Theory of random operators and applications to Anderson localization

Specialized M2 course in Fundamental Mathematics
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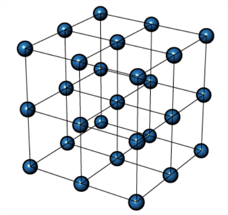
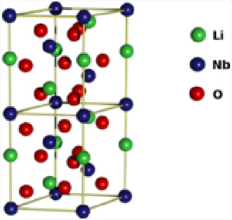
Professor: BOUMAZA Hakim
LAGA, Université Sorbonne Paris Nord
boumaza@math.univ-paris13.fr

Electronic transport

1. Goal: Study of electronic transport in a solid (metals, semiconductors, . . .).
2. In a perfect crystal: diffusion phenomenon for energies within spectral bands.

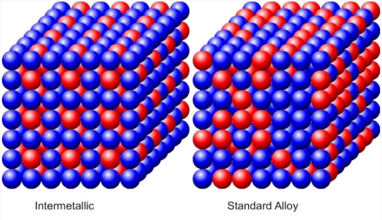


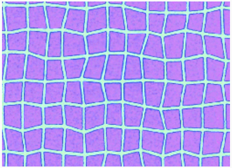


3. Periodic Schrödinger operators: $H = -\Delta + V$ where V is $\mathbb{Z}^d$-periodic. Band spectrum.
4. Reduction to a discrete problem – tight-binding approximation:

$$\text{on } \ell^2(\mathbb{Z}^d), \text{ for } u = (u_n)_{n\in\mathbb{Z}^d}, \ (-\Delta_{\mathrm{disc}} u)_n = \sum_{|m-n|=1} u_m.$$

Imperfect crystals – alloys

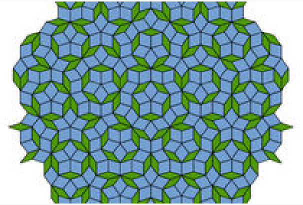


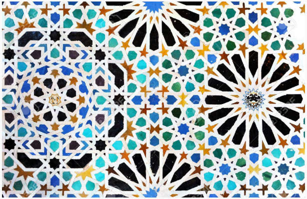


Family of random operators: $H_\omega = -\Delta + V_\omega$ where

1. Continuous Anderson model: $V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q_i(\omega) V(x - i)$ where V is supported in $[0,1]^d$ and the q_i are *i.i.d.*
2. Random displacement model: $V_\omega(x) = \sum_{i \in \mathbb{Z}^d} V(x - i - q_i(\omega))$.
3. Discrete model (Anderson model): $(V_\omega u)_n = q_n(\omega) u_n$.

Imperfect crystals – quasicrystals



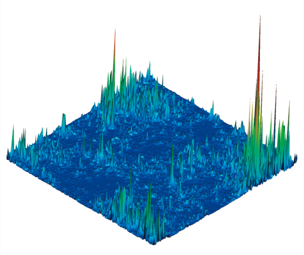


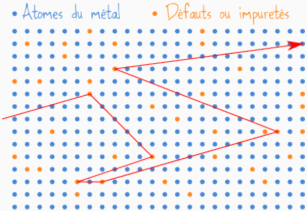
Family of quasiperiodic operators: $H_\omega = -\Delta + V_\omega$ where

1. $V_\omega(x) = V(\alpha_1 x_1 + \omega_1, \ldots, \alpha_d x_d + \omega_d)$ with $V : \mathbb{R}^d \to \mathbb{R}$ being $\mathbb{Z}^d$-periodic.
2. $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ with rationally independent coordinates.
3. $\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d/\mathbb{Z}^d$.

Physical phenomenon

1. At a fixed energy in the spectrum, beyond a certain amount of disorder in the crystal, the electron ceases to move freely and remains confined in a localized region.

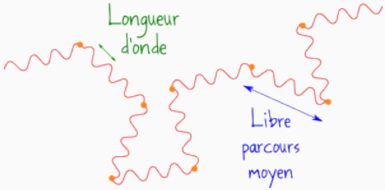




2. At each collision of the electron with an impurity in the crystal, its associated wave spreads.

Physical phenomenon

3. Mean free path: average distance traveled by the electron between two collisions.
4. When disorder increases, the mean free path does not decrease continuously.
5. Transition when the mean free path becomes shorter than the electron wavelength.



6. The localization phenomenon can be observed whenever a wave propagates in a disordered medium (light waves, microwaves, or acoustic waves).

Anderson localization

Definition
Let I be an interval of $\mathbb{R}$. We say that the family $\{H_\omega\}_{\omega\in\Omega}$ has the Anderson localization property in the interval I if:

(i) the spectrum of H_ω is purely point for P-almost every $\omega \in \Omega$,

(ii) the eigenfunctions associated with eigenvalues in I decay exponentially to 0 at infinity.

Anderson localization

For $d \geq 1$, we consider the operator

$$H_\omega = -\Delta_d + \sum_{i \in \mathbb{Z}^d} q_i(\omega) V(x - i)$$

acting on $H^2(\mathbb{R}^d)$. The $q_i(\omega)$ are Bernoulli *i.i.d.* random variables and V is supported in $[0, 1]^d$.

1. For $d = 1$, there is localization for all energies except those in a discrete set (Damanik–Sims–Stolz '02).
2. For $d \geq 2$, Anderson localization holds at the bottom of the spectrum (Bourgain–Kenig '05).
   a. For $d = 2$: it is believed that localization holds at all energies except possibly in a discrete set (but at the cost of a localization length that may become exponentially small at high energies).
   b. For $d \geq 3$, the existence of a localization/diffusion transition is conjectured.