N}$.

Since the neighborhood $\mathcal{O}$ depends only on $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$, hence only on N, we aim to construct an interval of values of E such that, for ℓ sufficiently small,

$$\forall \omega^{(0)} \in \{0,1\}^N,\ 0 < \ell ||X_{\omega^{(0)}}(E)|| < d_{\log \mathcal{O}}, \tag{5.18}$$

or equivalently

$$0 < \ell \max\left(1, \max_{\omega^{(0)}\in\{0,1\}^N} \max_{1\leq i\leq N} |\lambda_i^{\omega^{(0)}} - E|\right) < d_{\log \mathcal{O}}. \tag{5.19}$$

Assume that $\ell \leq d_{\log \mathcal{O}}$ and set $r_\ell = \frac{1}{\ell} d_{\log \mathcal{O}} \geq 1$. Then, since $r_\ell \geq 1$, the set

$$I(N,V,\ell) = \left\{ E \in \mathbb{R} \,\middle|\, \max\left(1, \max_{\omega^{(0)}\in\{0,1\}^N} \max_{1\leq i\leq N} |\lambda_i^{\omega^{(0)}} - E|\right) \leq r_\ell \right\} \tag{5.20}$$

can be written as the following intersection,

$$I(N,V,\ell) = \bigcap_{\omega^{(0)}\in\{0,1\}^N} \bigcap_{1\leq i\leq N} [\lambda_i^{\omega^{(0)}} - r_\ell, \lambda_i^{\omega^{(0)}} + r_\ell]. \tag{5.21}$$

If $\lambda_0 < r_\ell$, then $I(N,V,\ell) \neq \emptyset$ and more precisely, $I(N,V,\ell) = [\lambda_{\max} - r_\ell, \lambda_{\min} + r_\ell]$.

This interval is centered at $\frac{\lambda_{\min}+\lambda_{\max}}{2}$ and has length $2r_\ell - 2\lambda_0 > 0$, which tends to $+\infty$ as $\ell \to 0^+$.

Moreover, since $\lambda_{\min}$, $\lambda_{\max}$, and $d_{\log \mathcal{O}}$ depend only on N and V, $I(N,V,\ell)$ depends only on ℓ, V, and N, and the condition $\lambda_0 < r_\ell$ is equivalent to

$$\ell < \frac{d_{\log \mathcal{O}}}{\lambda_0} = \ell_C.$$

We have thus constructed ℓ_C and $I(N,V,\ell)$ such that

$$\forall \ell < \ell_C,\ \forall E \in I(N,V,\ell),\ 0 < \ell ||X_{\omega^{(0)}}(E)|| \leq d_{\log \mathcal{O}}. \tag{5.22}$$

Recall that, by the definition of $\mathcal{O}$ via Theorem 5.2.4, the map *exp* is a diffeomorphism from $\log \mathcal{O}$ onto $\mathcal{O}$. Therefore, for all $E \in I(N,V,\ell)$, $\log\, T_{\omega^{(0)}}(E) = \ell X_{\omega^{(0)}}(E)$, which immediately completes the second step of our proof.

For the third step, it suffices to apply Lemma 5.2.6 to obtain:

$$\forall \ell > 0,\ \forall E \in \mathbb{R},\ \mathrm{Lie}\{\ell X_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N\} = \mathfrak{sp}_{\mathrm{N}}(\mathbb{R})\,. \tag{5.23}$$

We can thus finally apply Theorem 5.2.4 to conclude that

$$\forall \ell < \ell_C,\ \forall E \in I(\ell,N),\ \overline{< T_{\omega^{(0)}}(E) \mid \omega^{(0)} \in \{0,1\}^N >} = \mathrm{Sp}_{\mathrm{N}}(\mathbb{R})\,. \tag{5.24}$$

The genericity argument

The use of the Breuillard and Gelander theorem to obtain the separation of Lyapunov exponents leads us to prove an algebraic property on a Lie algebra generated by a finite number of matrices. This is an open condition and, moreover, the n-tuples of elements in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ that do not generate a dense subgroup are contained in a closed analytic submanifold.

In Theorem 5.2.5, genericity is understood in the sense of the Lebesgue measure on $\mathrm{S}_{\mathrm{N}}(\mathbb{R})$, identified with the Lebesgue measure on $\mathbb{R}^{\frac{N(N+1)}{2}}$. However, the proof of this theorem implies a stronger result, which is Zariski genericity, meaning that one can choose V in a Zariski dense open subset of $\mathrm{S}_{\mathrm{N}}(\mathbb{R})$.

For $k \in \mathbb{N}^*$, let

$$\mathcal{V}_k = \Big\{(X_1,\dots,X_k) \in (\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}))^k \mid (X_1,\dots,X_k) \text{ does not generate } \mathfrak{sp}_{\mathrm{N}}(\mathbb{R})\Big\}. \tag{5.25}$$

Since generating the algebra $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$ is an algebraic condition of the type non-vanishing of a finite family of determinants (finite because, for any $m \in \mathbb{N}^*$, $\mathbb{R}[T_1,\dots,T_m]$ is Noetherian), there exist $Q_1,\dots,Q_{r_k} \in \mathbb{R}[(\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}))^k]$ such that:

$$\mathcal{V}_k = \Big\{(X_1,\dots,X_k) \in (\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}))^k \mid Q_1(X_1,\dots,X_k) = 0,\dots,Q_{r_k}(X_1,\dots,X_k) = 0\Big\}. \tag{5.26}$$

Here, we identify $\mathbb{R}[(\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}))^k] \simeq \mathbb{R}[T_1,\dots,T_{k(2N^2+N)}]$. Let $E \in \mathbb{R}$ and

$$\mathcal{V}_{(E)} = \{V \in \mathrm{S}_{\mathrm{N}}(\mathbb{R}) \mid \{X_1(E,V),\dots,X_{2^N}(E,V)\} \text{ does not generate } \mathfrak{sp}_{\mathrm{N}}(\mathbb{R})\}\,. \tag{5.27}$$

We show that $\mathrm{Leb}_{\frac{N(N+1)}{2}}(\mathcal{V}_{(E)}) = 0$. Indeed, let

$$f_E\ :\ \begin{array}{ccc} \mathrm{S}_{\mathrm{N}}(\mathbb{R}) & \to & (\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}))^{2^N} \\ V & \mapsto & (X_1(E,V),\dots,X_{2^N}(E,V)) \end{array}. \tag{5.28}$$

Then, f_E is polynomial in the $\frac{N(N+1)}{2}$ coefficients defining V, and we have:

$$V \in \mathcal{V}_{(E)} \ \Leftrightarrow\ (Q_1 \circ f_E)(V) = 0,\dots,(Q_{r_{2^N}} \circ f_E)(V) = 0, \tag{5.29}$$

each $Q_i \circ f_E$ being polynomial in the $\frac{N(N+1)}{2}$ coefficients defining V. Moreover, we have proven that $V_0 \notin \mathcal{V}_{(E)}$. Therefore, there exists $i_0 \in \{1,\dots,r_{2^N}\}$ such that $(Q_{i_0} \circ f_E)(V_0) \neq 0$ and, since the function $Q_{i_0} \circ f_E$ is polynomial and not identically zero,

$$\mathrm{Leb}_{\frac{N(N+1)}{2}}(\{V \in \mathrm{S}_{\mathrm{N}}(\mathbb{R}) \mid (Q_i \circ f_E)(V) = 0)\}) = 0, \tag{5.30}$$

and by inclusion,

$$\mathrm{Leb}_{\frac{N(N+1)}{2}}(\mathcal{V}_{(E)}) = 0. \tag{5.31}$$

Finally, let $\mathcal{V} = \cap_{E\in\mathbb{R}}\mathcal{V}_{(E)}$. Then $\mathcal{V}$ has Lebesgue measure zero, and if $V \notin \mathcal{V}$, there exists $E_0 \in \mathbb{R}$ such that the family $\{X_1(E_0,V),\dots,X_{2^N}(E_0,V)\}$ generates $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$. Therefore, there exists $i_0 \in \{1,\dots,r_{2^N}\}$ such that $(Q_{i_0}\circ f)(E_0,V) \neq 0$, where

$$f\ :\ \begin{array}{ccc} \mathbb{R}\times \mathrm{S_N}(\mathbb{R}) & \to & (\mathfrak{sp}_{\mathrm{N}}(\mathbb{R}))^{2^N} \\ (E,V) & \mapsto & (X_1(E,V),\dots,X_{2^N}(E,V)) \end{array}. \tag{5.32}$$

For fixed V, the map $E \mapsto (Q_{i_0}\circ f)(E,V)$ is polynomial and not identically zero, so it has only a finite set $\mathcal{S}_{\mathrm{V}}$ of zeros, and for all $E \in \mathbb{R}\setminus\mathcal{S}_{\mathrm{V}}$, $(Q_{i_0}\circ f)(E,V) \neq 0$, that is,

$$\forall E \in \mathbb{R}\setminus\mathcal{S}_{\mathrm{V}},\ \{X_1(E,V),\dots,X_{2^N}(E,V)\} \notin \mathcal{V}_{2^N}. \tag{5.33}$$

Hence, we have obtained that $\mathcal{V}$ has Lebesgue measure zero and that if $V \notin \mathcal{V}$, there exists a finite set $\mathcal{S}_{\mathrm{V}} \subset \mathbb{R}$ such that for all $E \in \mathbb{R}\setminus\mathcal{S}_{\mathrm{V}}$, $\{X_1(E,V),\dots,X_{2^N}(E,V)\}$ generates $\mathfrak{sp}_{\mathrm{N}}(\mathbb{R})$.

From this, the proof of Theorem 5.2.5 is completed. We fix $V \in \mathrm{S_N}(\mathbb{R})\setminus\mathcal{V}$ and apply Theorem 5.2.4, initially defining the real number $\ell_C(N,V)$ and the interval $I(N,V,\ell)$ so that the logarithms of the $T_{\omega^{(0)}}(E)$ equal $\ell X_{\omega^{(0)}}(E,V)$ and lie in $\log\mathcal{O}$.

As we have just seen, it is the algebraic nature of the objects involved that allows one to prove a generic result in V and the finiteness of the set of critical energies. The ideas used can be summarized by recalling that the set of zeros of a nonzero polynomial in one variable is finite, and more generally, the set of zeros of a nonzero polynomial in several variables has Lebesgue measure zero.

Chapter 6

Kotani's theory

6.1 Generalized eigenvalues and spectrum

In this section, we state results in both the discrete and continuous cases that allow the characterization of the spectrum using the notion of eigenvalues and generalized eigenfunctions.

All results will be stated for Schrödinger operators. These are deterministic results.

6.1.1 Discrete Case

We begin by stating the desired definitions and results for a discrete Schrödinger operator acting on $\ell^2(\mathbb{Z}^d)$.

Recall that a function $u : \mathbb{Z}^d \to \mathbb{R}$ is said to be polynomially bounded if there exist constants $C > 0$ and $p \geq 0$ such that

$$\forall n \in \mathbb{Z}^d,\ |u(n)| \leq C(1 + ||n||_\infty)^p.$$

In this section, $H = -\Delta_{\text{disc}} + V$ is a discrete Schrödinger operator with arbitrary potential V.

Definition 6.1.1. *We say that $E \in \mathbb{R}$ is a* generalized eigenvalue *of H if there exists a polynomially bounded function $u : \mathbb{Z}^d \to \mathbb{R}$ satisfying $Hu = Eu$. Such a function u is called a* generalized eigenfunction.

Let $\sigma_{p-g}(H)$ denote the set of generalized eigenvalues of H.

Note that a generalized eigenfunction is not necessarily an eigenfunction of H since it is not assumed to belong to $\ell^2(\mathbb{Z}^d)$.

Finally, we say that two Borel sets A and B of $\mathbb{R}$ are equal up to a set of spectral measure zero if $E_{A\setminus B}(H) = E_{B\setminus A}(H) = 0$. Using this notion, we can state the theorem characterizing the spectrum by generalized eigenvalues.

Theorem 6.1.2. *The spectrum of H coincides with $\sigma_{p-g}(H)$ up to a set of spectral measure zero.*

From this we deduce the following result.

Corollary 6.1.3. *We have $\sigma_{p-g}(H) \subset \sigma(H)$. Moreover,*

$$\sigma(H) = \overline{\sigma_{p-g}(H)}.$$

Let B be a Borel subset of $\mathbb{R}$ and let $\mu(B) = E_B(H)$ denote the projector-valued measure associated with H. For $\varphi, \psi \in \ell^2(\mathbb{Z}^d)$, define $\mu_{\varphi,\psi}(B) = (\varphi, \mu(B)\psi)$. In particular, for $m, n \in \mathbb{Z}^d$,

$$\mu_{m,n}(B) = (\delta_m, \mu(B)\delta_n).$$

Let $(\alpha_n)_{n\in\mathbb{Z}^d}$ be a family of strictly positive real numbers such that $\sum_{n\in\mathbb{Z}^d}\alpha_n = 1$. We then define the positive Borel measure

$$\rho(B) = \sum_{n\in\mathbb{Z}^d} \alpha_n \mu_{n,n}(B),$$

called a real-valued spectral measure of H. We have

$$\rho(B) = 0 \text{ if and only if } \mu(B) = 0.$$

In particular, two Borel sets are equal up to a set of spectral measure zero if and only if $\rho(A \setminus B) = \rho(B \setminus A) = 0$. Moreover, we have

$$\operatorname{supp}\rho = \sigma(H),$$

and this support does not depend on the choice of the family $(\alpha_n)_{n\in\mathbb{Z}^d}$.

The first inclusion in Theorem 6.1.2 can be stated as follows, also known as Sch'nol's lemma.

Lemma 6.1.4 (Sch'nol). *Let ρ be a real-valued spectral measure for $H = -\Delta_{\text{disc}} + V$. Then, for ρ-almost every $E \in \mathbb{R}$, there exists a polynomially bounded solution u of the equation $Hu = Eu$.*

Proof : Let B be a Borel subset of $\mathbb{R}$. By Cauchy-Schwarz, we have

$$\forall m, n \in \mathbb{Z}^d,\ |\mu_{m,n}(B)| \leq |\mu_{m,m}(B)|^{\frac{1}{2}} |\mu_{n,n}(B)|^{\frac{1}{2}}.$$

Since by construction of ρ, the $\mu_{n,n}$ are absolutely continuous with respect to ρ, the same holds for all $\mu_{m,n}$. We can thus apply the Radon-Nikodym theorem to obtain the existence of ρ-integrable functions $F_{m,n}$ such that

$$\mu_{m,n}(B) = \int_B F_{m,n}(\lambda)\mathrm{d}\rho(\lambda).$$

Since $\mu_{n,n}$ is a positive measure, the functions $F_{n,n}$ are ρ-almost surely positive. Moreover,

$$\rho(B) = \sum_{n\in\mathbb{Z}^d} \alpha_n \mu_{n,n}(B) = \sum_{n\in\mathbb{Z}^d} \alpha_n \int_B F_{n,n}(\lambda)\mathrm{d}\rho(\lambda) = \int_B \sum_{n\in\mathbb{Z}^d} \alpha_n F_{n,n}(\lambda)\mathrm{d}\rho(\lambda).$$

In particular, ρ-almost surely, $\sum_{n\in\mathbb{Z}^d}\alpha_n F_{n,n}(\lambda) = 1$. We deduce that ρ-almost surely, for all $n \in \mathbb{Z}^d$, $F_{n,n}(\lambda) \leq \frac{1}{\alpha_n}$. We then have

$$\begin{aligned}
\left| \int_B F_{m,n}(\lambda)\mathrm{d}\rho(\lambda) \right| &= |\mu_{m,n}(B)| \\
&\leq (\mu_{m,m}(B))^{\frac{1}{2}} (\mu_{n,n}(B))^{\frac{1}{2}} \\
&= \left(\int_B F_{m,m}(\lambda)\mathrm{d}\rho(\lambda) \right)^{\frac{1}{2}} \left(\int_B F_{n,n}(\lambda)\mathrm{d}\rho(\lambda) \right)^{\frac{1}{2}} \\
&\leq \alpha_m^{-\frac{1}{2}} \alpha_n^{-\frac{1}{2}} \rho(B).
\end{aligned}$$

Hence, ρ-almost surely,

$$|F_{m,n}| \leq \alpha_m^{-\frac{1}{2}} \alpha_n^{-\frac{1}{2}}. \tag{6.1}$$

By functional calculus, for any bounded measurable function f, we have

$$(\delta_m, f(H)\delta_n) = \int_{\mathbb{R}} f(\lambda) F_{m,n}(\lambda)\mathrm{d}\rho(\lambda).$$

In particular, if g is bounded with compact support, take $f(\lambda) = \lambda g(\lambda)$ and we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \lambda g(\lambda) F_{m,n}(\lambda) \mathrm{d}\rho(\lambda) &= (\delta_m, Hg(H)\delta_n) \\
&= (H\delta_m, g(H)\delta_n) \\
&= \sum_{|p|=1} -(\delta_{m+p}, g(H)\delta_n) + V(m)(\delta_m, g(H)\delta_n)) \\
&= \sum_{|p|=1} - \int_{\mathbb{R}} g(\lambda) F_{m+p,n}(\lambda) \mathrm{d}\rho(\lambda) + \int_{\mathbb{R}} \lambda g(\lambda) V(m) F_{m,n}(\lambda) \mathrm{d}\rho(\lambda) \\
&= \int_{\mathbb{R}} \lambda g(\lambda) H^{(m)} F_{m,n}(\lambda) \mathrm{d}\rho(\lambda),
\end{aligned}$$

where $H^{(m)} F_{m,n}(\lambda)$ denotes the application of the operator H to the sequence $m \mapsto F_{m,n}(\lambda)$. Thus, for any bounded compactly supported function g,

$$\int_{\mathbb{R}} \lambda g(\lambda) F_{m,n}(\lambda) \mathrm{d}\rho(\lambda) = \int_{\mathbb{R}} \lambda g(\lambda) H^{(m)} F_{m,n}(\lambda) \mathrm{d}\rho(\lambda).$$

Hence, for ρ-almost every $E \in \mathbb{R}$ and for any fixed $n \in \mathbb{Z}^d$, the sequence defined by $u_m = F_{m,n}(E)$ is a solution of $Hu = Eu$. Moreover, this sequence $(u_m)_{m \in \mathbb{Z}^d}$ satisfies by (6.1),

$$\forall m \in \mathbb{Z}^d, \ |u_m| \leq C_0 \alpha_m^{-\frac{1}{2}}.$$

For arbitrary $\beta > d$, choose $\alpha_m = c(1 + |m|)^{-\beta}$ with $c > 0$ a normalization constant. Let $\epsilon > 0$. For $\beta = d + 2\epsilon$ we obtain the estimate

$$\exists C > 0, \ \forall m \in \mathbb{Z}^d, \ |u_m| \leq C(1 + |m|)^{\frac{d}{2} + \epsilon}.$$

This shows that $(u_m)_{m \in \mathbb{Z}^d}$ is a polynomially bounded solution of $Hu = Eu$, constructed for ρ-almost every $E \in \mathbb{R}$.

□

The converse of Theorem 6.1.2 is given by:

Proposition 6.1.5. *If the equation $Hu = Eu$ admits a polynomially bounded solution, then $E \in \sigma(H)$.*

Proof : For $L \in \mathbb{N}$ we define $\Lambda_L = \{m \in \mathbb{Z}^d \mid |m| \leq L\}$, the cube centered at 0 with side length $2L + 1$. If $S \subset \mathbb{Z}^d$ and $u \in \ell^2(\mathbb{Z}^d)$, we denote

$$||u||_S = \sum_{m \in S} |u_m|^2.$$

Step 1. Let us show by contradiction that if u is polynomially bounded and nonzero and if $l \geq 1$, then there exists a sequence of integers $(L_n)_{n \in \mathbb{N}}$ tending to infinity such that

$$\frac{||u||_{\Lambda_{L_n + l}}}{||u||_{\Lambda_{L_n}}} \xrightarrow[n \to +\infty]{} 1.$$

Otherwise, there exists $a > 1$ and $L_0 \in \mathbb{N}$ such that for all $L \geq L_0$,

$$||u||_{\Lambda_{L+l}} \geq a ||u||_{\Lambda_L}.$$

Iterating this relation and starting with $L = L_0$, for all $k \in \mathbb{N}$,

$$||u||_{\Lambda_{L_0 + kl}} \geq a^k ||u||_{\Lambda_{L_0}}.$$

However, u is polynomially bounded, so there must exist $C, C_1 > 0$ and $p \in \mathbb{N}$ such that

$$||u||_{\Lambda_{L_0+kl}} \leq C_1(L_0 + lk)^p \leq Ck^p.$$

Hence, for all $k \in \mathbb{N}$, $a^k \leq \tilde{C}k^p$, which is a contradiction.

Step 2. Now, let u be a polynomially bounded solution of $Hu = Eu$. We truncate it by setting

$$(u_L)_m = \begin{cases} u_m & \text{if} \quad |m| \leq L \\ 0 & \text{otherwise} \end{cases}$$

Let $v_L = \frac{1}{||u_L||} u_L$ and $S_L = \{m \in \mathbb{Z}^d \mid L - 1 \leq |m| \leq L + 1\}$. Let $m \notin S_L$. If $|m| > L + 1$, then in the expression

$$((H - E)u_L)_m = -\sum_{|p|=1} (u_L)_{m+p} + (V_m - E)(u_L)_m$$

all terms vanish, so $((H - E)u_L)_m = 0$. If $|m| < L - 1$, then in the same expression, $u_L = u$ and we have $((H - E)u_L)_m = ((H - E)u)_m = 0$. Thus,

$$||(H - E)u_L||^2 \leq ||u||^2_{S_L} = \sum_{m \in S_L} |u_m|^2 = ||u||^2_{\Lambda_{L+1}} - ||u||^2_{\Lambda_{L-2}}.$$

By Step 1, there exists a sequence (L_n) tending to infinity such that

$$\frac{||u||_{\Lambda_{L_n+1}}}{||u||_{\Lambda_{L_n-2}}} \xrightarrow[n \to +\infty]{} 1.$$

Hence,

$$||(H - E)v_{L_n}||^2 \leq \frac{||u||^2_{\Lambda_{L_n+1}} - ||u||^2_{\Lambda_{L_n-2}}}{||u||_{\Lambda_{L_n-2}}} \xrightarrow[n \to +\infty]{} 0.$$

Thus, $(v_{L_n})_{n \in \mathbb{N}}$ is a Weyl sequence for H and E, so $E \in \sigma(H)$.

□

Proof : We prove the corollary. First, $\sigma_{p-g}(H) \subset \sigma(H)$ by Proposition 6.1.5. Since the spectrum is closed, we obtain the first inclusion: $\overline{\sigma_{p-g}(H)} \subset \sigma(H)$.

Then, by Theorem 6.1.2, we have $\rho(\mathbb{R} \setminus \sigma_{p-g}(H)) = 0$, hence

$$(\mathbb{R} \setminus \overline{\sigma_{p-g}(H)}) \cap \sigma(H) = (\mathbb{R} \setminus \overline{\sigma_{p-g}(H)}) \cap \operatorname{supp} \rho = \emptyset.$$

From this, $\sigma(H) \subset \overline{\sigma_{p-g}(H)}$.

□

6.1.2 Continuous Case

For proofs and further references for this section, we refer to [26]. We start by defining the class of functions in which the potential will be chosen.

Definition 6.1.6. *Let $V : \mathbb{R}^d \to \mathbb{R}$. We say that $V \in K_d$ if*

1. *If $d \geq 3$,*

$$\lim_{\alpha \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \alpha} |x - y|^{-(d-2)} |V(y)| \mathrm{d}y = 0.$$

2. *If $d = 2$,*
$$\lim_{\alpha \to 0} \sup_{x \in \mathbb{R}^2} \int_{|x-y| \leq \alpha} \ln(|x-y|^{-1}) |V(y)| \mathrm{d}y = 0.$$

3. *If $d = 1$,*
$$\sup_{x \in \mathbb{R}} \int_{|x-y| \leq 1} |V(y)| \mathrm{d}y < \infty.$$

Moreover, we say that $V \in K_d^{\mathrm{loc}}$ if $V \mathbf{1}_{B(0,R)} \in K_d$ for all $R > 0$.

Furthermore, we define $V_- = \max(-V, 0)$ and $V_+ = \max(V, 0)$. We then have the following result due to Sch'nol and independently to Simon. Recall that a function $u : \mathbb{R}^d \to \mathbb{R}$ is polynomially bounded if there exist $C > 0$ and $p \geq 0$ such that

$$\forall x \in \mathbb{R}^d, \ |u(x)| \leq C(1 + |x|)^p.$$

Theorem 6.1.7. *Let $H = -\Delta + V$ acting on $H^2(\mathbb{R}^d)$ with $V_+ \in K_d^{\mathrm{loc}}$ and $V_- \in K_d$. Let $E \in \mathbb{R}$ be such that there exists a polynomially bounded solution u of $Hu = Eu$. Then $E \in \sigma(H)$.*

In fact, the "polynomially bounded" condition on u can be weakened to:

$$\forall A > 0, \ \exists C > 0, \ \forall x \in \mathbb{R}^d, \ |u(x)| \leq C e^{A|x|}.$$

It is not true in general that conversely, if $E \in \sigma(H)$, then $Hu = Eu$ has a polynomially bounded solution. However, we have:

$$\sigma(H) = \overline{\{E \in \mathbb{R} | Hu = Eu \text{ has a polynomially bounded solution}\}}$$

where the overline denotes closure in the usual topology of $\mathbb{R}$.

More precisely, a continuous version of Sch'nol's lemma can be stated as in [26][Theorem C.5.4].

Theorem 6.1.8. *Let $V_+ \in K_d^{\mathrm{loc}}$ and $V_- \in K_d$ and $H = -\Delta + V$. Fix $\delta > \frac{d}{2}$. Then there exists a real-valued spectral measure ρ and a family $\{\Delta_n\}_{n \geq 1}$ of pairwise disjoint measurable sets whose union equals the support of ρ, such that if $E \in \Delta_n$, there exist n functions $u_1(\cdot, E), \ldots, u_n(\cdot, E)$ satisfying*

1. *$Hu_j = Eu_j$ in the weak sense with respect to the measure ρ,*
2. *$\exists C > 0, \ |u_j(x, E)| \leq C(1 + x^2)^{\frac{\delta}{2}}$,*
3. *if E is fixed, the functions $u_1(\cdot, E), \ldots, u_n(\cdot, E)$ are linearly independent.*

Proof : The main difficulty is to construct the measure ρ in this case and to verify the first point. It is obtained as a trace:

$$\rho(\Delta) = \mathrm{Tr}((1 + x^2)^{-\frac{\delta}{2}} E(\Delta)(1 + x^2)^{-\frac{\delta}{2}})$$

□

We then recover the equality stated above by noting that if $E_0 \in \sigma(H)$, then for all $\epsilon > 0$, $\rho(E_0 - \epsilon, E_0 + \epsilon) > 0$, and for any E arbitrarily close to E_0, $Hu = Eu$ admits a polynomially bounded solution (here we use the definition of the support of ρ).

6.2 Ishii-Pastur Theorem

6.2.1 Scalar-valued Schrödinger Operator

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space and let $T : \Omega \to \Omega$ be an invertible ergodic transformation. Let $f : \Omega \to \mathbb{R}$ be measurable and bounded. We define:

$$\forall \omega \in \Omega,\ \forall n \in \mathbb{Z},\ V_n^\omega = f(T^n \omega).$$

We consider the random Schrödinger operator h_ω acting on $\ell^2(\mathbb{Z})$ by:

$$(h_\omega u)_n = -(u_{n+1} + u_{n-1}) + V_n^\omega u_n.$$

We consider the equation $h_\omega u = Eu, \quad E \in \mathbb{R}$, and the associated transfer matrix sequence $(T_n^\omega(E))_{n\in\mathbb{Z}}$, given by

$$T_n^\omega(E) = \begin{pmatrix} V_n^\omega - E & -1 \\ 1 & 0 \end{pmatrix}.$$

To this sequence $(T_n^\omega(E))_{n\in\mathbb{Z}}$ is associated the Lyapunov exponent $\gamma(E) \geq 0$. Indeed, for $n \in \mathbb{Z}$, set $S_n(\omega, E) = T_n^\omega(E) \cdots T_1^\omega(E)$ if $n \geq 1$ and $S_{-n}(\omega, E) = (T_{-n+1}^\omega(E))^{-1} \cdots (T_0^\omega(E))^{-1}$ if $n \geq 1$. Then the two limits

$$\gamma_\pm(\omega, E) = \lim_{n\to\pm\infty} \frac{1}{|n|} \log(||S_n(\omega, E)||)$$

exist for P-almost every $\omega \in \Omega$. Moreover, since $S_{-n}(\omega, E) = (T_0^\omega(E) \cdots T_{-n+1}^\omega(E))^{-1}$, by the *i.i.d.* property,

$$\mathbb{E}(\log ||S_{-n}(\omega, E)||) = \mathbb{E}((S_n(E))^{-1}).$$

Now, if $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have ${}^t(JS_n(E)J^{-1}) = (S_n(E))^{-1}$, and since transposition, J and J^{-1} are isometries, it follows that $\gamma_-(\omega, E) = \gamma_+(\omega, E) = \gamma(E)$, P-almost surely.
Moreover, since $\det(S_n(\omega, E)) = 1$, at least one of its eigenvalues has modulus greater than 1, and therefore the norm of all these matrices is greater than 1. Taking logarithms, $\gamma(E)$ is positive.

We then define

$$\mathcal{Z} = \{E \in \mathbb{R} \mid \gamma(E) = 0\}.$$

We also recall that if $S \subset \mathbb{R}$, the essential closure of S is defined by:

$$\overline{S}^{\text{ess}} = \{E \in \mathbb{R} \mid \forall \epsilon > 0,\ \text{Leb}(S \cap (E - \epsilon, E + \epsilon)) > 0\}.$$

Theorem 6.2.1. *Let Σ_{ac} be the almost-sure absolutely continuous spectrum of the ergodic family $\{h_\omega\}_{\omega\in\Omega}$. Then,*

$$\Sigma_{\text{ac}} \subset \overline{\mathcal{Z}}^{\text{ess}}. \tag{6.2}$$

Proof : We prove this inclusion by contraposition. Let $(a, b) \subset \mathbb{R}$ be an interval and suppose that $\gamma(E) > 0$ for Lebesgue almost every $E \in (a, b)$.

Let $A_E = \{\omega \in \Omega \mid \gamma(\omega, E) = 0\}$. Then for Lebesgue almost every $E \in (a, b)$, $\mathsf{P}(A_E) = 0$. We then define

$$A = \{(\omega, E) \in \Omega \times (a, b) \mid \gamma_+(\omega, E) \neq \gamma_-(\omega, E) \text{ or the limits do not exist or } \gamma(\omega, E) = 0\}$$

and

$$A_\omega = \{E \in (a, b) \mid \gamma_+(\omega, E) \neq \gamma_-(\omega, E) \text{ or the limits do not exist or } \gamma(\omega, E) = 0\}.$$

Then, by Fubini's theorem,

$$(\mathsf{P}\otimes \mathrm{Leb})(A) = (\mathrm{Leb}\otimes \mathsf{P})(A) = \int_a^b \mathsf{P}(A_E)\mathrm{d}E = 0.$$

Hence, for P-almost every $\omega \in \Omega$, $\mathrm{Leb}(A_\omega) = 0$. Thus, for P-almost every $\omega \in \Omega$, $\gamma(\omega, E) > 0$ for all $E \in (a,b)$ outside a set of Lebesgue measure zero.

Moreover, by Theorem 6.2.1,

$$E_{\mathbb{R}\setminus\sigma_{p-g}(h_\omega)}(h_\omega) = 0.$$

Furthermore, since $\mathrm{Leb}(A_\omega) = 0$, by definition of the absolutely continuous spectrum, we have $E^{\mathrm{ac}}_{A_\omega}(h_\omega) = 0$. Now, if $E \notin A_\omega$, by Oseledets' theorem, the only polynomially bounded solutions of $h_\omega u = Eu$ are exponentially decaying at infinity and hence belong to $\ell^2(\mathbb{Z})$. Since $\ell^2(\mathbb{Z})$ is a separable Hilbert space, there can be at most countably many such eigenvectors, and we deduce that

$$\mathrm{Leb}(\sigma_{p-g}(h_\omega) \cap ((a,b) \setminus A_\omega)) = 0.$$

Finally, using that $E_{\mathbb{R}\setminus\sigma_{p-g}(h_\omega)}(h_\omega) = 0$, we obtain

$$E^{\mathrm{ac}}_{(a,b)}(h_\omega) = E^{\mathrm{ac}}_{(a,b)\cap\sigma_{p-g}(h_\omega)}(h_\omega) = E^{\mathrm{ac}}_{(a,b)\cap A_\omega}(h_\omega) + E^{\mathrm{ac}}_{(a,b)\cap\sigma_{p-g}(h_\omega)\cap A^c_\omega}(h_\omega) = 0.$$

This proves the desired inclusion.

□

Remark 6.2.2. *This proof does not use the fact that the Schrödinger operator is discrete. Indeed, assuming Sch'nol's lemma also holds in the continuous case, and provided the transfer matrices are properly defined in the continuous setting (which is also possible), this proof adapts directly.*

Corollary 6.2.3. *Let $(a,b) \subset \mathbb{R}$, $a < b$, be an interval. If for Lebesgue almost every $E \in (a,b)$, $\gamma(E) > 0$, then $\Sigma_{\mathrm{ac}} \cap (a,b) = \emptyset$.*

Example 6.2.4. *We again consider the almost-Mathieu operator $H^{\alpha,\lambda}_\omega$ acting on $\ell^2(\mathbb{Z})$ by:*

$$\forall u \in \ell^2(\mathbb{Z}),\ \forall n \in \mathbb{Z}, (H^{\alpha,\lambda}_\omega u)_n = u_{n+1} + u_{n-1} + \lambda\cos(\omega + 2\pi n\alpha)u_n,$$

where α is irrational, λ is real, and $\omega \in [0, 2\pi)$. Here the potential is ergodic by taking $\Omega = \mathbb{T}$ endowed with the Haar measure. We then show that

$$\forall E \in \mathbb{R},\ \gamma(E) \geq \log\left(\left|\frac{\lambda}{2}\right|\right). \tag{6.3}$$

For $E \in \mathbb{R}$, consider the transfer matrix sequence

$$T^{(n)}_\omega(E) = \begin{pmatrix} \lambda\cos(\omega + 2\pi n\alpha) - E & -1 \\ 1 & 0 \end{pmatrix}.$$

We rewrite it using trigonometric relations.
For $z \in \mathbb{C}$, set $z_n = z\mathrm{e}^{2\mathrm{i}\pi n\alpha}$ and consider the matrix

$$Z_n(z,E) = \begin{pmatrix} \frac{\lambda}{2}(z_n^2 + 1) - Ez_n & -z_n \\ z_n & 0 \end{pmatrix}.$$

For $z = \mathrm{e}^{\mathrm{i}\omega}$, $z_n = \mathrm{e}^{2\mathrm{i}\pi n\alpha + \omega}$ and

$$\cos(\omega + 2\pi n\alpha) = \frac{1}{2}(z_n + \tfrac{1}{z_n}) \ \Leftrightarrow\ 2z_n\cos(\omega + 2\pi n\alpha) = z_n^2 + 1.$$

Thus,

$$Z_n(e^{i\omega}, E) = \begin{pmatrix} \lambda z_n \cos(\omega + 2\pi n\alpha) - E z_n & -z_n \\ z_n & 0 \end{pmatrix} = z_n T_\omega^{(n)}(E).$$

Then,

$$||Z_n(e^{i\omega}, E) \cdots Z_0(e^{i\omega}, E)|| = |z_n \cdots z_0| \, ||T_\omega^{(n)}(E) \cdots T_0^{(\omega)}(E)|| = 1 \cdot ||S_n(\omega, E)||.$$

If we set

$$\gamma_n(E) = \mathbb{E}(\log ||S_n(\omega, E)||) = \frac{1}{2\pi} \int_0^{2\pi} \log ||T_\omega^{(n)}(E) \cdots T_\omega^{(n)}(E)||$$

then, since $f : z \mapsto \log ||Z_n(z, E) \cdots Z_0(z, E)||$ is subharmonic (it is upper semicontinuous and $f(z)$ is less than or equal to the average value of f over any circle centered at z), we have

$$\gamma_n(E) \geq f(0) = \log ||Z_n(0, E) \cdots Z_0(0, E)|| = \log \left(\left| \frac{\lambda}{2} \right| \right)^{n+1}$$

because

$$Z_n(0, E) = \begin{pmatrix} \frac{\lambda}{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, $\gamma_n(E) \geq (n+1) \log \left(\left| \frac{\lambda}{2} \right| \right)$, and taking the limit after dividing by $n+1$, we obtain

$$\gamma(E) \geq \log \left(\left| \frac{\lambda}{2} \right| \right).$$

We can then apply the corollary of the Ishii-Pastur theorem to conclude that for almost every $\omega \in [0, 2\pi)$, the absolutely continuous spectrum of $H_\omega^{\alpha,\lambda}$ is empty for $|\lambda| > 2$.

6.2.2 Matrix-valued Schrödinger Operator

Let $N \geq 1$. We consider operators of the form

$$h_\omega^N : \begin{array}{ccc} \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N & \to & \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N \\ (u_n) & \mapsto & (-(u_{n+1} + u_{n-1}) + V_{\omega^{(n)}} u_n) \end{array}$$

where the $V_{\omega^{(n)}}$ are real symmetric matrices that are *i.i.d.* on a complete probability space $(\Omega, \mathcal{B}, \mathsf{P})$. Thus, the family $\{h_\omega^N\}_{\omega \in \Omega}$ is ergodic.

To study the generalized eigenfunctions of the operator h_ω^N we solve the equation

$$-(u_{n+1} + u_{n-1}) + V_{\omega^{(n)}} u_n = E u_n,$$

which leads us to introduce the transfer matrices

$$T_{\omega^{(n)}}(E) = \begin{pmatrix} V_{\omega^{(n)}} - E & -I \\ I & 0 \end{pmatrix}.$$

These transfer matrices form an *i.i.d.* sequence in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ with common law μ_E, and we can therefore consider the associated Furstenberg subgroup,

$$G_{\mu_E} = \overline{\langle \operatorname{supp} \mu_E \rangle}.$$

The Lyapunov exponents at $\pm\infty$ are then well-defined, and they are equal and independent of ω for P-almost every $\omega \in \Omega$.

Notation. Let Ω_{Lyap} be a subset of Ω such that:

1. $\mathbb{P}(\Omega_{\text{Lyap}}) = 1$,
2. for all $\omega \in \Omega_{\text{Lyap}}$ the limits defining the Lyapunov exponents exist at $\pm\infty$ and for each p they are equal at $+\infty$ and $-\infty$,
3. for all $\omega \in \Omega_{\text{Lyap}}$, these limits are independent of ω.

In this setting of matrix-valued operators, an analogue of the Ishii-Pastur theorem is established on the characterization of the absolutely continuous spectrum via the zeros of the Lyapunov exponents.
Recall that for $\omega \in \Omega_{\text{Lyap}}$, the operator h_ω^N has $2N$ Lyapunov exponents which can be grouped in pairs of opposite real numbers:

$$\gamma_1(E) \geq \cdots \geq \gamma_N(E) \geq 0 \geq \gamma_{N+1}(E) = -\gamma_N(E) \geq \cdots \geq \gamma_{2N}(E) = -\gamma_1(E).$$

For $j \in \{1, \ldots, N\}$, we set

$$Z_j = \{E \in \mathbb{R} \mid \exists\, l_1, \ldots, l_{2j} \in \{1, \ldots, 2N\},\ \gamma_{l_1}(E) = \cdots = \gamma_{l_{2j}}(E) = 0\}.$$

Proposition 6.2.5. *Let $j \in \{1, \ldots, N\}$ and let $E \in Z_j$ be fixed. Let $\omega \in \Omega_{\text{Lyap}}$. Then, every subspace of*

$$\{\varphi \in (\mathbb{C}^N)^{\mathbb{Z}} \mid h_\omega^N \varphi = E\varphi,\ \varphi \notin \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N \text{ and } \varphi \text{ is polynomially bounded }\}$$

has dimension at most $2j$.

Proof : We define

$$V_{\text{sol}}(E) = \{\varphi \in (\mathbb{C}^N)^{\mathbb{Z}} \mid h_\omega^N \varphi = E\varphi\},$$

$$V_{\text{P}}(E) = \{\varphi \in V_{\text{sol}}(E) \mid \varphi \text{ is polynomially bounded }\},$$

and

$$V_{\ell^2}(E) = \{\varphi \in V_{\text{sol}}(z) \mid \varphi \in \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N\}.$$

To prove the proposition, it suffices to show that

$$\dim V_{\text{P}}(E) \leq 2j + \dim V_{\ell^2}(E).$$

For $\varphi = (\varphi_n)_{n\in\mathbb{Z}}$ and $\psi = (\psi_n)_{n\in\mathbb{Z}}$ in $V_{\text{sol}}(E)$, we define

$$W(\varphi, \psi) = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}^* J \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix},$$

where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Then W is an antisymmetric bilinear form on $V_{\text{sol}}(E)$. Moreover, since ψ is in $V_{\text{sol}}(E)$, it is uniquely determined by $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}$, and since J is invertible, W is non-degenerate. Indeed, if $\psi \in V_{\text{sol}}(E)$ satisfies $W(\varphi, \psi) = 0$ for all $\varphi \in V_{\text{sol}}(E)$, applying this equality to $2N$ distinct sequences φ such that $\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ are the $2N$ canonical basis vectors of $\mathbb{C}^{2N}$, we obtain a Cramer system with unique solution $J\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = 0$. Since J is invertible, we get $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = 0$ and hence $\psi = 0$.

Since W is a non-degenerate antisymmetric bilinear form on $V_{\text{sol}}(E)$, if V_1 and V_2 are two subspaces of $V_{\text{sol}}(E)$ such that

$$\forall \varphi \in V_1,\ \forall \psi \in V_2,\ W(\varphi, \psi) = 0, \tag{6.4}$$

then

$$\dim V_1 + \dim V_2 \leq 2N. \tag{6.5}$$

Indeed, V_1 and JV_2 are orthogonal for W.

We set:

$$D_\pm = \{\varphi \in V_{\mathrm{sol}}(E) \mid \varphi \text{ decays exponentially at } \pm\infty\}.$$

Since $E \in Z_j$, exactly $2j$ of the $2N$ Lyapunov exponents vanish at E. By Oseledets' theorem,

$$\dim D_\pm = N - j.$$

Also, $D_+ \cap D_- \subset V_{\ell^2}(E)$, so by the Grassmann formula,

$$\begin{aligned} \dim\,(D_+ + D_-) &= \dim D_+ + \dim D_- - \dim\,(D_+ \cap D_-) \\ &\geq 2N - 2j - \dim V_{\ell^2}(E) \end{aligned} \tag{6.6}$$

Moreover, if $\varphi \in V_{\mathrm{P}}(E)$ and $\psi \in D_\pm$, then, by domination,

$$\lim_{n\to\pm\infty} \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* J \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} = 0. \tag{6.7}$$

Moreover, the sequence $\left(\begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* J \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} \right)_{n\in\mathbb{Z}}$ is constant when φ and ψ are chosen in $V_{\mathrm{sol}}(E)$, since $T_{\omega^{(n)}}(E) \in \mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ for all $n \in \mathbb{Z}$. Hence, for all $n \in \mathbb{Z}$,

$$\begin{aligned} \begin{pmatrix} \varphi_{n+1} \\ \varphi_{n+2} \end{pmatrix}^* J \begin{pmatrix} \psi_{n+1} \\ \psi_{n+2} \end{pmatrix} &= \left(T_{\omega^{(n)}}(E) \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}\right)^* J T_{\omega^{(n)}}(E) \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* J \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix}. \end{aligned}$$

In particular,

$$\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}^* J \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \lim_{n\to\pm\infty} \begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix}^* J \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} = 0. \tag{6.8}$$

Hence, if $\varphi \in V_{\mathrm{P}}(E)$ and $\psi \in D_+ + D_-$, then $W(\varphi, \psi) = 0$ by linearity in the second argument. Therefore, by (6.5),

$$\dim V_{\mathrm{P}}(E) + \dim\,(D_+ + D_-) \leq 2N.$$

Combining with (6.6),

$$\dim V_{\mathrm{P}}(E) \leq 2j + \dim V_{\ell^2}(E),$$

which proves the proposition.

□

Let $\mathrm{P}_{\mathrm{bdd}}$ denote the set of real numbers E such that the equation $h_\omega^N \varphi = E\varphi$ admits a non-trivial polynomially bounded solution. We then have a matrix-valued analogue of Sch'nol's lemma.

Proposition 6.2.6. *For* P*-almost every* $\omega \in \Omega$*,*

$$\sigma(h_\omega^N) = \overline{\mathrm{P}_{\mathrm{bdd}}} = \overline{\{E \in \mathbb{R} \mid \exists \varphi \in V_{\mathrm{P}}(E) \setminus \{0\}\}}$$

and $\mathrm{E}_{\mathbb{R}\setminus\mathrm{P}_{\mathrm{bdd}}}(h_\omega^N) = 0$*, where* $\mathrm{E}_{\mathbb{R}\setminus\mathrm{P}_{\mathrm{bdd}}}(h_\omega^N)$ *is the spectral projector on* $\mathbb{R} \setminus \mathrm{P}_{\mathrm{bdd}}$ *associated with the self-adjoint operator* h_ω^N*.*

Using this proposition, one can prove a matrix-valued analogue of the Ishii-Pastur theorem. The ideas of the proof are the same as in the scalar case.

Theorem 6.2.7. *For every* $\omega \in \Omega_{\mathrm{Lyap}}$*, the multiplicity of the absolutely continuous spectrum of* h_ω^N *in* Z_j *is at most* $2j$*.*

Proof : Let $\omega \in \Omega_{\text{Lyap}}$. For Δ a Borel subset of $\mathbb{R}$, we denote by $\mathrm{E}_\Delta(h_\omega^N)$ the spectral projector on Δ and by $\mathrm{E}_\Delta^{\text{ac}}(h_\omega^N)$ the spectral projector onto the absolutely continuous part of $\sigma(h_\omega^N)$ in Δ.

To prove the theorem, we need to show that

$$\text{rg } \mathrm{E}_{Z_j}^{\text{ac}}(h_\omega^N) \leq 2j.$$

According to Proposition 6.2.6, $\mathrm{E}_{\mathbb{R}\setminus \mathrm{P}_{\text{bdd}}}(h_\omega^N) = 0$, so

$$\mathrm{E}_{Z_j}^{\text{ac}}(h_\omega^N) = \mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}}^{\text{ac}}(h_\omega^N) = \mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}\cap S}^{\text{ac}}(h_\omega^N) + \mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}\cap S^c}^{\text{ac}}(h_\omega^N),$$

where $S = \{E \in \mathbb{R} \mid \exists \varphi \in V_{\mathrm{P}}(E) \cap V_{\ell^2}(E),\ \varphi \neq 0\}$. If $E \in S$, then E is an eigenvalue of h_ω^N. Since there is only a countable number of eigenvectors in ℓ^2 for the self-adjoint operator h_ω^N, the Lebesgue measure of S is zero. Hence,

$$\mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}\cap S}^{\text{ac}}(h_\omega^N) = 0 \quad \text{and} \quad \mathrm{E}_{Z_j}^{\text{ac}}(h_\omega^N) = \mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}\cap S^c}^{\text{ac}}(h_\omega^N). \tag{6.9}$$

Since $E \in Z_j$, we can apply Proposition 6.2.5 to obtain directly that

$$\text{rg } \mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}\cap S^c}(h_\omega^N) \leq 2j$$

which implies

$$\text{rg } \mathrm{E}_{Z_j\cap \mathrm{P}_{\text{bdd}}\cap S^c}^{\text{ac}}(h_\omega^N) \leq 2j. \tag{6.10}$$

Combining (6.9) and (6.10), we have proved that

$$\text{rg } \mathrm{E}_{Z_j}^{\text{ac}}(h_\omega^N) \leq 2j,$$

which completes the proof.

□

Theorem 6.2.7 implies the following corollary.

Corollary 6.2.8. *If for Lebesgue-almost every $E \in \mathbb{R}$, $\gamma_1(E) \geq \cdots \geq \gamma_N(E) > 0$, then $\Sigma_{\text{ac}} = \emptyset$.*

Proof : If $E \in \mathbb{R}$ satisfies $\gamma_N(E) > 0$, then none of the Lyapunov exponents vanish at E, so $E \in Z_0$. By Theorem 6.2.7, $\mathrm{E}_{Z_0}^{\text{ac}}(h_\omega^N) = 0$. If Δ is a Borel subset of $\mathbb{R}$, then by hypothesis $\text{Leb}(\mathbb{R} \setminus Z_0) = 0$, so

$$\mathrm{E}_\Delta^{\text{ac}}(h_\omega^N) = \mathrm{E}_{\Delta\cap Z_0}^{\text{ac}}(h_\omega^N) = 0, \ \text{ for P-a.e. } \omega \in \Omega.$$

Hence $\Sigma_{\text{ac}} = \emptyset$ by the definition of Σ_{ac}.

□

6.2.3 Statement of the Ishii-Pastur-Kotani Theorem

The results linking the zeros of Lyapunov exponents and the absolutely continuous spectrum are actually deeper than the previously stated ones. We reuse the notations introduced earlier.

Theorem 6.2.9. 1. *The set Z_j is the essential support of the absolutely continuous spectrum of multiplicity $2j$.*

2. *There is no absolutely continuous spectrum of odd multiplicity.*

3. *We have*

$$\Sigma_{\text{ac}} = \overline{Z_N}^{\text{ess}} = \overline{\{E \in \mathbb{R} \mid \gamma_1(E) = \cdots = \gamma_{2N}(E) = 0\}}^{\text{ess}}.$$

Finally, we have a local result.

Proposition 6.2.10. *If Z_N contains an interval I, then the almost-sure spectrum of $\{h_\omega^N\}_{\omega\in\Omega}$ is purely absolutely continuous in I.*

The proofs of these results rely on the fine properties of the Weyl-Titchmarsh M-functions and Kotani's w-function (see [17]). They also allow to prove a partial characterization result for deterministic potentials.
Let $\mathbb{C}_+$ denote the upper complex half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\mathbb{C}_-$ the lower half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$.

Proposition 6.2.11. *Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. Fix $\omega \in \Omega$. Then there exists a unique function $x \mapsto F_+(x, E)$ with values in $\mathcal{M}_{\mathrm{N}}(\mathbb{C})$ (respectively $x \mapsto F_-(x, E)$) satisfying:*

$$-F_+'' + V_\omega F_+ = EF_+,\ F_+(0, E) = I, \text{ and } \int_0^\infty ||F_+(x, E)||^2 dx < +\infty$$

respectively:

$$-F_-'' + V_\omega F_- = EF_-,\ F_-(0, E) = I, \text{ and } \int_{-\infty}^0 ||F_-(x, E)||^2 dx < +\infty.$$

Definition 6.2.12. *For $E \in \mathbb{C}_+ \cup \mathbb{C}_-$, we define the functions m, M_+, and M_- by:*

$$M_+(E) = \frac{d}{dx} F_+(x, E)|_{x=0} \text{ and } M_-(E) = -\frac{d}{dx} F_-(x, E)|_{x=0}.$$

The function M_+ characterizes the potential on the positive real half-line.

Theorem 6.2.13. *M_+ characterizes $\{V(x)\}_{x\geq 0}$ in the sense that if V_1 and V_2 are two bounded potentials such that $M_{+,1} = M_{+,2}$, then $V_1(x) = V_2(x)$ for almost every $x \geq 0$.*

There is an analogous statement in the discrete case. See [17], Theorem 4.1 and Theorem 4.2.
To know the potential on the negative real half-line, we use the following result.

Proposition 6.2.14. *For almost every $E \in Z_N$ and for almost every $\omega \in \Omega$,*

$$M_-(E + \mathrm{i}0, \omega) = -M_+(E + \mathrm{i}0, \omega)^*.$$

Again, we refer to [17] for the proof and for the analogous statement in the discrete case. This finally allows to deduce:

Theorem 6.2.15. *If $\mathrm{Leb}(Z_N) > 0$, then V is deterministic.*

Proof : Knowing V on $(-\infty, 0]$ implies knowing M_-. The previous result implies that M_+ is then known on Z_N. Since $\mathrm{Leb}(Z_N) > 0$, this determines M_+ on all of $\mathbb{C}_+$ by analytic continuation of Herglotz functions. But M_+ determines V on $[0, +\infty)$. Finally, knowing V on $(-\infty, 0]$ implies knowing V on $[0, +\infty)$, which means that V is deterministic. □

Chapter 7

Regularity of the Lyapunov exponents and of the integrated density of states

7.1 Regularity of Lyapunov Exponents

7.1.1 Invariant Measures

We begin by giving a definition of the action of a group on a compact space. In this section, G denotes a locally compact group with identity element e, and B is a topological space.

Definition 7.1.1. *We say that G acts on B if one can continuously associate to each $(g, b) \in G \times B$ an element gb of B such that:*

1. $g_1(g_2 b) = (g_1 g_2)b$, $\forall g_1, g_2 \in G$ *and* $\forall b \in B$.
2. $eb = b$, $\forall b \in B$.

For such a pair (G, B), one may assume a measure μ on G and a measure ν on B are given. We can then define the pseudo-convolution of these two measures.

Definition 7.1.2. *The pseudo-convolution of a measure μ on G and a measure ν on B is the unique measure $\mu * \nu$ on B defined by:*

$$\mu * \nu(f) = \int_{G\times B} f(gb)\mathrm{d}\mu(g)\mathrm{d}\nu(b)$$

for any bounded measurable function f on B.

Remark 7.1.3. *If $B = G$, then this pseudo-convolution coincides with the ordinary convolution of two measures on a group. In this case, we denote μ^n as the n-th convolution of μ with itself: $\mu * \ldots * \mu$.*

Remark 7.1.4. *If μ_1 and μ_2 are measures on G and ν is a measure on B, then we have: $(\mu_1 * \mu_2) * \nu = \mu_1 * (\mu_2 * \nu)$.*

We can now define the notion of an invariant measure on a topological space on which a group acts. In the following, we fix a probability measure μ on G.

Definition 7.1.5. *A measure ν on B is called μ-invariant if: $\mu * \nu = \nu$.*

Theorem 7.1.6. *Let $(A_n^\omega(E))_{n\in\mathbb{N}}$ be a sequence of i.i.d. random symplectic matrices of order $2N$ depending on a real parameter E, and let $p \in \{1, \ldots, N\}$. Let μ_E be the common distribution of $A_n^\omega(E)$. Suppose that the Furstenberg subgroup G_{μ_E} associated with this sequence of symplectic matrices is p-contracting and L_p-strongly irreducible, and that $\mathbb{E}(\log ||A_0^\omega(E)||)$ is finite. Then the following assertions hold:*

1. $\gamma_p(E) > \gamma_{p+1}(E)$

2. *For any non-zero x in L_p:*

$$\lim_{n\to\infty} \frac{1}{n} \log || \wedge^p (A_{n-1}^{\omega}(E) \dots A_0^{\omega}(E))x|| = \sum_{i=1}^{p} \gamma_i(E)$$

3. *There exists a unique μ_E-invariant probability distribution, denoted $\nu_{p,E}$, on $\mathbb{P}(L_p) = \{\bar{x} \in \mathbb{P}(\wedge^p \mathbb{R}^{2N}) \mid x \in L_p\}$ such that:*

$$\int_{\mathrm{Sp_N}(\mathbb{R}) \times \mathbb{P}(L_p)} \log \frac{|| \wedge^p Mx||}{||x||} \mathrm{d}\mu_E(M) \mathrm{d}\nu_{p,E}(\bar{x}) = \sum_{i=1}^{p} \gamma_i(E)$$

The fact that we have an integral representation of the Lyapunov exponents involving the measure $\nu_{p,E}$ implies that, to study the regularity of these exponents as a function of E, it will suffice to study the regularity of this measure seen as a function of E.

7.1.2 Continuity

We will first prove the continuity of the Lyapunov exponents with respect to the energy parameter E.

Theorem 7.1.7. *Let $(A_n^{\omega}(E))_{n\in\mathbb{N}}$ be an i.i.d. sequence of symplectic matrices depending on a real parameter E. Let μ_E be the common law of $A_n^{\omega}(E)$. Fix a compact interval I in $\mathbb{R}$, and suppose that for all $E \in I$:*

1. *G_{μ_E} is p-contracting and L_p-strongly irreducible for all $p \in \{1, \dots, N\}$.*

2. *There exist constants $C_1 > 0$, $C_2 > 0$ independent of n, ω, E such that for all $p \in \{1, \dots, N\}$:*

$$|| \wedge^p A_n^{\omega}(E)||^2 \leq \exp(pC_1 + p|E| + p) \leq C_2. \tag{7.1}$$

3. *There exists a constant $C_3 > 0$ independent of n, ω, E such that for all $E, E' \in I$ and all $p \in \{1, \dots, N\}$:*

$$|| \wedge^p A_n^{\omega}(E) - \wedge^p A_n^{\omega}(E')|| \leq C_3|E - E'|. \tag{7.2}$$

Then, for all $p \in \{1, \dots N\}$, the map $E \mapsto \gamma_p(E)$ is continuous on I.

The methods to prove this result can be found in [8], Chapter V. In that reference, this regularity result is stated for a sequence of transfer matrices associated with discrete matrix-valued Schrödinger operators. However, this restriction concerns only the estimates (7.1) and (7.2). These are clearly satisfied in the case of transfer matrices associated with discrete operators and are also verified in the continuous case.

The main steps of the proof are as follows. First, one proves the continuity of the function

$$\Phi_{p,E} \ : \begin{array}{ccc} I \times \mathbb{P}(L_p) & \longrightarrow & \mathbb{R} \\ (E, \bar{x}) & \longmapsto & \Phi_{p,E}(\bar{x}) = \mathbb{E}\left(\log \frac{||(\wedge^p A_n^{\omega}(E))x||}{||x||}\right) \end{array}$$

for all $p \in \{1, \dots N\}$. At this stage, only the estimates (7.1) and (7.2) are used. Next, one proves the weak continuity of $E \mapsto \nu_{p,E}$ using the Banach-Alaoglu theorem and the uniqueness of the μ_E-invariant measure $\nu_{p,E}$. Combining these two continuity properties and noting that

$$\gamma_1(E) + \dots + \gamma_p(E) = \nu_{p,E}(\Phi_{p,E}),$$

one obtains the continuity of the Lyapunov exponents.
Fix an integer $p \in \{1, \dots, N\}$ and a compact interval I as in Theorem 7.1.7. For $E \in I$, define:

$$\forall \bar{x} \in \mathbb{P}(L_p),\ \Phi_{p,E}(\bar{x}) = \mathbb{E}\left(\log \frac{||(\wedge^p A_n^\omega(E))x||}{||x||}\right)$$

In the following proposition, we summarize the properties of the function $\Phi_{p,E}$.

Proposition 7.1.8. *The function $\Phi_{p,E}$ has the following properties:*

1. *$\bar{x} \mapsto \Phi_{p,E}(\bar{x})$ is continuous on $\mathbb{P}(L_p)$.*
2. *There exists $C > 0$ such that for all $E, E' \in I$, $\sup_{\bar{x} \in \mathbb{P}(L_p)} |\Phi_{p,E}(\bar{x}) - \Phi_{p,E'}(\bar{x})| \leq C|E - E'|$.*
3. *The function Φ : $\begin{array}{ccc} I \times \mathbb{P}(L_p) & \longrightarrow & \mathbb{R} \\ (E, \bar{x}) & \longmapsto & \Phi_{p,E}(\bar{x}) \end{array}$ is continuous.*

Proof : From (7.1) we have:

$$\log \frac{||(\wedge^p A_n^\omega(E))x||}{||x||} \leq \log ||(\wedge^p A_n^\omega(E))|| \leq \log \sqrt{C_2}.$$

Hence, if $\bar{x}_l \to \bar{x}$ in $\mathbb{P}(L_p)$, by the dominated convergence theorem of Lebesgue (a constant is integrable on a probability space!):

$$\Phi_{p,E}(\bar{x}_l) = \mathbb{E}\left(\log \frac{||(\wedge^p A_n^\omega(E))x_l||}{||x_l||}\right) \to \mathbb{E}\left(\log \frac{||(\wedge^p A_n^\omega(E))x||}{||x||}\right) = \Phi_{p,E}(\bar{x}).$$

This proves the first point.
For two matrices A and B, with B assumed invertible: $||Ax|| = ||AB^{-1}Bx|| \leq ||AB^{-1}||\,||Bx||$. Hence: $\frac{||Ax||}{||Bx||} \leq ||AB^{-1}||$. Then:

$$||AB^{-1}|| = ||(A - B + B)B^{-1}|| \leq ||A - B||\,||B^{-1}|| + ||I|| = ||A - B||\,||B^{-1}|| + 1.$$

Using this inequality and then (7.2), we obtain:

$$\begin{array}{rcl} |\Phi_{p,E}(\bar{x}) - \Phi_{p,E'}(\bar{x})| & = & \left|\mathbb{E}\left(\log \frac{||(\wedge^p A_n^\omega(E))x||}{||(\wedge^p A_n^\omega(E'))x||}\right)\right| \\ & \leq & \mathbb{E}\left(\log ||(\wedge^p A_n^\omega(E))(\wedge^p A_n^\omega(E'))^{-1}||\right) \\ & \leq & \mathbb{E}\left(\log(||(\wedge^p A_n^\omega(E)) - (\wedge^p A_n^\omega(E'))||\,||(\wedge^p A_n^\omega(E'))^{-1}|| + 1)\right) \\ & \leq & \mathbb{E}\left(||(\wedge^p A_n^\omega(E)) - (\wedge^p A_n^\omega(E'))||\,||(\wedge^p A_n^\omega(E'))^{-1}||\right) \\ & \leq & C_3|E - E'|\sqrt{C_2} \end{array}$$

This proves point (ii).

To prove the third point, we just need to combine points (i) and (ii). Let $\varepsilon > 0$ and choose $(E, \bar{x})$ and $(E', \bar{y})$ sufficiently close so that:

$$\begin{array}{rcl} |\Phi_{p,E}(\bar{x}) - \Phi_{p,E'}(\bar{y})| & \leq & |\Phi_{p,E}(\bar{x}) - \Phi_{p,E'}(\bar{x})| + |\Phi_{p,E'}(\bar{x}) - \Phi_{p,E'}(\bar{y})| \\ & \leq & C|E - E'| + \varepsilon \\ & \leq & \text{Cte } \varepsilon \end{array}$$

□

Using this proposition, we can now prove the continuity of the Lyapunov exponents.

Proposition 7.1.9. *The map $E \mapsto (\gamma_1 + \ldots + \gamma_p)(E)$ is continuous on I.*

Proof : Fix $E \in I$. Let E_l be a sequence of real numbers in I converging to E. By Theorem 7.1.6, there exists a unique μ_E-invariant measure $\nu_{p,E}$ on $\mathbb{P}(L_p)$, and for each $l \in \mathbb{N}$, there exists a μ_{E_l}-invariant measure ν_{p,E_l} on $\mathbb{P}(L_p)$. By (7.1), we have: $\mu_{E_l} \xrightarrow[l\to\infty]{w} \mu_E$.

By the Banach-Alaoglu theorem, the sequence $(\nu_{p,E_l})_{l\in\mathbb{N}}$ contains a subsequence, say $(\nu_{p,E_{l_i}})_{i\in\mathbb{N}}$, weakly converging to a limit $\tilde{\nu}$. Since convolution is weakly continuous:

$$\nu_{p,E_{l_i}} = \mu_{E_{l_i}} * \nu_{p,E_{l_i}} \xrightarrow[i\to\infty]{w} \mu_E * \tilde{\nu}.$$

Then, by uniqueness of the weak limit: $\tilde{\nu} = \mu_E * \tilde{\nu}$. Thus $\tilde{\nu}$ is a μ_E-invariant measure, and by uniqueness in Theorem 7.1.6: $\tilde{\nu} = \nu_{p,E}$. We deduce that $\nu_{p,E_l} \xrightarrow[l\to\infty]{w} \nu_{p,E}$, and hence $E \mapsto \nu_{p,E}$ is weakly continuous.

By the integral representation given in Theorem 7.1.6 we have:

$$(\gamma_1 + \ldots + \gamma_p)(E) = \nu_{p,E}(\Phi_{p,E}).$$

Then, using point (ii) of Proposition 7.1.8 and the weak continuity we just proved, we obtain:

$$\begin{aligned} \lim_{l\to\infty} (\gamma_1 + \ldots + \gamma_p)(E_l) &= \lim_{l\to\infty} \nu_{p,E_l}(\Phi_{p,E_l}) \\ &= \lim_{l\to\infty} \left(\nu_{p,E_l}(\Phi_{p,E}) + \nu_{p,E_l}(\Phi_{p,E_l} - \Phi_{p,E})\right) \\ &= \nu_{p,E}(\Phi_{p,E}) \\ &= (\gamma_1 + \ldots + \gamma_p)(E) \end{aligned}$$

This proves the continuity of the sums of Lyapunov exponents. □

We then have the following corollary:

Corollary 7.1.10. *For any integer $p \in \{1, \ldots, N\}$, the map $E \mapsto \gamma_p(E)$ is continuous.*

Proof : Indeed, one can write:

$$\gamma_p(E) = (\gamma_1 + \ldots + \gamma_p)(E) - (\gamma_1 + \ldots + \gamma_{p-1})(E).$$

□

7.1.3 Subharmonicity of sums of Lyapunov exponents

Throughout this section, we assume that the Lyapunov exponents under study are associated with a sequence $(A_n^\omega(E))_{n\in\mathbb{N}}$ of *i.i.d.* symplectic matrices that depend analytically on the parameter E. This is satisfied for the transfer matrices associated with a continuous (or discrete) model, since their expression involves only solutions of $-u'' + V_\omega u = Eu$ which are analytic in E.

First, we note that the definition of Lyapunov exponents allows us to define them for complex values of the energy E. Indeed, the formula:

$$(\gamma_1 + \ldots + \gamma_p)(E) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left(\log || \wedge^p (A_n^\omega(E) \ldots A_0^\omega(E))||\right)$$

also makes sense for $E \in \mathbb{C}$.

We now define subharmonic functions and state their basic properties.

Definition 7.1.11. *A function f $\mathbb{C} \to [-\infty, +\infty[$ is called subharmonic if:*

1. *f is upper semi-continuous, i.e.: $\forall E \in \mathbb{C},\ f(E) \geq \limsup_{y \to E} f(y)$*
2. *$\forall E \in \mathbb{C},\ \forall r > 0,\ f(E) \leq \frac{1}{2\pi} \int_0^{2\pi} f(E + re^{i\theta}) \mathrm{d}\theta$.*

Remark 7.1.12. *Of course, a continuous function is upper semi-continuous, and the first hypothesis of the definition is satisfied by the sums of Lyapunov exponents as proved in Proposition 7.1.9.*

We now prove a proposition giving the main stability properties of the set of subharmonic functions and the main example of a subharmonic function.

Proposition 7.1.13.
1. *If f and g are two subharmonic functions equal almost everywhere in the sense of Lebesgue measure on $\mathbb{R}^2$, then they are equal everywhere.*
2. *If f_n is a sequence of subharmonic functions locally bounded below, then the pointwise infimum of this sequence is subharmonic.*
3. *If $A(z)$ is an entire matrix-valued function, the function $z \mapsto \log ||A(z)||$ is subharmonic.*

Proof : (i) Fix $E \in \mathbb{C}$. By the maximum principle for subharmonic functions, we have:

$$\forall r > 0,\ f(E) \leq \frac{1}{\pi r^2} \int_{D(E,r)} f(z) dz.$$

Then, by upper semi-continuity:

$$f(E) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(E,r)} f(z) dz.$$

Hence, if $f = g$ almost surely, it follows that:

$$f(E) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(E,r)} f(z) dz = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(E,r)} g(z) dz = g(E),$$

and the first point is proved.

(ii) An upper semi-continuous function is bounded below on any compact subset $K \subset \mathbb{C}$. Fix a compact $K \subset \mathbb{C}$. We can then locally apply the dominated convergence theorem of Lebesgue to obtain:

$$\forall E \in K, \quad \inf_{n \in \mathbb{N}} f_n(E) \leq \inf_{n \in \mathbb{N}} \frac{1}{2\pi} \int_0^{2\pi} f_n(E + re^{i\theta}) \mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} (\inf_{n \in \mathbb{N}} f_n)(E + re^{i\theta}) \mathrm{d}\theta,$$

and the infimum of a family of upper semi-continuous functions is still upper semi-continuous.

(iii) This point follows from Jensen's formula on the logarithm of holomorphic functions.

□

We now have everything we need to prove the subharmonicity of sums of Lyapunov exponents.

Proposition 7.1.14. *For every $p \in \{1, \ldots, N\}$, the function $E \mapsto \gamma_1(E) + \ldots + \gamma_p(E)$ is subharmonic.*

Proof : First, the function $E \mapsto A_n^\omega(E)$ is entire by assumption. Then, since a product of entire functions is entire, $E \mapsto \wedge^p(A_n^\omega(E)\dots A_0^\omega(E))$ is also entire. Finally, by point (iii) of Proposition 7.1.13, $E \mapsto \log || \wedge^p (A_n^\omega(E)\dots A_0^\omega(E))||$ is subharmonic.

By Fatou's lemma, the map $E \mapsto \mathbb{E}\left(\log || \wedge^p (A_n^\omega(E)\dots A_0^\omega(E))||\right)$ is also subharmonic. Moreover, the sequence $\left(\frac{1}{n}\mathbb{E}\left(\log || \wedge^p (A_n^\omega(E)\dots A_0^\omega(E))||\right)\right)_{n\in\mathbb{N}}$ is a subadditive sequence of strictly positive numbers, so its limit is given by an infimum, and point (ii) of Proposition 7.1.13 applies to show that $\gamma_1(E) + \ldots + \gamma_p(E)$ is subharmonic.

□

To conclude this section, we recall without proof one of the most interesting properties of subharmonic functions: the existence of non-tangential limits.

Proposition 7.1.15. *For almost every $E \in \mathbb{R}$ and for every $p \in \{1,\dots,N\}$, the following limit exists and satisfies:*

$$\lim_{\varepsilon\to 0}(\gamma_1 + \ldots + \gamma_p)(E + \mathrm{i}\varepsilon) = (\gamma_1 + \ldots + \gamma_p)(E).$$

Proof : We use the fact that any subharmonic function on $\mathbb{C}$ is tangentially continuous at almost every point of the real line in the sense of Lebesgue measure.

□

7.1.4 Hölder regularity

With much more work, one can obtain the following general theorem on Hölder regularity of Lyapunov exponents.

Theorem 7.1.16. *Let $(A_n^\omega(E))_{n\in\mathbb{N}}$ be an* i.i.d. *sequence of symplectic matrices depending on a real parameter E. Let μ_E be the common law of $A_n^\omega(E)$. Fix a compact interval I in $\mathbb{R}$ and assume that for all $E \in I$:*

1. *G_{μ_E} is p-contracting and L_p-strongly irreducible for all $p \in \{1,\dots,N\}$.*
2. *There exist constants $C_1 > 0$, $C_2 > 0$ independent of n, ω, E such that for all $p \in \{1,\dots,N\}$:*
$$|| \wedge^p A_n^\omega(E)||^2 \leq \exp(pC_1 + p|E| + p) \leq C_2. \tag{7.3}$$
3. *There exists a constant $C_3 > 0$ independent of n, ω, E such that for all $E, E' \in I$ and all $p \in \{1,\dots,N\}$:*
$$|| \wedge^p A_n^\omega(E) - \wedge^p A_n^\omega(E')|| \leq C_3|E - E'|. \tag{7.4}$$

Then there exist constants $\alpha > 0$ and $0 < C < +\infty$ such that:

$$\forall p \in \{1,\dots N\},\ \forall E, E' \in I,\ |\gamma_p(E) - \gamma_p(E')| \leq C|E - E'|^\alpha.$$

To prove this result, one uses a result on negative cocycles as stated in [8], Proposition IV 3.5, p.187. One also needs estimates on Laplace operators on Hölder spaces as in Proposition V 4.13, p.277 in [8], which rely on estimates (7.3) and (7.4). Finally, using the decomposition given in Proposition IV 3.12, p.192 of [8], one can prove the Hölder continuity of $E \mapsto \nu_{p,E}$ on I.

7.1.5 Application to the continuous matrix-valued Anderson model

In an interval where it has already been shown that the Furstenberg group is p-contracting and L_p-irreducible for all p, it remains to prove estimates (7.3) and (7.4) in order to apply Theorem 7.1.16.

General Estimates

Lemma 7.1.17. *Let V be a matrix-valued function in $L^1_{\text{loc}}(\mathbb{R}, M_n(\mathbb{C}))$ and let u be a solution of $-u'' + Vu = 0$. Then for all $x, y \in \mathbb{R}$:*

$$||u(x)||^2 + ||u'(x)||^2 \leq (||u(y)||^2 + ||u'(y)||^2) \exp\left(\int_{\min(x,y)}^{\max(x,y)} ||V(t)+1|| \mathrm{d}t\right).$$

Proof : Set $R(t) = ||u(t)||^2 + ||u'(t)||^2$. Then, with this notation:

$$\begin{aligned} R'(t) &= < u(t), u'(t) > + < u'(t), u(t) > + < u''(t), u'(t) > + < u'(t), u''(t) > \\ &= 2\mathrm{Re}(< u(t), u'(t) >) + 2\mathrm{Re}(< u'(t), V(t)u(t) >) \\ &= 2\mathrm{Re}(< u'(t), (V(t)+1)u(t) >) \\ &\leq 2\mathrm{Re}(||u'(t)||\; ||V(t)+1||\; ||u(t)||) \\ &\leq 2||V(t)+1|| \left(\frac{||u(t)||^2 + ||u'(t)||^2}{2}\right) \\ &= ||V(t)+1|| R(t) \end{aligned}$$

We used the Cauchy-Schwarz inequality on the fourth line and the arithmetic-geometric inequality on the fifth. Finally, we have the inequality:

$$R'(t) \leq ||V(t)+1|| R(t),$$

which after integration gives the desired estimate.

□

Lemma 7.1.18. *For $i = 1, 2$, let $V_i \in L^1_{\text{loc}}(\mathbb{R}, M_n(\mathbb{C}))$ and let u_i be solutions of $-u'' + V_i u = 0$ such that:*

$$\exists y \in \mathbb{R},\ u_1(y) = u_2(y) \text{ and } u_1'(y) = u_2'(y).$$

Then for all $x \in \mathbb{R}$:

$$\begin{aligned} &(||u_1(x) - u_2(x)||^2 + ||u_1'(x) - u_2'(x)||^2)^{\frac{1}{2}} \\ \leq &\left(||u_1(y)||^2 + ||u_1'(y)||^2\right)^{\frac{1}{2}} \exp\left(\int_{\min(x,y)}^{\max(x,y)} ||V_1(t)|| + ||V_2(t)|| + 2\mathrm{d}t\right) \times \int_{\min(x,y)}^{\max(x,y)} ||V_1(t) - V_2(t)|| \mathrm{d}t. \end{aligned}$$

Proof : Without loss of generality, assume $y \leq x$. In $\mathbb{C}^{2N}$, by the hypotheses on u_1 and u_2:

$$\begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} = \int_x^y \begin{pmatrix} 0 \\ (V_1(t) - V_2(t))u_1(t) \end{pmatrix} \mathrm{d}t + \int_x^y \begin{pmatrix} 0 & I \\ V_2(t) & 0 \end{pmatrix} \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} \mathrm{d}t.$$

Taking norms on both sides, we get:

$$\left\| \begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} \right\| \leq \int_x^y ||V_1(t) - V_2(t)||\; ||u_1(t)|| \mathrm{d}t + \int_x^y (||V_2(t)|| + 1) \left\| \begin{pmatrix} u_1(t) - u_2(t) \\ u_1'(t) - u_2'(t) \end{pmatrix} \right\| \mathrm{d}t.$$

Then by Gronwall's lemma:

$$\left\| \begin{pmatrix} u_1(x) - u_2(x) \\ u_1'(x) - u_2'(x) \end{pmatrix} \right\| \leq \left(\int_x^y ||V_1(t) - V_2(t)||\; ||u_1(t)|| \mathrm{d}t\right) \exp\left(\int_x^y (||V_2(t)|| + 1) \mathrm{d}t\right). \quad (7.5)$$

Lemma 7.1.17 tells us that for all $t \in [y, x]$:

$$||u_1(t)||^2 \leq ||u_1(t)||^2 + ||u_1'(t)||^2 \leq (||u_1(y)||^2 + ||u_1'(y)||^2) \exp\left(\int_x^y (||V_1(s)|| + 1) \mathrm{d}s\right).$$

Hence:

$$||u_1(t)|| \leq \left(||u_1(y)||^2 + ||u_1'(y)||^2\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_x^y (||V_1(s)|| + 1)ds\right).$$

Plugging this into inequality (7.5) gives:

$$\left(||u_1(x) - u_2(x)||^2 + ||u_1'(x) - u_2'(x)||^2\right)^{\frac{1}{2}}$$

$$\leq \left(||u_1(y)||^2 + ||u_1'(y)||^2\right)^{\frac{1}{2}} \exp\left(\int_{\min(x,y)}^{\max(x,y)} \frac{1}{2}||V_1(t)|| + \frac{1}{2} + ||V_2(t)|| + 1\mathrm{d}t\right) \times \int_{\min(x,y)}^{\max(x,y)} ||V_1(t) - V_2(t)||\mathrm{d}t.$$

This proves the desired inequality since $\frac{1}{2}||V_1(t)|| + \frac{1}{2} \leq ||V_1(t)|| + 1$.

□

Estimates for the norm of the transfer matrices

From the general estimates we have just proved, one can obtain estimates for the norm of the transfer matrices, uniform in the parameter E when it varies in a compact interval of $\mathbb{R}$. Fix a compact interval $I \subset \mathbb{R}$. Let $E \in I$. First, $u^1, \dots, u^{2N}$ denote the solutions of $-u'' + V_\omega u = Eu$ with initial conditions:

$$\widetilde{u}^1(n,E) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \widetilde{u}^2(n,E) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \widetilde{u}^{2N}(n,E) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{7.6}$$

where $\widetilde{u}^l(n,E) = {}^t(u^l(n,E) \ \ (u^l)'(n,E))$.

The transfer matrix is the matrix whose columns are the vectors $\widetilde{u}^l(n+1,E)$:

$$A_n^\omega(E) = \begin{pmatrix} u_1^1(n{+}1,E) & u_1^2(n{+}1,E) & \dots & u_1^{2N}(n{+}1,E) \\ \vdots & \vdots & & \vdots \\ u_N^1(n{+}1,E) & u_N^2(n{+}1,E) & \dots & u_N^{2N}(n{+}1,E) \\ (u_1^1)'(n{+}1,E) & (u_1^2)'(n{+}1,E) & \dots & (u_1^{2N})'(n{+}1,E) \\ \vdots & \vdots & & \vdots \\ (u_N^1)'(n{+}1,E) & (u_N^2)'(n{+}1,E) & \dots & (u_N^{2N})'(n{+}1,E) \end{pmatrix}$$

Lemma 7.1.19. *There exist constants $C_1 > 0$ and $C_2 > 0$ independent of n, ω, E such that:*

$$||A_n^\omega(E)||^2 \leq \exp(C_1 + |E| + 1) \leq C_2$$

Proof : Let $\widetilde{u}^i(n+1,E)$ be the column of $A_n^\omega(E)$ with maximal norm. Then:

$$||A_n^\omega(E)||^2 = ||\widetilde{u}^i(n+1,E)||^2 = ||u^i(n+1,E)||^2 + ||(u^i)'(n+1,E)||^2.$$

Applying Lemma 7.1.17 with $x = n+1$ and $y = n$:

$$||u^i(n+1,E)||^2 + ||(u^i)'(n+1,E)||^2$$

$$\leq \left(||u^i(n,E)||^2 + ||(u^i)'(n,E)||^2\right) \exp\left(\int_n^{n+1} ||V_\omega(t) - E|| + 1\mathrm{d}t\right)$$

Now, $||u^i(n,E)||^2 = 1$ and $||(u^i)'(n,E)||^2 = 0$ or vice versa. In all cases: $||\widetilde{u}^i(n,E)||^2 = 1$. Hence

$$||A_n^\omega(E)||^2 \leq \exp\left(\int_n^{n+1} ||V_\omega(t) - E|| + 1\mathrm{d}t\right).$$

Since $V_\omega(x) = \sum_{k\in\mathbb{Z}} V_\omega^{(n)}(x-k)$ is invariant under translation by 1:

$$\int_n^{n+1} ||V_\omega(t) - E|| + 1\mathrm{d}t = \int_0^1 ||V_\omega(t) - E|| + 1\mathrm{d}t \leq \left(\sup_{t\in[0,1]} ||V_\omega(t)||\right) + |E| + 1.$$

Moreover, as in V_ω the ω_i take values in $\{0,1\}$, there exists a constant $C_1 > 0$ independent of ω, n, E such that:

$$\left(\sup_{t\in[0,1]} ||V_\omega(t)||\right) \leq C_1.$$

Thus:

$$||A_n^\omega(E)||^2 \leq \exp(C_1 + |E| + 1).$$

Finally, since I is bounded, $|E|$ is bounded and there exists a constant $C_2 > 0$ independent of ω, n, E such that:

$$\exp(C_1 + |E| + 1) \leq C_2.$$

□

Remark 7.1.20. *Since $A_n^\omega(E)$ is symplectic, its norm is the same as that of its inverse, hence:*

$$||A_n^\omega(E)^{-1}||^2 \leq C_2.$$

We now prove an estimate giving the variation of the norm of $A_n^\omega(E)$ when E varies in I.

Lemma 7.1.21. *For all $E, E' \in I$, there exists a constant $C_3 > 0$ independent of n, ω, E such that:*

$$||A_n^\omega(E) - A_n^\omega(E')|| \leq C_3|E - E'|.$$

Proof : We have:

$$||A_n^\omega(E) - A_n^\omega(E')|| = ||\widetilde{u}^i(n+1,E) - \widetilde{u}^i(n+1,E')||.$$

By Lemma 7.1.18:

$$||\widetilde{u}^i(n+1,E) - \widetilde{u}^i(n+1,E')||$$

$$\leq ||\widetilde{u}^i(n,E)|| \left(\int_n^{n+1} ||V_\omega(t) - E - (V_\omega(t) - E')||\mathrm{d}t\right) \exp\left(\int_n^{n+1} ||V_\omega(t) - E|| + ||(V_\omega(t) - E')|| + 2\,\mathrm{d}t\right)$$

Hence:

$$||A_n^\omega(E) - A_n^\omega(E')|| \leq |E - E'| \exp\left(\int_0^1 2||V_\omega(t)|| + |E| + |E'| + 2\,\mathrm{d}t\right).$$

As in the proof of Lemma 7.1.19, $\sup_{t\in[0,1]} ||V_\omega(t)|| \leq C_1$, and $|E|$ and $|E'|$ are bounded since I is, say by a constant M. Then there exists $C_3 > 0$ independent of n, ω, E such that:

$$\exp\left(\int_n^{n+1} ||V_\omega(t) - E|| + ||(V_\omega(t) - E')|| + 2\,\mathrm{d}t\right) \leq \exp(2C_1 + 2 + 2M) \leq C_3.$$

Our estimate is proved.

□

We conclude by giving estimates for the p-th exterior powers of the transfer matrices.

Lemma 7.1.22. *There exist constants $C_1' > 0$, $C_2' > 0$ and $C_3' > 0$ independent of n, ω, E such that for all integers $p \in \{1, \ldots, N\}$:*

$$|| \wedge^p A_n^{\omega}(E)||^2 \leq \exp(pC_1' + p|E| + p) \leq C_2',$$

and for all $E, E' \in I$:

$$|| \wedge^p A_n^{\omega}(E) - \wedge^p A_n^{\omega}(E')|| \leq C_3'|E - E'|.$$

Proof : For the first inequality, we simply use the general fact: if $M \in \mathrm{GL}_{2N}(\mathbb{R})$ then $|| \wedge^p M|| \leq ||M||^p$, which follows from the polar decomposition of M. A proof can be found in [2], Lemmas 5.3 and 5.4, p.62. Applying this result and Lemma 7.1.19, we obtain:

$$|| \wedge^p A_n^{\omega}(E)||^2 \leq (\exp(C_1 + |E| + 1))^p = \exp(pC_1 + p|E| + p) \leq C_2'.$$

For the second inequality, the result is slightly more technical. We show that for two invertible matrices M and N:

$$|| \wedge^p M - \wedge^p N|| \leq ||N - M|| \big(||N||^{p-1} + ||M|| \, ||N||^{p-2} + \ldots + ||M||^{p-1}\big).$$

Indeed, we start from:

$$\wedge^p M - \wedge^p N = \wedge^p M(I - \wedge^p(M^{-1}N)) = \wedge^p M(I - \wedge^p(M^{-1}(N - M) + I)).$$

We then need to compute $\wedge^p(M^{-1}(N - M) + I)$. For a decomposable p-vector $u_1 \wedge \ldots \wedge u_p$:

$$\begin{aligned}
&\wedge^p(M^{-1}(N-M)+I)(u_1 \wedge \ldots \wedge u_p) \\
&= (u_1 + M^{-1}(N-M)u_1) \wedge \ldots \wedge (u_p + M^{-1}(N-M)u_p) \\
&= u_1 \wedge \ldots \wedge (u_p + M^{-1}(N-M)u_p) + M^{-1}(N-M)u_1 \wedge \ldots \wedge (u_p + M^{-1}(N-M)u_p) \\
&\vdots \\
&= u_1 \wedge \ldots \wedge u_p + \ldots + u_1 \wedge \ldots \wedge M^{-1}(N-M)u_p
\end{aligned}$$

Hence, for $\wedge^p M(I - \wedge^p(M^{-1}(N - M) + I))(u_1 \wedge \ldots \wedge u_p)$, we obtain:

$$\begin{aligned}
&\wedge^p M(I - \wedge^p(M^{-1}(N-M)+I))(u_1 \wedge \ldots \wedge u_p) \\
&\quad = -((N-M)u_1 \wedge Nu_2 \wedge \ldots \wedge Nu_p + Mu_1 \wedge (N-M)u_2 \wedge \ldots \wedge Nu_p + \ldots \\
&\qquad \ldots + Mu_1 \wedge \ldots \wedge Mu_{p-1} \wedge (N-M)u_p)
\end{aligned}$$

Taking norms, we get the estimate:

$$|| \wedge^p M - \wedge^p N|| \leq ||N - M|| \big(||N||^{p-1} + ||M|| \, ||N||^{p-2} + \ldots + ||M||^{p-1}\big).$$

Finally, applying Lemmas 7.1.19 and 7.1.21 and this estimate with $M = A_n^{\omega}(E)$ and $N = A_n^{\omega}(E')$, we obtain:

$$|| \wedge^p A_n^{\omega}(E) - \wedge^p A_n^{\omega}(E')|| \leq pC_2^{p-1}C_3|E - E'|,$$

and $C_3' = pC_2^{p-1}C_3$ is independent of n, ω and E, E'.

□

7.2 The Integrated Density of States

We aim to study the existence of the integrated density of states and its regularity for continuous, matrix-valued Anderson operators of the form:

$$H_\omega = -\Delta_d \otimes I_N + \sum_{n \in \mathbb{Z}^d} V_\omega^{(n)}(x-n) \tag{7.7}$$

acting on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$, where d and N are natural numbers, I_N is the identity matrix of order N, and Δ_d is the continuous Laplacian in dimension d.

Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a complete probability space and $\omega \in \Omega$. For each $n \in \mathbb{Z}$, the functions $x \mapsto V_\omega^{(n)}(x)$ take values in symmetric matrices, are supported in $[0,1]^d$, and are uniformly bounded in x, n, and ω. We also set:

$$\forall x \in \mathbb{R}^d,\ V_\omega(x) = \sum_{n \in \mathbb{Z}^d} V_\omega^{(n)}(x-n),$$

and denote by V_ω the maximal multiplication operator by $x \mapsto V_\omega(x)$. The function $x \mapsto V_\omega(x)$ is uniformly bounded on $\mathbb{R}$ in x and ω. The potential V_ω will also be chosen so that $\{H_\omega\}_{\omega\in\Omega}$ is $\mathbb{Z}^d$-ergodic. As a bounded perturbation of $-\Delta_d \otimes I_N$, H_ω is self-adjoint on the Sobolev space $H^2(\mathbb{R}^d) \otimes \mathbb{C}^N$.

7.2.1 Existence of the Integrated Density of States

We want to define a function of the real variable that counts the number of spectral values of H_ω below a fixed energy E. To define this function, we start by restricting H_ω to cubes of finite volume in $\mathbb{R}^d$. Let L be an integer greater than 1 and let $D = [-L, L]^d \subset \mathbb{R}^d$ be the cube centered at 0 with side length $2L$. We set:

$$H_\omega^{(L)} = -\Delta_d^{(L)} \otimes I_N + \sum_{n \in \mathbb{Z}^d} V_\omega^{(n)}(x-n) \tag{7.8}$$

the restriction of H_ω acting on $L^2(D) \otimes \mathbb{C}^N$ with Dirichlet boundary conditions on ∂D.

Definition 7.2.1. *The integrated density of states of $\{H_\omega\}_{\omega\in\Omega}$ is the function from $\mathbb{R}$ to $\mathbb{R}_+$, $E \mapsto N(E)$, where for $E \in \mathbb{R}$:*

$$N(E) = \lim_{L \to +\infty} \frac{1}{|D|} \#\{\lambda \leq E \mid \lambda \in \sigma(H_\omega^{(L)})\}, \tag{7.9}$$

and $|D|$ is the volume of D.

There is a double issue of existence in expression (7.9). First, one must prove that the cardinality $\#\{\lambda \leq E \mid \lambda \in \sigma(H_\omega^{(L)})\}$ is finite for any fixed E, and second, one must prove the existence of the limit. The answer to both issues is provided by the existence of an L^2 kernel for the one-parameter semigroup $(\mathrm{e}^{-tH_\omega^{(L)}})_{t>0}$. This kernel is obtained using a Feynman-Kac formula involving a time-ordered exponential.

A matrix-valued Feynman-Kac formula

We first present how to obtain a matrix-valued Feynman-Kac formula for the one-parameter semigroup $(\mathrm{e}^{-tH_\omega})_{t>0}$. We then deduce a Feynman-Kac formula for $(\mathrm{e}^{-tH_\omega^{(L)}})_{t>0}$.

Let $\mathsf{W} = C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. For each $t \geq 0$, consider the coordinate function:

$$X_t \,:\, \begin{array}{ccc} \mathsf{W} & \longrightarrow & \mathbb{R} \\ \mathsf{w} & \longmapsto & X_t(\mathsf{w}) = \mathsf{w}(t) \end{array}$$

Let $\mathcal{W}$ be the smallest σ-algebra on W for which the maps X_t are measurable. For $s,t \geq 0$ and $x,y \in \mathbb{R}^d$, denote by $W_{s,x,t,y}$ the conditional Wiener measure, defined on $(\mathsf{W}, \mathcal{W})$, associated with Brownian motion starting at x at time s and arriving at y at time t. Also denote by $\mathbb{E}_{s,x,t,y}$ the expectation associated with $W_{s,x,t,y}$.

We now study the one-parameter semigroup $(\mathrm{e}^{-tH_\omega})_{t>0}$. Fix $t > 0$ and $\omega \in \Omega$. By the Lie-Trotter formula, we have:

$$\forall f \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^N,\ \mathrm{e}^{-tH_\omega} f = \lim_{n\to+\infty} \left(\mathrm{e}^{-(-\Delta_d \otimes I_N)\frac{t}{n}} \mathrm{e}^{-V_\omega \frac{t}{n}} \right)^n f. \tag{7.10}$$

For fixed $n \in \mathbb{N}$, the operator:

$$\left(\mathrm{e}^{-(-\Delta_d \otimes I_N)\frac{t}{n}} \mathrm{e}^{-V_\omega \frac{t}{n}} \right)^n$$

has an integral kernel given by the path integral:

$$\int \prod_{j=1}^{n} \mathrm{e}^{-(\frac{jt}{n})V_\omega(\mathsf{w}(\frac{jt}{n}))}\, \mathrm{d}W_{0,x,t,y}(\mathsf{w}). \tag{7.11}$$

As n tends to infinity, by the definition of the Dyson time-ordered exponential:

$$\lim_{n\to+\infty} \prod_{j=1}^{n} \mathrm{e}^{-(\frac{jt}{n})V_\omega(\mathsf{w}(\frac{jt}{n}))} = \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(\mathsf{w}(s))\, \mathrm{d}s \right). \tag{7.12}$$

Then, by dominated convergence:

$$\forall f \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^N,\ \forall x \in \mathbb{R}^d,\ \mathrm{e}^{-tH_\omega} f(x) = \int_{\mathbb{R}^d} K_t(x,y) f(y) \mathrm{d}x \tag{7.13}$$

where

$$\forall x,y \in \mathbb{R}^d,\ \forall t>0,\ K_t(x,y) = \int \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(\mathsf{w}(s))\, \mathrm{d}s \right) \mathrm{d}W_{0,x,t,y}(\mathsf{w}). \tag{7.14}$$

We have thus shown that e^{-tH_ω} has an integral kernel $K_t(x,y)$. Let us see how to deduce the existence of an integral kernel for $\mathrm{e}^{-tH_\omega^{(L)}}$. Denote by $T_D(\mathsf{w})$ the first exit time of D for the path $\mathsf{w} \in \mathsf{W}$:

$$T_D(\mathsf{w}) = \inf\{t > 0 \mid X_t(\mathsf{w}) \notin D\}. \tag{7.15}$$

Since Dirichlet boundary conditions are used on D, to define $H_\omega^{(L)}$ one can use results on killed Brownian motions, which lead to the formula:

$$\forall t>0,\ \forall f \in L^2(\mathbb{R}^d) \otimes \mathbb{C}^N,\ \forall x \in \mathbb{R}^d,\ \mathrm{e}^{-tH_\omega^{(L)}} f(x) =$$
$$\frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}^d} \int \chi_{\{t<T_D(\mathsf{w})\}}(\mathsf{w}) \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(X_s(\mathsf{w}))\, \mathrm{d}s \right) \mathrm{d}W_{0,x,t,y}(\mathsf{w})\, \mathrm{e}^{-\frac{|x-y|^2}{2t}} f(y) \mathrm{d}y. \tag{7.16}$$

We have thus established the following result.

Proposition 7.2.2. *For all $t>0$, $\mathrm{e}^{-tH_\omega^{(L)}}$ has an integral kernel given by:*

$$\forall x,y \in \mathbb{R}^d,\ \forall t>0,\ K_t^{(L)}(x,y) =$$
$$\frac{1}{\sqrt{2\pi t}} \left(\int \chi_{\{t<T_D(\mathsf{w})\}}(\mathsf{w}) \exp_{\mathrm{ord}} \left(-\int_0^t V_\omega(X_s(\mathsf{w}))\, \mathrm{d}s \right) \mathrm{d}W_{0,x,t,y}(\mathsf{w})\, \mathrm{e}^{-\frac{|x-y|^2}{2t}} \right) \tag{7.17}$$

and $K_t^{(L)}$ belongs to $L^2(D^2) \otimes \mathcal{M}_{\mathrm{N}}(\mathbb{C})$ for all $t>0$.

Existence of the Integrated Density of States

Since the kernel is in L^2, it follows that for any $t > 0$, the operator $\mathrm{e}^{-tH_\omega^{(L)}}$ is Hilbert-Schmidt on $L^2(D) \otimes \mathbb{C}^N$. In particular, it is compact and its spectrum has the form:

$$\{\mathrm{e}^{-t\lambda_{j,\omega}^{(L)}},\ j \geq 0\},$$

where $(\lambda_{j,\omega}^{(L)})_{j\geq 0}$ is an increasing sequence of real numbers, bounded from below and tending to $+\infty$. This sequence coincides with the spectrum of $H_\omega^{(L)}$. In particular, for fixed $E \in \mathbb{R}$:

$$\#\{\lambda \leq E \mid \lambda \in \sigma(H_\omega^{(L)})\} = \#\{\lambda_{j,\omega}^{(L)} \leq E\} < +\infty.$$

This addresses the first issue in proving the existence of $N(E)$.

It remains to show that the sequence $\left(\frac{1}{|D|}\#\{\lambda_{j,\omega}^{(L)} \leq E\}\right)_{L\geq 1}$ converges to a real number independent of ω: $N(E)$. To do this, we introduce the eigenvalue counting measure of $H_\omega^{(L)}$:

$$\mathfrak{n}_{D,\omega} = \frac{1}{|D|}\sum_{j\geq 0}\delta_{\lambda_{j,\omega}^{(L)}}, \tag{7.18}$$

where $\delta_{\lambda_{j,\omega}^{(L)}}$ is the Dirac measure at $\lambda_{j,\omega}^{(L)}$. Then we have:

Proposition 7.2.3. *The sequence of measures $(\mathfrak{n}_{D,\omega})_{L\geq 1}$ converges weakly to a measure $\mathfrak{n}$ independent of ω as L tends to infinity, for P-almost every ω in Ω. Moreover, the Laplace transform of this limiting measure is given by: $\forall t > 0$,*

$$L(\mathfrak{n})(t) = \frac{1}{\sqrt{2\pi t}}\int\int_\Omega \mathrm{Tr}_{\mathbb{C}^N}\exp_{\mathrm{ord}}\left(-\int_0^t V_\omega(X_s(\mathrm{w}))\,\mathrm{d}s\right)\,\mathrm{d}\omega\,\mathrm{d}W_{0,0,t,0}(\mathrm{w}). \tag{7.19}$$

Corollary 7.2.4. *For any $E \in \mathbb{R}$, the limit:*

$$N(E) = \lim_{L\to+\infty}\frac{1}{|D|}\#\{\lambda \leq E \mid \lambda \in \sigma(H_\omega^{(L)})\}$$

exists and is P-almost surely independent of ω. The function $E \mapsto N(E)$ is the distribution function of $\mathfrak{n}$:

$$\forall E \in \mathbb{R},\ N(E) = \mathfrak{n}((-\infty, E]).$$

We conclude by giving a formula linking the measure $\mathfrak{n}$ and the spectral measure E_{H_ω} associated with the self-adjoint operator H_ω.

Proposition 7.2.5. *Let f be a continuous, compactly supported, positive function on $\mathbb{R}^d$ such that $||f||_{L^2(\mathbb{R}^d)} = 1$. Let M_f be the maximal multiplication operator by f. Then, for any bounded Borel set B in $\mathbb{R}$, the operator $M_f E_{H_\omega}(B) M_f$ is of trace class P-almost surely in ω, and:*

$$\mathfrak{n}(B) = \mathbb{E}(\mathrm{Tr}(M_f E_{H_\omega}(B) M_f)), \tag{7.20}$$

where $\mathbb{E}$ is the expectation associated with the probability measure P.

We now want, in the case $d = 1$, to link $N(E)$ with the sum of the Lyapunov exponents at E. Before that, we introduce Kotani's w-function.

From now on, we assume $d = 1$.

7.2.2 Kotani's w-function

Before defining Kotani's w-function, we define the m-functions. Let $\mathbb{C}_+$ be the upper complex half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\mathbb{C}_-$ the lower half-plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$.

Proposition 7.2.6. *Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. Fix $\omega \in \Omega$. Then there exists a unique function $x \mapsto F_+(x,E)$ with values in $\mathcal{M}_\mathrm{N}(\mathbb{C})$ (respectively $x \mapsto F_-(x,E)$) satisfying:*

$$-F_+'' + V_\omega F_+ = EF_+,\ F_+(0,E) = I, \text{ and } \int_0^\infty ||F_+(x,E)||^2 dx < +\infty,$$

respectively:

$$-F_-'' + V_\omega F_- = EF_-,\ F_-(0,E) = I, \text{ and } \int_{-\infty}^0 ||F_-(x,E)||^2 dx < +\infty.$$

Definition 7.2.7. *For $E \in \mathbb{C}_+ \cup \mathbb{C}_-$, we define the m-functions M_+ and M_- associated with $\{H_\omega\}_{\omega\in\Omega}$ by:*

$$M_+(E) = \frac{d}{dx} F_+(x,E)|_{x=0} \text{ and } M_-(E) = -\frac{d}{dx} F_-(x,E)|_{x=0}.$$

These functions allow the computation of the Green kernel of the resolvent of H_ω off the real axis.

Proposition 7.2.8. *Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. Then $(H_\omega - E)^{-1}$ has a continuous integral kernel $G_E(x,y,\omega)$ given by:*

$$G_E(x,y,\omega) = \begin{cases} -F_-(x)(M_+ + M_-)^{-1}\, {}^t F_+(y) & \text{if} \quad x \le y, \\ -F_+(x)(M_+ + M_-)^{-1}\, {}^t F_-(y) & \text{if} \quad y \le x. \end{cases}$$

We can now define Kotani's w-function. This function links the Lyapunov exponents to the integrated density of states. Indeed, its real part will (in the limit) be the sum of the N positive Lyapunov exponents, while its imaginary part will (again in the limit) equal $\pi N(E)$.

Definition 7.2.9. *Let $E \in \mathbb{C}_+ \cup \mathbb{C}_-$. Kotani's w-function is defined by:*

$$w(E) = \frac{1}{2}\mathbb{E}(\mathrm{Tr}(M_+(E) + M_-(E))).$$

The w-function has the following properties:

Proposition 7.2.10. *For $E \in \mathbb{C}_+ \cup \mathbb{C}_-$:*

1. $w(E) = \mathbb{E}(\mathrm{Tr}(M_+(E))) = \mathbb{E}(\mathrm{Tr}(M_-(E)))$.
2. $\frac{\mathrm{d}}{\mathrm{d}E} w(E) = \mathbb{E}(\mathrm{Tr}(G_E(0,0,\omega)))$.
3. $-\mathrm{Re}\, w(E) = (\gamma_1 + \ldots + \gamma_N)(E)$.
4. $\mathbb{E}\left(\mathrm{Tr}(\mathrm{Im}\ M_\pm(E,\omega)^{-1})\right) = -\frac{2\mathrm{Re}\ w(E)}{\mathrm{Im} E} = \frac{2(\gamma_1+\ldots+\gamma_N)(E)}{\mathrm{Im} E}$.

In point 3, let us clarify that the formula:

$$\gamma_1(E) + \ldots + \gamma_N(E) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log || \wedge^N (A_{n-1}^\omega(E) \ldots A_0^\omega(E))||)$$

makes sense for all $E \in \mathbb{C}$.

One can then use results from harmonic analysis valid for matrix-valued Schrödinger operators. First, we introduce the space of Herglotz functions:

$$\mathcal{H} = \{h \mid h \text{ is holomorphic on } \mathbb{C}_+ \text{ and } h\ :\ \mathbb{C}_+ \to \mathbb{C}_+\}.$$

We then consider the subspace of $\mathcal{H}$:

$$\mathcal{W} = \{w \in \mathcal{H} \mid w,\ w',\ -\mathrm{i}w \in \mathcal{H}\}.$$

Proposition 7.2.11. *Kotani's function w belongs to $\mathcal{W}$.*

Proof : First, since H_ω is self-adjoint, its spectrum lies in $\mathbb{R}$ and $E \mapsto M_+(E)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. The same holds for $E \mapsto \mathrm{Tr}(M_+(E))$. Moreover, if $\mathrm{Im}E > 0$, we have:

$$\mathrm{Im}\ M_+(E) = (\mathrm{Im}E) \int_0^{+\infty} F_+(x,E)^* F_+(x,E) > 0.$$

Thus, $E \mapsto \mathrm{Tr}(M_+(E))$ is in $\mathcal{H}$ and $w \in \mathcal{H}$.

Next, by point 2 of Proposition 7.2.10, $w'(E) = \mathbb{E}(\mathrm{Tr}(G_E(0,0,\omega)))$. But $G_E(0,0,\omega)$ is holomorphic off the spectrum of H_ω, so $\mathrm{Tr}(G_E(0,0,\omega))$ is also holomorphic. If $\mathrm{Im}E > 0$, the operator $\mathrm{Im}(H_\omega - E)^{-1}$ is positive, and $\mathrm{Im}\mathrm{Tr}(G_E(0,0,\omega)) > 0$. Therefore, $\mathrm{Im}\, w'(E) = \mathrm{Im}\mathrm{Tr}(G_E(0,0,\omega)) > 0$ and $w' \in \mathcal{H}$.

Finally, $-\mathrm{i}w$ is holomorphic on $\mathbb{C}_+$ since w is. If $E \in \mathbb{C}_+$:

$$\mathrm{Im}(-\mathrm{i}w(E)) = -\mathrm{Re}\ w(E) = (\mathrm{Im}E)\mathbb{E}(\mathrm{Tr}(\mathrm{Im}\ M_+(E,\omega)^{-1}))$$

by point 4 of Proposition 7.2.10. Since $\mathrm{Tr}(\mathrm{Im}\ M_+(E,\omega)^{-1}) > 0$ for $E \in \mathbb{C}_+$, we have $\mathrm{Im}(-\mathrm{i}w(E)) > 0$. Hence, $-\mathrm{i}w \in \mathcal{H}$.

□

7.2.3 Thouless formula

Let $\mathfrak{n}$ be the measure defined in Proposition 7.2.3.

Proposition 7.2.12.

$$\forall E \in \mathbb{C} \setminus \mathbb{R},\ \mathbb{E}(\mathrm{Tr}\ G_E(0,0,\omega)) = \int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-E}. \tag{7.21}$$

Proof : Since $\mathbb{R}$ is a countable union of bounded Borel sets and the Dirac distribution at 0, δ_0, can be approximated by continuous, compactly supported, positive functions with L^2-norm equal to 1, using Proposition (7.2.12), we have:

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-E} = \int_{\mathbb{R}} \frac{1}{E'-E} \mathrm{d}\mathbb{E}\left(\mathrm{Tr}(< \delta_0, E_{H_\omega}((-\infty, E'])\delta_0 >)\right).$$

Then, by the spectral theorem,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\mathrm{d}\mathfrak{n}(E')}{E'-E} &= \mathbb{E}\left(\mathrm{Tr}(\int_{\mathbb{R}} \frac{1}{E'-E} \mathrm{d} < \delta_0, E_{H_\omega}((-\infty, E'])\delta_0 >)\right) \\
&= \mathbb{E}\left(\mathrm{Tr}(< \delta_0, \left(\int_{\mathbb{R}} \frac{1}{E'-E} \mathrm{d}E_{H_\omega}((-\infty, E'])\right) \delta_0 >)\right) \\
&= \mathbb{E}\left(\mathrm{Tr}(< \delta_0, (H_\omega - E)^{-1}\delta_0 >)\right) \\
&= \mathbb{E}\left(\mathrm{Tr}(G_E(0,0,\omega))\right).
\end{aligned}$$

□

From this formula, we deduce the link between the imaginary part of w and $E \mapsto N(E)$.

Proposition 7.2.13.

$$\forall E \in \mathbb{R},\ \lim_{a \to 0^+} \mathrm{Im}\ w(E + \mathrm{i}a) = \pi N(E). \tag{7.22}$$

Proof : First, by point 2 of Proposition 7.2.10:

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ w'(z) = \mathbb{E}(\mathrm{Tr}(G_z(0,0,\omega))).$$

Then, by Proposition 7.2.12:

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ w'(z) = \int_{\mathbb{R}} \frac{\mathrm{d}n(E')}{E'-z} = \int_{\mathbb{R}} \frac{N(E')}{(E'-z)^2}\,\mathrm{d}E' \tag{7.23}$$

by integration by parts. Integrating this expression, there exists a constant $c \in \mathbb{C}$ such that:

$$w(z) = c + \int_{\mathbb{R}} \frac{1+E'z}{(E'-z)(1+E'^2)} N(E')\mathrm{d}E'. \tag{7.24}$$

If $z \in \mathbb{R}$ is not in the spectrum of H_ω, then $w(z) \in \mathbb{R}$ (see Carmona Saint-Flour, Lemma 5.10, p84). Hence, $c \in \mathbb{R}$. Taking the imaginary part in (7.24) and writing $z \in \mathbb{C}_+$ as $z = E + \mathrm{i}a$, $E \in \mathbb{R}$, $a > 0$:

$$\begin{aligned} \mathrm{Im}\, w(E+\mathrm{i}a) &= a \int_{\mathbb{R}} \frac{N(E')}{(E'-E)^2+a^2}\,\mathrm{d}E' \\ &= \int_{\mathbb{R}} \frac{N(E+au)}{1+u^2}\,\mathrm{d}u \end{aligned}$$

where $u = \frac{E'-E}{a}$. Since $N(E)$ is a distribution function, it is right-continuous, and

$$\forall E \in \mathbb{R},\ \lim_{a\to 0^+} \mathrm{Im}\, w(E+\mathrm{i}a) = N(E) \int_{\mathbb{R}} \frac{1}{1+u^2}\,\mathrm{d}u = \pi N(E).$$

□

There is an analogous result for the real part of $w(E)$.

Proposition 7.2.14. *For Lebesgue-almost every E in $\mathbb{R}$, we have:*

$$\lim_{a\to 0^+} \mathrm{Re}\, w(E+\mathrm{i}a) = -(\gamma_1 + \ldots + \gamma_N)(E). \tag{7.25}$$

Moreover, if $I \subset \mathbb{R}$ is an interval on which $E \mapsto -(\gamma_1 + \ldots + \gamma_N)(E)$ is continuous, then (7.25) holds for all $E \in I$.

Proof : First, by point 3 of Proposition 7.2.10, we have:

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ \mathrm{Re}\, w(z) = -(\gamma_1 + \ldots + \gamma_N)(z). \tag{7.26}$$

The function $z \mapsto -(\gamma_1 + \ldots + \gamma_N)(z)$ is subharmonic, and for almost every $E \in \mathbb{R}$, the tangential limit exists:

$$\lim_{a\to 0}(\gamma_1 + \ldots + \gamma_N)(E+\mathrm{i}a) = (\gamma_1 + \ldots + \gamma_N)(E). \tag{7.27}$$

Let E be a real number for which (7.27) holds. Setting $z = E + \mathrm{i}a$ with $a > 0$ in (7.26), we obtain the existence of the limit:

$$\lim_{a\to 0^+} \mathrm{Re}\, w(E+\mathrm{i}a) = -(\gamma_1 + \ldots + \gamma_N)(E). \tag{7.28}$$

Moreover, if I is an interval on which $E \mapsto (\gamma_1 + \ldots + \gamma_N)(E)$ is continuous, the relation (7.28) holds for all E in I since it holds for almost every $E \in I$.

□

We now have all the necessary tools to prove a Thouless formula adapted to the case of continuous, matrix-valued random Schrödinger operators. Since $(\gamma_1 + \ldots + \gamma_N)(E)$ and $N(E)$ are respectively the real and imaginary parts of the function w, which belongs to the space $\mathcal{W}$, the harmonic analysis developed for this space by Kotani gives that these two functions are related by an integral formula.

Theorem 7.2.15 (Thouless formula). *For Lebesgue-almost every $E \in \mathbb{R}$, we have:*

$$(\gamma_1 + \ldots + \gamma_N)(E) = -\alpha + \int_{\mathbb{R}} \log\left(\left|\frac{E' - E}{E' - \mathrm{i}}\right|\right) \mathrm{d}\mathfrak{n}(E') \tag{7.29}$$

where α is a real number independent of E and $\mathfrak{n}$ is the measure whose integrated density of states is the distribution function. Moreover, if $I \subset \mathbb{R}$ is an interval on which $E \mapsto -(\gamma_1 + \ldots + \gamma_N)(E)$ is continuous, then (7.29) holds for all $E \in I$.

Proof : Since $w \in \mathcal{W}$, by Kotani's results and also using Proposition 7.2.13,

$$\forall z \in \mathbb{C} \setminus \mathbb{R},\ w(z) = w(\mathrm{i}) + \int_{\mathbb{R}} \log\left(\frac{E' - \mathrm{i}}{E' - z}\right) \mathrm{d}\mathfrak{n}(E'). \tag{7.30}$$

Then:

$$\mathrm{Re}\, w(z) = \mathrm{Re}\, w(\mathrm{i}) + \int_{\mathbb{R}} \log\left(\left|\frac{E' - \mathrm{i}}{E' - z}\right|\right) \mathrm{d}\mathfrak{n}(E'). \tag{7.31}$$

Let $z = E + \mathrm{i}a$ with $E \in \mathbb{R}$ such that (7.25) holds and $a > 0$. Then, as a tends to 0, by Proposition 7.2.14,

$$-(\gamma_1 + \ldots + \gamma_N)(E) = \mathrm{Re}\, w(\mathrm{i}) + \int_{\mathbb{R}} \log\left(\left|\frac{E' - \mathrm{i}}{E' - E}\right|\right) \mathrm{d}\mathfrak{n}(E'). \tag{7.32}$$

Setting $\alpha = \mathrm{Re}\, w(\mathrm{i})$ gives (7.29) for all $E \in \mathbb{R}$ such that (7.25) holds, *i.e.* for Lebesgue-almost every E in $\mathbb{R}$. Finally, if I is an interval on which $E \mapsto (\gamma_1 + \ldots + \gamma_N)(E)$ is continuous, by Proposition 7.2.14, (7.29) holds for all $E \in I$.

□

We can then use this Thouless formula to show that the function $E \mapsto N(E)$ has the same regularity as the Lyapunov exponents.

7.2.4 Hölder regularity

We begin with some reminders on the Hilbert transform and its main properties.

Definition 7.2.16. *If $\psi \in L^2(\mathbb{R})$, its Hilbert transform is the function defined on $\mathbb{R}$ by:*

$$(T\psi)(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{\psi(t)}{x - t}\, \mathrm{d}t.$$

Proposition 7.2.17. *Let $\psi \in L^2(\mathbb{R})$.*

1. *$T^2\psi(x) = -\psi(x)$ for Lebesgue-almost every $x \in \mathbb{R}$.*

2. *If ψ is Hölder continuous on the interval $[x_0 - a, x_0 + a]$, $a > 0$, then $T\psi$ is Hölder continuous on the interval $[x_0 - \frac{a}{2}, x_0 + \frac{a}{2}]$.*

From these properties, we deduce the following regularity result.

Theorem 7.2.18. *Let I be a compact interval in $\mathbb{R}$ and $\tilde{I}$ an open interval containing I. Assume that the potential V_ω in H_ω for $d=1$ and $N\geq 1$ is such that the Furstenberg group G_{μ_E} of $\{H_\omega\}_{\omega\in\Omega}$ is p-contracting and L_p-strongly irreducible for all $p\in\{1,\dots,N\}$ and all $E\in\tilde{I}$. Then the integrated density of states associated with $\{H_\omega\}_{\omega\in\Omega}$ is Hölder continuous on I.*

Proof : First, the map $E'\mapsto \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)$ is $\mathfrak{n}$-integrable on $\mathbb{R}$. Indeed, the renormalization term $E'-\mathrm{i}$ in the denominator compensates for the fact that the support of $\mathfrak{n}$ is non-compact. Hence, we have:

$$\forall E\in\mathbb{R},\ \lim_{\varepsilon\to 0^+}\int_{E-\varepsilon}^{E+\varepsilon}\left|\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\right|\,d\mathfrak{n}(E')=0 \tag{7.33}$$

which implies:

$$\forall E\in\mathbb{R},\ \lim_{\varepsilon\to 0^+}|\log(\varepsilon)|(N(E+\varepsilon)-N(E-\varepsilon))=0. \tag{7.34}$$

This shows that $E\mapsto N(E)$ is continuous on $\mathbb{R}$. Let $E_0\in I$ be fixed and $a>0$ such that $[E_0-4a,E_0+4a]\subset\tilde{I}$. By Theorem 7.2.15, for $E\in]E_0-4a,E_0+4a[$:

$$(\gamma_1+\ldots+\gamma_N)(E)+\alpha-\int_{|E'-E_0|>4a}\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,d\mathfrak{n}(E')=\int_{E_0-4a}^{E_0+4a}\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,d\mathfrak{n}(E').$$

Then:

$$\int_{E_0-4a}^{E_0+4a}\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,d\mathfrak{n}(E')=\lim_{\varepsilon\to 0^+}\left(\int_{E_0-4a}^{E-\varepsilon}\log|E'-E|\,d\mathfrak{n}(E')\right.$$
$$\left.+\int_{E+\varepsilon}^{E_0+4a}\log|E'-E|\,d\mathfrak{n}(E')\right)-\frac{1}{2}\int_{E_0-4a}^{E_0+4a}\log(1+(E')^2)\,d\mathfrak{n}(E').$$

Set:

$$\mathcal{I}(E_0)=\frac{1}{2}\int_{E_0-4a}^{E_0+4a}\log(1+(E')^2)\,d\mathfrak{n}(E').$$

Integrating by parts the first two integrals, we get:

$$\begin{aligned}
&\int_{E_0-4a}^{E_0+4a}\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,d\mathfrak{n}(E')\\
=&\lim_{\varepsilon\to 0^+}\left[[N(E')\log|E'-E|]_{E_0-4a}^{E-\varepsilon}-\int_{E_0-4a}^{E-\varepsilon}\frac{N(E')}{E-E'}dE'+[N(E')\log|E'-E|]_{E+\varepsilon}^{E_0+4a}\right.\\
&\left.-\int_{E+\varepsilon}^{E_0+4a}\frac{N(E')}{E'-E}dE'\right]-\mathcal{I}(E_0).
\end{aligned}$$

Set $\psi(E)=N(E)\chi_{\{|E-E_0|\leq 4a\}}\in L^2(\mathbb{R})$. By the definition of the Hilbert transform:

$$\begin{aligned}
&\int_{E_0-4a}^{E_0+4a}\log|E'-E|\,d\mathfrak{n}(E')\\
=&\ \pi(T\psi)(E)+\lim_{\varepsilon\to 0^+}[(N(E-\varepsilon)-N(E+\varepsilon))\log\varepsilon+N(E_0+4a)\log|E_0-E+4a|\\
&\qquad -N(E_0-4a)\log|E_0-E-4a|]-\mathcal{I}(E_0)\\
=&\ \pi(T\psi)(E)+N(E_0+4a)\log|E_0-E+4a|-N(E_0-4a)\log|E_0-E-4a|-\mathcal{I}(E_0)
\end{aligned}$$

by (7.34). Finally:

$$\begin{aligned}
\pi(T\psi)(E) =&\ (\gamma_1+\ldots+\gamma_N)(E)+\alpha-\int_{|E'-E_0|>4a}\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,d\mathfrak{n}(E')-\\
&\ N(E_0+4a)\log|E_0-E+4a|+N(E_0-4a)\log|E_0-E-4a|+\mathcal{I}(E_0)\\
=&\ (\gamma_1+\ldots+\gamma_N)(E)+\alpha-\int_{|E'-E_0|\geq 4a}\log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right)\,d\mathfrak{n}(E')+\mathcal{I}(E_0).
\end{aligned}$$

Since $[E_0 - 4a, E_0 + 4a] \subset I \subset \tilde{I}$, $E \mapsto (\gamma_1 + \ldots + \gamma_N)(E)$ is Hölder continuous on $[E_0 - 4a, E_0 + 4a]$ by Theorem 7.1.16. Moreover, $E \mapsto \int_{|E'-E_0| \geq 4a} \log\left(\left|\frac{E'-E}{E'-\mathrm{i}}\right|\right) \,\mathrm{d}\mathfrak{n}(E')$ is Hölder continuous of order 1 on the interval $]E_0 - 4a, E_0 + 4a[$.

Hence $T\psi$ is Hölder continuous on any compact interval contained in $]E_0 - 4a, E_0 + 4a[$, in particular on $[E_0 - 2a, E_0 + 2a]$. By point 2 of Proposition 7.2.17, $T^2\psi$ is Hölder continuous on $[E_0 - a, E_0 + a]$. By point 1 of Proposition 7.2.17 and the continuity of $E \mapsto N(E)$ (by (7.34)),

$$\forall E \in [E_0 - a, E_0 + a],\ (T^2\psi)(E) = -N(E).$$

Thus $E \mapsto N(E)$ is Hölder continuous on $[E_0 - a, E_0 + a]$. Since I is compact, it can be covered by a finite number of intervals $]E_0 - a, E_0 + a[\subset \tilde{I}$ with $E_0 \in I$. Therefore, $E \mapsto N(E)$ is Hölder continuous on I.

□

Chapter 8

Introduction to multi-scale analysis

8.1 Mathematical definitions of localization

There are several mathematical definitions to describe the phenomenon of Anderson localization. Throughout, $(\Omega, \mathcal{A}, \mathsf{P})$ is a complete probability space.

Definition 8.1.1. *Let I be an interval of $\mathbb{R}$. The family of self-adjoint random operators $\{H_\omega\}_{\omega\in\Omega}$ is said to be **spectrally localized in** I if the spectrum of H_ω in I is non-empty and purely point for P-almost every $\omega \in \Omega$.*

Definition 8.1.2. *Let I be an interval of $\mathbb{R}$. The family of self-adjoint random operators $\{H_\omega\}_{\omega\in\Omega}$ is said to have the **Anderson localization property in** I or equivalently is **exponentially localized** if:*

1. *the spectrum of H_ω in I is non-empty and purely point for P-almost every $\omega \in \Omega$,*
2. *the (generalized) eigenfunctions associated with the (generalized) eigenvalues in I decay exponentially to 0 at infinity.*

Definition 8.1.3. *Let I be an interval of $\mathbb{R}$. The family of self-adjoint random operators $\{H_\omega\}_{\omega\in\Omega}$ on $\mathcal{H}$ is said to be dynamically localized in I if:*

1. *the spectrum of H_ω in I is non-empty for P-almost every $\omega \in \Omega$,*
2. *for every compact interval $I_0 \subset I$ and every $\psi \in \mathcal{H}$,*

$$\forall n \geq 0,\ \mathbb{E}\left(\sup_{t\in\mathbb{R}} ||(1+|x|^2)^{\frac{n}{2}} e^{-\mathrm{i}tH_\omega}\mathbf{1}_{I_0}(H_\omega)\psi||^2\right) < +\infty.$$

This definition is of a dynamic nature and follows the evolution of wave packets over time. It indicates that the particle remains near its initial position uniformly in time.
To understand the evolution of an electron in the crystal, one actually looks at the quantity:

$$(r(t))^2 = \int_{\mathbb{R}^d} |x\psi(t,x)|^2 \mathrm{d}x.$$

If it diverges, it means that x goes to infinity; if it converges, it means that x remains bounded. There can be several behaviors for this quantity. Among them, the ballistic behavior where $(r(t))^2 \sim t^2$ corresponds to a conducting medium, with the electron behaving like a moving particle. When $(r(t))^2$ is bounded, one speaks of a localized state.

The study of the spectrum for the Anderson model is not easy. Indeed, if the Laplacian has a purely absolutely continuous spectrum, the multiplication operator by V_ω is (in the discrete case) a random diagonal matrix, and its spectrum is therefore discrete. The two effects counterbalance each other when one looks at the spectrum of the sum of these two operators. Many mathematical results are perturbative in nature and include a coefficient in front of the term V_ω. When this parameter is large, the random potential dominates over the Laplacian, and a priori, there will be localization. When this parameter is small, the opposite occurs. Intuitively, the greater the disorder, the more likely localized states will appear.

Surprisingly, in dimension 1, even a very small random term induces Anderson localization. The idea is that in dimension 1, there is no "room" to bypass an impurity in the crystal, whereas in higher dimensions there is.

In general, localization occurs at the bottom of the spectrum or at the edges of spectral bands when there is a band spectrum (for example in the discrete case with a finite common law support, we then have a finite number of translates of $[0, 4d]$). Inside the bands, however, there will be absolutely continuous spectrum and thus diffusion for $d \geq 3$. In dimension 2, it is believed that localization occurs everywhere as in dimension 1 (except possibly on a discrete set of energies).

Finally, another interpretation of Anderson localization is that it describes the absence of tunneling effects between two eigenvalues.

8.2 Localization for quasi-$1d$ Anderson models

In what follows, we present a localization criterion for operators of the form:

$$H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}} + \sum_{n \in \mathbb{Z}} V_\omega^{(n)}(x - \ell n), \tag{8.1}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^N$, where $N \geq 1$ is an integer, I_{N} is the identity matrix of order N, and $\ell > 0$ is a real number. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a complete probability space and let $\omega \in \Omega$. For all $n \in \mathbb{Z}$, the functions $x \mapsto V_\omega^{(n)}(x)$ take values in the space of real symmetric matrices, are supported in $[0, \ell]$, and are uniformly bounded in x, n, and ω. The sequence $(V_\omega^{(n)})_{n \in \mathbb{Z}}$ is an *i.i.d.* sequence on Ω. We also assume that the potential $x \mapsto \sum_{n \in \mathbb{Z}} V_\omega^{(n)}(x - \ell n)$ is such that $\{H_\omega\}_{\omega \in \Omega}$ is $\mathbb{Z}$-ergodic.
As a bounded perturbation of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}}$, the operator H_ω is self-adjoint on the Sobolev space $H^2(\mathbb{R}) \otimes \mathbb{C}^N$, and thus, for all $\omega \in \Omega$, the spectrum of H_ω is contained in $\mathbb{R}$.
By $\mathbb{Z}$-ergodicity, there exists $\Sigma \subset \mathbb{R}$ such that for P-almost every $\omega \in \Omega$, $\Sigma = \sigma(H_\omega)$. There also exist Σ_{pp}, Σ_{ac}, and Σ_{sc}, subsets of $\mathbb{R}$, such that for P-almost every $\omega \in \Omega$, $\Sigma_{\mathrm{pp}} = \sigma_{\mathrm{pp}}(H_\omega)$, $\Sigma_{\mathrm{ac}} = \sigma_{\mathrm{ac}}(H_\omega)$, and $\Sigma_{\mathrm{sc}} = \sigma_{\mathrm{sc}}(H_\omega)$.
We will show that under certain assumptions on the Furstenberg group of $\{H_\omega\}_{\omega \in \Omega}$, this operator exhibits Anderson localization on a certain interval of $\mathbb{R}$.

For $E \in \mathbb{R}$, let G_{μ_E} be the Furstenberg group of $\{H_\omega\}_{\omega \in \Omega}$.

Theorem 8.2.1. *Let $I \subset \mathbb{R}$ be a compact interval such that $\Sigma \cap I \neq \emptyset$ and let $\tilde{I}$ be an open interval containing I such that for all $E \in \tilde{I}$, G_{μ_E} is p-contracting and L_p-strongly irreducible. Then, $\{H_\omega\}_{\omega \in \Omega}$ has the Anderson localization property and is dynamically localized on I.*

In dimension 1, to prove a theorem such as Theorem 8.2.1, one can follow the following plan:

1. Show that the Lyapunov exponents of $\{H_\omega\}_{\omega \in \Omega}$ are separated.

2. Prove the Hölder continuity of these exponents.
3. Deduce the same regularity for the integrated density of states.
4. From this, deduce a Wegner estimate.
5. Then, one can apply a multi-scale analysis scheme.

8.3 Wegner estimate

The Hölder continuity of the integrated density of states is a key ingredient in proving a Wegner estimate adapted to operators whose randomness may be singular, such as H_ω. Let $L \in \mathbb{N}^*$ and let $H_\omega^{(L)}$ be the restriction of H_ω to $L^2([-\ell L, \ell L]) \otimes \mathbb{C}^N$ with Dirichlet boundary conditions.

Theorem 8.3.1. *Let $I \subset \mathbb{R}$ be a compact interval and $\tilde{I}$ an open interval, $I \subset \tilde{I}$, such that for all $E \in \tilde{I}$, G_{μ_E} is p-contracting and L_p-strongly irreducible. Then, for any $\beta \in (0,1)$ and any $\kappa > 0$, there exist $L_0 \in \mathbb{N}$ and $\xi > 0$ such that*

$$(\mathbf{W}) \quad \mathsf{P}\left(d\left(E, \sigma(H_\omega^{(L)})\right) \leq \mathrm{e}^{-\kappa(\ell L)^\beta}\right) \leq \mathrm{e}^{-\xi(\ell L)^\beta}, \tag{8.2}$$

for all $E \in I$ and all $L \geq L_0$.

This statement can be compared with classical Wegner estimates, valid for operators whose randomness is regular, involving random variables with densities.

For example, one can prove for a discrete scalar Anderson operator h_ω with random variables having densities an optimal Wegner estimate:

$$\mathbb{E}\left(\mathrm{Tr}\left(\mathbf{1}_I\left(h_\omega^\Lambda\right)\right)\right) \leq C_W \cdot |I| \cdot |\Lambda| \tag{8.3}$$

with $C_W > 0$, $|I|$ the length of the interval I, and $|\Lambda|$ the volume of the cube Λ to which h_ω is restricted to define h_ω^Λ.

One can also obtain an improved Wegner estimate,

$$\mathbb{E}\left(\mathrm{Tr}\left(\mathbf{1}_I\left(h_\omega^\Lambda\right)\right)\right) \leq C_W \cdot N(I) \cdot |\Lambda| \tag{8.4}$$

where $N(\cdot)$ is the integrated density of states associated with $\{h_\omega\}_{\omega\in\Omega}$.

We can now prove the following proposition upon which will be based the proof of Theorem 8.3.1.

Proposition 8.3.2. *Let $I \subset \mathbb{R}$ be a compact interval and $\tilde{I}$ be an open interval, $I \subset \tilde{I}$, such that, for every $E \in \tilde{I}$, G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1, \dots, N\}$. Then, there exist $\alpha > 0$, $L_0 \in \mathbb{N}$ and $C > 0$ such that, for every $E \in I$ and every $\varepsilon > 0$:*

$$\begin{aligned} \forall L \geq L_0, \qquad & \mathsf{P}\big(\{\exists E' \in (E-\varepsilon, E+\varepsilon), \exists \phi \in D(H_\omega^{(L)}) \mid (H_\omega^{(L)} - E')\phi = 0, \\ & ||\phi|| = 1 \text{ and } ||\phi'(-\ell L)||^2 + ||\phi'(\ell L)||^2 \leq \varepsilon^2\}\big) \leq C\,\ell\,L\,\varepsilon^\alpha. \end{aligned} \tag{8.5}$$

Proof : The proof will mostly relies on the Hölder continuity of the integrated density of states of H_ω. Let $\tilde{I}$ be an open interval such that, for every $E \in \tilde{I}$, G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1, \dots, N\}$. Let $I \subset \tilde{I}$ be a compact interval. There exist $\alpha > 0$ and $C_1 > 0$ such that :

$$\forall E, E' \in I, \ |N(E) - N(E')| \leq C_1 |E - E'|^\alpha. \tag{8.6}$$

Let E be in the interior of I, $\varepsilon > 0$ and $L \in \mathbb{N}$. For every $k \in \mathbb{Z}$, let I_k be the interval $2k\ell L + [-\ell L, \ell L] = [(2k-1)\ell L, (2k+1)\ell L]$ and denote by $H_\omega^{(I_k)}$ the restriction of H_ω to $L^2(I_k) \otimes \mathbb{C}^N$ with Dirichlet boundary conditions. We define the event $A_k \in \mathcal{A}$ as :

$A_k = \{\omega \in \Omega \mid H_\omega^{(I_k)}$ has an eigenvalue $\lambda_k \in (E-\varepsilon, E+\varepsilon)$ such that the corresponding normalized eigenfunction ϕ_k satisfies $||\phi'(-\ell L)||^2 + ||\phi'(\ell L)||^2 \leq \varepsilon^2\}$.

Then, because the $V_\omega^{(n)}$ are *i.i.d.* random variables, and because of the form of the potential in H_ω as a ℓ-periodization of $V_\omega^{(0)}$, we deduce that $\mathsf{P}(A_k)$ is independent of k, that is, $\forall k \in \mathbb{Z}$, $\mathsf{P}(A_k) = \mathsf{P}(A_0)$. Moreover, $\mathsf{P}(A_0)$ is equal to the probability in (8.5).

Let $n \in \mathbb{N}$ and let $J_n = \bigcup_{k=-n}^{n} I_k = [-(2n+1)\ell L, (2n+1)\ell L]$. Let $H_\omega^{(J_n)}$ be the restriction of H_ω to $L^2(J_n) \otimes \mathbb{C}^N$ with Dirichlet boundary conditions. For a fixed $\omega \in \Omega$, let $k_1, \ldots, k_j \in \{-n, \ldots, n\}$ be distinct and such that $\omega \in A_{k_i}$ for every $i \in \{1, \ldots, j\}$. Let $i \in \{1, \ldots, j\}$. Let ϕ_i be defined on I_{k_i}, $||\phi_i|| = 1$, $\phi_i((2k_i-1)\ell L) = \phi_i((2k_i+1)\ell L) = 0$. We also assume that there exists $\lambda_{k_i} \in (E-\varepsilon, E+\varepsilon)$ such that $H_\omega^{(I_{k_i})}\phi_i = \lambda_{k_i}\phi_i$.

Let χ be a smooth function on $\mathbb{R}$, $0 \leq \chi \leq 1$, $\chi(x) = 0$ on $(-\infty, 0]$, $\chi(x) = 1$ on $[\ell, +\infty)$ and $\int_0^\ell \chi(x)\mathrm{d}x = 1$. Let $x_i^\pm = (2k_i \pm 1)\ell L$, so that $I_{k_i} = [x_i^-, x_i^+]$. We extend ϕ_i to J_n by defining $\hat{\phi}_i$, for $x \in J_n$, by :

$$\hat{\phi}_i(x) = \begin{cases} 0 & \text{if} \quad x \notin [x_i^-, x_i^+] \\ \chi(x - x_i^-)\phi_i(x) & \text{if} \quad x \in [x_i^-, x_i^- + \ell] \\ \phi_i(x) & \text{if} \quad x \in [x_i^- + \ell, x_i^+ - \ell] \\ \chi(x_i^+ - x)\phi_i(x) & \text{if} \quad x \in [x_i^+ - \ell, x_i^+]. \end{cases} \tag{8.7}$$

Then, $\hat{\phi}_i \in D(H_\omega^{(J_n)})$ and $||\hat{\phi}_i|| \leq ||\phi_i|| = 1$. As $H_\omega^{(I_{k_i})}\phi_i = \lambda_{k_i}\phi_i$ and $\lambda_{k_i} \in (E-\varepsilon, E+\varepsilon)$, we have :

$$\begin{aligned} ||(H_\omega^{(J_n)} - E)\hat{\phi}_i|| &\leq ||(H_\omega^{(J_n)} - \lambda_{k_i})\hat{\phi}_i|| + ||(\lambda_{k_i} - E)\hat{\phi}_i|| \\ &= ||H_\omega^{(J_n)} - \lambda_{k_i})\hat{\phi}_i|| + |\lambda_{k_i} - E|\,||\hat{\phi}_i|| \\ &\leq ||H_\omega^{(J_n)} - \lambda_{k_i})\hat{\phi}_i|| + \varepsilon. \end{aligned} \tag{8.8}$$

We want to estimate $||H_\omega^{(J_n)} - \lambda_{k_i})\hat{\phi}_i||$. For every $x \in [x_i^-, x_i^- + \ell]$,

$$(\chi(x - x_i^-)\phi_i(x))'' = \chi''(x - x_i^-)\phi_i(x) + 2\chi'(x - x_i^-)\phi_i'(x) + \chi(x - x_i^-)\phi_i''(x),$$

and, for every $x \in [x_i^+ - \ell, x_i^+]$,

$$(\chi(x_i^+ - x)\phi_i(x))'' = \chi''(x_i^+ - x)\phi_i(x) - 2\chi'(x_i^+ - x)\phi_i'(x) + \chi(x_i^+ - x)\phi_i''(x).$$

Thus, using $H_\omega^{(I_{k_i})}\phi_i = \lambda_{k_i}\phi_i$,

$$H_\omega^{(J_n)} - \lambda_{k_i})\hat{\phi}_i = \begin{cases} 0 & \text{if} \quad x \notin [x_i^-, x_i^+] \\ -\chi''(x - x_i^-)\phi_i(x) - 2\chi'(x - x_i^-)\phi_i'(x) & \text{if} \quad x \in [x_i^-, x_i^- + \ell] \\ 0 & \text{if} \quad x \in [x_i^- + \ell, x_i^+ - \ell] \\ -\chi''(x_i^+ - x)\phi_i(x) + 2\chi'(x_i^+ - x)\phi_i'(x) & \text{if} \quad x \in [x_i^+ - \ell, x_i^+]. \end{cases}$$

Hence we have, applying twice Lemma 7.1.17 for x_i^- and for x_i^+ at the second inequality and using the fact that potential is uniformly bounded,

$$\begin{aligned}
||H_\omega^{(J_n)} - \lambda_{k_i})\hat{\phi}_i||^2 &= \int_{x_i^-}^{x_i^-+\ell} ||\chi''(x-x_i^-)\phi_i(x) + 2\chi'(x-x_i^-)\phi_i'(x)||^2 \mathrm{d}x \\
&\quad + \int_{x_i^+-\ell}^{x_i^+} ||\chi''(x_i^+-x)\phi_i(x) - 2\chi'(x_i^+-x)\phi_i'(x)||^2 \mathrm{d}x \\
&\leq \left|\left|\begin{pmatrix}\phi_i \\ \phi_i'\end{pmatrix}\right|\right|^2_{L^\infty([x_i^-, x_i^-+\ell])} \times \int_{x_i^-}^{x_i^-+\ell} \left|\left|\begin{pmatrix}\chi''(x-x_i^-) \\ 2\chi'(x-x_i^-)\end{pmatrix}\right|\right|^2 \mathrm{d}x \\
&\quad + \left|\left|\begin{pmatrix}\phi_i \\ \phi_i'\end{pmatrix}\right|\right|^2_{L^\infty([x_i^+-\ell, x_i^+])} \times \int_{x_i^+-\ell}^{x_i^+} \left|\left|\begin{pmatrix}\chi''(x_i^+-x) \\ 2\chi'(x_i^+-x)\end{pmatrix}\right|\right|^2 \mathrm{d}x \\
&\leq C_2 \left(\left|\left|\begin{pmatrix}\phi_i(x_i^-) \\ \phi_i'(x_i^-)\end{pmatrix}\right|\right| + \left|\left|\begin{pmatrix}\phi_i(x_i^+) \\ \phi_i'(x_i^+)\end{pmatrix}\right|\right|\right)^2 \\
&= C_2 \left(||\phi_i'(x_i^-)||^2 + ||\phi_i'(x_i^+)||^2\right) \leq C_2\varepsilon^2, \qquad (8.9)
\end{aligned}$$

using the fact that $\omega \in A_{k_i}$ and using the Dirichlet boundary conditions of $H_\omega^{(I_{k_i})}$ at x_i^- and x_i^+ to say that $\phi_i(x_i^-) = \phi_i(x_i^+) = 0$. The constant C_2 depends only on χ and the parameters of the potential of $\{H_\omega\}_{\omega\in\Omega}$. We normalize $\hat{\phi}_i$ by setting $\tilde{\phi}_i = \hat{\phi}_i/||\hat{\phi}_i||$. We also have $||\hat{\phi}_i|| \geq \frac{1}{2}$ because $||\phi_i|| = 1$ and $\int_0^\ell \chi(x)\mathrm{d}x = 1$, and thus, by (8.8) and (8.9),

$$\begin{aligned}
||H_\omega^{(J_n)} - E)\tilde{\phi}_i|| &= ||\hat{\phi}_i||^{-1}||H_\omega^{(J_n)} - E)\hat{\phi}_i|| \\
&\leq 2\sqrt{C_2}\varepsilon := C_3\varepsilon. \qquad (8.10)
\end{aligned}$$

We have construct, for each $i \in \{1, \dots, j\}$, a normalized function $\tilde{\phi}_i$ in $D(H_\omega^{(J_n)})$, supported in I_{k_i}, such that :

$$\forall i \in \{1, \dots, j\},\ ||(H_\omega^{(J_n)} - E)\tilde{\phi}_i|| \leq C_3\varepsilon, \qquad (8.11)$$

where C_3 depends only on the choice of χ and on the parameters of the potential of $\{H_\omega\}_{\omega\in\Omega}$. Moreover, as $\tilde{\phi}_i$ is supported in I_{k_i} and the intervals $I_{k_1}, \dots, I_{k_j}$ are disjoints, $(\tilde{\phi}_1, \dots, \tilde{\phi}_j)$ is an orthonormal set and :

$$\forall i \neq i',\ (\tilde{\phi}_i, H_\omega^{(J_n)}\tilde{\phi}_{i'}) = 0 = (H_\omega^{(J_n)}\tilde{\phi}_i, H_\omega^{(J_n)}\tilde{\phi}_{i'}). \qquad (8.12)$$

We recall that, the spectrum of $H_\omega^{(J_n)}$ is a discrete set of eigenvalues with only $+\infty$ as accumulation point, and thus, its number of eigenvalues in any compact interval is finite (as already used in the proof of the well-definedness of the IDS). As we have (8.11) and (8.12), we can apply to $(\tilde{\phi}_1, \dots, \tilde{\phi}_j)$ and $H_\omega^{(J_n)}$ the version of Temple's inequality given in [29, Lemma A.3.2] to obtain that the number of eigenvalues of $H_\omega^{(J_n)}$ in $[E - C_3\varepsilon, E + C_3\varepsilon]$, counted with multiplicity, is at least j. So we have, for a fixed $\omega \in \Omega$,

$$j = \#\{k \in \{-n, \dots, n\} \mid \omega \in A_k\} \leq \#\{\lambda \in [E - C_3\varepsilon, E + C_3\varepsilon] \mid \lambda \in \sigma_{\mathrm{p}}(H_\omega^{(J_n)})\}.$$

Moreover, applying the law of large numbers to the random variables $\mathbf{1}_{A_{-n}}, \dots, \mathbf{1}_{A_n}$, we get that, for P-almost every $\omega \in \Omega$,

$$\frac{1}{2n+1}\#\{k \in \{-n, \dots, n\} \mid \omega \in A_k\} = \frac{1}{2n+1}\left(\mathbf{1}_{A_{-n}} + \dots + \mathbf{1}_{A_n}\right) \xrightarrow[n\to+\infty]{} \mathbb{E}(\mathbf{1}_{A_0}),$$

with $\mathbb{E}(\mathbf{1}_{A_0}) = \mathsf{P}(A_0)$. Now, we assume that ε is small enough to ensure that $[E - C_3\varepsilon, E + C_3\varepsilon] \subset I \subset \tilde{I}$ and to apply (8.6) on $[E - C_3\varepsilon, E + C_3\varepsilon]$. Then we have, for P-almost every $\omega \in \Omega$,

$$\begin{aligned}
\mathsf{P}(A_0) &= \lim_{n\to+\infty} \frac{1}{2n+1} \#\{k \in \{-n,\dots,n\} \mid \omega \in A_k\} \\
&\leq \lim_{n\to+\infty} \frac{1}{2n+1} \#\{\lambda \in [E - C_3\varepsilon, E + C_3\varepsilon] \mid \lambda \in \sigma_{\mathrm{p}}(H_\omega^{(J_n)})\} \\
&= 2\ell L \lim_{n\to+\infty} \frac{1}{2(2n+1)\ell L} \#\{\lambda \in [E - C_3\varepsilon, E + C_3\varepsilon] \mid \lambda \in \sigma_{\mathrm{p}}(H_\omega^{(J_n)})\} \\
&= 2\ell L (N(E + C_3\varepsilon) - N(E - C_3\varepsilon)) \\
&\leq 2\ell L C_1 (2C_3\varepsilon)^\alpha := C\ell L\varepsilon^\alpha.
\end{aligned}$$

It finishes the proof.

□

We remark that the exponent α in (8.5) is the same as the Hölder exponent of the Lyapounov exponent and the integrated density of states.

Before proving the Wegner estimate, we need three more results on the products of transfer matrices. The first is a result of large deviations which is also used in the proof of the Initial Length Scale Estimate which we will discuss later.

Notation. For every $n \in \mathbb{Z}$, let $U^{(n)}(E) = T_{\omega^{(n-1)}}(E) \cdots T_{\omega^{(0)}}(E)$.

Lemma 8.3.3. *We assume that $E \in \mathbb{R}$ is such that G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1,\dots,N\}$. Let $p \in \{1,\dots,N\}$. Then, there exists $\kappa_0 > 0$ such that, for every $\varepsilon > 0$, $x \in L_p$, $x \neq 0$,*

$$\limsup_{n\to+\infty} \frac{1}{\ell n} \log \mathsf{P}\left(\left| \log ||(\wedge^p U^{(n)}(E))x|| - \ell n(\gamma_1 + \dots + \gamma_p)(E) \right| > \ell n\varepsilon \right) < -\kappa_0. \tag{8.13}$$

The second is just a result of uniform convergence.

Lemma 8.3.4. *Let $I \subset \mathbb{R}$ be an open interval such that, for every $E \in I$, G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1,\dots,N\}$.*

$$\frac{1}{n}\mathbb{E}\left(\log \frac{||(\wedge^p U^{(n)}(E))x||}{||x||} \right) \xrightarrow[n\to+\infty]{} \gamma_1(E) + \ldots + \gamma_p(E),$$

uniformly in $E \in I$ and $\bar{x} \in \mathbb{P}(L_p)$.

The third lemma is about exponential decay of some power of the products of transfer matrices. This estimate is also crucial in the Fractional Moments Method as one can see for example in the proofs of localization for unitray models which will be discussed at Chapter 9.

Lemma 8.3.5. *Let $I \subset \mathbb{R}$ be an open interval such that, for every $E \in I$, G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1,\dots,N\}$. Let $p \in \{1,\dots,N\}$. There exist $\xi_1 > 0$, $\delta > 0$ and $n_1 \in \mathbb{N}$ such that, for every $E \in I$, $n \geq n_1$ and $x \in \wedge^p(\mathbb{R}^{2N})$, $||x|| = 1$, we have :*

$$\mathbb{E}\left(|| \wedge^p U^{(n)}(E)x||^{-\delta} \right) \leq \mathrm{e}^{-\xi_1 n}. \tag{8.14}$$

Proof : We fix $p \in \{1,\dots,N\}$. We set $N_U = || \wedge^p U^{(n)}(E)x||$. We start by writing

$$N_U^{-\delta} = \mathrm{e}^{-\delta \log N_U},$$

and using the inequality $e^y \leq 1 + y + y^2 e^{|y|}$, for any $y \in \mathbb{R}$. Then, for every $E \in I$ and every $\delta > 0$,

$$\mathbb{E}\left(N_U^{-\delta}\right) \leq 1 - \delta \mathbb{E}\left(\log N_U\right) + \delta^2 \mathbb{E}\left(\left(\log N_U\right)^2 e^{\delta \log N_U}\right). \tag{8.15}$$

But, as $e^{\delta \log N_U} = N_U^{\delta}$, $||x|| = 1$ and the $T_{\omega^{(n)}}(E)$ are *i.i.d.*, by Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left(\left(\log N_U\right)^2 e^{\delta \log N_U}\right) &\leq \mathbb{E}\left[\left(\log N_U\right)^4\right]^{\frac{1}{2}} \mathbb{E}\left(N_U^{2\delta}\right)^{\frac{1}{2}} \\
&\leq \mathbb{E}\left[\left(\sum_{i=0}^{n-1} p \log ||T_{\omega^{(i)}}(E)||\right)^4\right]^{\frac{1}{2}} \mathbb{E}\left[\prod_{i=1}^{n-1} ||T_{\omega^{(i)}}(E)||^{2p\delta}\right]^{\frac{1}{2}} \\
&\leq n^2 p^2 \mathbb{E}\left[\left(\log ||T_{\omega^{(0)}}(E)||\right)^4\right]^{\frac{1}{2}} \mathbb{E}\left[||T_{\omega^{(0)}}(E)||^{2p\delta}\right]^{\frac{n}{2}}.
\end{aligned}$$

Thus, there exist constants $C_1 = C_1(I)$ and $C_2 = C_2(I)$ such that,

$$\mathbb{E}\left(|| \wedge^p U^{(n)}(E)x||^{-\delta}\right) \leq 1 - \delta \mathbb{E}\left(\log || \wedge^p U^{(n)}(E)x||\right) + \delta^2 n^2 C_1 (C_2)^n. \tag{8.16}$$

If $C > 0$ is such that, for every $E \in I$, $T_{\omega^{(0)}}(E) \leq C$, we can choose $C_1 = (p \log C)^2$ and $C_2 = C^{p\delta}$. Moreover, by Lemma 8.3.4, there exists $n_0 \geq 1$, uniform in $E \in I$, and x normalized, such that,

$$\begin{aligned}
\mathbb{E}\left(|| \wedge^p U^{(n_0)}(E)x||^{-\delta}\right) &\leq 1 - \frac{1}{2} n_0 \delta (\gamma_1(E) + \cdots + \gamma_p(E)) + \delta^2 n_0^2 C_1 C_2^{n_0} \\
&\leq 1 - \varepsilon,
\end{aligned}$$

for $\varepsilon > 0$, if we choose δ small enough. Then, for $n \geq 1$, we set $[\frac{n}{n_0}]$ the largest integer less than or equal to $\frac{n}{n_0}$, and we write the euclidian division of n by n_0, $n = [\frac{n}{n_0}] n_0 + r$, $0 \leq r < n_0$. Then, there exist $n_1 \geq 1$, a constant $\tilde{C}$ and $\xi_1 > 0$ such that for every $p \in \{1, \ldots, N\}$,

$$\forall n \geq n_1,\ \forall E \in I,\ \mathbb{E}\left(|| \wedge^p U^{(n_0)}(E)x||^{-\delta}\right) \leq \tilde{C}(1-\varepsilon)^{[\frac{n}{n_0}]} \leq e^{-\xi_1 n}.$$

□

We can now use Proposition 8.3.2, Lemma 8.3.5 and Lemma 7.1.18 to prove Theorem 8.3.1.

Proof : [of Theorem 8.3.1] Let $I \subset \mathbb{R}$ be a compact interval and $\tilde{I}$ be an open interval, $I \subset \tilde{I}$, such that, for every $E \in \tilde{I}$, G_{μ_E} is p-contracting and L_p-strongly irreducible for every $p \in \{1, \ldots, N\}$. Let $\beta \in (0,1)$ and $\kappa > 0$. For $L \in \mathbb{N}$, we set $n_L = [\tau(\ell L)^{\beta}] + 1$ with some arbitrary $\tau > 0$, where $[\tau(\ell L)^{\beta}]$ is the largest integer less or equal to $\tau(\ell L)^{\beta}$. For every $E \in I$ and $\theta_0 > 0$, we define the events :

$$A_{\theta_0}^{(L)}(E) = \left\{\omega \in \Omega \mid ||T_{\omega^{(n_L - L - 1)}}(E) \ldots T_{\omega^{(-L)}}(E) \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)|| > e^{\theta_0 (\ell L)^{\beta}}\right\}, \tag{8.17}$$

$$B_{\theta_0}^{(L)}(E) = \left\{\omega \in \Omega \mid ||T_{\omega^{(L+1-n_L)}}(E) \ldots T_{\omega^{(L)}}(E) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)|| > e^{\theta_0 (\ell L)^{\beta}}\right\}. \tag{8.18}$$

Let $\xi_1 > 0$ and $\delta > 0$ be the constants given by Lemma 8.3.5. Let $\theta = \frac{\tau \xi_1}{2\delta}$ and let $C^{(L)}(E) \in \mathcal{A}$ be the event :

$$\left\{\omega \in \Omega \,\middle|\, d\left(E, \sigma(H_{\omega}^{(L)})\right) \leq e^{-\kappa (\ell L)^{\beta}}\right\}.$$

If we set :

$$\begin{aligned}
(a) &:= \mathsf{P}\left(C^{(L)}(E) \cap \bigcap_{\{E' \mid |E-E'| \leq \mathrm{e}^{-\kappa(\ell L)^\beta}\}} \left[A^{(L)}_{\frac{\theta}{2}}(E') \cap B^{(L)}_{\frac{\theta}{2}}(E') \right] \right), \\
(b) &:= \mathsf{P}\left(A^{(L)}_{\theta}(E) \cap B^{(L)}_{\theta}(E) \cap \bigcup_{\{E' \mid |E-E'| \leq \mathrm{e}^{-\kappa(\ell L)^\beta}\}} \left(A^{(L)}_{\frac{\theta}{2}}(E') \right)^c \right), \\
(c) &:= \mathsf{P}\left(A^{(L)}_{\theta}(E) \cap B^{(L)}_{\theta}(E) \cap \bigcup_{\{E' \mid |E-E'| \leq \mathrm{e}^{-\kappa(\ell L)^\beta}\}} \left(B^{(L)}_{\frac{\theta}{2}}(E') \right)^c \right), \\
(d) &:= \mathsf{P}\left(\left(A^{(L)}_{\theta}(E) \right)^c \right) + \mathsf{P}\left(\left(B^{(L)}_{\theta}(E) \right)^c \right),
\end{aligned}$$

then we have :

$$\mathsf{P}\left(\left\{ \omega \in \Omega \,\middle|\, d\left(E, \sigma(H^{(L)}_\omega)\right) \leq \mathrm{e}^{-\kappa(\ell L)^\beta} \right\} \right) \leq (a) + (b) + (c) + (d). \tag{8.19}$$

Using Tchebychev's inequality and Lemma 8.3.5, applied for $p = 1$, we directly get, for L large enough,

$$(d) \leq 2\mathrm{e}^{-\xi_1 n_L - \delta\theta(\ell L)^\beta} \leq 2\mathrm{e}^{-\xi_1 \tau(\ell L)^\beta + \frac{\tau\xi_1}{2}(\ell L)^\beta} = 2\mathrm{e}^{-\frac{\tau\xi_1}{2}(\ell L)^\beta}. \tag{8.20}$$

To estimate $(b) + (c)$, we use the fact that there exists a constant $C_0 > 0$ independent of n, ω, E such that, for every $E, E' \in I$,

$$\forall n \in \mathbb{Z},\ ||T_{\omega^{(n)}}(E) - T_{\omega^{(n)}}(E')|| \leq C_0|E - E'|. \tag{8.21}$$

From this, we deduce that the event $\left(A^{(L)}_{\frac{\theta}{2}}(E') \right)^c \cap A^{(L)}_{\theta}(E)$ occurs for at least one E' such that $|E - E'| \leq \mathrm{e}^{-\kappa(\ell L)^\beta}$. Then, we prove that for this E', there exists $\alpha_0 > 0$ such that, if $\tau > 0$ is small enough,

$$\mathsf{P}\left(A^{(L)}_{\theta}(E) \cap \left(A^{(L)}_{\frac{\theta}{2}}(E') \right)^c \right) \leq \mathrm{e}^{-\alpha_0(\ell L)^\beta}. \tag{8.22}$$

We have a similar inequality for $B^{(L)}_{\theta}(E) \cap \left(B^{(L)}_{\frac{\theta}{2}}(E'') \right)^c$ for at least one E'' such that $|E - E''| \leq \mathrm{e}^{-\kappa(\ell L)^\beta}$. Thus, by inclusions of the events :

$$\begin{aligned}
(b) + (c) &\leq \mathsf{P}\left(A^{(L)}_{\theta}(E) \cap \left(A^{(L)}_{\frac{\theta}{2}}(E') \right)^c \right) + \mathsf{P}\left(B^{(L)}_{\theta}(E) \cap \left(B^{(L)}_{\frac{\theta}{2}}(E'') \right)^c \right) \qquad (8.23) \\
&\leq 2\mathrm{e}^{-\alpha_0(\ell L)^\beta}.
\end{aligned}$$

It remains to estimate (a). Let ω be in the event in the probability (a). Let $E' \in (E - \mathrm{e}^{-\kappa(\ell L)^\beta}, E + \mathrm{e}^{-\kappa(\ell L)^\beta})$ be an eigenvalue of $H^{(L)}_\omega$ with a normalized eigenvector ϕ. As ω is in the event in the probability (a), we have, using Lemma 7.1.18,

$$||\phi(-\ell L)||^2 + ||\phi'(-\ell L)||^2 = ||\phi'(-\ell L)||^2 \leq 2\mathrm{e}^{-\theta(\ell L)^\beta}, \tag{8.24}$$

and

$$||\phi(\ell L)||^2 + ||\phi'(\ell L)||^2 = ||\phi'(\ell L)||^2 \leq 2\mathrm{e}^{-\theta(\ell L)^\beta}. \tag{8.25}$$

Now, using Proposition 8.3.2 with $\varepsilon = \mathrm{e}^{-\kappa(\ell L)^\beta}$, we get :

$$(a) \leq C\ell L \max\left(\mathrm{e}^{-\kappa(\ell L)^\beta}, 2\sqrt{2}\mathrm{e}^{-\frac{\theta}{2}(\ell L)^\beta} \right)^\alpha. \tag{8.26}$$

Putting (8.20), (8.23) and (8.26) in (8.19), we finally obtain (8.2) for a suitable $\xi > 0$ and L large enough.

□

8.4 A quick tour of Multi-Scale Analysis

We start by stating a property that ensures the existence of generalized eigenfunctions in a sense to be specified for $\{H_\omega\}_{\omega\in\Omega}$. Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R})\otimes\mathbb{C}^N$, and let $\nu>\frac{1}{4}$. We define the weighted spaces $\mathcal{H}_\pm$ by:

$$\mathcal{H}_\pm = L^2(\mathbb{R}, < x >^{\pm 4\nu}\,\mathrm{d}x)\otimes\mathbb{C}^N,$$

where $< x >=\sqrt{1+|x|^2}$ for all $x\in\mathbb{R}$. On $\mathcal{H}_+\times\mathcal{H}_-$, we define the sesquilinear form $<\,,\,>_{\mathcal{H}_+,\mathcal{H}_-}$ by:

$$\forall(\phi,\psi)\in\mathcal{H}_+\times\mathcal{H}_-,\ <\phi,\psi>_{\mathcal{H}_+,\mathcal{H}_-}=\int_{\mathbb{R}} {}^t\phi(x)\overline{\psi(x)}\mathrm{d}x.$$

Let also T be the self-adjoint operator on $\mathcal{H}$ given by multiplication by $< x >^{2\nu}$. Recall that $E_\omega(.)$ denotes the spectral projector of H_ω, and we define the property of "Strong Generalized Eigenfunction Expansion".

Definition 8.4.1. *Let $I\subset\mathbb{R}$ be an open interval. We say that $\{H_\omega\}_{\omega\in\Omega}$ satisfies the (SGEE) property on I if, for some $\nu>\frac{1}{4}$,*

(i) for P-almost every $\omega\in\Omega$, the set $\mathcal{D}_+(\omega)=\{\phi\in D(H_\omega)\cap\mathcal{H}_+\mid H_\omega\phi\in\mathcal{H}_+\}$ is dense in $\mathcal{H}_+$ and is a core for H_ω,

(ii) there exists a continuous and bounded function f on $\mathbb{R}$, strictly positive on $\sigma(H_\omega)$, such that:

$$\mathbb{E}\left(\left(\mathrm{tr}_{\mathcal{H}}(T^{-1}f(H_\omega)E_\omega(I)T^{-1})\right)^2\right)<\infty.$$

Definition 8.4.2. *A measurable function $\psi:\mathbb{R}\to\mathbb{C}^N$ is a generalized eigenfunction for H_ω associated with the generalized eigenvalue λ if $\psi\in\mathcal{H}_-\setminus\{0\}$ and:*

$$\forall\phi\in\mathcal{D}_+(\omega),\ < H_\omega\phi,\psi>_{\mathcal{H}_+,\mathcal{H}_-}=\overline{\lambda}<\phi,\psi>_{\mathcal{H}_+,\mathcal{H}_-}.$$

We introduce notations for restrictions of H_ω to finite-length intervals of $\mathbb{R}$ not necessarily centered at 0. For $x\in\mathbb{Z}$ and $L\geq 1$, let $I_L(x)=[x-\ell L,x+\ell L]$, centered at x and of length $2\ell L$. Let $\mathbf{1}_{x,L}$ be the characteristic function of $I_L(x)$ and $\mathbf{1}_x$ the characteristic function of $I_1(x)$. For $L\in 3\mathbb{N}^*$, we also define:

$$\mathbf{1}_{x,L}^{\mathrm{out}}=\mathbf{1}_{x,L}-\mathbf{1}_{x,L-2}\quad\text{and}\quad\mathbf{1}_{x,L}^{\mathrm{in}}=\mathbf{1}_{x,\frac{L}{3}}.$$

For all $x\in\mathbb{Z}$ and $L\geq 1$, let $H_\omega^{(x,L)}$ be the restriction of H_ω to $L^2(I_L(x))\otimes\mathbb{C}^N$ with Dirichlet boundary conditions, and for $E\notin\sigma(H_\omega^{(x,L)})$, let $R_\omega^{(x,L)}(E)$ be the resolvent of $H_\omega^{(x,L)}$ at E, $R_\omega^{(x,L)}(E)=(H_\omega^{(x,L)}-E)^{-1}$. Finally, let $E_\omega^{(x,L)}$ be the spectral projector of $H_\omega^{(x,L)}$. With these notations, we can state a Simon-Lieb type inequality.

Definition 8.4.3. *Let $I\subset\mathbb{R}$ be a compact interval. We say that $\{H_\omega\}_{\omega\in\Omega}$ has the (SLI) property if there exists a constant C_I such that, given $L,L',L''\in\mathbb{N}$ and $x,y,y'\in\mathbb{Z}$ with $I_{L''}(y)\subset I_{L'-2}(y')\subset I_{L-2}(x)$, for P-almost every $\omega\in\Omega$, if $E\in I$, $E\notin\sigma(H_\omega^{(x,L)})\cup\sigma(H^{(y',L')}(\omega))$, we have:*

$$||\mathbf{1}_{x,L}^{\mathrm{out}}R_\omega^{(x,L)}(E)\mathbf{1}_{y,L''}||\leq C_I||\mathbf{1}_{y',L'}^{\mathrm{out}}R_\omega^{(y',L')}(E)\mathbf{1}_{y,L''}||\,||\mathbf{1}_{x,L}^{\mathrm{out}}R_\omega^{(x,L)}(E)\mathbf{1}_{y',L'}^{\mathrm{out}}||.$$

The (SLI) property allows estimating how the restricted resolvents $R_\omega^{(x,L)}(E)$ vary in norm when passing from a given interval to a longer one containing it. This type of estimate is also called the "Geometric Resolvent Inequality" in the literature. We can now state an estimate for generalized eigenfunctions in terms of restricted resolvents, called the "Eigenfunction Decay Inequality".

Definition 8.4.4. *Let $I \subset \mathbb{R}$ be a compact interval. We say that $\{H_\omega\}_{\omega\in\Omega}$ has the (EDI) property if there exists a constant $\tilde{C}_I$ such that, for P-almost every $\omega \in \Omega$, given a generalized eigenvalue $E \in I$, we have for all $x \in \mathbb{Z}$ and $L \in \mathbb{N}$ with $E \notin \sigma(H_\omega^{(x,L)})$,*

$$||\mathbf{1}_x\psi|| \leq \tilde{C}_I ||\mathbf{1}^{\text{out}}_{x,L} R_\omega^{(x,L)}(E)\mathbf{1}_x|| \, ||\mathbf{1}^{\text{out}}_{x,L}\psi||.$$

The following property is an estimate on the average number of eigenvalues of $H_\omega^{(x,L)}$.

Definition 8.4.5. *Let $I \subset \mathbb{R}$ be a compact interval. We say that $\{H_\omega\}_{\omega\in\Omega}$ has the (NE) property if there exists a finite constant $\hat{C}_I$ such that, for all $x \in \mathbb{Z}$ and $L \in \mathbb{N}$,*

$$\mathbb{E}\left(\text{tr}_{\mathcal{H}}(E_\omega^{(x,L)}(I))\right) \leq \hat{C}_I \ell L.$$

The last property needed for multi-scale analysis is of a different nature. It is a probabilistic property of independence of intervals far from each other. An event $A \in \mathcal{A}$ is said to be based on $I_L(x)$ if it is determined by conditions on $H_\omega^{(x,L)}$. Given $d_0 > 0$, $I_L(x)$ and $I_{L'}(x')$ are said to be d_0-non-overlapping if $d(I_L(x), I_{L'}(x')) > d_0$.

Definition 8.4.6. *We say that $\{H_\omega\}_{\omega\in\Omega}$ has the (IAD) property if there exists $d_0 > 0$ such that any pair of events based on d_0-non-overlapping intervals are independent.*

Before giving the definition of the multi-scale analysis set Σ_{MSA}, we need one last definition.

Definition 8.4.7. *Let $\gamma, E \in \mathbb{R}$ and $\omega \in \Omega$. For $x \in \mathbb{Z}$ and $L \in 3\mathbb{N}^*$, the interval $I_L(x)$ is said to be (ω,γ,E)-good if $E \notin \sigma(H_\omega^{(x,L)})$ and*

$$||\mathbf{1}^{\text{out}}_{x,L} R_\omega^{(x,L)}(E)\mathbf{1}^{\text{in}}_{x,L}|| \leq \mathrm{e}^{-\gamma\ell\frac{L}{3}}.$$

We assume that $\{H_\omega\}_{\omega\in\Omega}$ satisfies the (IAD) property.

Definition 8.4.8. *The set Σ_{MSA} for $\{H_\omega\}_{\omega\in\Omega}$ is the set of $E \in \Sigma$ for which there exists an open interval I such that $E \in I$ and, given ζ, $0 < \zeta < 1$, and $\alpha_0 \in (1, \zeta^{-1})$, there exists a length scale $L_0 \in 6\mathbb{N}$ and a real number $\gamma > 0$, such that if we set $L_{k+1} = \max\{L \in 6\mathbb{N} \mid L \leq L_k^{\alpha_0}\}$ for all $k \in \mathbb{N}$, we have:*

$$\mathsf{P}\left(\{\omega \in \Omega \mid \forall E' \in I,\ I_L(x) \text{ or } I_L(y) \text{ is } (\omega,\gamma,E') - \text{good}\}\right) \geq 1 - \mathrm{e}^{-L_k^{\zeta}}.$$

for all $k \in \mathbb{N}$ and $x, y \in \mathbb{Z}$ such that $|x - y| > L_k + d_0$.

We conclude by stating the multi-scale analysis theorem of Germinet and Klein for operators involving singular probability measures such as $\{H_\omega\}_{\omega\in\Omega}$.

Theorem 8.4.9. *Suppose that $\{H_\omega\}_{\omega\in\Omega}$ has the properties (IAD), (SLI), (NE) and satisfies a Wegner estimate (W) as in (8.2) on an open interval $I \subset \mathbb{R}$. Given $\gamma > 0$, for all $E \in I$, there exists an integer $L_\gamma(E)$, bounded on compact subintervals of I, such that for a given $E_0 \in \Sigma \cap I$, we have:*

$$\mathsf{P}\left(\{\omega \in \Omega \mid I_{L_0}(0) \text{ is } (\omega,\gamma,E_0) - \text{good}\}\right) \geq 1 - \mathrm{e}^{-\delta\ell L}, \tag{8.27}$$

for $L_0 \in \mathbb{N}$, $L_0 > L_\gamma(E)$ and $\delta > 0$, then $E_0 \in \Sigma_{\text{MSA}}$.

The hypothesis (8.27) is also known as the initial length scale estimate or “Initial Length Scale Estimate” (ILSE).

Finally, one can summarize the ingredients of a multi-scale analysis proof of Anderson and dynamical localization:

$$\underbrace{(\mathrm{IAD}) + (\mathrm{SLI}) + (\mathrm{NE}) + (\mathrm{W}) + (\mathrm{ILSE})} \tag{8.28}$$
$$\Downarrow$$
$$\underbrace{(\mathrm{MSA}) + (\mathrm{SGEE}) + (\mathrm{EDI})}$$
$$\Downarrow$$
$$\overbrace{\text{Anderson and dynamical localization}}$$

8.5 Initial Length Scale Estimate in the quasi-1d continuous case

8.5.1 Model and Notations

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. We consider, for all $\omega \in \Omega$,

$$H_{\omega,\ell} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \otimes I_{\mathrm{N}} + \sum_{n\in\mathbb{Z}} V_\omega^{(n)}(x - \ell n), \tag{8.29}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{R}^N$, where $N \geq 1$ is an integer and $\ell > 0$ is a real number. For all $n \in \mathbb{Z}$, the functions $x \mapsto V_\omega^{(n)}(x)$ are symmetric matrix-valued functions, supported in $[0, \ell]$ and uniformly bounded in x, n, and ω. The sequence $(V_\omega^{(n)})_{n\in\mathbb{Z}}$ is a sequence of *i.i.d.* random variables on Ω. We also assume that the potential $x \mapsto \sum_{n\in\mathbb{Z}} V_\omega^{(n)}(x - \ell n)$ is such that the family $\{H_{\omega,\ell}\}_{\omega\in\Omega}$ is $\ell\mathbb{Z}$-ergodic.

We consider the generalized eigenvalue equation, for all $\omega \in \Omega$,

$$H_{\omega,\ell} u = Eu, \quad \text{where } E \in \mathbb{R} \text{ and } u = \left(\begin{smallmatrix} u' \\ u \end{smallmatrix}\right) : \mathbb{R} \to \mathbb{R}^{2N}. \tag{8.30}$$

For $E \in \mathbb{R}$ and for all $x, y \in \mathbb{R}$, we introduce the transfer matrix $T_x^y(E)$ associated with $\{H_{\omega,\ell}\}_{\omega\in\Omega}$ from x to y, which maps a solution (u', u) at position x to the same solution at position y. It is defined by

$$\begin{pmatrix} u'(y) \\ u(y) \end{pmatrix} = T_x^y(E) \begin{pmatrix} u'(x) \\ u(x) \end{pmatrix} \tag{8.31}$$

and, in particular, $T_x^x(E) = I_{2\mathrm{N}}$ for all $x \in \mathbb{R}$. The transfer matrices are elements of the real symplectic group

$$\mathrm{Sp}_{\mathrm{N}}(\mathbb{R}) = \{M \in \mathcal{M}_{2\mathrm{N}}(\mathbb{R}) \mid {}^tMJM = J\} \tag{8.32}$$

with $J = \left(\begin{smallmatrix} 0 & -I_N \\ I_N & 0 \end{smallmatrix}\right)$.

For $E \in \mathbb{R}$, we denote $\gamma_1(E), \ldots, \gamma_{2N}(E)$ the *Lyapunov exponents* associated with the sequence $(T_{\ell n}^{\ell(n+1)}(E))_{n\in\mathbb{Z}}$. We also define the *Furstenberg group* G_{μ_E} of $\{H_{\omega,\ell}\}_{\omega\in\Omega}$ at E as the closed group generated by the support of μ_E, where μ_E denotes the common law of the random matrices $T_{\ell n}^{\ell(n+1)}(E)$, and where the closure is taken with respect to the usual topology of $\mathcal{M}_{2N}(\mathbb{R})$:

$$G_{\mu_E} = \overline{< \operatorname{supp} \mu_E >}.$$

8.5.2 A large deviations result

To estimate blocks of products of transfer matrices as in [19], we introduce, given a vector subspace F of $\mathbb{R}^{2N}$, the orthogonal projection onto F, $\pi_F : \mathbb{R}^{2N} \to F$, and we set

$$\pi_F^* : \begin{array}{ccc} F & \to & \mathbb{R}^{2N} \\ x & \mapsto & x \end{array}.$$

Recall that a subspace $F \subset \mathbb{R}^{2N}$ is called *Lagrangian* if it is orthogonal to itself with respect to J and of dimension N.

Let $s_p(\cdot)$ denote the p-th singular value of the matrix under consideration. We prove a large deviations property for the singular values of products of transfer matrices.

Proposition 8.5.1. *Fix a compact interval $I \subset \mathbb{R}$. Assume that for all $E \in I$:*

1. *the Furstenberg group G_{μ_E} is included in $\mathrm{Sp_N}(\mathbb{R})$;*
2. *for all $p \in \{1, \dots, N\}$, G_{μ_E} is L_p-strongly irreducible.*

Then, for all $\epsilon > 0$ and all $E \in I$, there exist constants $C(\epsilon, E) > 0$ and $c(\epsilon, E) > 0$ such that, for all $p \in 1, \dots, N$, all Lagrangian subspaces F, and all integers m, n,

$$\mathcal{P}\left(\left\{\left|\frac{1}{\ell(n-m)} \log s_p\left(T_{\ell m}^{\ell n}(E)\right) - \gamma_p(E)\right| \geq \epsilon\right\}\right) \leq C(\epsilon, E)\mathrm{e}^{-c(\epsilon,E)\ell|n-m|} \tag{8.33}$$

and

$$\mathcal{P}\left(\left\{\left|\frac{1}{\ell(n-m)} \log s_p\left(T_{\ell m}^{\ell n}(E)\pi_F^*\right) - \gamma_p(E)\right| \geq \epsilon\right\}\right) \leq C(\epsilon, E)\mathrm{e}^{-c(\epsilon,E)\ell|n-m|}. \tag{8.34}$$

Remark 8.5.2. *The constants $C(\epsilon, E)$ and $c(\epsilon, E)$ depend* a priori *on E and ϵ, but they can be chosen uniformly as ϵ tends to* 0 *and uniformly in E over the compact interval I. This is one of the reasons why we need to assume that the interval I is compact.*

Proof : Recall that for a symplectic matrix $M \in \mathrm{Sp_N}(\mathbb{R})$, we have $s_p(M^{-1}) = s_p(M)$ for all $p \in 1, \dots, 2N$. Hence, we may assume without loss of generality that $m \leq n$, since $(T_{\ell m}^{\ell n}(E))^{-1} = T_{\ell n}^{\ell m}(E)$.

Let $p \in 1, \dots, N$. For each $M \in \mathrm{Sp_N}(\mathbb{R})$, denote by $\widehat{M}$ the matrix in $\mathrm{GL}_k(\mathbb{R})$ defined by

$$\widehat{M}_{ij} = \langle f_i, \Lambda^p M f_j \rangle. \tag{8.35}$$

where k is the dimension of Lp and $(f_1, \dots, f_k)$ is an orthonormal basis of L_p, with $f_1 = e_1 \wedge \cdots \wedge e_p$. Let $\widehat{G_{\mu_E}}$ be the subgroup of $\mathrm{GL}_k(\mathbb{R})$ generated by the matrices $\widehat{M}$ for $M \in G_{\mu_E}$. Since G_{μ_E} is L_p-strongly irreducible, $\widehat{G_{\mu_E}}$ is strongly irreducible. Applying then [2, Theorem A.V.6.2], we obtain the existence of $\alpha > 0$ such that, for all $\epsilon > 0$ and all $\bar{x} \in \mathsf{P}(L_p)$,

$$\limsup_{|n-m|\to+\infty} \frac{1}{\ell|n-m|} \log \mathbb{P}\left(\left|\frac{1}{\ell|n-m|} \log(||\Lambda^p T_{\ell m}^{\ell n}(E)\bar{x}||) - (\gamma_1 + \cdots + \gamma_p)(E)\right| > \epsilon\right) \leq -\alpha \tag{8.36}$$

and

$$\limsup_{|n-m|\to+\infty} \frac{1}{\ell|n-m|} \log \mathbb{P}\left(\left|\frac{1}{\ell|n-m|} \log(||\Lambda^p T_{\ell m}^{\ell n}(E)||) - (\gamma_1 + \cdots + \gamma_p)(E)\right| > \epsilon\right) \leq -\alpha. \tag{8.37}$$

Indeed, since the function $x \mapsto V_\omega^{(n)}(x)$ is uniformly bounded in x, n, and ω, and since the support of the law of the transfer matrices is bounded, the integrability assumption in [2, Theorem A.V.6.2] is satisfied.

Let now F be a Lagrangian subspace of $\mathbb{R}^{2N}$. Then

$$\begin{aligned} ||\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)|| &= \sup_{\substack{u_1\wedge\cdots\wedge u_p \in \mathsf{P}(L_p)\\ u_i \in F}} ||(\Lambda^p T_{\ell m}^{\ell n}(E))(u_1 \wedge \cdots \wedge u_p)|| \\ &= ||\Lambda^p T_{\ell m}^{\ell n}(E)\, \bar{u}||, \text{ for some } \bar{u} \in \mathsf{P}(L_p), \end{aligned}$$

since the supremum is attained by compactness of $\mathsf{P}(L_p)$. Hence, (8.36) rewrites as

$$\limsup_{|n-m|\to+\infty} \frac{1}{\ell|n-m|} \log \mathbb{P}\left(\left| \frac{1}{\ell|n-m|} \log(||\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)||) - (\gamma_1+\cdots+\gamma_p)(E) \right| > \epsilon \right) \leq -\alpha \tag{8.38}$$

for all Lagrangian subspaces F.

For $n, m \in \mathbb{Z}$, $p \in 1,\ldots,N$, $\epsilon > 0$ and Lagrangian F, set

$$A_{n,m,p}(\epsilon, F) = \left\{ \left| \frac{1}{\ell|n-m|} \log(||\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)||) - (\gamma_1+\cdots+\gamma_p)(E) \right| > \epsilon \right\},$$

$$A_{n,m,p}(\epsilon) = \left\{ \left| \frac{1}{\ell|n-m|} \log(||\Lambda^p T_{\ell m}^{\ell n}(E)||) - (\gamma_1+\cdots+\gamma_p)(E) \right| > \epsilon \right\},$$

$$B_{n,m,p}(\epsilon, F) = \left\{ \left| \frac{1}{\ell|n-m|} \left(\log(||\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)||) - \log(||\Lambda^{p-1}(T_{\ell m}^{\ell n}(E)\pi_F^*)||) \right) - \gamma_p(E) \right| > \epsilon \right\}$$

and

$$B_{n,m,p}(\epsilon) = \left\{ \left| \frac{1}{\ell|n-m|} \left(\log(||\Lambda^p T_{\ell m}^{\ell n}(E)||) - \log(||\Lambda^{p-1} T_{\ell m}^{\ell n}(E)||) \right) - \gamma_p(E) \right| > \epsilon \right\}.$$

Then

$$B_{n,m,p}(2\epsilon) \subset A_{n,m,p}(\epsilon) \cap A_{n,m,p-1}(\epsilon) \text{ and } B_{n,m,p}(2\epsilon, F) \subset A_{n,m,p}(\epsilon, F) \cap A_{n,m,p-1}(\epsilon, F). \tag{8.39}$$

Since for all $p \in 1,\ldots,N$, $||\Lambda^p T_{\ell m}^{\ell n}(E)|| = s_1(T_{\ell m}^{\ell n}(E))\cdots s_p(T_{\ell m}^{\ell n}(E))$, combining (8.39) and (8.37), we obtain (8.33).

Also, for any Lagrangian subspace F,

$$||\Lambda^p(T_{\ell m}^{\ell n}(E)\pi_F^*)|| = s_1(T_{\ell m}^{\ell n}(E)\pi_F^*)\cdots s_p(T_{\ell m}^{\ell n}(E)\pi_F^*).$$

Combining (8.39) and (8.38), we obtain (8.34). This concludes the proof.

□

For any integer $n \in [-L/3, L/3]$, any matrix $T \in \mathcal{M}_{2\mathrm{N}}(\mathbb{R})$, and any vector subspace $F \subset \mathbb{R}^{2N}$, we define

$$\Omega_\epsilon^F[T] := \left\{ \max_{1\leq p\leq N} \left(\left| \frac{1}{\ell|n-L|} \log s_p\,(T) - \gamma_p(E) \right| + \left| \frac{1}{\ell|n-L|} \log s_p\,(T\pi_F^*) - \gamma_p(E) \right| \right) \leq \frac{\epsilon}{100N} \right\}. \tag{8.40}$$

We set

$$F_+ := \left\{ \left(\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right) \ |u \in \mathbb{R}^N \right\} \subset \mathbb{R}^{2N} \text{ and } F_- := \left\{ \left(\begin{smallmatrix} 0 \\ v \end{smallmatrix}\right) \ |v \in \mathbb{R}^N \right\} \subset \mathbb{R}^{2N} \tag{8.41}$$

and, for all $n \in \mathbb{Z}$,

$$F_n := \left\{ \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathbb{C}^{2N} \;\middle|\; u = -\Phi_+^{\downarrow}(\ell n)(\Phi_+^{\uparrow}(\ell n))^{-1} v \right\}. \tag{8.42}$$

Note that, since the transfer matrices belong to $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$, the vector subspace F_n is Lagrangian.

Finally, we define

$$\Omega_\epsilon(n) := \Omega_\epsilon^{F_n}[{}^tT_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_+}[{}^tT_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_+}[T_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_-}[{}^tT_{\ell L}^{\ell n}(E)] \cap \Omega_\epsilon^{F_-}[T_{\ell L}^{\ell n}(E)]. \tag{8.43}$$

8.5.3 Initial Length Scale Estimate (ILSE)

We denote $I_L := I_L(0)$.

Proposition 8.5.3 (ILSE for Schrödinger). *Let $I \subset \mathbb{R}$ be an open interval such that, for all $E \in I$, the Furstenberg group associated with $\{H_{\omega,\ell}\}_{\omega\in\Omega}$ is p-contracting and L_p-strongly irreducible. Let $E \in I$. For any $\varepsilon > 0$, there exist constants $C, c > 0$ and $L_0 \in \mathbb{N}$ such that, for all $L \geq L_0$,*

$$\mathsf{P}\left(\left\{I_L \text{ is } (\omega, \gamma_N(E) - \varepsilon, E)\text{-}good\right\}\right) \geq 1 - \mathrm{C}e^{-c\ell L}. \tag{8.44}$$

To prove the good ILSE for $\{H_{\omega,\ell}\}_{\omega\in\Omega}$, the first step is to give an explicit formula for the Green kernel of $H^{(0,L)}_{\omega,\ell}$ in terms of the solutions $\Phi_\pm$ of $H_{\omega,\ell}\Phi_\pm = E\Phi_\pm$ satisfying

$$\Phi_-(-\ell L) = \begin{pmatrix} 0 \\ I_N \end{pmatrix} \text{ and } \Phi_+(\ell L) = \begin{pmatrix} 0 \\ I_N \end{pmatrix}. \tag{8.45}$$

Lemma 8.5.4. *Let $\omega \in \Omega$ and let $x, y \in I_L$. Assume that $\Phi_+(x)$ and $\Phi_-(x)$ are invertible, as well as $\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}$. The Green kernel of $H^{(0,L)}_{\omega,\ell}$ is given by*

$$G^\omega_{I_L}(E,x,y) = \begin{cases} \Phi_+(y)(\Phi_+(x))^{-1}\left(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}\right)^{-1} & \text{if } x \leq y \\ \Phi_-(y)(\Phi_-(x))^{-1}\left(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}\right)^{-1} & \text{if } x \geq y \end{cases} \tag{8.46}$$

Proof : We want to have

$$G^\omega_{I_L}(E,x,y) = \begin{cases} \Phi_+(y)\alpha_+(x) & \text{if } x \leq y \\ \Phi_-(y)\alpha_-(x) & \text{if } x \geq y \end{cases} \tag{8.47}$$

Moreover, we want $G^\omega_{I_L}$ to be continuous, *i.e.* for all $x \in I_L$

$$\Phi_+(x)\alpha_+(x) = \Phi_-(x)\alpha_-(x). \tag{8.48}$$

By definition, we must have, for any $\psi \in L^2(I_L)$ and almost all $x \in I_L$,

$$(H^{(0,L)}_{\omega,\ell} - E)\int_{I_L} G^\omega_{I_L}(E,x,y)\psi(x)\mathrm{d}x = \psi(y).$$

Computing this expression explicitly and using the fact that $(H_{\omega,\ell} - E)\Phi_\pm = 0$, we obtain that for all $x \in I_L$

$$-\Phi_+(x)\alpha'_+(x) + \Phi_-(x)\alpha'_-(x) = I_N. \tag{8.49}$$

Differentiating (8.48), combined with (8.49), implies that

$$\Phi'_+(x)\alpha_+(x) - \Phi'_-(x)\alpha_-(x) = I_N. \tag{8.50}$$

We can solve the system formed by (8.48) and (8.50) to obtain

$$\begin{cases} \alpha_+(x) &= (\Phi_+(x))^{-1}\left(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}\right)^{-1} \\ \alpha_-(x) &= (\Phi_-(x))^{-1}\left(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}\right)^{-1} \end{cases} \tag{8.51}$$

which concludes the proof.

□

Our next goal is to bound $\sup_{x,y\in I_{L_0}} |\Gamma_{0,L_0}(x)G^\omega_{I_{L_0}}(E,x,y)\chi_{0,L_0/3}(y)|$ with high probability. To this end, for any integer $n \in [-L/3, L/3]$, we consider the event $\Omega_\epsilon(n)$ as defined in (8.43).

Proposition 8.5.5. *On $\Omega_\epsilon := \cap_{n\in[-L/3,L/3]}\Omega_\epsilon(n)$, we have, for all $x \in [-\ell L/3, \ell L/3]$ and all $y \in [\ell L - \ell, \ell L]$,*

$$||G^\omega_{I_L}(E,x,y)|| \leq Ce^{-2(\gamma_N(E)-\epsilon)\ell L}. \tag{8.52}$$

Proof : We begin by showing that on Ω_ϵ, for all $x \in [-\ell L/3, \ell L/3]$, the matrix $\Phi_+(x)$ is invertible, and we estimate its inverse. This amounts to bounding from below its N-th singular value.

Let n be the unique integer such that $x \in [n\ell, (n+1)\ell)$. First, note that for all $p \in \{1, \dots, 2N\}$,

$$s_p(T^x_{\ell L}) \geq s_p(T^{\ell n}_{\ell L}) s_{2N}(T^x_{\ell n}) = s_p(T^{\ell n}_{\ell L}) ||T^x_{\ell n}||^{-1}, \tag{8.53}$$

where the last equality comes from the fact that $T^x_{\ell n}$ is symplectic. By [3, Lemma 6], there exists a constant $C > 0$, independent of x and ω, such that $||T^x_{\ell n}|| \leq C$. Consequently, $s_p(T^x_{\ell L}) \geq C^{-1} s_p(T^{\ell n}_{\ell L})$.

We have the following singular value decomposition:

$$T^x_{\ell L}(E) = U \Sigma V, \tag{8.54}$$

where U and V are unitary and, since the matrix is symplectic, we can write $\Sigma = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & \Sigma_- \end{pmatrix}$ with $\Sigma_+ = \text{diag}(s_1, \dots, s_N)$ and $\Sigma_- = \text{diag}(1/s_1, \dots, 1/s_N)$, where $s_i \geq 1$ for all $i \in \{1, \dots, N\}$. We can write a block decomposition of U and V: $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ and $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$. We then obtain

$$\Phi_+(x) = \begin{pmatrix} I_N & 0 \end{pmatrix} \begin{pmatrix} \Phi_+(x) \\ \Phi'_+(x) \end{pmatrix} = \begin{pmatrix} I_N & 0 \end{pmatrix} T^x_{\ell L}(E) \begin{pmatrix} 0 \\ I_N \end{pmatrix} = U_{11} \Sigma_+ V_{12} + U_{12} \Sigma_- V_{22}.$$

On the one hand, the blocks U_{ij} and V_{ij} have norm less than 1, and on the other hand, on the event $\Omega_\epsilon(n)$, we have $||\Sigma_-|| \leq C e^{-(\gamma_N(E) - \epsilon)\ell |L-n|}$. Therefore,

$$\begin{aligned} s_N(\Phi_+(x)) &\geq s_N(U_{11} \Sigma_+ V_{12}) - C e^{-(\gamma_N(E)-\epsilon)\ell|L-n|} \\ &\geq s_N(U_{11}) s_N(\Sigma_+) s_N(V_{12}) - C e^{-(\gamma_N(E)-\epsilon)\ell|L-n|}. \end{aligned}$$

Since $s_N(\Sigma_+) \geq C^{-1} e^{(\gamma_N(E)-\epsilon)\ell|L-n|}$ (on the event $\Omega_\epsilon(n)$), it remains to control $s_N(U_{11})$ and $s_N(V_{12})$. As in Claim 3.4 and Remark 3.5 of [19], on $\Omega_\epsilon(n)$,

$$s_N(V_{12}) \geq e^{-\frac{\epsilon}{25}\ell|L-n|} \text{ and } s_N(U_{11}) \geq e^{-\frac{\epsilon}{25}\ell|L-n|}. \tag{8.55}$$

Consequently, on $\Omega_\epsilon(n)$,

$$|s_N(\Phi_+(x))| \geq C^{-1} e^{(\gamma_N(E) - \frac{27\epsilon}{25})\ell|L-n|} - C e^{-(\gamma_N(E)-\epsilon)\ell|L-n|}. \tag{8.56}$$

For L sufficiently large, we obtain on Ω_ϵ,

$$|s_N(\Phi_+(x))| \geq e^{(\gamma_N(E)-2\epsilon)\frac{2\ell L}{3}} > 0. \tag{8.57}$$

In particular, $\Phi_+(x)$ is invertible.

The next step, in order to apply Lemma 8.5.4, is to show that $\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}$ is invertible. As in Equation (33) of [19], one shows that

$$s_N(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1}) \geq e^{-\frac{\epsilon}{2}\ell L}. \tag{8.58}$$

Then, by Lemma 8.5.4,

$$G^\omega_{I_L}(E, x, y) = \Phi_+(y)(\Phi_+(x))^{-1}(\Phi'_+(x)\Phi_+(x)^{-1} - \Phi'_-(x)\Phi_-(x)^{-1})^{-1}. \tag{8.59}$$

Finally, note that for such y, by [3, Lemma 6],

$$||\Phi_+(y)||^2 \leq N \exp\left(2\int_y^{\ell L} |V_\omega(t)|\mathrm{d}t\right) \leq C, \tag{8.60}$$

where C is independent of ω and L.

Combining (8.60), (8.56), and (8.58), we obtain that on Ω_ϵ

$$||G_{I_L}^\omega(E,x,y)|| \leq Ce^{-2(\gamma_N(E)-\frac{7}{4}\epsilon)\ell L}. \tag{8.61}$$

□

The last step is to estimate the probability of Ω_ϵ. Since the sequence of transfer matrices of $\{H_{\omega,\ell}\}_{\omega\in\Omega}$ and its Furstenberg group satisfy the hypotheses of Proposition 8.5.1, there exist constants $C, c > 0$ such that $\mathbb{P}(\Omega_\epsilon(n)) \leq Ce^{-c\ell|n-L|}$. Consequently,

$$\mathbb{P}(^c(\Omega_\epsilon)) \geq 1 - \sum_{n\in[-L/3,L/3]} Ce^{-c\ell|n-L|} \geq 1 - C'e^{-c'\ell L} \tag{8.62}$$

which proves the ILSE for $\{H_{\omega,\ell}\}_{\omega\in\Omega}$.

Chapter 9

Quasi-$1d$ models of unitary and Dirac types

9.1 Quasi-$1d$ models of unitary type

In this setting, the role played by the symplectic group for the quasi-one-dimensional models of Schrödinger type will be played by the so-called Lorentz group.

The Lorentz group $\mathrm{U}(D,D)$ of signature (D,D) is defined as the set of matrices of size $2D\times 2D$ which preserve the form $\mathcal{L}=\begin{pmatrix} \mathrm{I_D} & 0 \\ 0 & -\mathrm{I_D}\end{pmatrix}$ in the sense that T is in $\mathrm{U}(D,D)$ if and only if $T^*\mathcal{L}T=\mathcal{L}$.

To pass from the results on Lyapunov exponents in the symplectic framework to the Lorentz group framework, one uses the Cayley transform. By the Cayley transform, the group $\mathrm{U}(D,D)$ is unitarily equivalent to the complex symplectic group. More precisely, if $C=\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{I_D} & -i\mathrm{I_D} \\ \mathrm{I_D} & i\mathrm{I_D}\end{pmatrix}\in\mathcal{M}_{2D}(\mathbb{C})$ and if $J=\begin{pmatrix} 0 & -\mathrm{I_D} \\ \mathrm{I_D} & 0\end{pmatrix}$, then $\mathrm{U}(D,D)=C\mathrm{Sp_D}(\mathbb{C})C^*$, where

$$\mathrm{Sp_D}(\mathbb{C})=\{M\in\mathcal{M}_{2D}(\mathbb{C})|M^*JM=J\}.$$

In order to apply directly the results of[2], we have to pass from the complex symplectic group to the real symplectic group. For that we introduce the application which separates the real and imaginary parts of a matrix with complex coefficients and place them in blocks:

$$\pi\ :\ \begin{array}{ccc} \mathcal{M}_{2D}(\mathbb{C}) & \to & \mathcal{M}_{4D}(\mathbb{R}) \\ A+iB & \mapsto & \begin{pmatrix} A & -B \\ B & A\end{pmatrix}. \end{array}$$

Finally, $\pi(C^*\cdot\mathrm{U}(D,D)\cdot C)\subset\mathrm{Sp_{2D}}(\mathbb{R})$ which allows to use the results on Lyapunov exponents in the symplectic group to study the Lyapunov exponents in the unitary setting.

9.1.1 The unitary Anderson model

The first example of unitary model in dimension one for which we present a localization result is the unitary analog of the Anderson model. It was studied by Hamza, Joye and Stolz in [11]. We present their result and use their notations.

First, consider two 2×2 unitary matrices $B_1=\begin{pmatrix} r & t \\ -t & r\end{pmatrix}$ and $B_2=\begin{pmatrix} r & -t \\ t & r\end{pmatrix}$ with $(r,t)\in\mathbb{R}^2$ satisfying $r^2+t^2=1$. These real numbers correspond to reflection and transition coefficients. Then, let U_e the unitary matrix operator in $\ell^2(\mathbb{Z})$ defined as the direct sum of identical B_1-blocks with blocks starting at even indices. Construct also U_o the unitary matrix operator in $\ell^2(\mathbb{Z})$ defined as the direct sum of identical B_2-blocks with blocks starting at odd indices. Let $S=U_eU_o$, unitary operator on $\ell^2(\mathbb{Z})$ with band structure.

Then, introduce the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \mathbb{T}^{\mathbb{Z}}$ (with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$), $\mathcal{F}$ is the σ-algebra generated by cylinders of Borel sets and $\mathbb{P} = \bigotimes_{k\in\mathbb{Z}} \mu$ where μ is a non-trivial probability measure on $\mathbb{T}$. Assume that μ is absolutely continuous with bounded density. Then, define a sequence of random variables $(\theta_k)_{k\in\mathbb{Z}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ which are $\mathbb{T}$-valued and *i.i.d.* with common law μ. With these random variables one defines the diagonal random operator D_ω acting on $\ell^2(\mathbb{Z})$ and defined by :

$$\forall \omega \in \Omega,\ \forall k \in \mathbb{Z}, D_\omega e_k = \mathrm{e}^{-i\theta_k(\omega)} e_k$$

where $(e_k)_{k\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$.

The **unitary Anderson model** is the family $\{U_\omega\}_{\omega\in\Omega}$ of unitary operators acting on $\ell^2(\mathbb{Z})$ where for every $\omega \in \Omega$, $U_\omega = D_\omega S$.

The family $\{U_\omega\}_{\omega\in\Omega}$ is ergodic with respect to the 2-shift in Ω and one can show that its almost-sure spectrum is equal to: $\Sigma = \{\mathrm{e}^{ia} \mid a \in [-\lambda_0, \lambda_0] - \operatorname{supp}\mu\} \subset \mathbb{S}^1$ where $\lambda_0 = \arccos(r^2 - t^2)$[11].

For $\omega \in \Omega$ and $z \in \mathbb{C} \setminus \mathbb{S}^1$ let $G_\omega(z) = (U_\omega - z)^{-1}$ and for $k, l \in \mathbb{Z}$, let $G_\omega(k, l, z) = \langle e_k | G_\omega(z) e_l \rangle$ be the Green function of U_ω.

Using the Fractional Moments Method [1], Hamza, Joye and Stolz proved the following result:

Theorem 9.1.1 ([11])**.** *For every $t < 1$ there exists $s > 0$, $C < \infty$ and $\alpha > 0$ such that*

$$\mathbb{E}(|G_\omega(k, l, z)|^s) \leq C\mathrm{e}^{-\alpha|k-l|}$$

for all $z \in \mathbb{C}$ such that $0 < ||z| - 1| < \frac{1}{2}$ and all $k, l \in \mathbb{Z}$. Therefore, $\{U_\omega\}_{\omega\in\Omega}$ exhibits dynamical localization throughout Σ.

The proof of Theorem 9.1.1 relies on the formalism of transfer matrices and on the use of the Furstenberg theorem to get positivity of the Lyapunov exponent which allows to prove results of exponential decay of some products of transfer matrices.

Let $\omega \in \Omega$. Consider the equation $U_\omega \psi = z\psi$ for $z \in \mathbb{C} \setminus \{0\}$ and with ψ not necessarily in $\ell^2(\mathbb{Z})$. For all $(\theta, \eta) \in \mathbb{T}^2$, let

$$T_z(\theta, \eta) = \begin{pmatrix} -\frac{1}{z}\mathrm{e}^{-i\eta} & \frac{r}{t}\left(\mathrm{e}^{i(\theta-\eta)} - \frac{1}{z}\mathrm{e}^{-i\eta}\right) \\ \frac{r}{t}\left(1 - \frac{1}{z}\mathrm{e}^{-i\eta}\right) & -\frac{z}{t^2}\mathrm{e}^{i\theta} + \frac{r^2}{t^2}\left(1 + \mathrm{e}^{i(\theta-\eta)} - \frac{1}{z}\mathrm{e}^{-i\eta}\right) \end{pmatrix}.$$

Then, one has

$$\forall k \in \mathbb{Z},\ \begin{pmatrix} \psi_{2k+1} \\ \psi_{2k+2} \end{pmatrix} = T_z(\theta_{2k}(\omega), \theta_{2k+1}(\omega)) \begin{pmatrix} \psi_{2k-1} \\ \psi_{2k} \end{pmatrix}.$$

Let $\gamma(z)$ be the Lyapunov exponent associated to the sequence of *i.i.d.* matrices in $\mathrm{SL}_2(\mathbb{R})$, $(T_z(\theta_{2k}(\omega), \theta_{2k+1}(\omega)))_{k\in\mathbb{Z}}$. Using Furstenberg theorem in a very similar way as for the discrete scalar-valued one-dimensional Anderson model, it is proven that

(i) if $\operatorname{supp}\mu = \{a, b\}$ with $|a - b| = \pi$ then $\gamma(-a) = \gamma(-b) = 0$ and for all $z \in \mathbb{T} \setminus \{-a, -b\}$, $\gamma(z) > 0$.

(ii) If $\{a, b\} \subset \operatorname{supp}\mu$ with $|a - b| \notin \{0, \pi\}$ then for every $z \in \mathbb{T}$, $\gamma(z) > 0$.

This result of positivity of Lyapunov exponents implies that for every compact $K \subset \mathbb{C}$ there exist $\alpha > 0$, $\delta \in (0, 1)$ and $C < \infty$ such that: $\forall z \in K$, $\forall n \in \mathbb{N}, \forall v \in \mathbb{C}^2$, $||v|| = 1$,

$$\mathbb{E}(||T_z(\theta_{2(n-1)}(\cdot), \theta_{2(n-1)+1}(\cdot)) \cdots T_z(\theta_0(\cdot), \theta_1(\cdot))v||^{-\delta}) \leq C\mathrm{e}^{-\alpha n}.$$

This estimate is the core of the proof of Theorem 9.1.1.

9.1.2 CMV matrices

The unitary Anderson model is a particular case of more general one-dimensional unitary random operators, the so-called CMV matrices.

Let $(\alpha_n)_{n\in\mathbb{N}}$ a sequence of complex numbers such that for every $n \in \mathbb{N}$, $|\alpha_n| < 1$. If $\rho_n = (1-|\alpha_n|^2)^{\frac{1}{2}}$ then the CMV matrix associated to the Verblunsky sequence $(\alpha_n)_{n\in\mathbb{N}}$ is the operator acting on $\ell^2(\mathbb{N},\mathbb{C})$ given by the semi-infinite matrix,

$$\mathcal{C} = \begin{pmatrix} \overline{\alpha_0} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & & & \\ \rho_0 & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha 0 & & & & & \\ & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & & & \\ & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & & & \\ & & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \overline{\alpha_5}\rho_4 & \rho_5\rho_4 & \\ & & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\overline{\alpha_5}\alpha_4 & -\rho_5\alpha_4 & \\ & & & & \ddots & \ddots & \ddots & \end{pmatrix} \tag{9.1}$$

One also define the extended CMV matrix associated to $(\alpha_n)_{n\in\mathbb{Z}}$, a sequence of complex numbers such that for every $n \in \mathbb{Z}$, $|\alpha_n| < 1$, as the operator acting on $\ell^2(\mathbb{Z},\mathbb{C})$ given by the infinite matrix,

$$\mathcal{E} = \begin{pmatrix} \ddots & \ddots & \ddots & & & & & & \\ \overline{\alpha_0}\rho_{-1} & -\overline{\alpha_0}\alpha_{-1} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & & & & \\ \rho_0\rho_{-1} & -\rho_0\alpha_{-1} & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha 0 & & & & & \\ & & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 & & & \\ & & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 & & & \\ & & & & \overline{\alpha_4}\rho_3 & -\overline{\alpha_4}\alpha_3 & \overline{\alpha_5}\rho_4 & \rho_5\rho_4 & \\ & & & & \rho_4\rho_3 & -\rho_4\alpha_3 & -\overline{\alpha_5}\alpha_4 & -\rho_5\alpha_4 & \\ & & & & & \ddots & \ddots & \ddots & \end{pmatrix} \tag{9.2}$$

The CMV matrices are the unitary analog of Jacobi matrices and were originally introduced in the study of orthogonal polynomials on the unit circle. Indeed they arise in the representation of the map $f \mapsto zf$ on $L^2(\mathbb{S}^1, \mathrm{d}\mu)$ in the basis given by the orthonormalization of the set of Laurent polynomials $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ according to the usual scalar product on $L^2(\mathbb{S}^1, \mathrm{d}\mu)$. Here $\mathbb{S}^1$ denote the unit circle in $\mathbb{C}$ and μ denotes a probability measure on $\mathbb{S}^1$ that does not admit a finite support. See [27, 28] for a comprehensive review of this vast subject.

In order to introduce randomness in extended CMV matrices, let ν be a Borel probability measure supported on a compact set S of the open unit disk in $\mathbb{C}$ which contains at least two points. Let $\Omega = S^{\mathbb{Z}}$ and consider $(\alpha_n(\omega))_{n\in\mathbb{Z}} = (\omega_n)_{n\in\mathbb{Z}} \in \Omega$ a sequence of *i.i.d.* random variables of common law ν. In particular, the ω_n could be Bernoulli variables.

The random Verblunsky sequence $(\alpha_n(\omega))_{n\in\mathbb{Z}}$ defines a random extended CMV matrix $\mathcal{E}_\omega$. The family $\{\mathcal{E}_\omega\}_{\omega\in\Omega}$ is $\mathbb{Z}$-ergodic. For such an ergodic family, there is no multi-scale analysis or Kunz-Souillard approach available and the proof of the first localization result in the Bernoulli case is based upon the Furstenberg theorem and large deviations estimates.

9.1.3 Scattering zippers

Let $L \geq 1$ an integer. A scattering zipper is a system obtained by concatenation of elementary unitary diffusion with a fixed number $2L$ of incoming and outgoing channels each.

Let $\|\cdot\|$ any subordinate norm on $\mathcal{M}_L(\mathbb{C})$. Let $U(L)$ be the unitary subgroup of the linear group $\mathrm{GL_L}(\mathbb{C})$ of order L. We introduce the set

$$\mathrm{U}(2L)_{\mathrm{inv}} = \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{U}(2L) \;\middle|\; \alpha, \gamma, \delta \in \mathcal{M}_L(\mathbb{C}) \text{ and } \beta \in \mathrm{GL_L}(\mathbb{C}) \right\}. \tag{9.3}$$

which also has the representation:

$$\mathrm{U}(2L)_{\mathrm{inv}} = \{ S(\alpha, U, V) \in \mathrm{U}(2L) \mid \|\alpha^*\alpha\| < 1 \text{ and } U, V \in \mathrm{U}(L) \}, \tag{9.4}$$

where for any $U, V \in \mathrm{U}(L)$ and any $\alpha \in \mathcal{M}_L(\mathbb{C})$ with $\|\alpha^*\alpha\| < 1$,

$$S(\alpha, U, V) \;=\; \begin{pmatrix} \alpha & \rho(\alpha)U \\ V\widetilde{\rho}(\alpha) & -V\alpha^* U \end{pmatrix} \quad \text{with} \quad \rho(\alpha) = (\mathrm{I_D} - \alpha\alpha^*)^{\frac{1}{2}} \quad \text{and} \quad \tilde{\rho}(\alpha) = (\mathrm{I_D} - \alpha^*\alpha)^{\frac{1}{2}}. \tag{9.5}$$

Let $(S_n)_{n\in\mathbb{Z}}$ be a sequence of matrices in the set $\mathrm{U}(2L)_{\mathrm{inv}}$. The scattering zipper operator $\mathbb{U}$ associated to the sequence $(S_n)_{n\in\mathbb{Z}}$ is the operator acting on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ and defined by :

$$\mathbb{U} \;=\; \mathbb{V}\,\mathbb{W}\,, \tag{9.6}$$

where the two unitary operators $\mathbb{V}$ and $\mathbb{W}$ act on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ and are given by

$$\mathbb{V} \;=\; \begin{pmatrix} \ddots & & & \\ & S_0 & & \\ & & S_2 & \\ & & & \ddots \end{pmatrix} \circ s_g^L\,, \quad \mathbb{W} \;=\; \begin{pmatrix} \ddots & & & \\ & S_{-1} & & \\ & & S_1 & \\ & & & \ddots \end{pmatrix} \tag{9.7}$$

where s_g is the shift operator to the left $(v_n)_{n\in\mathbb{Z}} \mapsto (v_{n+1})_{n\in\mathbb{Z}}$.

We now introduce randomness which allow to prove a dynamical localization result for the random scattering zipper.

Let

$$\tilde{\Omega} = \Big(U(L) \times \{-1, 1\}^L \times [0, 2\pi]^L\Big)^2 \tag{9.8}$$

endowed with the probability measure

$$\tilde{\mathbb{P}} = \Big(\nu_L \otimes (\mathcal{B}(p))^{\otimes L} \otimes (\mathcal{L}eb_{[0,2\pi]})^{\otimes L}\Big)^{\otimes 2}, \tag{9.9}$$

defined on $\tilde{\mathcal{B}}$, the Borel σ-algebra on $\tilde{\Omega}$ for the usual topology on the Lie group. Here ν_L denotes the Haar measure on the compact Lie group $U(L)$, $\mathcal{B}(p)$ denotes the Bernoulli distribution of parameter $p \in (0, 1)$ and $\mathcal{L}eb_{[0,2\pi]}$ denotes the Lebesgue measure on the interval $[0, 2\pi]$.

We then define the product probability space:

$$(\Omega, \mathcal{B}, \mathbb{P}) = \left(\tilde{\Omega}^{\mathbb{Z}}, \bigotimes_{n\in\mathbb{Z}} \tilde{\mathcal{B}}, \bigotimes_{n\in\mathbb{Z}} \tilde{\mathbb{P}}\right). \tag{9.10}$$

Let $\omega \in \Omega$ and $n \in \mathbb{Z}$. Then $\omega_n \in \tilde{\Omega}$ and we set

$$\omega_n = (\widetilde{V}_\omega^{(n)}, d_\omega^{(n)}, \theta_\omega^{(n)}, \widetilde{U}_\omega^{(n)}, D_\omega^{(n)}, \Theta_\omega^{(n)}) \tag{9.11}$$

with :

(i) $\widetilde{U}_\omega^{(n)} \in U(L)$ and $\widetilde{V}_\omega^{(n)} \in U(L)$;

(ii) $d_\omega^{(n)} = (d_{\omega,1}^{(n)}, \ldots, d_{\omega,L}^{(n)}) \in \{-1, 1\}^L$ which will represent indifferently the L-uple or the diagonal matrix whose diagonal elements are the $d_{\omega,1}^{(n)}, \ldots, d_{\omega,L}^{(n)}$ and the same for $D_\omega^{(n)} = (D_{\omega,1}^{(n)}, \ldots, D_{\omega,L}^{(n)}) \in \{-1, 1\}^L$;

(iii) $\theta_\omega^{(n)} = (\theta_{\omega,1}^{(n)}, \ldots, \theta_{\omega,L}^{(n)}) \in [0, 2\pi]^L$ which will represent indifferently the L-uple or the diagonal matrix whose diagonal elements are the $\theta_{\omega,1}^{(n)}, \ldots, \theta_{\omega,L}^{(n)}$ and the same for $\Theta_\omega^{(n)} = (\Theta_{\omega,1}^{(n)}, \ldots, \Theta_{\omega,L}^{(n)}) \in [0, 2\pi]^L$.

With these notations we define, for every $\omega \in \Omega$ and every $n \in \mathbb{Z}$, the following phases

$$U_\omega^{(n)} = e^{i\Theta_\omega^{(n)}} \widetilde{U}_\omega^{(n)} D_\omega^{(n)} (\widetilde{U}_\omega^{(n)})^* := e^{i\Theta_\omega^{(n)}} \widehat{U}_\omega^{(n)} \quad \text{and} \quad V_\omega^{(n)} = \widetilde{V}_\omega^{(n)} d_\omega^{(n)} (\widetilde{V}_\omega^{(n)})^* e^{i\theta_\omega^{(n)}} := \widehat{V}_\omega^{(n)} e^{i\theta_\omega^{(n)}} \tag{9.12}$$

where the reduced notations $\widehat{U}_\omega^{(n)}$ and $\widehat{V}_\omega^{(n)}$ are used in the proofs of Theorem 9.1.5, Lemma 9.1.9 and Theorem 9.1.12.

Remark 9.1.2. *In view of the definition of the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ one can see $(\widetilde{U}_\omega^{(n)})_{n\in\mathbb{Z}}$ and $(\widetilde{V}_\omega^{(n)})_{n\in\mathbb{Z}}$ as sequences of independent and identically distributed* (i.i.d. *for short) random variables in $U(L)$ of uniform law according to the Haar measure on $U(L)$. The sequences $(d_{\omega,i}^{(n)})_{n\in\mathbb{Z}}$ and $(D_{\omega,j}^{(n)})_{n\in\mathbb{Z}}$, for $i,j \in \{1,\dots,L\}$, can be considered as sequences of* i.i.d. *random variables in $\{-1,1\}$ of common law the Bernoulli law $\mathcal{B}(p)$ of parameter $p \in (0,1)$ and the sequences $(\theta_{\omega,i}^{(n)})_{n\in\mathbb{Z}}$ and $(\Theta_{\omega,j}^{(n)})_{n\in\mathbb{Z}}$ for $i,j \in \{1,\dots,L\}$ can be considered as sequences of* i.i.d. *random variables in $[0,2\pi]$ of common law the uniform law on $[0,2\pi]$. Moreover, all these sequences of random variables are independent from each others.*

Remark 9.1.3. *Note that for every $\omega \in \Omega$ and every $n \in \mathbb{Z}$, $\widehat{U}_\omega^{(n)}$ and $\widehat{V}_\omega^{(n)}$ are elements of $U(L) \cap H_L(\mathbb{C})$ where $H_L(\mathbb{C})$ is the vector space of the Hermitian matrices of size $L \times L$. This property will be used in the proof of Lemma 9.1.9 where will need that some squareroots of unitary matrices are also Hermitian. This property is the reason why we introduce the randomness in this particular form. The presence of the Bernoulli variables as diagonal elements of the matrices $D_\omega^{(n)}$ and $d_\omega^{(n)}$ is therefore necessary in order to have that $\widehat{U}_\omega^{(n)}$ and $\widehat{V}_\omega^{(n)}$ can take all the possible values in $U(L) \cap H_L(\mathbb{C})$. Then we add the diagonal matrices $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$ to ensure that the random phases $U_\omega^{(n)}$ and $V_\omega^{(n)}$ can take all the possible values in $U(L)$.*

The presence of separates terms $e^{i\Theta_\omega^{(n)}}$ and $e^{i\theta_\omega^{(n)}}$ also simplify the formulation of the finite rank perturbation argument in the proofs of Theorem 9.1.5 and Theorem 9.1.12.

With these notations, we now introduce the model of random scattering zipper. Consider the random family of unitary operators $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ where, for every $\omega \in \Omega$, the operator $\mathbb{U}_\omega$ acts on $\ell^2(\mathbb{Z}, \mathbb{C}^L)$ and is the scattering zipper associated to the sequence of random scattering matrices $(S_\omega^{(n)})_{n\in\mathbb{Z}}$ defined by

$$S_\omega^{(n)} = S(\alpha, U_\omega^{(n)}, V_\omega^{(n)}) = \begin{pmatrix} \alpha & \rho(\alpha) U_\omega^{(n)} \\ V_\omega^{(n)} \tilde{\rho}(\alpha) & -V_\omega^{(n)} \alpha^* U_\omega^{(n)} \end{pmatrix}, \tag{9.13}$$

for a fixed $\alpha \in \mathcal{M}_L(\mathbb{C})$ such that $\|\alpha\| < 1$.

Note that the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ has a very important property of $2\mathbb{Z}$-ergodicity which implies the existence of an almost sure spectrum for this family. Note that if Σ denotes the almost-sure spectrum of $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$, $\Sigma \subset \mathbb{S}^1$ since the operators $\mathbb{U}_\omega$ are unitary. The $2\mathbb{Z}$-ergodicity also implies the existence of almost-sure pure point, absolutely continuous and singular continuous spectra.

We have the following diagram in which the horizontal arrows represent the transition from a scalar-valued operator to a matrix-valued operator, and the vertical arrows represent the transition from a self-adjoint model to its unitary analogue.

$$\begin{array}{ccc} \text{Jacobi matrices} & \longrightarrow & \text{Jacobi matrices with matrix coefficients} \\ \downarrow & & \downarrow \\ \text{CMV matrices} & \longrightarrow & \text{Scattering zipper} \end{array}$$

9.1.4 Ingredients of the Fractional Moments Method

Notation. Let $\{e_k\}_{k\in\mathbb{Z}}$ be the canonical basis of $\ell^2(\mathbb{Z})\otimes\mathbb{C}^L$. For $i,j\in\mathbb{Z}$, we set $e_{\{i,j\}}:=e_{iL+j}$, which corresponds to the j-th component in the i-th L-block.

Theorem 9.1.4. *There exists $r_0>0$ such that, for every $\alpha\in\mathrm{GL_L}(\mathbb{C})$ with $\|\alpha\|\le r_0$, there exists $C_{r_0}>0$ and $b>0$ such that for all $\{k,p\}$ and $\{l,q\}$ in $\mathbb{Z}\times\{1,\dots,L\}$,*

$$\mathbb{E}\left[\sup_{n\in\mathbb{Z}}\left|\left\langle e_{\{k,p\}},(\mathbb{U}_\omega)^n e_{\{l,q\}}\right\rangle\right|\right]\le C_{r_0}e^{-b|k-l|}. \tag{9.14}$$

The estimate (9.14) means that the family $\{\mathbb{U}_\omega\}_{\omega\in\Omega}$ satisfies the condition of dynamical localization. This property is dynamic in nature and follows the evolution of wave packets over discrete time $n\in\mathbb{Z}$. It tells us that the solutions of the Schrödinger equation are localized in space in the vicinity of their initial position and this, uniformly over time. This reflects the absence of quantum transport.

The proof of Theorem 9.1.4 is based upon a number of intermediate results. First, following the result of positivity of the Lyapunov exponents on the unit circle $\mathbb{S}^1$, we prove their strict positivity on an annulus

$$\mathbb{S}_\epsilon:=\{z\in\mathbb{C};1-\epsilon<|z|<1+\epsilon\} \tag{9.15}$$

for some $\epsilon\in(0,1]$. The proof is done by combining the positivity of the Lyapunov exponents on the unit circle and their continuity on $\mathbb{C}\setminus\{0\}$.

Notation. For $j\in\mathbb{Z}$, let $E_j=(\dots,0,\mathrm{I_D},0,\dots)$ where $\mathrm{I_D}$ is in the j^{th} position.

For $z\in\mathbb{C}\setminus\mathbb{S}^1$ let $G_\omega(z)$ be the resolvent at z of the operator $\mathbb{U}_\omega$ and let $G_\omega(z,\cdot,\cdot)$ be its block Green kernel defined for every $k,l\in\mathbb{Z}$ by:

$$G_\omega(z,k,l)=\langle E_k,(\mathbb{U}_\omega-z)^{-1}E_l\rangle. \tag{9.16}$$

For $a,b\in\mathbb{Z}\cup\{\pm\infty\}$, $a<b$, we denote by $G_\omega^{[a,b]}(z)$ the resolvent of the restriction of the operator $\mathbb{U}_\omega$ to the interval $[a,b]$ and by $G_\omega^{[a,b]}(z,\cdot,\cdot)$ its block Green kernel :

$$G_\omega^{[a,b]}(z)=\left(\mathbb{U}_\omega^{[a,b]}-z\right)^{-1}, \tag{9.17}$$

and for $k,l\in\mathbb{Z}$,

$$G_\omega^{[a,b]}(z,k,l)=\langle E_k,\left(\mathbb{U}_\omega^{[a,b]}-z\right)^{-1}E_l\rangle. \tag{9.18}$$

See [5, 11] for a precise definition of the boundary conditions used to define properly $\mathbb{U}_\omega^{[a,b]}$.

We study the fractional moments of $G_\omega^{[a,b]}(z,\cdot,\cdot)$ for $z\in\mathbb{C}\setminus\mathbb{S}^1$ and we first prove that they are uniformly bounded.

Theorem 9.1.5. *For every $s\in(0,\frac14)$ there exists $C(s)>0$ such that, for every $z\in\mathbb{C}\setminus\mathbb{S}^1$, every $a,b\in\mathbb{Z}\cup\{\pm\infty\}$, $a<b$, and every $k,l\in\mathbb{Z}$ such that $|k-l|>4$,*

$$\mathbb{E}\left(\|G_\omega^{[a,b]}(z,k,l)\|^s\right)\le C(s). \tag{9.19}$$

Once we have obtained this uniform bound, combining it with the positivity of the smallest Lyapunov exponent and proving estimates on the blocks of the products of transfer matrices, we prove the exponential decay of some blocks of the Green kernel restricted to suitable intervals.

Lemma 9.1.6. *Let $K \subset \mathbb{C}$ a compact set. There exist constants $C > O$, $c > 0$ and $s \in (0,1)$, such that:*

$$\mathbb{E}\left(\| \left(T_\omega^{(m)}(z) \dots T_\omega^{(n)}(z)\right)^{-1} v\|^s\right) \leq Ce^{-c|n-m|} \tag{9.20}$$

for all $z \in K$, for all unit vector v , and for $n, m \in \mathbb{Z}$ such that $|n-m|$ is sufficiently large.

Theorem 9.1.7. *There exist $r_0 > 0$, $\epsilon_0 > 0$, $s_0 \in (0,1)$, $p_0 > 1$, $C_{s_0,r_0} > 0$ and $\gamma > 0$ such that, for every $\alpha \in \mathrm{GL}_{\mathrm{L}}(\mathbb{C})$ with $\|\alpha\| \leq r_0$, every $s \in (0, s_0]$ and every $\epsilon \in (0, \epsilon_0]$,*

$$\mathbb{E}\left(\|G_\omega^{[2n,2m+1]}(z, 2n, 2m+1)\|^s\right) \leq Ce^{-\gamma|m-n|} \tag{9.21}$$

for every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and for every m and n in $\mathbb{Z}$ such that $|m-n| > p_0$.

Theorem 9.1.7 is central in the proof of Theorem 9.1.4. In order to be able to reduce our analysis to the blocks for which we have the exponential decay (9.21), we prove two reduction results. We start by showing that it suffices to deal with even-sized scattering zipper operators.

Proposition 9.1.8. *Assume $\alpha \in \mathrm{GL}_{\mathrm{L}}(\mathbb{C})$ is such that $\|\alpha\| < 1$. Let $s \in (0, \frac{1}{4})$, $\epsilon \in (0,1)$ and let $k, l \in \mathbb{Z}$ such that $|k-l| > 4$. There exists $C(s,\epsilon) > 0$ such that:*

$$\mathbb{E}\left(\|G_\omega^{[a,b]}(z,k,l)\|^s\right)^2 \leq C(s,\epsilon) \sum_{i,j=0}^{1} \mathbb{E}(\|G_\omega^{[a,b]}(z, 2n+2i, 2m+1+2j)\|^{4s})^{\frac{1}{2}}, \tag{9.22}$$

for all $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and n, m such that $k \in \{2n, 2n+1\}$ and $l \in \{2m, 2m+1\}$.

The proof involves bounding the norm of the "even" blocks $\|G_\omega^{[a,b]}(z, 2n, 2m)\|$ and $\|G_\omega^{[a,b]}(z, 2n+1, 2m)\|$ by the norms of the "odd" blocks $\|G_\omega^{[a,b]}(z, 2n, 2m+1)\|$ and $\|G_\omega^{[a,b]}(z, 2n+1, 2m+1)\|$. A necessary condition for establishing this is the invertibility of α. This invertibility is necessary to prove the following lemma.

Lemma 9.1.9. *If $\alpha \in \mathrm{GL}_{\mathrm{L}}(\mathbb{C})$, then for every $\epsilon > 0$, for every $n \in \mathbb{Z}$ and for every $z \in \mathbb{S}_\epsilon$, the matrix $\alpha + zV_\omega^{(n)}\alpha U_\omega^{(n)}$ is invertible almost surely, and for every $s \in (0,1)$, there exists $C(s) > 0$ such that*

$$\forall n \in \mathbb{Z},\ \mathbb{E}\left(\|(\alpha + zV_\omega^{(n)}\alpha U_\omega^{(n)})^{-1}\|^s\right) \leq C(s). \tag{9.23}$$

Once we get Lemma 9.1.9, using Hölder inequality and Theorem 9.1.5, one gets Proposition 9.1.8.

The second reduction result show that it suffices to control the Green kernel of the restricted operator to some finite interval in order to control the Green kernel of $\mathbb{U}_\omega$.

Proposition 9.1.10. *Let $s \in (0, \frac{1}{2})$ and $\epsilon > 0$. There exists $C(s) > 0$ such that*

$$\mathbb{E}\left(\|G_\omega(z,k,l)\|^s\right)^2 \leq C(s)\mathbb{E}\left(\|G_\omega^{[k,l]}(z,k,l)^{2s}\|\right) \tag{9.24}$$

for every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$ and every $k, l \in \mathbb{Z}$ such that $|k-l| > 4$.

The proof of Lemma 9.1.10 relies on the geometric resolvent identity.

Combining Proposition 9.1.8 and Proposition 9.1.10, as well as Theorem 9.1.7 on the exponential decay of the reduced case, we obtain the exponential decay of the fractional moments.

Theorem 9.1.11. *There exist $r_0 > 0$, $s \in (0,1)$, $\epsilon_0 > 0$, $C_{s,r_0} > 0$ and $\gamma > 0$, such that for every $\alpha \in \mathrm{GL}_{\mathrm{L}}(\mathbb{C})$ with $\|\alpha\| < r_0$,*

$$\mathbb{E}\left(\|G_\omega(z,k,l)\|^s\right) \leq C_{s,r_0}e^{-\gamma|k-l|} \tag{9.25}$$

for every $\epsilon \in (0, \epsilon_0]$, every $k, l \in \mathbb{Z}$ and every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$.

The proof of Theorem 9.1.4 follows from Theorem 9.1.11 and from the following estimate on the moments of order two of the coefficients of the resolvent.

Theorem 9.1.12. *There exist $\epsilon_0 > 0$, $r_0 > 0$, $C_{r_0} > 0$ and $\gamma > 0$ such that, for every $\epsilon \in (0, \epsilon_0]$ and every $z \in \mathbb{S}_\epsilon \setminus \mathbb{S}^1$, for every $\alpha \in GL_L(\mathbb{C})$ with $\|\alpha\| \leq r_0$, and every $\{k,p\}$ and $\{l,q\}$ in $\mathbb{Z} \times \{1,\dots,L\}$:*

$$\mathbb{E}\left((1-|z|^2) \left| \langle e_{\{k,p\}} | \left(\mathbb{U}_\omega - z\right)^{-1} e_{\{l,q\}} \rangle \right|^2 \right) \leq C_{r_0} e^{-\gamma(k-l)}$$

Theorem 9.1.12 is a consequence of Theorem 9.1.11 using second order perturbation theory.

Remark 9.1.13. *The parameter ϵ_0 is chosen to guarantee strictly positive Lyapunov exponents over the annulus $\mathbb{S}_{\epsilon_0}$.*

9.2 Quasi-1d models of Dirac type

Given an integer $N \geq 1$, the free Dirac operator on N parallel straight lines is

$$D_0^{(N)} := J\frac{\mathrm{d}}{\mathrm{d}x}, \text{ with } J := \begin{pmatrix} 0 & -\mathrm{I_N} \\ \mathrm{I_N} & 0 \end{pmatrix} \text{ and } \mathrm{Dom}(D_0^{(N)}) = H^1(\mathbb{R}) \otimes \mathbb{C}^{2N}. \tag{9.26}$$

This operator is self-adjoint.

We add to this free operator a random potential. Let $(\Omega, \mathcal{A}, \mathbb{P})$ a complete probability space and $\ell > 0$ be a disorder parameter: the smaller ℓ is, the "more disordered" the system is. The random potential $(V_\omega^{(n)})_{n\in\mathbb{Z}}$ is a sequence of independent and identically distributed (i.i.d. for short) random variables such that, for every $n \in \mathbb{Z}$, the function $x \mapsto V_\omega^{(n)}(x)$ takes values in the Hermitian matrices, is supported in $[0, \ell]$ and is uniformly bounded in x, n and ω.

We consider the random family $\{D_\omega\}_{\omega\in\Omega}$ of quasi-one-dimensional Dirac operators defined for every realization $\omega \in \Omega$ by:

$$D_\omega := D_0^{(N)} + \sum_{n\in\mathbb{Z}} V_\omega^{(n)}(\cdot - \ell n). \tag{9.27}$$

Under such conditions, for each $\omega \in \Omega$, the operator D_ω is self-adjoint on the Sobolev space $H^1(\mathbb{R}) \otimes \mathbb{C}^{2N}$ and thus, for every $\omega \in \Omega$, the spectrum of D_ω, denoted by $\sigma(D_\omega)$, is included in $\mathbb{R}$.

9.2.1 Transfer matrices and Lyapunov exponents

In order to determine the almost-sure spectrum of $\{D_\omega\}_{\omega\in\Omega}$ and to study the asymptotic behaviour of the corresponding generalized eigenfunctions, one considers, for every $\omega \in \Omega$, the equation for the generalized eigenvalues,

$$D_\omega u = Eu, \quad \text{where } E \in \mathbb{C} \text{ and } u = \begin{pmatrix} u^\uparrow \\ u_\downarrow \end{pmatrix} : \mathbb{R} \to \mathbb{C}^{2N}. \tag{9.28}$$

The notation $u = \begin{pmatrix} u^\uparrow \\ u_\downarrow \end{pmatrix}$ refers to the decomposition spin up / spin down of the solution of the Dirac equation.

Equation (9.28) is a linear differential system of order 1. We introduce, for $E \in \mathbb{C}$ and every $x, y \in \mathbb{R}$, the transfer matrix $T_x^y(E)$ of D_ω from x to y which maps a solution $(u^\uparrow, u_\downarrow)$ at position x to the same solution at position y. It is defined by the relation

$$\begin{pmatrix} u^\uparrow(y) \\ u_\downarrow(y) \end{pmatrix} = T_x^y(E) \begin{pmatrix} u^\uparrow(x) \\ u_\downarrow(x) \end{pmatrix} \tag{9.29}$$

and in particular, $T_x^x(E) = \mathrm{I}_{2\mathrm{N}}$ for every $x \in \mathbb{R}$. For real E, the transfer matrices are elements of the hermitian symplectic group

$$\mathrm{Sp}_{\mathrm{N}}^*(\mathbb{C}) = \{M \in \mathcal{M}_{2\mathrm{N}}(\mathbb{C}) \mid M^*JM = J\} \tag{9.30}$$

with $J = \begin{pmatrix} 0 & -\mathrm{I}_{\mathrm{N}} \\ \mathrm{I}_{\mathrm{N}} & 0 \end{pmatrix}$. Indeed, for $E \in \mathbb{R}$ and $x \in \mathbb{R}$ fixed, $y \mapsto T_x^y(E)$ satisfies $D_\omega(T_x^y(E)) = E\ T_x^y(E)$ on $\mathbb{R}$. It implies $\left(\frac{d}{dy}T_x^y(E)\right)^* JT_x^y(E) + (T_x^y(E))^*J\frac{d}{dy}T_x^y(E) = 0$. Hence, the function $y \mapsto (T_x^y(E))^*JT_x^y(E)$ is constant on $\mathbb{R}$. Taking the value at $y = x$ one obtains J and $(T_x^y(E))^*JT_x^y(E) = J$ for every $y \in \mathbb{R}$.

Remark 9.2.1. *Note that* $\mathrm{Sp}_{\mathrm{N}}^*(\mathbb{C})$ *is a real Lie group since it is a* C^∞ *manifold and not a holomorphic manifold because of the presence of a conjugation in its definition.*

For $E \in \mathbb{C}$ fixed and two couples (x, y) and (x', y') in $\mathbb{R}^2$, the random matrices $T_x^y(E)$ and $T_{x'}^{y'}(E)$ are not necessarily independent. In order to apply the results of the theory of products of i.i.d. random matrices, we also introduce, for every $n \in \mathbb{Z}$, the transfer matrices $T_\omega^{(n)}(E) = T_{\ell n}^{\ell(n+1)}(E)$ from ℓn to $\ell(n+1)$. The transfer matrix $T_\omega^{(n)}(E)$ is therefore defined by the relation

$$\begin{pmatrix} u^\uparrow(\ell(n+1)) \\ u_\downarrow(\ell(n+1)) \end{pmatrix} = T_\omega^{(n)}(E) \begin{pmatrix} u^\uparrow(\ell n) \\ u_\downarrow(\ell n) \end{pmatrix} \tag{9.31}$$

for all $n \in \mathbb{Z}$.

The sequence $(T_\omega^{(n)}(E))_{n\in\mathbb{Z}}$ is a sequence of i.i.d. matrices because of the i.i.d. character of the $V_\omega^{(n)}$'s and the disjointness of their supports for different values of n.

9.2.2 Localization criteria for quasi-one-dimensional operators of Dirac type

The formalism of transfer matrices, Lyapunov exponents and the Furstenberg group enables to state criteria of dynamical localization for quasi-one-dimensional operators of Dirac type.

Before that, we introduce several definitions in order to fix the framework in which we are able to obtain such criteria of dynamical localization. For this, we generalize the notion of L_p-strong irreducibility.

Let $L \geq 1$ an integer. For $l \in \{1, \dots, L\}$ we denote indifferently by b_l a bilinear form on $\mathbb{C}^{2N}$ or its matrix in the canonical basis of $\mathbb{C}^{2N}$. We also denote by b_0, or most simply by J, the symplectic bilinear form on $\mathbb{C}^{2N}$ associated with the matrix $J = \begin{pmatrix} 0 & -\mathrm{I}_{\mathrm{N}} \\ \mathrm{I}_{\mathrm{N}} & 0 \end{pmatrix}$.

For any $p \in \{1, \dots, N\}$, let $(J, b_1, \dots, b_L)$-L_p be the vector subspace of $\Lambda^p\mathbb{C}^{2N}$ whose elements are p-decomposable vectors $u_1 \wedge \cdots \wedge u_p$ such that

$$\forall i, j \in \{1, \dots, p\},\ \forall l \in \{0, \dots, L\},\ b_l(u_i, u_j) = 0$$

i.e, the family $(u_1, \dots, u_p)$ is orthogonal for all the bilinear forms b_l (and in particular each u_i is orthogonal to itself for all b_l).

The second property is called the $(J, b_1, \dots, b_L)$-L_p-strong irreducibility. It generalizes the notion of L_p-strong irreducibility as defined in [2] in the setting of the real symplectic group.

Definition 9.2.2. *We say that a subset* T *of* $\mathrm{GL}_{2\mathrm{N}}(\mathbb{C})$ *is* $(J, b_1, \dots, b_L)$-L_p-strongly irreducible *if there does not exist any* W*, finite union of proper vector subspaces of* $(J, b_1, \dots, b_L)$-L_p*, such that* $(\Lambda^p M)(W) = W$ *for all* M *in* T*.*

We now state two localization criteria for $\{D_\omega\}_{\omega\in\Omega}$. The first one states for the group $\mathrm{Sp}_{\mathrm{N}}^*(\mathbb{C})$.

Theorem 9.2.3. *We fix a compact interval* $I \subset \mathbb{R}$*. We assume that there exists an open interval* $\tilde{I}$ *containing* I *and such that for every* $E \in \tilde{I}$*:*

1. *the Furstenberg group G_{μ_E} is included in $\mathrm{Sp}^*_{\mathrm{N}}(\mathbb{C})$;*
2. *for every $p \in \{1, \ldots, N\}$, G_{μ_E} is p-contracting and J-L_p-strongly irreducible.*

Then $\{D_\omega\}_{\omega\in\Omega}$ exhibits dynamical localization in $\Sigma \cap I$.

In order to state a second theorem, we introduce the matrix

$$S := \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \in \mathcal{M}_{2\mathrm{N}}(\mathbb{R}), \tag{9.32}$$

where K is the diagonal matrix with $(-1)^{i+1}$ in position i. We define the group

$$\mathrm{SpO}_{\mathrm{N}}(\mathbb{R}) := \left\{ M \in \mathcal{M}_{2\mathrm{N}}(\mathbb{R}),\ {}^tMJM = J,\ {}^tMSM = S \right\}. \tag{9.33}$$

The second localization criterion states for the group $\mathrm{SpO}_{\mathrm{N}}(\mathbb{R})$.

Theorem 9.2.4. *Assume that N is even. We fix a compact interval $I \subset \mathbb{R}$. We assume that there exists an open interval $\tilde{I}$ containing I and such that for every $E \in \tilde{I}$:*

1. *the Furstenberg group G_{μ_E} is included in $\mathrm{SpO}_{\mathrm{N}}(\mathbb{R})$;*
2. *for every $2p \in \{1, \ldots, N\}$, G_{μ_E} is $2p$-contracting and (J,S)-L_{2p}-strongly irreducible.*

Then $\{D_\omega\}_{\omega\in\Omega}$ exhibits dynamical localization in $\Sigma \cap I$.

The proofs of Theorems 9.2.3 and 9.2.4 involve several steps:

1. The assumptions of the two theorems lead to an integral formula for the Lyapunov exponents which implies their Hölder regularity.
2. We then deduce the same Hölder regularity for the integrated density of states, using a Thouless formula.
3. From this regularity of the integrated density of states, we get a weak Wegner's estimate adapted to Bernoulli randomness.
4. Finally we apply a multiscale analysis scheme which involves the proof of an Initial Length Scale Estimate.

Actually, most of these steps except the proof of the Initial Length Scale Estimate in the last one are true with more general hypothesis. Let us state them now.

Assumption 9.2.5 ($\mathbf{L}^{(\mathbf{N})}$). *We fix a compact interval $I \subset \mathbb{R}$. We assume that there exists L_N a vector subspace of $\Lambda^N\mathbb{C}^{2N}$ and an open interval $\tilde{I}$ containing I such that, for all $E \in \tilde{I}$:*

($\mathbf{L}_1^{(\mathbf{N})}$) *for all $g \in G_{\mu_E}$, $(\Lambda^N g)(L_N) \subset L_N$;*

($\mathbf{L}_2^{(\mathbf{N})}$) *for all $x \neq 0$ in L_N,*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log \|\Lambda^N \Phi_E(n,\cdot)x\|) = \sum_{i=1}^{N} \gamma_i(E);$$

($\mathbf{L}_3^{(\mathbf{N})}$) *there exists a unique probability measure $\nu_{N,E}$ on $\mathsf{P}(L_N)$ which is μ_E-invariant, such that*

$$\gamma_1(E) + \cdots + \gamma_N(E) = \int_{G_{\mu_E}\times \mathsf{P}(L_N)} \log \frac{\|(\Lambda^N g)x\|}{\|x\|} \mathrm{d}\mu_E(g)\mathrm{d}\nu_{N,E}(\bar{x});$$

($\mathbf{L}_4^{(\mathbf{N})}$) $\gamma_1(E) + \cdots + \gamma_N(E) > 0$.

The properties of N-contractivity and J or (J, S)-L_N-strong irreducibility imply Assumption $(\mathbf{L^{(N)}})$. It is a consequence of [2, Proposition A.IV.3.4] in the case of $\mathrm{Sp}_{\mathrm{N}}^*(\mathbb{C})$ since one has the identification between $\mathrm{Sp}_{\mathrm{N}}^*(\mathbb{C})$ and $\mathrm{Sp}_{2\mathrm{N}}(\mathbb{R})$ using the following application which split the real and imaginery parts of the matrices in $\mathcal{M}_{2\mathrm{N}}(\mathbb{C})$:

$$\pi \; : \; \begin{array}{ccc} \mathcal{M}_{2\mathrm{N}}(\mathbb{C}) & \to & \mathcal{M}_{4\mathrm{N}}(\mathbb{R}) \\ A + iB & \mapsto & \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right). \end{array}$$

In the following Section, we will present some explicit cases of the model $\{D_\omega\}_{\omega\in\Omega}$ for which we are able to verify the assumptions of Theorem 9.2.3 or Theorem 9.2.4.

9.2.3 Application of the localization criteria to a class of splitting potentials

We introduce a particular case of quasi-one-dimensional operators of Dirac type whose potentials split in a sum of two tensorized Pauli matrices. Recall the usual notations for Pauli matrices :

$$\sigma_0 := \mathrm{I}_2,\ \sigma_1 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),\ \sigma_2 := \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right),\ \sigma_3 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Consider the particular family $\{D_{\omega,\ell}^{(N)}\}_{\omega\in\Omega}$ where the potential split into a periodic part and a random part :

$$D_{\omega,\ell}^{(N)} := D_0^{(N)} + V_{\mathrm{per}} + V_\omega. \tag{9.34}$$

The potential V_{per} is a ℓ-periodic function, linear combination of tensorized Pauli matrices of the form

$$V_{\mathrm{per}} := (\alpha_0\sigma_0 + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3) \otimes \hat{V}_{\mathrm{per}}, \tag{9.35}$$

where $\alpha_0, \dots, \alpha_3$ are real numbers and $\hat{V}_{\mathrm{per}}$ is a ℓ-periodic function with value in the space of the N-by-N real symmetric matrices denoted by $\mathrm{S_N}(\mathbb{R})$. Note that :

$$\sigma_0 \otimes V := \left(\begin{smallmatrix} V & 0 \\ 0 & V \end{smallmatrix}\right) \in \mathcal{M}_{2\mathrm{N}}(\mathbb{C})$$

and the same for the other tensor products.
We construct the random potential V_ω in the following way. Let $(\tilde{\Omega}_1, \tilde{\mathcal{A}}_1, \tilde{\mathbb{P}}_1), \dots, (\tilde{\Omega}_N, \tilde{\mathcal{A}}_N, \tilde{\mathbb{P}}_N)$ be N complete probability spaces. We take

$$(\Omega, \mathcal{A}, \mathbb{P}) = \left(\mathop{\times}_{n\in\mathbb{Z}} \tilde{\Omega}_1 \times \cdots \times \tilde{\Omega}_N, \bigotimes_{n\in\mathbb{Z}} \tilde{\mathcal{A}}_1 \otimes \cdots \otimes \tilde{\mathcal{A}}_N, \bigotimes_{n\in\mathbb{Z}} \tilde{\mathbb{P}}_1 \otimes \cdots \otimes \tilde{\mathbb{P}}_N \right).$$

For $i \in \{1, \dots, N\}$, the sequences $(\omega_i^{(n)})_{n\in\mathbb{Z}}$ are independent of each other and each one is a sequence of i.i.d. real-valued random variables on $(\widetilde{\Omega}_i, \widetilde{\mathcal{A}}_i, \widetilde{\mathbb{P}}_i)$. Let ν_i be the common law of the $\omega_i^{(n)}$'s. We assume that $\{0, 1\} \subset \mathrm{supp}\, \nu_i$ and $\mathrm{supp}\, \nu_i$ is bounded. In particular, the $\omega_i^{(n)}$'s can be Bernoulli random variables which is the most difficult case of randomness to deal with since it will imply the smallest possible Furstenberg group. We also set, for every $n \in \mathbb{Z}$, $\omega^{(n)} = (\omega_1^{(n)}, \dots, \omega_N^{(n)})$, which is a random variable on $(\widetilde{\Omega}_1 \times \cdots \times \widetilde{\Omega}_N, \widetilde{\mathcal{A}}_1 \otimes \cdots \otimes \widetilde{\mathcal{A}}_N, \widetilde{\mathbb{P}}_1 \otimes \cdots \otimes \widetilde{\mathbb{P}}_N)$ of law $\nu = \nu_1 \otimes \cdots \otimes \nu_N$. For each n in $\mathbb{Z}$, we introduce the random function

$$\hat{V}_{\omega^{(n)}} := \begin{pmatrix} \omega_1^{(n)} v_1(\cdot - \ell n) & & 0 \\ & \ddots & \\ 0 & & \omega_N^{(n)} v_N(\cdot - \ell n) \end{pmatrix},$$

where the v_i are measurable functions from $[0, \ell)$ to $\mathbb{R}$. We take

$$V_\omega = (\beta_0\sigma_0 + \beta_1\sigma_1 + \beta_2\sigma_2 + \beta_3\sigma_3) \otimes \sum_{n\in\mathbb{Z}} \hat{V}_{\omega^{(n)}} \tag{9.36}$$

where $\beta_0, \ldots, \beta_3$ are real numbers.

We will restrict ourselves to particular combinations of non-vanishing α_i's and β_j's. For simplicity we will only consider real-valued potentials V_{per} and V_ω, which corresponds to the absence of magnetic field and to the following assumption.

Assumption 9.2.6. *We assume that* $\alpha_2 = \beta_2 = 0$.

Hence we only consider potentials which are on σ_0, σ_1 and σ_3 which implies in particular that, for real energies E, the corresponding Furstenberg groups will be included in $\mathrm{Sp}_{\mathrm{N}}(\mathbb{R})$ instead of $\mathrm{Sp}^*_{\mathrm{N}}(\mathbb{C})$. Next, we consider only splitting potentials with one deterministic term and one random term which allows to reduce the number of cases in which we should compute the Furstenberg group from 43 to 9.

Assumption 9.2.7. *We assume that one and only one among* α_0*,* α_1 *and* α_3 *is different from zero and one and only one among* β_0*,* β_1 *and* β_3 *is different from zero.*

Since we have to choose one random potential and one deterministic one, there are *a priori* nine possibilities. Nevertheless, it is possible to reduce this number to five in the following way. If one sets for $(V_0, V_1, V_3) \in (\mathcal{M}_{\mathrm{N}}(\mathbb{R}))^3$,

$$D(V_0, V_1, V_3) = D_0^{(N)} + \sigma_0 \otimes V_0 + \sigma_1 \otimes V_1 + \sigma_3 \otimes V_3$$

acting on $H^1(\mathbb{R}) \otimes \mathbb{R}^2$, then

$$\forall (V_0, V_1, V_3) \in (\mathcal{M}_{\mathrm{N}}(\mathbb{R}))^3,\ D(V_0, V_1, V_3) = P(-D(-V_0, -V_3, -V_1))P^*$$

with P the unitary matrix defined by

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{I}_{\mathrm{N}} & \mathrm{I}_{\mathrm{N}} \\ \mathrm{I}_{\mathrm{N}} & -\mathrm{I}_{\mathrm{N}} \end{pmatrix}. \tag{9.37}$$

Hence, for any $(V_0, V_1, V_3) \in (\mathcal{M}_{\mathrm{N}}(\mathbb{R}))^3$, the operators $D(V_0, V_1, V_3)$ and $-D(-V_0, -V_3, -V_1)$ have the same spectrum and also the same pure point, absolutely continuous and singular continuous spectra. These two operators have transfer matrices which are unitarily equivalent (through P defined at (9.37)), hence their Lyapunov exponents are equal. They also have Furstenberg groups which are unitarily equivalent (again through P) and there is localization for $D(V_0, V_1, V_3)$ if and only if there is localization for $-D(-V_0, -V_3, -V_1)$. As a consequence, there are five cases, as explained in the following table.

$V_\omega \setminus V_{per}$	σ_0	σ_1	σ_3
σ_0	1	5	5
σ_1	3	2	4
σ_3	3	4	2

Table 9.1: The five possible cases

For each of these cases, we prove either localization or delocalization.

Moreover, in order to compute the Furstenberg group associated with $\{D_\omega\}_{\omega\in\Omega}$, we will express the transfer matrices as matrix exponentials. This is possible only when the potentials are constant on each interval $(n\ell, (n+1)\ell)$. Else, we would have to deal with time-ordered exponentials instead of matrix exponentials.

Assumption 9.2.8.

1. *We assume that* $\hat{\mathbf{V}}_{\rm per}$ ***is a constant function*** *equal to some fixed real symmetric matrix. We will denote by* $\hat{V}_{\rm per}$ *the unique value of the function* $\hat{V}_{\rm per}$ *as a small abuse of notation.*

2. *We also assume that the functions* $v_1, \dots, v_N$ *in the definition of* $\hat{V}_{\omega^{(n)}}$ *are all equal to the characteristic function of the interval* $[0, \ell]$ *:* $\mathbf{v_1} = \cdots = \mathbf{v_N} = \mathbf{1}_{[\mathbf{0},\ell]}$.

Notation. Let Δ the tridiagonal matrix with 0 on the diagonal and 1 on the upper and lower diagonal.
Notation. Let us denote by $\{\cdot,\cdot\}$ the anti-commutator of two matrices defined for every $A, B \in \mathcal{M}_{\rm N}(\mathbb{R})$ by $\{A, B\} = AB + BA$.

Theorem 9.2.9.

1. *In Case* 1*, for any symmetric matrix* $\hat{V}_{\rm per} \in {\rm S_N}(\mathbb{R})$ *,* $\Sigma = \mathbb{R}$ *and it is purely absolutely continuous of multiplicity* $2N$*.*

2. *In cases* 2*,* 3 *and* 4*, if* $\hat{\mathbf{V}}_{\rm per} = \mathbf{\Delta}$*, there exists* $\ell_{\rm C} := \ell_{\rm C}(N) > 0$ *such that, for every* $\ell \in (0, \ell_{\rm C})$*, there exists a compact interval* $I(N,\ell) \subset \mathbb{R}$ *such that if* $I \subset I(N,\ell)$ *is an open interval with* $\Sigma \cap I \neq \emptyset$*, then* $\{D^{(N)}_{\omega,\ell}\}_{\omega\in\Omega}$ *exhibits Anderson and dynamical localization in* $\Sigma \cap I$*.*

3. *In Case* 5 *:*

   (i) if $\{\hat{\mathbf{V}}_{\rm per}, \mathbf{K}\} = \mathbf{0}$ *and if* N *is odd, there is presence of a.c. spectrum of multiplicity at least* 2*.*

   (ii) In particular, if $\hat{\mathbf{V}}_{\rm per} = \mathbf{\Delta}$*,* $\{D^{(N)}_{\omega,\ell}\}_{\omega\in\Omega}$ *exhibits Anderson and dynamical localization in* $\Sigma \cap I$ *if* N *is even (for* I *as in Point* (2)*), and there is presence of a.c. spectrum of multiplicity exactly* 2 *if* N *is odd.*

   (iii) On the contrary, if $\hat{\mathbf{V}}_{\rm per} = \mathbf{\Delta} + {\rm I_N}$*, there exists* $\tilde{\ell}_{\rm C} > 0$ *such that, for every* $\ell \in (0, \tilde{\ell}_{\rm C})$*, there exists a compact interval* $\tilde{I}(N,\ell) \subset \mathbb{R}$ *such that if* $\tilde{I} \subset \tilde{I}(N,\ell)$ *is an open interval with* $\Sigma \cap \tilde{I} \neq \emptyset$*, then* $\{D^{(N)}_{\omega,\ell}\}_{\omega\in\Omega}$ *exhibits Anderson and dynamical localization in* $\Sigma \cap \tilde{I}$ *for any* N*.*

The localization results come from applications of our localization criteria in Theorems 9.2.3 and 9.2.4. The presence of a.c. spectrum in Case 1 and in Case 5 when N is odd is a consequence of the following theorem of Sadel and Schulz-Baldes.

Theorem 9.2.10. *Let* $\{D_\omega\}_{\omega\in\Omega}$ *be as in* (9.27)*. Then, for* $k \in \{1, \dots, N\}$*, the set*

$$S_k := \{E \in \mathbb{R},\ \text{exactly } 2k \text{ Lyapunov exponents vanish at } E\}$$

is an essential support of the almost-sure absolutely continuous spectrum of multiplicity $2k$*.*

This result of Kotani's theory for quasi-one-dimensional operators of Dirac type has to be seen as a *delocalization* result for $\{D_\omega\}_{\omega\in\Omega}$.

Finally, one gets from the results of Theorem 9.2.9, localization and delocalization results for generic choices of the value of the matrix $\hat{V}_{\rm per}$. To state these results we need to precise the notions of genericity. We identify the space ${\rm S_N}(\mathbb{R})$ of real symmetric matrices to $\mathbb{R}^{\frac{N(N+1)}{2}}$ and we consider the Lebesgue measure ${\rm Leb}_{\frac{N(N+1)}{2}}$ on it. The genericity on ${\rm S_N}(\mathbb{R})$ is defined according to ${\rm Leb}_{\frac{N(N+1)}{2}}$.

We also introduce the space of symmetric matrices which anticommute to K :

$$\mathrm{S}_{\mathrm{N}}^{\{K\}}(\mathbb{R}) = \{V \in \mathrm{S}_{\mathrm{N}}(\mathbb{R}) \mid \{V, K\} = 0\}. \tag{9.38}$$

This space is of dimension $\frac{N^2}{4}$ if N is even and of dimension $\frac{(N-1)(N+1)}{4}$ if N is odd. Hence, the genericity on $\mathrm{S}_{\mathrm{N}}^{\{K\}}(\mathbb{R})$ is defined according to $\mathrm{Leb}_{\frac{N^2}{4}}$ if N is even and according to $\mathrm{Leb}_{\frac{(N-1)(N+1)}{4}}$ if N is odd.

Corollary 9.2.11. *Denote by V the unique value of the function $\hat{V}_{\mathrm{per}}$.*

1. *For almost-every real symmetric matrix $V \in \mathrm{S}_{\mathrm{N}}(\mathbb{R})$, there exist a finite set $\mathcal{S}_V \subset \mathbb{R}$ and $\ell_C := \ell_C(N, V) > 0$ such that, for every $\ell \in (0, \ell_C)$, there exists a compact interval $I(N, V, \ell) \subset \mathbb{R}$ such that if $I \subset I(N, V, \ell) \setminus \mathcal{S}_V$ is an open interval with $\Sigma \cap I \neq \emptyset$, then, in cases 2, 3, 4 and 5, $\{D_{\omega,\ell}^{(N)}\}_{\omega\in\Omega}$ exhibits Anderson and dynamical localization in $\Sigma \cap I$.*

2. *For almost-every $\tilde{V} \in \mathrm{S}_{\mathrm{N}}^{\{K\}}(\mathbb{R})$, there exist a finite set $\mathcal{S}_{\tilde{V}} \subset \mathbb{R}$ and $\ell_C := \ell_C(N, \tilde{V}) > 0$ such that, for every $\ell \in (0, \ell_C)$, there exists a compact interval $I(N, \tilde{V}, \ell) \subset \mathbb{R}$ such that if $I \subset I(N, \tilde{V}, \ell) \setminus \mathcal{S}_{\tilde{V}}$ is an open interval with $\Sigma \cap I \neq \emptyset$, then, in Case 5, $\{D_{\omega,\ell}^{(N)}\}_{\omega\in\Omega}$ exhibits Anderson and dynamical localization in $\Sigma \cap I$ if N is even and there is presence of a.c. spectrum of multiplicity 2 if N is odd.*

Remark 9.2.12. *Both results of genericity in the Case 5 are consistent since $\mathrm{Leb}_{\frac{N(N+1)}{2}}(\mathrm{S}_{\mathrm{N}}^{\{K\}}(\mathbb{R})) = 0$ in $\mathrm{S}_{\mathrm{N}}(\mathbb{R})$. In Case 5, the generic result of localization in point (1) corresponds to Theorem 9.2.9 (3)(iii) and the generic result in point (2) corresponds to Theorem 9.2.9 (3)(ii).*

Appendix A

Self-adjoints operators and spectral theorem

$(\mathcal{H}, (\cdot|\cdot))$ denotes a Hilbert space on $\mathbb{R}$ or on $\mathbb{C}$. By convention, when $(\cdot|\cdot)$ is a Hermitian product, it will be semilinear on the right.

A.1 Bounded operators

We begin by defining the space of bounded operators between normed vector spaces and then define several topologies on this space.

Definition A.1.1. *If $(E, \| \|_E)$ and $(F, \| \|_F)$ are normed vector spaces, a bounded operator from E to F is a continuous linear mapping $T : E \to F$,* i.e, *such that*

$$\exists C > 0,\ \forall u \in E,\ \|Tu\|_F \leq C \|u\|_E .$$

Notation. We denote by $\mathcal{L}(E, F)$ the set of bounded operators from E to F. When $E = F$, we write $\mathcal{L}(E) = \mathcal{L}(E, E)$.

$\mathcal{L}(E, F)$ is a vector space on which we introduce the norm,

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{u \in E \setminus \{0\}} \frac{\|Tu\|_F}{\|u\|_E} = \sup_{\|u\|_E = 1} \|Tu\|_F .$$

The topology induced by this norm on $\mathcal{L}(E, F)$ is called the *uniform operator topology*. If $(F, \| \|_F)$ is a Banach space, then $(\mathcal{L}(E, F), \| \|_{\mathcal{L}(E,F)})$ is also a Banach space. Furthermore, the norm $\| \|_{\mathcal{L}(E,E)}$ is an algebra norm on $(\mathcal{L}(E), +, ., \circ)$ and, more generally, if $(E, \| \|_E)$, $(F, \| \|_F)$, and $(G, \| \|_G)$ are normed vector spaces and $T_1 \in \mathcal{L}(E, F)$ and $T_2 \in \mathcal{L}(F, G)$, then $T_2 \circ T_1 \in \mathcal{L}(E, G)$ and

$$\|T_2 \circ T_1\|_{\mathcal{L}(E,G)} \leq \|T_2\|_{\mathcal{L}(F,G)} \ \|T_1\|_{\mathcal{L}(E,F)} .$$

Notation. Throughout, we denote $T_2 T_1$ as the composition $T_2 \circ T_1$ of two operators $T_1 \in \mathcal{L}(E, F)$ and $T_2 \in \mathcal{L}(F, G)$.

We now introduce two weaker topologies on $\mathcal{L}(E, F)$. Firstly, the *strong operator topology*. It is the smallest topology making the maps $\mathrm{ev}_u : \mathcal{L}(E, F) \to F$, $\mathrm{ev}_u(T) = Tu$ continuous. For this topology, a sequence of bounded operators $(T_n)_{n \in \mathbb{N}}$ converges to a bounded operator T if and only if, for every $u \in E$, $\|T_n u - Tu\|_F \xrightarrow[n \to +\infty]{} 0$. We write $T_n \to T$ in this case.

The second topology is the *weak operator topology*. We could define it for arbitrary Banach spaces E and F, but in what follows we restrict ourselves to bounded operators on $(\mathcal{H}, (\cdot|\cdot))$, so we

shall assume $E = F = \mathcal{H}$. Then the weak topology on $\mathcal{L}(\mathcal{H})$ is the smallest topology making the maps ev$u, v : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$, ev$u, v(T) = (Tu|v)$ continuous. For this topology, a sequence of bounded operators $(T_n)_{n\in\mathbb{N}}$ converges to a bounded operator T if and only if, for all $u, v \in \mathcal{H}$, $(T_n u|v) \xrightarrow[n\to+\infty]{} (Tu|v)$ in $\mathbb{C}$. We then write $T_n \rightharpoonup T$.
The weak operator topology is weaker than the strong operator topology, which in turn is weaker than the uniform operator topology. The following examples illustrate the differences between these topologies on $\mathcal{L}(\ell^2(\mathbb{N}))$.
The following examples illustrate the differences between these topologies on $\mathcal{L}(\ell^2(\mathbb{N}))$.

Example A.1.2. *Let $T_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $T_n(x_0, x_1, \ldots) = (\frac{1}{n}x_0, \frac{1}{n}x_1, \ldots)$. Then $(T_n)_{n\in\mathbb{N}}$ uniformly converges to* 0*, the zero operator.*

Example A.1.3. *Let $S_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, $S_n(x_0, x_1, \ldots) = (0, \ldots, 0, x_n, x_{n+1}, \ldots)$. Then $(S_n)_{n\in\mathbb{N}}$ strongly converges to* 0 *but not uniformly.*

Indeed, for any $x \in \ell^2(\mathbb{N})$,

$$||S_n x||_{\ell^2}^2 = \sum_{k=n}^{+\infty} |x_k|^2 \xrightarrow[n\to+\infty]{} 0.$$

Then, for any $x \in \ell^2(\mathbb{N})$, $||S_n x||_{\ell^2} \leq ||x||_{\ell^2}$ hence $||S_n||_{\mathcal{L}(\ell^2(\mathbb{N}))} \leq 1$. Furthermore, for all $n \in \mathbb{N}$, $||S_n e_n||_{\ell^2} = 1$ where e_n is the sequence being 0 *for all $k \neq n$ and* 1 *at the n-th term. Therefore, for all n, $||S_n||_{\mathcal{L}(\ell^2(\mathbb{N}))} = 1$, and (S_n) does not uniformly converge to* 0*.*

Example A.1.4. *Let $W_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by $W_n(x_0, x_1, \ldots) = (0, \ldots, 0, x_0, x_1, \ldots)$ with n zeros at the beginning of the sequence. Then $(W_n)_{n\in\mathbb{N}}$ converges weakly to* 0*, but neither strongly nor uniformly.*

Throughout, we will often consider bounded operators between Hilbert spaces. In this Hilbert framework, we provide a characterization of the operator norm.

Proposition A.1.5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. Then,*

$$\|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} = \sup\{|(Tu|v)_{\mathcal{H}_2}| \mid \|u\|_{\mathcal{H}_1} \leq 1 \text{ and } \|v\|_{\mathcal{H}_2} \leq 1\}.$$

Proof : Let S be the right-hand side of the equality. By the Cauchy-Schwarz inequality,

$$|(Tu|v)| \leq \|Tu\|_{\mathcal{H}_2} \|v\|_{\mathcal{H}_2} \leq \|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \|u\|_{\mathcal{H}_1} \|v\|_{\mathcal{H}_2} \leq \|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$$

when $\|u\|_{\mathcal{H}_1} \leq 1$ and $\|v\|_{\mathcal{H}_2} \leq 1$. Hence, $S \leq \|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$. Conversely, let M be a positive real number; suppose $S \leq M$. Then, for any $u \in \mathcal{H}_1$, $\|Tu\|_{\mathcal{H}_2} \leq M\|u\|_{\mathcal{H}_1}$. Indeed, if $u = 0$ or $Tu = 0$, the inequality holds. Otherwise, $u' = u/\|u\|_{\mathcal{H}_1}$ and $v' = Tu/\|Tu\|_{\mathcal{H}_2}$ have norm 1, and as $S \leq M$, $|(Tu'|v')| \leq M$. Now, $|(Tu'|v')| = \|Tu\|_{\mathcal{H}_2}/\|u\|_{\mathcal{H}_1}$, hence $\|Tu\|_{\mathcal{H}_2} \leq M\|u\|_{\mathcal{H}_1}$. By the definition of $\|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)}$, we get $\|T\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leq S$.

□

A.2 Adjoint of a Bounded Operator

We will now define the adjoint of a bounded operator, which generalizes to any dimension the transpose of a real matrix or the conjugate transpose of a complex matrix.

Proposition A.2.1. *Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. There exists a unique operator $T^* \in \mathcal{L}(\mathcal{H})$ such that*

$$\forall u \in \mathcal{H},\ \forall v \in \mathcal{H},\ (Tu|v) = (u|T^*v). \tag{A.1}$$

Proof: Let $v \in \mathcal{H}$. Then, $\ell_v : u \mapsto (Tu|v)$ is a continuous linear form on $\mathcal{H}$. By the Riesz representation theorem for continuous linear forms, there exists a unique vector $w \in \mathcal{H}$ such that, for all $u \in \mathcal{H}$, $\ell_v(u) = (u|w)$. Let $T^* : \mathcal{H} \to \mathcal{H}$, $T^*v = w$.

T^* is linear. Indeed, for $v_1, v_2 \in \mathcal{H}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, let $v = \lambda_1 v_1 + \lambda_2 v_2$ and $w_1 = T^*(v_1)$, $w_2 = T^*(v_2)$, $T^*(v) = w$. Then,

$$\begin{aligned}\forall u \in \mathcal{H},\ (u|w) &= (Tu|v) = (Tu|\lambda_1 v_1 + \lambda_2 v_2) = \overline{\lambda}_1 (Tu|v_1) + \overline{\lambda}_2 (Tu|v_2) \\ &= \overline{\lambda}_1 (u|w_1) + \overline{\lambda}_2 (u|w_2) = (u|\lambda_1 w_1 + \lambda_2 w_2).\end{aligned}$$

So, $w - \lambda_1 w_1 - \lambda_2 w_2 \in \mathcal{H}^\perp = \{0\}$ and $w = \lambda_1 w_1 + \lambda_2 w_2$, proving linearity of T^*.

T^* is bounded. Indeed, for $u, v \in \mathcal{H}$, $\|u\|_{\mathcal{H}} \leq 1$ and $\|v\|_{\mathcal{H}} \leq 1$, then

$$|(u|T^*v)| = |(Tu|v)| \leq \|Tu\|_{\mathcal{H}} \, \|v\|_{\mathcal{H}} \leq \|T\|_{\mathcal{L}(\mathcal{H})} .$$

Thus, taking $u = \frac{T^*v}{\|T^*v\|_{\mathcal{H}}}$, for all $v \in \mathcal{H}$, $\|v\|_{\mathcal{H}} \leq 1$ and $T^*v \neq 0$, $\|T^*v\|_{\mathcal{H}} \leq \|T\|_{\mathcal{L}(\mathcal{H})}$. If $v \in \mathcal{H}$ is such that $T^*v = 0$, the inequality is still valid. This gives $\|T^*\|_{\mathcal{L}(\mathcal{H})} \leq \|T\|_{\mathcal{L}(\mathcal{H})}$ and T^* is bounded.

Finally, for uniqueness, if T_1^* and T_2^* satisfy (A.1), then for all $u, v \in \mathcal{H}$, $(u|(T_1^* - T_2^*)v) = 0$, thus $T_1^* - T_2^* = 0$.

□

Definition A.2.2 (Adjoint). *The bounded operator $T^* \in \mathcal{L}(\mathcal{H})$ is called the adjoint of the operator T.*

Example A.2.3. *For any Hilbert space $\mathcal{H}$, $\mathrm{Id}_{\mathcal{H}}^* = \mathrm{Id}_{\mathcal{H}}$.*

We state the first properties verified by the adjoint of a bounded operator.

Proposition A.2.4 (Algebraic Properties of the Adjoint). *Let $T, T_1, T_2 \in \mathcal{L}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. Then,*

1. $(T_1 + T_2)^* = T_1^* + T_2^*$;
2. $(\lambda T)^* = \overline{\lambda} T^*$;
3. $(T_1 T_2)^* = T_2^* T_1^*$;
4. $(T^*)^* = T$;
5. *if T has a bounded inverse T^{-1}, T^* also has a bounded inverse, and $(T^*)^{-1} = (T^{-1})^*$.*

Proof: The first two points come from the right semilinearity of the inner product. For the third point, for all $u, v \in \mathcal{H}$, write $(T_1 T_2 u|v) = (T_2 u|T_1^* v) = (u|T_2^* T_1^* v)$. The fourth point is obtained by noticing that in (A.1), vectors u and v play the same role, and $(Tu|v) = (u|T^*v)$ for all $u, v \in \mathcal{H}$ if and only if $(T^*u|v) = (u|Tv)$ for all $u, v \in \mathcal{H}$ by taking conjugates. Finally, for the last point, from $TT^{-1} = I = T^{-1}T$, we deduce by taking the adjoint that $T^*(T^{-1})^* = I^* = I = I^* = (T^{-1})^* T^*$.

□

Proposition A.2.5 (Metric Properties of the Adjoint)**.** *Let $T \in \mathcal{L}(\mathcal{H})$. Then,*

1. $\|T^*\|_{\mathcal{L}(\mathcal{H})} = \|T\|_{\mathcal{L}(\mathcal{H})}$*;*
2. $\|T^*T\|_{\mathcal{L}(\mathcal{H})} = \|T\|^2_{\mathcal{L}(\mathcal{H})}$.

Proof : From the proof of proposition A.2.1, $\|T^*\|_{\mathcal{L}(\mathcal{H})} \leq \|T\|_{\mathcal{L}(\mathcal{H})}$. Then, applying this inequality to the bounded operator T^* and using the fact that $(T^*)^* = T$, we obtain $\|T\|_{\mathcal{L}(\mathcal{H})} \leq \|T^*\|_{\mathcal{L}(\mathcal{H})}$, proving the first point. For the second point, we initially have $\|T^*T\|_{\mathcal{L}(\mathcal{H})} \leq \|T^*\|_{\mathcal{L}(\mathcal{H})}\|T\|_{\mathcal{L}(\mathcal{H})} = \|T\|^2_{\mathcal{L}(\mathcal{H})}$. Conversely, if $u \in \mathcal{H}$, $\|u\|_{\mathcal{H}} = 1$,

$$\|Tu\|^2_{\mathcal{H}} = (Tu|Tu) = (T^*Tu|u) \leq \|T^*T\|_{\mathcal{L}(\mathcal{H})},$$

hence $\|T\|^2_{\mathcal{L}(\mathcal{H})} \leq \|T^*T\|_{\mathcal{L}(\mathcal{H})}$. □

Proposition A.2.6 (Geometric Properties of the Adjoint)**.** *Let $T \in \mathcal{L}(\mathcal{H})$. Then,*

1. $\operatorname{Ker} T^* = (\operatorname{Im} T)^\perp$ *and* $(\operatorname{Ker} T^*)^\perp = \overline{\operatorname{Im} T}$*;*
2. *if $F \subset \mathcal{H}$ is a subspace stable under T, then $F^\perp$ is stable under T^*.*

Proof : u belongs to $(\operatorname{Im} T)^\perp$ if and only if, for all $v \in \mathcal{H}$, $(u|Tv) = 0$, which is equivalent to saying that, for all $v \in \mathcal{H}$, $(T^*u|v) = 0$. This is equivalent to $T^*u = 0$, or equivalently $u \in \operatorname{Ker} T^*$. The second property arises from properties of orthogonal spaces in Hilbert spaces.

For the second point, let $v \in F^\perp$ and $u \in F$. Then $Tu \in F$, so $(T^*v|u) = (v|Tu) = 0$. Therefore, $T^*v \in F^\perp$. □

Definition A.2.7. *An operator $T \in \mathcal{L}(\mathcal{H})$ is called self-adjoint when $T = T^*$.*

Self-adjoint operators are a generalization of symmetric matrices to infinite dimensions. They play a major role in functional analysis and mathematical physics. A structural theorem for these operators asserts that every self-adjoint operator is diagonalizable, in a sense to be specified in infinite dimensions. A primary example of a self-adjoint operator is that of an orthogonal projector.

Definition A.2.8. *An operator $P \in \mathcal{L}(\mathcal{H})$ is called a projector when $P^2 = P$. Moreover, if $P^* = P$, P is called an orthogonal projector.*

It is noted that the image of a projector is a closed subspace on which P acts as the identity. Furthermore, if P is orthogonal, P acts as the null operator on $(\operatorname{Im} T)^\perp$. Then, the projection theorem for closed subspaces in Hilbert spaces assures that there is a bijection between orthogonal projectors in a Hilbert space $\mathcal{H}$ and closed subspaces of $\mathcal{H}$.

Example A.2.9. ***(Multiplication Operator).*** *Let (X, μ) be a σ-finite measure space and let $\mathcal{H} = L^2(\mu)$. If $\varphi \in L^\infty(\mu)$, we define the multiplication operator by φ, $M_\varphi : L^2(\mu) \to L^2(\mu)$ such that, for all $u \in \mathcal{H}$, $M_\varphi u = \varphi u$.*

Then, M_φ is in $\mathcal{L}(L^2(\mu))$ and $\|M_\varphi\| = \|\varphi\|_\infty$. Here, $\|\varphi\|_\infty$ denotes the essential supremum, $\|\varphi\|_\infty = \inf\{c > 0 \mid \mu(\{x \in X \mid |\varphi(x)| > c\}) = 0\}$. Therefore, by changing the representative within the class of φ, we can assume that φ is a bounded function.

Moreover, since $\|\varphi u\|_2 \leq \|\varphi\|_\infty \|u\|_2$, M_φ is a bounded operator and $\|M_\varphi\| \leq \|\varphi\|_\infty$. For any $\varepsilon > 0$, as μ is σ-finite, there exists a measurable set A with $0 < \mu(A) < +\infty$ such that $|\varphi(x)| \geq \|\varphi\|_\infty - \varepsilon$ for all $x \in A$. Taking $u = \mu(A)^{-\frac{1}{2}}\mathbf{1}_A$, then $u \in L^2(\mu)$ and $\|u\|_2 = 1$. Thus, $\|M_\varphi\|^2 \geq \|\varphi u\|_2^2 = \mu(A)^{-1}\int_A |\varphi|^2 \mathrm{d}\mu \geq (\|\varphi\|_\infty - \varepsilon)^2$. As ε tends to 0, it holds that $\|M_\varphi\| \geq \|\varphi\|_\infty$.

It is observed that, for any $\varphi \in L^\infty(\mu)$, $M_\varphi^* = M_{\overline{\varphi}}$, where $\overline{\varphi}(x) = \overline{\varphi(x)}$ for all x in X. In particular, if φ takes real values, $M_\varphi^* = M_\varphi$ and M_φ is self-adjoint.

For bounded self-adjoint operators, proposition A.1.5 can be refined.

Proposition A.2.10. *Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Then, for all $u \in \mathcal{H}$, $(Tu|u) \in \mathbb{R}$, and*

$$\|T\|_{\mathcal{L}(\mathcal{H})} = \sup\{|(Tu|u)| \mid \|u\|_{\mathcal{H}} = 1\}.$$

Proof : Let S be the right-hand side of the equality. According to Proposition A.1.5, $S \leq \|T\|_{\mathcal{L}(\mathcal{H})}$. To prove the other inequality, let's begin by showing that, for all $u \in \mathcal{H}$, $(Tu|u) \in \mathbb{R}$. Indeed, as $T = T^*$, $(Tu|u) = (u|Tu) = \overline{(Tu|u)}$ and $(Tu|u) \in \mathbb{R}$. Then, using the polarization identity, we have

$$\forall u, v \in \mathcal{H}, \text{ Re } (Tu|v) = \frac{1}{4}\left((T(u+v)|u+v) - (T(u-v)|u-v)\right).$$

Now, for all $u \in \mathcal{H}$, $|(Tu|u)| \leq S\|u\|^2$. Therefore, for all $u, v \in \mathcal{H}$,

$$|\text{Re } (Tu|v)| \leq \frac{S}{4}\left(\|u+v\|^2 + \|u-v\|^2\right).$$

Then, by the parallelogram identity, for all $u, v \in \mathcal{H}$, $|\text{Re } (Tu|v)| \leq \frac{S}{2}\left(\|u\|^2 + \|v\|^2\right)$. Hence, if we assume $\|u\| \leq 1$ and $\|v\| \leq 1$, we obtain $|\text{Re } (Tu|v)| \leq S$. By replacing v with $e^{-i\theta}v$, where $e^{i\theta}(Tu|v) = |(Tu|v)|$, we then get that, for all $u, v \in \mathcal{H}$, $|(Tu|v)| = (Tu|e^{-i\theta}v) = |\text{Re } (Tu|e^{-i\theta}v)| \leq S$. Therefore, by Proposition A.1.5, $\|T\|_{\mathcal{L}(\mathcal{H})} \leq S$, which concludes the proof.

□

We conclude this section with a result that paves the way for the formalism of unbounded operators.

Theorem A.2.11 (Hellinger-Toeplitz)**.** *Let $T : \mathcal{H} \to \mathcal{H}$ be an operator such that, for all $u, v \in \mathcal{H}$, $(u|Tv) = (Tu|v)$. Then $T \in \mathcal{L}(\mathcal{H})$.*

Proof : By the closed graph theorem, it suffices to demonstrate that $\Gamma(T)$, the graph of T, is closed. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{H}$ converging to $u \in \mathcal{H}$, such that $(Tu_n)_{n\in\mathbb{N}}$ converges to $v \in \mathcal{H}$. We only need to show that $v = Tu$. For any $w \in \mathcal{H}$,

$$(w|v) = \lim_{n\to\infty}(w|Tu_n) = \lim_{n\to\infty}(Tw|u_n) = (Tw|u) = (w|Tu),$$

thus $v = Tu$.

□

This result asserts that there cannot be an unbounded operator defined on the entire space $\mathcal{H}$ that is self-adjoint (or symmetric in general). This poses a problem in quantum mechanics where one wishes to define operators like energy (involving a derivative) that are unbounded while being symmetric in the sense of $(u|Tv) = (Tu|v)$.

A.3 Unbounded operators on a Hilbert space

A.3.1 Unbounded operators

To bypass the Hellinger-Toeplitz result, we define unbounded operators on subspaces of $\mathcal{H}$. Such a subspace, called the domain of the operator T, is assumed to be dense in $\mathcal{H}$.

Definition A.3.1. *An unbounded operator T on a Hilbert space $\mathcal{H}$ is a linear map $T : D(T) \to \mathcal{H}$ where $D(T)$ is a dense vector subspace of $\mathcal{H}$, called the domain of T.*

Example A.3.2. *We return to the multiplication operator. Let (X, μ) be a σ-finite measured space and let $\mathcal{H} = L^2(\mu)$. We now assume that φ is only measurable and nonzero almost everywhere, and we define the multiplication operator by φ via*

$$D(M_\varphi) = \{u \in L^2(\mu) \mid u\varphi \in L^2(\mu)\} \subset L^2(\mu)$$

and $M_\varphi : D(M_\varphi) \to L^2(\mu)$ such that, for all $u \in \mathcal{H}$, $M_\varphi u = \varphi u$.
Then $D(M_\varphi)$ is dense in $L^2(\mu)$. Likewise, the image of M_φ, $M_\varphi(D(M_\varphi))$, is also dense in $L^2(\mu)$.

Example A.3.3. *We wish to study directly the following Sturm-Liouville problem in terms of operators:*

$$\begin{cases} -u'' + qu &= \lambda u, \quad u \in C^2([0,1]) \\ u(0) = u(1) &= 0 \end{cases} \tag{A.2}$$

where $q : [0,1] \to \mathbb{R}$ is a continuous function and $\lambda \in \mathbb{C}$. To this end, we set $T : u \mapsto -u'' + qu$, which is linear but not defined on the whole of $L^2([0,1])$. We then introduce

$$D(T) = \{u \in C^2([0,1]) \mid u(0) = u(1) = 0\}$$

which is dense in $L^2([0,1])$, and we view T as an operator on $L^2([0,1])$ with domain $D(T)$.

Definition A.3.4. *The graph of an unbounded operator $(T, D(T))$ is the vector subspace of $\mathcal{H} \times \mathcal{H}$*

$$\Gamma(T) = \{(u, Tu) \mid u \in D(T)\}.$$

We can then define a class of unbounded operators whose properties are not too far from those of bounded operators, namely closed operators.

Definition A.3.5. *An unbounded operator $(T, D(T))$ is said to be closed if its graph $\Gamma(T)$ is closed in $\mathcal{H} \times \mathcal{H}$.*

Remark A.3.6. *In other words, $(T, D(T))$ is closed if for every sequence $(u_n)_{n\in\mathbb{N}}$ in $D(T)$ that converges to $u \in \mathcal{H}$ and such that (Tu_n) converges to $v \in \mathcal{H}$, then $u \in D(T)$ and $Tu = v$.*

Remark A.3.7. *By the closed graph theorem, if $D(T) = \mathcal{H}$, then $(T, D(T))$ is closed if and only if it is bounded.*

The Sturm-Liouville operator presented above is not closed. Indeed, one can show that the limit u in $L^2([0,1])$ of a sequence in $D(T)$ is not necessarily C^2. The domain $D(T)$ in this example is too small for the operator to be closed. By considering instead the domain

$$\{u \in L^2([0,1]) \mid u'' \in L^2([0,1]),\ u(0) = u(1) = 0\}$$

where the derivative u'' is taken in the sense of distributions (and where Sobolev embedding is used to make sense of the boundary values), one then defines a closed operator. This leads us to the following definition.

Definition A.3.8. *Let $(S, D(S))$ and $(T, D(T))$ be two unbounded operators. We say that T extends S (or is an extension of S), and we write $S \subset T$, when $D(S) \subset D(T)$ and $T|_{D(S)} = S$.*
An unbounded operator is said to be closable if it admits a closed extension.

Remark A.3.9. *An operator T is closable if and only if for every sequence (u_n) in $D(T)$ such that $u_n \to 0$ and $Tu_n \to v \in \mathcal{H}$, one has $v = 0$.*

Proposition A.3.10.

1. *$(T, D(T))$ is closable if and only if $\overline{\Gamma(T)} \cap (\{0\} \times \mathcal{H}) = \{(0,0)\}$.*
2. *If $(T, D(T))$ is closable, then it admits a smallest closed extension whose graph is $\overline{\Gamma(T)}$.*

3. *This smallest closed extension is called the closure of $(T, D(T))$ and is denoted $\overline{T}$. One then has: $\Gamma(\overline{T}) = \overline{\Gamma(T)}$.*

Proof : If T is closable, there exists a closed operator S such that $T \subset S$, hence $\Gamma(T) \subset \Gamma(S)$ with $\Gamma(S)$ closed. Therefore, $\overline{\Gamma(T)} \subset \Gamma(S)$. Then,

$$\overline{\Gamma(T)} \cap (\{0\} \times \mathcal{H}) \subset \Gamma(S) \cap (\{0\} \times \mathcal{H}) = \{(u, Su) \,|u = 0\} = \{(0,0)\}$$

since S is linear.

Conversely, if $\overline{\Gamma(T)} \cap (\{0\} \times \mathcal{H}) = \{(0,0)\}$, then $p_1 : \overline{\Gamma(T)} \to \mathcal{H},\ (u,v) \mapsto u$ is injective since its kernel is $\{(0,0)\}$. Hence it is bijective onto its image. Denote by $D(\overline{T})$ its image and define, for $u \in D(\overline{T})$, $(u, \overline{T}u) = p_1^{-1}(u)$. Then $D(\overline{T})$ is dense because $\overline{\Gamma(T)}$ is closed and p_1 is linear and continuous, and $\Gamma(\overline{T}) = \overline{\Gamma(T)}$ is closed, *i.e.* T is closable.

Finally, if S is a closed extension of T, then $\Gamma(T) \subset \Gamma(S)$ and since $\Gamma(S)$ is closed, $\overline{\Gamma(T)} \subset \Gamma(S)$, *i.e.* $\overline{T} \subset S$.

□

Definition A.3.11. *Let T be a closed operator. A set $D \subset D(T)$ is called a core for T if $\overline{T|_D} = T$.*

Example A.3.12. *Let Ω be an open subset of $\mathbb{R}^d$ and $(a_\alpha)_{0 \le |\alpha| \le m}$ a family of functions in $C^\infty(\Omega)$. We work in $\mathcal{H} = L^2(\Omega)$ and set*

$$D(T) = C_0^\infty(\Omega) \ \text{ and } \ \forall u \in D(T),\ Tu = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha u.$$

Let (u_n) be a sequence in $\mathcal{H}$ converging to 0 and such that (Tu_n) converges to v in $\mathcal{H}$. We want to show that $v = 0$. To do so, we use the Dubois-Reymond lemma. Let $\varphi \in D(T)$. Since convergence in L^2 implies convergence in $\mathcal{D}'(\Omega)$,

$$\int_\Omega v\varphi = \lim_{n \to +\infty} \int_\Omega Tu_n\, \varphi.$$

But

$$\int_\Omega Tu_n\, \varphi = \sum_{|\alpha| \le m} \int_\Omega a_\alpha \partial^\alpha u_n \varphi$$

and since φ has compact support,

$$\int_\Omega a_\alpha \partial^\alpha u_n \varphi = \int_{\mathbb{R}^d} \partial^\alpha u_n a_\alpha \varphi = \int_{\mathbb{R}^d} u_n (-1)^{|\alpha|} \partial^\alpha (a_\alpha \varphi)$$

by integration by parts. Since (u_n) tends to 0 in $L^2(\Omega)$, the resulting sequence of integrals also tends to 0 (either in $\mathcal{D}'$ or by Cauchy-Schwarz). Finally, $\int_\Omega v\varphi = 0$ and thus $v = 0$. Hence T is closable.

Example A.3.13. *For $\mathcal{H} = L^2(\mathbb{R})$ and $D(T) = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, if $f \in L^2(\mathbb{R})$ is nonzero, we define*

$$\forall u \in D(T),\ Tu = \left(\int_{-\infty}^{+\infty} u(x) \mathrm{d}x \right) f.$$

Then $(T, D(T))$ is not closable.
Indeed, let $\rho \in D(T)$ with integral equal to 1 and set, for $n \ge 1$, $u_n = \frac{1}{n}\rho(\frac{\cdot}{n})$. Then, for all n, u_n has integral equal to 1 and $Tu_n = f$. But we also have $||u_n||^2 \to 0$ whereas $(u_n, Tu_n) \to (0, f) \ne (0,0)$.

A.3.2 Adjoint of an unbounded operator

Let $(T, D(T))$ be an unbounded operator. As in the bounded case, we would like to define an operator $(T^*, D(T^*))$ such that : $\forall u \in D(T),\ \forall v \in D(T^*),\ (Tu|v) = (u|T^*v)$.
The first thing to do is to properly define $D(T^*)$.

Definition A.3.14. *Let $(T, D(T))$ be an unbounded operator. We define*

$$D(T^*) = \{v \in \mathcal{H},\ |\ \exists C > 0,\ \forall u \in D(T),\ |(Tu|v)| \leq C||u||\}.$$

Then, for $v \in D(T^*)$ we define $T^*v = w$ where $w \in \mathcal{H}$ is the unique vector such that

$$\forall u \in D(T),\ (Tu|v) = (u|w).$$

Indeed, w exists by the Riesz representation theorem for continuous linear forms on a Hilbert space, which applies here since, $D(T)$ being dense, any continuous linear form on $D(T)$ can be extended to a continuous linear form on the whole space $\mathcal{H}$.

Lemma A.3.15. *With the notations of Definition A.3.14 and if $R : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H},\ (u, v) \mapsto (-v, u)$ (rotation by angle $\frac{\pi}{2}$), then*

$$\Gamma(T^*) = \{(v, T^*v)\ |v \in D(T^*)\} = (R(\Gamma(T)))^{\perp}.$$

Proof : Let $(v, w) \in \mathcal{H} \times \mathcal{H}$. Then,

$$(v, w) \in (R(\Gamma(T)))^{\perp} \Leftrightarrow \forall u \in D(T),\ ((v, w)|R((u, Tu))) = 0.$$

Equivalently, $((v, w)|(-Tu, u)) = 0 \Leftrightarrow -(v|Tu) + (w|u) = 0$. Hence,

$$(v, w) \in (R(\Gamma(T)))^{\perp} \Leftrightarrow \forall u \in D(T),\ (u|w) = (Tu|v).$$

□

Lemma A.3.16. *With the notations of Definition A.3.14, $D(T^*)$ is dense in $\mathcal{H}$ if and only if T is closable.*

Proof : We use the fact that T is closable if and only if $\overline{\Gamma(T)} \cap (\{0\} \times \mathcal{H}) = \{(0, 0)\}$. First, by Lemma A.3.15 and since $R^2 = -Id_{\mathcal{H}}$, we have

$$\Gamma(T^*)^{\perp} = ((R(\Gamma(T)))^{\perp})^{\perp} = \overline{R(\Gamma(T))} = R(\overline{\Gamma(T)}).$$

Thus, $(0, u) \in \overline{\Gamma(T)}$ if and only if $(-u, 0) \in \Gamma(T^*)^{\perp}$, *i.e.* $0 = ((-u, 0)|(v, T^*v)) = -(u|v)$ for all $v \in D(T^*)$. Hence, $(0, u) \in \overline{\Gamma(T)}$ if and only if $u \in D(T^*)^{\perp}$. Therefore, T is closable if and only if $D(T^*)^{\perp} = \{0\}$, which is equivalent to $D(T^*)$ being dense in $\mathcal{H}$.

□

Remark A.3.17. *By Lemma A.3.15, we obtain that T^* is always closed. Indeed, its graph is the orthogonal of a subspace, hence it is closed.*

Proposition A.3.18. *If T is closable, then $(T^*)^* = \overline{T}$.*

Proof : We again use Lemma A.3.15 and the fact that R, being an isometry, commutes with $\perp$. Thus,

$$\Gamma(T^{**}) = R^2(\Gamma(T)^{\perp\perp}) = -\overline{\Gamma(T)} = \overline{\Gamma(T)}.$$

□

Proposition A.3.19. *Let T be a bijective operator. Then T^* is also bijective and $(T^*)^{-1} = (T^{-1})^*$.*

Definition A.3.20. *An operator T is said to be symmetric when*

$$\forall u, v \in D(T),\ (Tu|v) = (u|Tv).$$

An unbounded operator T is said to be ***self-adjoint*** *when $T = T^*$.*

Proposition A.3.21. *Let T be a self-adjoint operator. Then T is closed and symmetric.*

Remark A.3.22. *Warning: the converse is only true for bounded operators. Indeed, one can show that if we consider $(A_0, D(A_0))$ defined by $D(A_0) = C_0^\infty(]0,1[)$ and $A_0 u = iu''$, then A_0 is symmetric, as well as its closure $\overline{A_0}$, which is moreover closed. However, one can show that $\overline{A_0}$ is not self-adjoint since $D(\overline{A_0}) = H_0^1(]0,1[)$ whereas $D((\overline{A_0})^*) = H^1(]0,1[)$.*

Remark A.3.23. *Since T is symmetric if and only if $T \subset T^*$, every symmetric operator is closable.*

Example A.3.24.
1. *As in the bounded case, a multiplication operator is self-adjoint if and only if the function φ defining it is real-valued.*
2. *The free Schrödinger operator, defined by $D(H_0) = H^2(\mathbb{R}^d)$ (Sobolev space) and for all $\psi \in H^2(\mathbb{R}^d)$, $H_0\psi = -\Delta\psi$, is self-adjoint.*

   This second point reduces to the first via the Fourier transform, which is unitary.

Since a closed and symmetric operator is not necessarily self-adjoint, one can give a characterization result for self-adjoint operators among symmetric operators. This is important since the main results, namely the spectral theorem and Stone's theorem, are only valid for self-adjoint operators.

Theorem A.3.25. *Let T be a symmetric operator on $\mathcal{H}$. The following conditions are equivalent.*

1. *T is self-adjoint.*
2. *T is closed and* $\mathrm{Ker}\,(T^* - i) = \mathrm{Ker}\,(T^* + i) = \{0\}$.
3. *There exists $\lambda \in \mathbb{C}$ such that* $\mathrm{Im}\,(T - \lambda) = \mathrm{Im}\,(T - \overline{\lambda}) = \mathcal{H}$.

Proof : See [22], Theorem VIII.3, page 256.

□

In many cases, the operators under consideration will be symmetric but not necessarily closed, and therefore not self-adjoint. Since a symmetric operator is closable, one would then like at least its closure to be self-adjoint. This leads to the following definition.

Definition A.3.26. *A symmetric operator T is said to be essentially self-adjoint if its closure $\overline{T}$ is a self-adjoint operator.*

Proposition A.3.27. *If T is essentially self-adjoint, then it admits a unique self-adjoint extension, T^{**}.*

Proof : Indeed, let S be such an extension of T. Then S is closed and since $T \subset S$, we have $T^{**} \subset S$. Hence, $S = S^* \subset (T^{**})^* = T^{**}$ and therefore $S = T^{**}$.

□

Remark A.3.28. *In general, a symmetric operator may admit several self-adjoint extensions, and even infinitely many. Indeed, if $D(T)$ is the set of absolutely continuous functions on $[0,1]$ vanishing at 0 and 1, and if T is given by $i\frac{d}{dx}$ on $D(T)$, then for any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, the operator $(T_\alpha, D(T_\alpha))$, defined by the same formula as T but on the set of absolutely continuous functions φ on $[0,1]$ such that $\varphi(0) = \alpha\varphi(1)$, is a self-adjoint extension of T.*

Finally, one can state the analogue of Theorem A.3.25 for essentially self-adjoint operators.

Corollary A.3.29. *Let T be a symmetric operator on $\mathcal{H}$. The following conditions are equivalent.*

1. *T is essentially self-adjoint.*
2. $\mathrm{Ker}\,(T^* - i) = \mathrm{Ker}\,(T^* + i) = \{0\}$.
3. $\mathrm{Im}\,(T \pm i)$ *are dense in $\mathcal{H}$.*

These criteria for self-adjointness make it possible to prove an essential result concerning the self-adjoint or essentially self-adjoint character of the sum of two self-adjoint operators, namely the Kato-Rellich theorem.

Let $(T, D(T))$ and $(S, D(S))$ be two self-adjoint operators. We set $D(T+S) = D(T) \cap D(S)$ and define $T+S$ on this space. It is clear that if T and S are bounded, then $T+S$ is bounded and self-adjoint. In the case where one of the two is unbounded, an additional criterion is needed. In particular, this makes it possible to obtain conditions for self-adjointness of an operator of the form $-\Delta + V$ (see [18], p57-62.)

Definition A.3.30. *The operator S is said to be T-bounded if $D(T) \subset D(S)$ and if there exist strictly positive constants a and b such that*

$$\forall u \in D(T),\ ||Su|| \leq a||Tu|| + b||u||.$$

The infimum of the real numbers a for which such a b exists is called the relative bound of S with respect to T.

Theorem A.3.31 (Kato-Rellich). *Let T be self-adjoint and let S be symmetric and relatively bounded with respect to T with relative bound $a < 1$. Then $T+S$ is self-adjoint on $D(T)$ and essentially self-adjoint on any core for T.*

Proof : According to Theorem A.3.25, it suffices to prove that for a suitably chosen $\lambda \in \mathbb{R}$, $\mathrm{Im}\,(T + S \pm \mathrm{i}\lambda) = \mathcal{H}$. Since T is self-adjoint, its spectrum is contained in $\mathbb{R}$, so we can consider the operator $S(T \pm \mathrm{i}\lambda)^{-1}$ from $\mathcal{H}$ into $\mathcal{H}$. Moreover, since T is self-adjoint, for all $u \in D(T)$,

$$||(T \pm \mathrm{i}\lambda)u||^2 = ||Tu||^2 + \lambda^2||u||^2.$$

Applying this equality to $u = (T \pm \mathrm{i}\lambda)^{-1}v$ for arbitrary $v \in \mathcal{H}$, we obtain

$$||v||^2 = ||T(T \pm \mathrm{i}\lambda)^{-1}v||^2 + \lambda^2||(T \pm \mathrm{i}\lambda)^{-1}v||^2 \geq ||T(T \pm \mathrm{i}\lambda)^{-1}v||^2$$

as well as

$$\lambda^2||(T \pm \mathrm{i}\lambda)^{-1}v||^2 \leq ||v||^2,$$

which implies that $||(T \pm \mathrm{i}\lambda)^{-1}v|| \leq \frac{1}{|\lambda|}||v||$.

Moreover, since S is T-bounded with relative bound $a < 1$, for any $a < a' < 1$, there exists $b' > 0$ such that, for all $u \in \mathcal{H}$,

$$||S(T \pm \mathrm{i}\lambda)^{-1}u|| \leq a'||T(T \pm \mathrm{i}\lambda)^{-1}u|| + b'||(T \pm \mathrm{i}\lambda)^{-1}u|| \leq (a' + \frac{b'}{|\lambda|})||u||.$$

One can then choose λ such that $a' + \frac{b'}{|\lambda|} < 1$. This implies, by the Neumann series lemma, that $\mathrm{Id} + S(T \pm \mathrm{i}\lambda)^{-1}$ is invertible and that its inverse is given by

$$(\mathrm{Id} + S(T \pm \mathrm{i}\lambda)^{-1})^{-1} = \sum_{n \geq 0} (-1)^n (S(T \pm \mathrm{i}\lambda)^{-1})^n,$$

and is bounded. Since $T \pm i\lambda$ is invertible from $D(T)$ into $\mathcal{H}$ with bounded inverse, it follows that the same is true for $T + S \pm i\lambda$, whose bounded inverse is given by

$$(T + S \pm i\lambda)^{-1} = (T \pm i\lambda)^{-1}(\mathrm{Id} + S(T \pm i\lambda)^{-1})^{-1}.$$

In particular, $\mathrm{Im}\,(T + S \pm i\lambda) = \mathcal{H}$, and $T + S$ is self-adjoint.

Let D be a core for T. Since $\mathrm{Id} + S|_D(T \pm i\lambda)^{-1}$ extends to a bounded operator and since this operator is invertible for sufficiently large λ, it follows from

$$(T + S)_D \pm i\lambda = (\mathrm{Id} + S|_D(T \pm i\lambda)^{-1})(T|_D \pm i\lambda)^{-1},$$

by passing to the inverse, that this inverse has a bounded closure and in particular that $\overline{\mathrm{Im}\,((T + S)_D \pm i\lambda)} = \mathcal{H}$. Thus, $T + S$ is essentially self-adjoint on D.

□

A.4 Spectrum of a closed operator

Let $(T, D(T))$ be an unbounded operator. We begin by defining the resolvent set of T as

$$\rho(T) = \{\lambda \in \mathbb{C},\ T - \lambda \text{ is bijective from } D(T) \text{ into } \mathcal{H} \text{ and } (T - \lambda)^{-1} \text{ is bounded}\}.$$

Definition A.4.1. *The spectrum of an operator T is the complement of its resolvent set:*

$$\sigma(T) = \mathbb{C} \setminus \rho(T).$$

In infinite dimension, the spectrum of an operator is not composed solely of its eigenvalues. The **point spectrum** is defined as the set of eigenvalues of T and is denoted by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \exists u \neq 0,\ Tu = \lambda u\}.$$

We then have

$$\sigma_p(T) \subset \sigma(T).$$

When T is moreover closed, by the bijection theorem (a corollary of the closed graph theorem), if $T - \lambda$ is bijective, then its inverse is automatically continuous. Thus, for T closed,

$$\rho(T) = \{\lambda \in \mathbb{C},\ T - \lambda \text{ is bijective from } D(T) \text{ into } \mathcal{H}\}.$$

We then introduce the resolvent of T:

Definition A.4.2. *Let T be a closed operator and let $\lambda \in \rho(T)$. The mapping*

$$R_\lambda(T)\ :\ \begin{array}{ccc} \mathcal{H} & \to & D(T) \\ v & \mapsto & (T - \lambda)^{-1} v \end{array}$$

is called the resolvent of T at the point λ.

For closed operators, the spectrum has good properties, as does the resolvent.

Proposition A.4.3. *Let T be a closed operator. Then*

1. *$\sigma(T)$ is closed.*

2. *The mapping* $\begin{array}{ccc} \rho(T) & \to & \mathcal{L}(\mathcal{H}) \\ \lambda & \mapsto & (T - \lambda)^{-1} \end{array}$ *is holomorphic on the open set $\rho(T)$.*

3. *If T^* denotes the adjoint of T, then $\sigma(T^*) = \overline{\sigma(T)}$ and $\rho(T^*) = \overline{\rho(T)}$.*

4. *If T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$.*

Proof : **1.** Let $\lambda_0 \in \rho(T)$. Then $R_{\lambda_0}(T)$ is linear and

$$\Gamma(R_{\lambda_0}(T)) = \{(u, (T-\lambda_0)^{-1}u) \mid u \in \mathcal{H}\} = \{(u,v) \mid (v,u) \in \Gamma(T-\lambda_0)\}.$$

Now, $\Gamma(T-\lambda_0)$ is closed since $\Gamma(T)$ is closed because T is closed, hence $\Gamma(R_{\lambda_0}(T))$ is closed. Since $R_{\lambda_0}(T)$ is defined on the whole space $\mathcal{H}$, by the closed graph theorem we have $R_{\lambda_0}(T) \in \mathcal{L}(\mathcal{H})$. Moreover,

$$T - \lambda = (T-\lambda_0) - (\lambda - \lambda_0) = (Id - (\lambda-\lambda_0)R_{\lambda_0}(T))(T-\lambda_0).$$

Since $T-\lambda_0$ is bijective from $D(T)$ onto $\mathcal{H}$, $T-\lambda$ is bijective if and only if $Id - (\lambda - \lambda_0)R_{\lambda_0}(T) : \mathcal{H} \to \mathcal{H}$ is bijective. As $R_{\lambda_0}(T)$ is bounded, if $|\lambda-\lambda_0|||R_{\lambda_0}(T)||_{\mathcal{L}(\mathcal{H})} < 1$ then $\lambda \in \rho(T)$ (Neumann series lemma). Hence $\rho(T)$ is open and $\sigma(T)$ is closed.

2. Moreover, if $|\lambda-\lambda_0|||R_{\lambda_0}(T)||_{\mathcal{L}(\mathcal{H})} < 1$, then

$$(Id - (\lambda-\lambda_0)R_{\lambda_0}(T))^{-1} = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n (R_{\lambda_0}(T))^n$$

and

$$(T-\lambda)^{-1} = (T-\lambda_0)^{-1}(Id - (\lambda-\lambda_0)R_{\lambda_0}(T))^{-1} = \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n (R_{\lambda_0}(T))^{n+1},$$

which proves the second point.

3. We prove the equality of the resolvent sets by double inclusion. Assume $\lambda \in \rho(T)$. Then $T-\lambda$ is bijective and closed. Hence $(T-\lambda)^*$ is injective (since $\mathrm{Ker}\,((T-\lambda)^*) = (\mathrm{Im}\,(T-\lambda))^{\perp} = \mathcal{H}^{\perp} = \{0\}$) and therefore bijective. Indeed, it is surjective since $T-\lambda$ is injective and closed. Moreover, $((T-\lambda)^*)^{-1} = ((T-\lambda)^{-1})^*$. Since $(T-\lambda)^{-1}$ is bounded, so is $((T-\lambda)^{-1})^*$. Furthermore $(T-\lambda)^* = T^* - \overline{\lambda}$, hence $\overline{\lambda} \in \rho(T^*)$, which yields the inclusion $\rho(T) \subset \overline{\rho(T^*)}$. Then

$$\overline{\rho(T^*)} \subset \overline{\overline{\rho((T^*)^*)}} = \rho(\overline{T}) = \rho(T),$$

which gives the reverse inclusion, and by taking complements, the desired equality of spectra.

4. Let λ and μ be two real numbers. Then, by a direct computation, for all $u \in \mathcal{H}$,

$$\|(T-(\lambda+\mathrm{i}\mu))u\|^2 = \|(T-\lambda)u\|^2 + \mu^2\|u\|^2.$$

Thus, for all $u \in \mathcal{H}$, $\|(T-(\lambda+\mathrm{i}\mu))u\|^2 \geq \mu^2\|u\|^2$. If $\mu \neq 0$, $T-(\lambda+\mathrm{i}\mu)$ is injective.

Assume that $\overline{\mathrm{Im}\,(T-(\lambda+\mathrm{i}\mu))} \neq \mathcal{H}$. Then $\mathrm{Ker}\,(T^* - \overline{(\lambda+\mathrm{i}\mu)})^{\perp} = \overline{\mathrm{Im}\,(T-(\lambda+\mathrm{i}\mu))} \neq \mathcal{H}$ and $\mathrm{Ker}\,(T^* - \overline{(\lambda+\mathrm{i}\mu)}) = \overline{\mathrm{Im}\,(T-(\lambda+\mathrm{i}\mu))}^{\perp} \neq \{0\}$, so that $\lambda - \mathrm{i}\mu \in \sigma_p(T^*) = \sigma_p(T)$ since $T = T^*$.

But this is impossible since, for all $u \in \mathcal{H}$, $\|(T-(\lambda-\mathrm{i}\mu))u\|^2 \geq \mu^2\|u\|^2$.

Hence $\overline{\mathrm{Im}\,(T-(\lambda+\mathrm{i}\mu))} = \mathcal{H}$ and since T is closed, we deduce that $T-(\lambda+\mathrm{i}\mu)$ is bijective and has a bounded inverse on its image, which is closed. Therefore, if $\mu \neq 0$, $\lambda + \mathrm{i}\mu \in \rho(T)$, which proves that $\sigma(T) \subset \mathbb{R}$ by contraposition.

□

Proposition A.4.4. *Let T be a closed operator. The family $\{R_\lambda(T) \mid \lambda \in \rho(T)\}$ is commutative and satisfies:*

$$\forall \lambda, \mu \in \rho(T),\ R_\lambda(T) - R_\mu(T) = (\lambda - \mu) R_\lambda(T) R_\mu(T).$$

Moreover, for all $\lambda \in \mathbb{C}$, $(R_\lambda(T))^* = R_{\overline{\lambda}}(T)$.

Proof : We write

$$R_\lambda(T) - R_\mu(T) = R_\lambda(T)(T-\mu)R_\mu(T) - R_\lambda(T)(T-\lambda)R_\mu(T) = (\lambda-\mu)R_\lambda(T)R_\mu(T).$$

Then, by exchanging λ and μ, we see that $R_\mu(T)$ and $R_\lambda(T)$ commute.

□

This relation is called the first resolvent identity.

Theorem A.4.5 (Weyl's criterion)**.** *Let T be a self-adjoint operator. Then $\lambda \in \sigma(T)$ if and only if there exists a family $\{u_n\}_{n\in\mathbb{N}}$ of elements of $D(T)$ such that $||u_n|| = 1$ and $||(T-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0$.*

Proof : Let us prove the converse implication. Suppose that there exists a family $\{u_n\}_{n\in\mathbb{N}}$ of elements of $D(T)$ such that $||u_n|| = 1$ and $||(T-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0$. Assume, by contradiction, that $\lambda \in \rho(T)$. Then

$$\forall n \in \mathbb{N},\ 1 = ||u_n|| = ||(T-\lambda)^{-1}(T-\lambda)u_n|| \leq |||(T-\lambda)^{-1}||| \cdot ||(T-\lambda)u_n|| \xrightarrow[n\to+\infty]{} 0.$$

This yields a contradiction, and thus $\lambda \in \sigma(T)$.

For the other implication, we will need the spectral theorem proved later, which is why we must assume that T is self-adjoint.

□

A.5 On compact (self-adjoints) operators

Recall that an operator $T : E \to F$ is said to be compact when $\overline{T(B_E)}$ is compact in F, where B_E is the unit ball of E. If T is compact, T is bounded.
We have the following results for the spectrum of compact operators.

Theorem A.5.1 (Riesz-Schauder)**.** *Let E be a Banach space, and $T \in \mathcal{B}_\infty(E)$. Then, $\sigma(T) \setminus \{0\}$ is a discrete set in $\mathbb{C}$ consisting of finite multiplicity eigenvalues of T. Additionally, if E is of infinite dimension, $0 \in \sigma(T)$.*

Note that when $0 \in \sigma(T)$, 0 might not be an eigenvalue of T. Moreover, 0 could be an accumulation point of $\sigma(T)$.

Theorem A.5.2 (Spectral Theorem for Self-adjoint Compact Operators)**.** *Let $T \in \mathcal{L}(H)$ be a self-adjoint compact operator. Denote the non-zero eigenvalues of T by $\{\lambda_1, \lambda_2, \dots\}$ and P_n as the projection of H onto* Ker $(T - \lambda_n)$*. Then, for $n \neq m$, $P_n P_m = P_m P_n = 0$, and P_n is of finite rank. Moreover, if T has finite rank, the set of eigenvalues of T is finite, and if T does not have finite rank, $\lambda_n \to 0$ as n tends to infinity. Finally,*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n$$

where the series converges in the operator norm (or is a finite sum in the case where T has finite rank).

Corollary A.5.3. *Let $T \in \mathcal{L}(H)$ be a compact self-adjoint operator. There exists a Hilbert basis $(\phi_n)_{n\in\mathbb{N}}$ of H such that, for all $n \in \mathbb{N}$, there exists a real number λ_n such that $T\phi_n = \lambda_n \phi_n$ and $\lambda_n \to 0$ as n tends to infinity.*

A.6 The spectral theorem for self-adjoints operators

We will generalize the classical result asserting that any real symmetric matrix is diagonalizable in an orthonormal basis to the framework of self-adjoints operators on a Hilbert space.
A good way to state this theorem for matrices is to write that for any real symmetric matrix $A \in M_n(\mathbb{R})$, there exist real numbers $\lambda_1, \ldots, \lambda_n$ and orthogonal projectors $P_1, \ldots, P_n$ such that:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

It is this formulation that we will generalize to infinite dimension by transforming the sum into an integral against measures with projector values.

A.6.1 Spectral Families

Definition A.6.1. *A spectral family (or identity resolution) on $\mathcal{H}$ is a function $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ such that:*

1. *For all $t \in \mathbb{R}$, $E(t)$ is an orthogonal projection, i.e., $E(t)^2 = E(t)$ and $E(t)^* = E(t)$.*
2. *Monotonicity: $\forall s \le t$, $E(s) \le E(t)$, i.e., $\forall u \in \mathcal{H}, (E(s)u|u) \le (E(t)u|u)$.*
3. *Right-continuous: $\forall u \in \mathcal{H},\ E(t+\varepsilon)u \xrightarrow[\varepsilon \to 0^+]{} E(t)u$.*
4. *Normalization at infinity: $\forall u \in \mathcal{H},\ E(t)u \xrightarrow[t \to -\infty]{} 0$ and $E(t)u \xrightarrow[t \to +\infty]{} u$.*

In particular, points 1 and 2 imply that $E(t)E(s) = E(s)E(t)$ for all s, t and if $s \le t$, $E(s)E(t) = E(s)$.
Also: $\forall u \in \mathcal{H}, \forall t \in \mathbb{R},\ (E(t)u|u) = ||E(t)u||^2 \ge 0$ (or with 2 and letting s tend to $-\infty$ for a fixed t).

Remark A.6.2. *The concept of a spectral family is analogous to the cumulative distribution function of a random variable in probabilities.*

Example A.6.3. *Let $M \subset \mathbb{R}^d$ be measurable and $g : M \to \mathbb{R}$ be measurable. We define $M(t) = \{x \in M \mid g(x) \le t\}$. Then $M(t)$ increases towards M in terms of inclusion. We then define for $u \in L^2(M)$ and $t \in \mathbb{R}$, $E(t)u = \chi_{M(t)}u$. Then, $E : t \mapsto E(t)$ is a spectral family.*

Example A.6.4. *If T is a self-adjoint operator, with a discrete spectrum and such that for all $u \in \mathcal{H}$, $(Tu|u) \ge C||u||^2$, then there exists a sequence λ_i of real numbers increasing towards infinity and an orthonormal basis $\{u_i\}_{i \in \mathbb{N}}$ of $\mathcal{H}$ such that*

$$\forall u \in \mathcal{H},\ Tu = \sum_{i=0}^{+\infty} \lambda_i (u|u_i) u_i.$$

This resembles the spectral theorem for self-adjoint compact operators. We then define for all $t \in \mathbb{R}$, $E(t)$ as the orthogonal projector onto $Vect\{u_0, \ldots u_j | \lambda_j \le t\}$. Then $t \mapsto E(t)$ is a spectral family.

A.6.2 Spectral Theorem

Let $u, v \in \mathcal{H}$. By the polarization identity, the function $F_{u,v}(\lambda) = (E(\lambda)u|v)$ is a complex linear combination of four right-continuous, non-decreasing functions at every point:

$$F_{u,v}(\lambda) = \frac{1}{4}\left(\|E(\lambda)(u+v)\|^2 - \|E(\lambda)(u-v)\|^2 + \mathrm{i}\,\|E(\lambda)(u+\mathrm{i}v)\|^2 - \mathrm{i}\,\|E(\lambda)(u-\mathrm{i}v)\|^2\right),$$

and we note this expression as $F_{u,v}(\lambda) = \alpha_1 F_1(\lambda) + \cdots + \alpha_4 F_4(\lambda)$. According to the Stieljes integration theory, there exist four Borel measures $\mu_1, \ldots, \mu_4$ corresponding to F_i such that for any function ϕ in $\mathcal{L}^1(\mathbb{R}, \mu_1 + \cdots + \mu_4)$,

$$\int_{\mathbb{R}} \phi(\lambda) \mathrm{d}F_{u,v}(\lambda) = \alpha_1 \int_{\mathbb{R}} \phi(\lambda) \mathrm{d}\mu_1 + \cdots + \alpha_4 \int_{\mathbb{R}} \phi(\lambda) \mathrm{d}\mu_4.$$

The measures μ_i depend on u and v, and, by the normalization property of spectral families, each μ_i is a finite measure. Indeed, we have $\mu_1(\mathbb{R}) \leq \|u+v\|^2, \ldots, \mu_4(\mathbb{R}) \leq \|u - \mathrm{i}v\|^2$.

Example A.6.5. *Let's revisit the second example from the previous section. If $u \in \mathcal{H}$, then $F_{u,u}(\lambda) = (E(\lambda)u|u)$. If $u = u_0$, then for $\lambda < \lambda_0$, $F_{u_0,u_0}(\lambda) = 0$ and for $\lambda \geq \lambda_0$, $F_{u_0,u_0}(\lambda) = ||u_0||^2 = 1$. Therefore, $\mathrm{d}F_{u_0,u_0} = \delta_{\lambda_0}$. If $u = au_0 + bu_1$, then $\mathrm{d}F_{u,u} = |a|^2\delta_{\lambda_0} + |b|^2\delta_{\lambda_1}$. More generally, if $u = \sum_{i=0}^{+\infty} a_i u_i$ with $\sum |a_i|^2 < +\infty$, then $\mathrm{d}F_{u,u} = \sum_{i=0}^{+\infty} |a_i|^2 \delta_{\lambda_i}$.*

We can now state the spectral theorem for self-adjoint operators.

Theorem A.6.6 (Spectral Theorem for Bounded Operators)**.** *Let T be a self-adjoint operator. There exists a unique spectral family $E : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ such that*

$$T = \int_{\mathbb{R}} \lambda \, \mathrm{d}E(\lambda) = \int_{\sigma(T)} \lambda \, \mathrm{d}E(\lambda)$$

where, for all $u, v \in \mathcal{H}$,

$$(Tu|v) = \int_{\sigma(T)} \lambda \, \mathrm{d}F_{u,v}(\lambda).$$

Proof : We outline the main steps of the construction.

We start by defining for $z \in \mathbb{C}$, $\operatorname{Im} z \neq 0$, and $u \in \mathcal{H}$, $F(z) = (R_z(T)u|u)$. Then F is holomorphic in the upper complex half-plane, and we verify that $\operatorname{Im} F(z) > 0$. It is therefore a Herglotz function that satisfies the inequality

$$|F(z)| \leq \frac{C}{|\operatorname{Im} z|}.$$

We can thus associate it with a positive Borel measure of finite mass, with a distribution function w_u such that

$$F(z) = \int_{-\infty}^{+\infty} \frac{1}{z - \lambda} dw_u(\lambda).$$

Through polarization, we then obtain a complex Borel measure $\mathrm{d}w_{u,v}$ that similarly represents $(R_z(T)u|v)$ for all $u, v \in \mathcal{H}$. Moreover, by harmonic analysis results,

$$w_{u,v}(\lambda) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{-\infty}^{\lambda+\delta} ((R_{s-i\varepsilon}(T) - R_{s+i\varepsilon})u, v) \mathrm{d}s.$$

Now, the map $(u, v) \mapsto w_{u,v}(\lambda)$ is a continuous sesquilinear form, so for any $\lambda \in \mathbb{R}$, there exists a unique operator $E(\lambda) \in \mathcal{L}(\mathcal{H})$ such that

$$w_{u,v}(\lambda) = (E(\lambda)u|v).$$

We then demonstrate that $\lambda \mapsto E(\lambda)$ is a spectral family that satisfies the desired representation formula for T.

□

A.6.3 Functional Calculus

If $\phi : \mathbb{R} \to \mathbb{C}$ is a locally bounded Borel map on $\mathbb{R}$ and T is a self-adjoint operator, we can define the operator $\phi(T)$ as follows:

$$\forall u, v \in \mathcal{H}, \ (\phi(T)u|v) = \int_{\sigma(T)} \phi(\lambda) \mathrm{d}F_{u,v}(\lambda),$$

where $F_{u,v}$ comes from the spectral family associated with T via the spectral theorem. This allows the development of a functional calculus on self-adjoint operators.
Note that if ϕ takes real values, then $\phi(T)$ is also self-adjoint. Then we have the following property.

Proposition A.6.7. *Let f and g be two bounded Borel functions and T a self-adjoint operator. For all $u, v \in \mathcal{H}$,*

$$(f(T)u|g(T)v) = \int_{\mathbb{R}} f(\lambda)\overline{g(\lambda)} \mathrm{d}F_{u,v}(\lambda),$$

where $F_{u,v}(\lambda) = (E(\lambda)u|v)$ with E being the spectral family associated with T.

Proof : This is demonstrated by considering f and g as indicator functions of Borel sets, then by linear combinations of such functions (step functions), and eventually passing to the limit.

□

An initial application of functional calculus is the following formula for the resolvent of a self-adjoint operator.

Proposition A.6.8. *Let T be a self-adjoint operator. Let $z \in \mathbb{C}$, $z \notin \sigma(T)$. Then*

$$R_z(T) = (z - T)^{-1} = \int_{\mathbb{R}} \frac{1}{z - \lambda} \mathrm{d}E(\lambda)$$

where E is the spectral family associated with T. Furthermore,

$$||(z - T)^{-1}|| \leq \frac{1}{\mathrm{dist}(z, \sigma(T))}.$$

Proof : The first point follows immediately from the definition of functional calculus. Then, for $u \in \mathcal{H}$,

$$\begin{aligned}
||(z - T)^{-1}u||^2 &= ((z - T)^{-1}u|(z - T)^{-1}u) \\
&= \int_{\sigma(T)} (z - \lambda)^{-1}\overline{(z - \lambda)^{-1}} \mathrm{d}(E(\lambda)u|u) \\
&= \int_{\sigma(T)} |(z - \lambda)|^{-2} \mathrm{d}(E(\lambda)u|u) \\
&\leq \sup_{\lambda \in \sigma(T)} |z - \lambda|^{-2} \int_{\mathbb{R}} \mathrm{d}(E(\lambda)u|u) = \frac{1}{(\mathrm{dist}(z, \sigma(T)))^2} ||u||^2.
\end{aligned}$$

□

The spectral theorem also allows us to define the notion of a spectral projector on a Borel set B in $\mathbb{R}$ using the formula:

$$E_B = \mathbb{1}_B(T).$$

In particular, if B is an interval and if E is the spectral family associated with T, let's denote

$$E_{(a,b)} = E(b^-) - E(a^+) \quad \text{and} \quad E_{[a,b]} = E(b^+) - E(a^-).$$

Proposition A.6.9 (Stone's formula.)**.** *Let T be a self-adjoint operator. For all $a < b$,*

$$s - \lim_{\varepsilon \to 0} \frac{1}{2i\pi} \int_a^b (R_{s-i\varepsilon}(T) - R_{s+i\varepsilon}(T)) \mathrm{d}s = \frac{1}{2} \left(E_{[a,b]} + E_{(a,b)} \right).$$

Proof : For a complete and detailed proof, see [16], Theorem 2.13, page 37.

□

Using the spectral theorem and the functional calculus it induces, we can define for $t \in \mathbb{R}$ and T a self-adjoint operator, the unitary operator $U(t) = \mathrm{e}^{\mathrm{i}tT}$. Let's summarize the properties of this operator.

Proposition A.6.10. *1. For any $t \in \mathbb{R}$, $U(t)$ is unitary, and if $s, t \in \mathbb{R}$, $U(t+s) = U(t)U(s)$.*

2. If $\psi \in \mathcal{H}$ then $U(t)\psi \xrightarrow[t \to t_0]{} U(t_0)\psi$.

3. If $\psi \in \mathcal{H}$, then $\frac{U(t)\psi - \psi}{t} \xrightarrow[t \to 0]{} \mathrm{i}T\psi$.

Proof : See Theorem VIII.7 in [22].

□

The unitary operator $U(t)$ allows to solve the Schrödinger equation:

$$\left\{ \begin{array}{rcl} \partial_t \psi & = & \mathrm{i}T\psi \\ \psi|_{t=0} & = & \psi_0 \end{array} \right. \quad \text{with } \psi_0 \in D(T).$$

Indeed, for any $t \geq 0$, $\psi(t) = U(t)\psi_0$.

This result of existence for a Schrödinger equation admits a reciprocal, the Stone's theorem.

Definition A.6.11. *An operator-valued function $t \mapsto U(t)$ which satisfies properties 1 and 2 of Proposition A.6.10 is called a strongly continuous one-parameter unitary group.*

Theorem A.6.12. *Let $t \mapsto U(t)$ be a strongly continuous one-parameter unitary group on a Hilbert space $\mathcal{H}$. Then there exists a self-adjoint operator T on $\mathcal{H}$ such that for all $t \in \mathbb{R}$, $U(t) = \mathrm{e}^{\mathrm{i}tT}$. The operator T is called the infinitesimal generator of U.*

Proof : The construction of this infinitesimal generator is given in Theorem VIII.8 in [22].

□