IHARA'S LEMMA FOR GL_d: THE LIMIT CASE

by

Pascal Boyer

Abstract. — Clozel, Harris and Taylor proposed in [**CHT08**] conjectural generalizations of the classical Ihara's lemma for GL_2, to higher dimensional similitude groups. We prove these conjectures in the so called *limit case*, which after base change is the essential one, under any hypothesis allowing level raising as for example theorem 5.1.5 in [**Gee11**].

Contents

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1. Introduction

1.1. Ihara's lemma: origin and proofs. — In the Taylor-Wiles method Ihara's lemma is the key ingredient to extend a $R = T$ property from the minimal case to a non minimal one. It is usually formulated by the injectivity of some map as follows.

Let $\Gamma = \Gamma_0(N)$ be the usual congruence subgroup of $SL_2(\mathbb{Z})$ for some $N > 1$, and for a prime p not dividing N let $\Gamma' := \Gamma \cap \Gamma_0(p)$. We then have two degeneracy maps

$$\pi_1, \pi_2 : X_{\Gamma'} \longrightarrow X_\Gamma$$

between the compactified modular curves of levels Γ' and Γ respectively, induced by the inclusion

$$\Gamma' \hookrightarrow \Gamma \text{ and } \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma' \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \hookrightarrow \Gamma.$$

For $l \neq p$, we then have a map

$$\pi^* := \pi_1^* + \pi_2^* : H^1(X_\Gamma, \mathbb{F}_l)^2 \longrightarrow H^1(X_{\Gamma'}, \mathbb{F}_l).$$

Theorem 1.1.1. — *Let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra acting on these cohomology groups which is non Eisenstein, i.e. that corresponds to an irreducible Galois representation. Then after localizing at $\mathfrak{m}$, the map π^* is injective.*

Diamond and Taylor in [**DT94**] proved an analogue of Ihara's lemma for Shimura curves over $\mathbb{Q}$. For a general totally real number field F with ring of integers $\mathcal{O}_F$, Manning and Shotton in [**MS21**] succeeded to prove it under some large image hypothesis. Their strategy is entirely different from those of [**DT94**]but consists roughly

- to carry Ihara's lemma for a compact Shimura curve $Y_{\bar{K}}$ associated to a definite quaternion algebra $\overline{D}$ ramified at some auxiliary place v of F, in level $\bar{K} = \bar{K}^v \bar{K}_v$ an open compact subgroup of $D \otimes \mathbb{A}_{F,f}$ unramified at v,
- to the indefinite situation X_K relatively to a quaternion division algebra D ramified at all but one infinite place of F, and isomorphic to $\bar{D}$ at all finite places of F different to v, and with level K agreeing with $\bar{K}^v$ away from v.

Indeed in the definite case Ihara's statement is formulated by the injectivity of

$$\pi^* = \pi_1^* + \pi_2^* : H^0(Y_{\bar{K}}, \mathbb{F}_l)_{\mathfrak{m}} \oplus H^0(Y_{\bar{K}}, \mathbb{F}_l)_{\mathfrak{m}} \longrightarrow H^0(Y_{\bar{K}_0(v)}, \mathbb{F}_l)_{\mathfrak{m}}$$

where both $\overline{D}$ and $\bar{K}$ are unramified at the place v and $\bar{K}_0(v)_v$ is the subgroup of $\mathrm{GL}_2(F_v)$ of elements which are upper triangular modulo p.

The proof goes like this, cf. [**MS21**] theorem 6.8. Suppose $(f, g) \in \ker \pi^*$. Regarding f and g as K^v-invariant function on $\overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})$, then $f(x) = -g(x\omega)$ where $\omega = \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix}$, ϖ_v being an uniformizer for F_w and $\overline{G}$ being the algebraic group over $\mathcal{O}_F$ associated to $\mathcal{O}_{\overline{D}}^\times$ the inversible group of the maximal order $\mathcal{O}_{\overline{D}}$ of $\overline{D}$: note that $\overline{G}(F_v) \cong \mathrm{GL}_2(F_v)$. Then f is invariant under K^v and $\omega^{-1}K^v\omega$ so that, using the strong approximation theorem for the subgroup of $\overline{G}$ of elements of reduced norm 1, then f factors through the reduced norm map, and so is supported on Eisenstein maximal ideals.

The link between X_K and Y_{K^v} is given by the geometry of the integral model of the Shimura curve $X_{K_0(v)}$ with $\Gamma_0(v)$-level structure. The main new ingredient of [**MS21**] to carry this geometric link to Ihara's lemma goes through the patching technology which allows to obtain maximal Cohen-Macaulay modules over deformation rings. Using a flatness property and Nakayama's lemma, there are then able to extend a surjective property, dual to the injectivity in the Ihara's lemma, from the maximal unipotent locus on the deformation space to the whole space, and recover the Ihara's statement reducing by the maximal ideal of the deformation ring.

Recently Caraiani and Tamiozzo following closely [**MS21**] also obtained Ihara's lemma for Hilbert varieties essentially because Galois deformations rings are the same and so regular which is not the case beyond GL_2.

1.2. Generalisations of Ihara's Lemma. — To generalize the classical Ihara's lemma for GL_d, there are essentially two approaches.

The first natural one developed by Clozel, Harris and Taylor in their first proof of the Sato-Tate theorem [**CHT08**], focuses on the H^0 with coefficients in $\mathbb{F}_l$ of a zero dimensional Shimura variety associated to higher dimensional definite division algebras. More precisely consider a totally real field F^+ and a imaginary quadratic extension $E/\mathbb{Q}$ and define $F = F^+E$. We then consider $\overline{G}/\mathbb{Q}$ an unitary group with $\overline{G}(\mathbb{Q})$ compact so that $\overline{G}$ becomes an inner form of GL_d over F. This means, cf. §2.3, we have fixed a division algebra $\overline{B}$ with center F, of dimension d^2, provided with an involution of the second kind such that its restriction to F is the complex conjugation. We moreover suppose that at every place w of F, either $\overline{B}_w$ is split or a local division algebra.

Let v be a place of F above a prime number p split in E and such that $\overline{B}_v^\times \cong \mathrm{GL}_d(F_v)$ where F_v is the associated local field with ring of integers $\mathcal{O}_v$ and residue field $\kappa(v)$.

Notation 1.2.1. — Let q_v be the order of the residue field $\kappa(v)$.

Consider then an open compact subgroup $\overline{K}^v$ infinite at v in the following sense: $\overline{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{v_i^+} \overline{B}_{v_i^+}^{op,\times}$ where $p = \prod_i v_i^+$ in F^+ and we identify places of F^+ over $p = uu^c \in E$ with places of F over u. We then ask $\overline{K}_p^v = \mathbb{Z}_p^\times \times \prod_{w|u} \overline{K}_w$ to be such that $\overline{K}_v$ is restricted to the identity element.

The associated Shimura variety with level $\overline{K} = \overline{K}^v \overline{K}_v$ for some finite level $\overline{K}_v$ at v, denoted by $\overline{\mathrm{Sh}}_{\overline{K}}$, is then such that its $\mathbb{C}$-points are $\overline{G}(\mathbb{Q})\backslash \overline{G}(\mathbb{A}_{\mathbb{Q}}^\infty)/\overline{K}$ and for l a prime not divisible by v, its H^0 with coefficients in $\overline{\mathbb{F}}_l$ is then identified with the space

$$S_{\overline{G}}(\overline{K}, \overline{\mathbb{F}}_l) = \{f : \overline{G}(\mathbb{Q})\backslash \overline{G}(\mathbb{A}_{\mathbb{Q}}^\infty)/\overline{K} \longrightarrow \overline{\mathbb{F}}_l \text{ locally constant}\}.$$

Replacing $\overline{K}$ by $\overline{K}^v$, we then obtain an admissible smooth representation of $\mathrm{GL}_d(F_v)$ equipped with an action of the Hecke algebra $\mathbb{T}(\overline{K}^v)$ defined as the image of the abstract unramified Hecke algebra, cf. definition 3.2.1, inside $\mathrm{End}\big(S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l)\big)$.

To a maximal ideal $\mathfrak{m}$ of $\mathbb{T}(\overline{K}^v)$ is associated a Galois $\overline{\mathbb{F}}_l$-representation $\overline{\rho}_{\mathfrak{m}}$, cf. §5.2. We consider the case where this representation is irreducible. Note in particular that such an $\mathfrak{m}$ is then non pseudo-Eisenstein in the usual terminology.

Conjecture 1.2.2. — *(cf. conjecture B in* [**CHT08**]*)*
Any irreducible $\mathrm{GL}_d(F_v)$*-submodule of* $S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ *is generic.*

For rank 2 unitary groups, we recover the previous statement as the characters are exactly those representations which do not have a Whittaker model, i.e. are the non generic ones. For $d \geq 2$, over $\overline{\mathbb{Q}}_l$, the generic representations of $\mathrm{GL}_d(F_v)$ are the irreducible parabolically induced representations $\mathrm{st}_{t_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{t_r}(\pi_{v,r})$ where for $i = 1, \cdots, r$,

- $\pi_{v,i}$ is an irreducible cuspidal representation of $\mathrm{GL}_{g_i}(F_v)$,
- $\mathrm{st}_{t_i}(\pi_{v,i})$ is a Steinberg representations, cf. definition 2.1.2,
- $\sum_{i=1}^r t_i g_i = d$ where the Zelevinsky segments $[\pi_{v,i}\{\frac{1-t_i}{2}\}, \pi_{v,i}\{\frac{t_i-1}{2}\}]$are not linked in the sense of [**Zel80**].

Over $\overline{\mathbb{F}}_l$ every irreducible generic representation is obtained as the unique generic subquotient of the modulo l reduction of a generic representation. It can also be characterized intrinsically using representation of the mirabolic subgroup, cf. §2.1.

Definition 1.2.3. — (cf. definition of [**CHT08**] 5.1.9)
An admissible smooth $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module M is said to have the weak Ihara property

if for every $m \in M^{\mathrm{GL}_d(\mathcal{O}_v)}$ which is an eigenvector of $\overline{\mathbb{F}}_l[\mathrm{GL}_d(\mathcal{O}_v)\backslash \mathrm{GL}_d(F_v)/ \mathrm{GL}_d(\mathcal{O}_v)]$, every irreducible submodule of the $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module generated by m, is generic.

Remark. If we ask $S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ to verify the weak Ihara property, then it should have non trivial unramified vectors so that the supercuspidal support of the restriction $\overline{\rho}_{\mathfrak{m},v}$ of $\overline{\rho}_{\mathfrak{m}}$ to the decomposition subgroup at v, is made of unramified characters.

The second approach asks to find a map playing the same role as $\pi^* = \pi_1^* + \pi_2^*$. It is explained in section 5.1 of [**CHT08**] with the help of the element

$$\theta_v \in \mathbb{Z}_l[K_1(v^n)\backslash \mathrm{GL}_d(F_v)/ \mathrm{GL}_d(\mathcal{O}_{F_v})]$$

constructed by Russ Mann, cf. proposition 5.1.7 of [**CHT08**], where F_v is here a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_v$.

Definition 1.2.4. — An admissible smooth $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module M is said to have the almost Ihara property if $\theta_v : M^{\mathrm{GL}_d(\mathcal{O}_v)} \longrightarrow M$ is injective.

Recall that l is called quasi-banal for $\mathrm{GL}_d(F_v)$ if either $l \nmid \sharp \mathrm{GL}_d(\kappa_v)$ (the banal case) or $l > d$ and $q_v \equiv 1 \mod l$ (the limit case).

Proposition 1.2.5. — *(cf.* [**CHT08**] *lemma 5.1.10)*
Suppose that l is quasi-banal and M is a $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module verifying the Ihara property. If $\ker(\theta_v : M^{\mathrm{GL}_d(\mathcal{O}_v)} \longrightarrow M)$ is a $\mathbb{F}_l[\mathrm{GL}_d(\mathcal{O}_{F_v})\backslash \mathrm{GL}_d(F_v)/ \mathrm{GL}_d(\mathcal{O}_{F_v})]$-module, then M has the almost Ihara property.

Applications: the generalizations of the classical Ihara's lemma were introduced in [**CHT08**] to prove a non minimal $R = \mathbb{T}$ theorem. The weaker statement $R^{red} = \mathbb{T}$ where R^{red} is the reduced quotient of R, was later obtained unconditionally using Taylor's *Ihara avoidance* method, cf. [**Tay08**] which was enough to prove the Sato-Tate conjecture. However, the full $R = \mathbb{T}$ theorem would have applications to special values of the adjoint L-function and would imply that R is a complete intersection. It should also be useful for generalizing the local-global compatibility results of [**Eme**].

In [**Mos21**], the author also proved that Ihara's property in the quasi-banal case is equivalent to the following result.

Proposition 1.2.6. — *(cf.* [**Mos21**] *corollary 9.5)*
Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^S$ and $f \in S_{\overline{G}}(\overline{K}^v \mathrm{GL}_d(\mathcal{O}_v), \overline{\mathbb{F}}_l)$. Let K_v be the Iwahori subgroup of $\mathrm{GL}_d(\mathcal{O}_v)$, then the $\overline{\mathbb{F}}_l[\mathrm{K}_v \backslash \mathrm{GL}_d(F_v)/ \mathrm{GL}_d(\mathcal{O}_v)]$-submodule of $S_{\overline{G}}(\overline{K}^v \mathrm{K}_v, \overline{\mathbb{F}}_l)$ generated by f is of dimension $d!$.

1.3. Main results. — With the previous notations, let q_v be the order of the residue field of F_v. We fix some prime number l unramified in F^+ and split in E and we place ourself in the limit case where $q_v \equiv 1 \mod l$ with $l > d$, which is, after by base change, the crucial case to consider.

Definition 1.3.1. — As in definition 2.5.1 of [**CHT08**], we say that a subgroup $H \subseteq \mathrm{GL}_d(\overline{\mathbb{F}}_l)$ is big if :

- H has no l-power order quotients;
- $H^i(H, \mathfrak{g}_d^0(\overline{\mathbb{F}}_l)) = (0)$ for $i = 0, 1$ and where $\mathfrak{g}_d := \mathrm{Lie}\, \mathrm{GL}_d$ and $\mathfrak{g}_d^0$ is the trace zero subspace of $\mathfrak{g}_d$;
- for all irreducible $\overline{\mathbb{F}}_l[H]$-submodules W of $\mathfrak{g}_d(\overline{\mathbb{F}}_l)$, we can find $h \in H$ and $\alpha \in \overline{\mathbb{F}}_l$ satisfying the following properties.
  - The α-generalized eigenspace $V(h,\alpha)$ of h on $\overline{\mathbb{F}}_l^d$ is one dimensional.
  - Let $\pi_{h,\alpha} : \overline{\mathbb{F}}_l^d \twoheadrightarrow V(h,\alpha)$ be the h-equivariant projection of $\overline{\mathbb{F}}_l^d$ to $V(h,\alpha)$ and let $i_{h,\alpha} : V(h,\alpha) \hookrightarrow \overline{\mathbb{F}}_l^d$ be the h-equivariant injection of $V(h,\alpha)$ into $\overline{\mathbb{F}}_l^d$. Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.

Theorem 1.3.2. — *In the limit case, suppose that there exists a prime $p_0 = u_0\bar{u}_0$ split in E with a place $v_0|u_0$ of F such that $\overline{B}_{v_0}$ is a division algebra. Consider $\mathfrak{m}$ such that*

$$\overline{\rho}_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_l)$$

is an irreducible representation which is unramified at all places of F lying above primes which do not split in E and which satisfies the following hypothesis:

- $\overline{F}^{\ker \mathrm{ad}\, \overline{\rho}}$ *does not contain $F(\zeta_l)$ where ζ_l is any primitive l-root of 1;*
- $\overline{\rho}(G_{F^+(\zeta_l)})$ *is big.*

Then Ihara's lemma of the conjecture 1.2.2 is true, i.e. every irreducible $\mathrm{GL}_d(F_v)$-submodule of $S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ is generic.

- The last two hypothesis come from theorem 5.1.5 of [**Gee11**] which is some level raising and lowering statement, cf. theorem 5.2.2. Any other similar statement, for example theorem 4.4.1 of [**BLGGT14**], with different hypothesis can then be used to modify the hypothesis of the theorem above.
- Our techniques work also in the banal case as soon as you avoid cuspidal $\overline{\mathbb{F}}_l$-representations which are not supercuspidal which is for example the case if you suppose that, after semi-simplification, $\overline{\rho}_{\mathfrak{m},v}$ is a direct sum of characters. In particular the resulting statement is more general than those of [**Boy22**].

The basic idea(1), cf. §5.1, as in [**Boy22**], is to introduce geometry and move from the Shimura variety associated to $\overline{G}$ which is of dimension zero, to another Shimura variety Sh_K associated to some reductive group G and level K, of strictly positive dimension, so that $S_{\overline{G}}(\overline{K}, \overline{\mathbb{F}}_l)$ appears in a certain cohomology group of some sheaf over Sh_K so that Ihara's lemma is a corollary of the following result.

(1)this explains the hypothesis on the existence of p_0 in the statement

Theorem 1.3.3. — *With the same hypothesis as in theorem 1.3.2, every irreducibe* $\mathrm{GL}_d(F_v)$*-submodule of* $H^0(\mathrm{Sh}_{K^v(\infty),\bar\eta}, \overline{\mathbb{Z}}_l[d-1])_{\mathfrak{m}}$ *is generic.*(2)

The first point in our strategy is to work in characteristic zero and then consider modulo l reduction. For this we will need to fix the following data.

Notation 1.3.4. — Consider a coefficient field L which is a large enough finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}_L$ and residue field $\mathcal{O}_L/\varpi_F\mathcal{O}_L\mathbb{F}_L$ for some fixed uniformizer ϖ_L.

We then want to construct a filtration of the middle cohomology group of Sh_K with coefficients in $\mathcal{O}_L$ so that the graded parts, which are expected to be more easy to handle with, all verify the genericity property of their irreducible sub-spaces. More explicitly we study the middle degree cohomology group with coefficients in $\overline{\mathbb{Z}}_l$, of the KHT Shimura variety $\mathrm{Sh}_{K^v(\infty)}$ associated to some similitude group $G/\mathbb{Q}$ such that $G(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \cong \overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$, cf. §2.3 for more details, and with level $K^v(\infty) := \overline{K}^v$ meaning finite level outside v and infinite level at v. The localization at $\mathfrak{m}$ of the cohomology groups of $\mathrm{Sh}_{K^v(\infty)}$ can be computed as the cohomology of the geometric special fiber $\mathrm{Sh}_{K^v(\infty),\bar s_v}$ of $\mathrm{Sh}_{K^v(\infty)}$, with coefficient in the complex of nearby cycles $\Psi_{K^v(\infty),v}$.

The Newton stratification of $\mathrm{Sh}_{K^v(\infty),\bar s_v}$ gives us a filtration of $\Psi_{K^v(\infty),v}$, cf. [**Boy20**], and so a filtration $\mathrm{Fil}^\bullet(K^v(\infty))$ of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar\eta_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ and the main point of [**Boy22**] is to prove that the modulo l reduction of each graded part of this filtration verifies the Ihara property, i.e. each of their irreducible sub-space are generic. To realize this strategy

- we need first the cohomology groups of $\mathrm{Sh}_{K^v(\infty)}$ to be torsion free: this point is now essentially settled by the main result of [**Boy23a**].
- More crucially the previous filtration $\mathrm{Fil}^\bullet(K^v(\infty))$ should be strict, i.e. its graded parts have to be torsion free, cf. theorem 3.2.3.
- For each of the graded parts, the $\overline{\mathbb{Q}}_l$-cohomology can be described by a set of automorphic representations, each of them giving an automorphic contribution. We choose any numbering of this discrete set and for each n, we choose a coefficient field L large enough to be able to separate the first n automorphic contributions. The cohomology over $\mathcal{O}_L$ gives us a lattice which then allows us to deal with the modulo l reduction meaning reduction modulo $\varpi_L\mathcal{O}_L$.

Remark. In the following we will just write modulo l reduction for this construction.

It appears that the graded parts of $\mathrm{Fil}^\bullet(K^v(\infty))$ are parabolically induced and in the limit case when the order q_v of the residue field is such that $q_v \equiv 1 \mod l$, the socle of the modulo l reduction of these parabolic induced representations are no more irreducible and do not fulfill the Ihara property, i.e. some of their subspaces are not generic. It then appears that we have at least

(2) cf. (8) for the meaning of this notation

- to verify that the modulo l reduction of the first non trivial graded part of $\mathrm{Fil}^\bullet(K^v(\infty))$ verifies the genericity property of its irreducible submodule. For this we need a level raising statement as theorem 5.1.5 in [**Gee11**], cf. theorem 5.2.2, or theorem 4.4.1 of [**BLGGT14**].
- Then we have to understand that the extensions between the graded parts of $\mathrm{Fil}^\bullet(K^v(\infty)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ are non split.

One problem about this last point is that the $\overline{\mathbb{Q}}_l$-cohomology is split. For any irreducible automorphic representation Π of $G(\mathbb{A})$ cohomological for, say, the trivial coefficients, the $\overline{\mathbb{Z}}_l$-cohomology defines a lattice $\Gamma(\Pi)$ of $(\Pi^\infty)^{K^v(\infty)} \otimes \sigma(\Pi)_v$ whose modulo l reduction gives a subspace of the $\overline{\mathbb{F}}_l$-cohomology: Ihara's lemma predicts that the socle of this subspace is still generic, i.e. it gives informations about the lattice $\Gamma(\Pi)$. We then see that non splitness of $\mathrm{Fil}^\bullet(K^v(\infty)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ should be understood in a very flexible point of view. We then naturally face to prove the following result.

Proposition 1.3.5. — *(cf. 5.5.1) The contribution $\Gamma(\Pi)$ of an automorphic representation Π to the cohomology viewed as a subrepresentation, defines a stable lattice of Π_v uniquely defined by the property that the socle of its modulo l reduction is irreducible and generic.*

In particular this lattice should depend only on Π_v and not on the global representation Π. This statement looks similar to the Breuil lattice conjecture which is stated when $l = p$ and K_v-types. We then first prove a simple version of this result, cf. proposition 4.3.2 which can be stated as follows.

- Consider some fixed K_v-type $\sigma_{v,\overline{\mathbb{Q}}_l}$
- and a system λ of Hecke eigenvalues associated to some automorphic representation Π as above
- such that $\sigma_{v,\overline{\mathbb{Q}}_l}$ appears with multiplicity one in Π_v.

Then the lattice of $\sigma_{v,\overline{\mathbb{Q}}_l} \cap \Gamma(\Pi)$ depends only on the modulo l reduction of λ. One combinatorial problem is then to recover the information on the $\mathrm{GL}_d(F_v)$-lattices from this vague obversation, cf. §5.3.2. The idea to do so is to start from the filtration $\mathrm{Fil}^\bullet(K^v(\infty))$ coming from a filtration of the nearby cycles perverse sheaf and modify it step by step, cf. §5.3, until we arrive at an automorphic filtration, cf. §5.3.1, i.e. where the graded parts correspond to the contribution of some automorphic representation. The main ingredient to construct modifications of filtrations is to consider situations as illustrated in the figure 1.

- A filtration $\mathrm{Fil}^\bullet$ of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ whose graded parts $\mathrm{gr}^\bullet$ are torsion free;
- let k and $X := \mathrm{Fil}^k / \mathrm{Fil}^{k-2}$ such that $X \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong (gr^{k-1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l) \oplus (\mathrm{gr}^k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$.
- We can then define $\widetilde{\mathrm{gr}^k} := X \cap (\mathrm{gr}^k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$ and the quotient $\widetilde{\mathrm{gr}^{k-1}}$: note that T is torsion.

– We choose a finite extension L over $\mathbb{Q}_l$, so that this diagram is defined over $\mathcal{O}_L$ and which allows to look modulo l, meaning modulo $\varpi_L \mathcal{O}_L$, where we then obtained a priori two distinct filtrations. Note that the subquotients of these filtrations which are not involved in $T[\varpi_L]$ are identified, through the modulo ϖ_L reduction of the first square in the figure 1.

Let us first explain why something interesting should happen during this process.
(a) We can define a $\overline{\mathbb{F}}_l$-monodromy operator for the Galois action at the place v.[(3)] We are looking for a geometric monodromy operator N_v^{geo} which then exists whatever are the coefficients, $\overline{\mathbb{Q}}_l$, $\overline{\mathbb{Z}}_l$ and $\overline{\mathbb{F}}_l$, compatible with tensor products. One classical construction is known in the semi-stable reduction case, cf. [**Ill94**] §3, which corresponds to the case where the level at v of our Shimura variety is of Iwahori type.[(4)] Using our knowledge of the $\overline{\mathbb{Z}}_l$-nearby cycles described completely in [**Boy23b**], we can construct such a geometric nilpotent monodromy operator which generalizes the semi-stable case, cf. §3.3.
(b) Taking this geometric monodromy operator, we then obtain a cohomological monodromy operator $N_{v,\mathfrak{m}}^{coho}$ acting on $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$ One of the main point, cf theorem 3.2.3, is that the graded parts of the filtration of $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$ induced by the Newton filtration on the nearby cycles spectral sequence, are all torsion free, so that in particular we are in position to understand quite enough the action of $\overline{N}_{v,\mathfrak{m}}^{coho} := N_{v,\mathfrak{m}}^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ on $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, and prove that its nilpotency order is as large as possible.
(c) Note that as $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, then the modulo l reduction of the monodromy operator acting on $\rho_{\widetilde{\mathfrak{m}}}$ does not depend on the choice of the prime ideal $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$ so that it is usually trivial. Finally, as $N_v^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is far from being trivial,

$$\begin{tikzcd}
& \widetilde{\mathrm{gr}^k} \arrow[d, hook] \arrow[dr, hook] & & \\
\mathrm{gr}^{k-1} \arrow[r, hook] \arrow[dr, hook] & X \arrow[r, two heads] \arrow[d] & \mathrm{gr}^k \arrow[dr] & \\
& \widetilde{\mathrm{gr}^{k-1}} \arrow[dr] & & T \arrow[dl, equal] \\
& & T &
\end{tikzcd}$$

FIGURE 1. Exchange process

(3) Note that over $\overline{\mathbb{F}}_l$ the usual arithmetic approach for defining the nilpotent monodromy operator, is hopeless because, up to consider a finite extension of F_v, such a $\overline{\mathbb{F}}_l$-representation has a trivial action of the inertia group.
(4) This corresponds to automorphic representations Π such that the cuspidal support of Π_v is made of unramified characters, and so with the weak form of Ihara's lemma of definition 1.2.3.

there should be non split extensions between the graded parts of The heart of our proof is then divided in three main steps:

- prove that the lattice of Π_v induced by $\Gamma(\Pi)$ depends only on the modulo l reduction of the system of Hecke eigenvalues of a globalization Π of Π_v;
- using the modulo l monodromy operator $\overline{N}_v$, prove that this lattice verifies the Ihara property;
- prove that the graded parts of our final filtration, which do not verify Ihara's property can not give a non generic subspace of the all cohomology.

To conclude this long introduction, note that Ihara's lemma in Clozel-Harris-Taylor formulation, was stated in order to be able to do level raising. In our proof we use level raising statements, proved thanks to Taylor's Ihara avoidance in [**Tay08**], in order to prove Ihara's lemma. Then we can see our arguments as the proof that *level raising implies Ihara's lemma in the limit case.*

2. Preliminaries

2.1. Representations of GL_d**.** — Consider a finite extension $M/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We denote by $|-|$ its absolute value. For a representation π of $\mathrm{GL}_d(M)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \mathrm{val} \circ \det}.$$

Notation 2.1.1. — For π_1 and π_2 representations of respectively $\mathrm{GL}_{n_1}(M)$ and $\mathrm{GL}_{n_2}(M)$, we will denote by

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(M)}^{\mathrm{GL}_{n_1+n_2}(M)} \pi_1\{\frac{n_2}{2}\} \otimes \pi_2\{-\frac{n_1}{2}\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$\mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2-r_1} \times \cdots \times \mathrm{GL}_{r_k-r_{k-1}}.$$

Recall that a representation ϱ of $\mathrm{GL}_d(M)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero, these two notions coincides, but this is no more true over $\overline{\mathbb{F}}_l$.

Definition 2.1.2. — (see [**Zel80**] §9 and [**Boy10**] §1.4) Let g be a divisor of $d = sg$ and π an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_g(M)$. The induced representation

$$\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\} \tag{1}$$

holds an unique irreducible quotient (resp. subspace) denoted $\mathrm{st}_s(\pi)$ (resp. $\mathrm{Speh}_s(\pi)$); it is a generalized Steinberg (resp. Speh) representation. Their cuspidal support is the Zelevinsky segment

$$[\pi\{\frac{1-s}{2}\},\pi\{\frac{s-1}{2}\}] := \Big\{\pi\{\frac{1-s}{2}\},\pi\{\frac{3-s}{2}\},\cdots,\pi\{\frac{s-1}{2}\}\Big\}.$$

More generally the set of sub-quotients of the induced representation (1) is in bijection with the following set.

$$\mathrm{Dec}(s)=\{(t_1,\cdots,t_r),\ \text{such that } t_i\geq 1 \text{ and } \sum_{i=1}^r t_i=s\}.$$

For any $\underline{s}\in\mathrm{Dec}(s)$, we then denote by $\mathrm{st}_{\underline{s}}(\pi)$ the associated irreducible sub-quotient of (1). Following Zelevinsky, we fix this bijection such that $\mathrm{Speh}_s(\pi)$ corresponds to (s) and $\mathrm{st}_s(\pi)$ to $(1,\cdots,1)$. The Lubin-Tate representation $LT_{h,s}(\pi)$ will also appear in the following, it corresponds with $(\overbrace{1,\cdots,1}^{h},s-h)$.

Definition 2.1.3. — For $1\leq s-1$, we say that $\underline{s}\in\mathrm{Dec}(s)$ is t-small if the number of consecutive 1 is less than t. We say that an irreducible subquotient of (1) is (π,t)-small if its parameter $\underline{s}\in\mathrm{Dec}(s)$ is t-small.

Proposition 2.1.4. — *(cf.* [**Vig96**] *III.5.10) Let π be an irreducible cuspidal representation of $\mathrm{GL}_g(M)$ with a stable $\overline{\mathbb{Z}}_l$-lattice*[5]*, then its modulo l reduction is irreducible and cuspidal (but not necessary supercuspidal).*

We now suppose as explained in the introduction that

$$q\equiv 1 \mod l \quad \text{and} \quad l>d$$

so the following facts are verified (cf. [**Vig96**] §III):

- the modulo l reduction of every irreducible cuspidal representation of $\mathrm{GL}_g(M)$ for $g\leq d$, is supercuspidal[6]: with the notation of [**Boy11**] proposition 1.3.5, $m(\varrho)=l>d$ for any irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation ϱ.
- For a $\overline{\mathbb{F}}_l$-irreducible supercuspidal representation ϱ of $\mathrm{GL}_g(M)$, the parabolic induced representation $\varrho\times\cdots\times\varrho$, with s copies of ϱ, is semi-simple with irreducible constituants[7] the modulo l reduction of the set of elements of $\{\mathrm{st}_{\underline{s}}(\pi)$ such that $\underline{s}\in\mathrm{Dec}(s)\}$, where π is any cuspidal representation whose modulo l reduction is isomorphic to ϱ.

[5] As before, we fix $L/\mathbb{Q}_l$ large enough and consider an $\mathcal{O}_L$-lattice: we say that π is integral.
[6] In the banal case this is not always the case but it is when the cuspidal support contains only characters.
[7] some of them might be isomorphic to each others

Concerning the notion of genericity, consider the mirabolic subgroup $\mathbb{M}_d(M)$ of $\mathrm{GL}_d(M)$ as the set of matrices with last row $(0,\cdots,0,1)$: we denote by

$$\mathbb{V}_d(M)=\{(m_{i,j})\in\mathbb{M}_d(M):\ m_{i,j}=\delta_{i,j} \text{ for } j<d\}.$$

its unipotent radical. We fix a non trivial character ψ of L and let θ be the character of $\mathbb{V}_d(M)$ defined by $\theta((m_{i,j}))=\psi(m_{d-1,d})$. For $G=\mathrm{GL}_r(M)$ or $\mathbb{M}_r(M)$, we denote by $\mathrm{alg}(G)$ the abelian category of smooth representations of G and, following [**BZ77**], we introduce

$$\Psi^-:\mathrm{alg}(\mathbb{M}_d(M))\longrightarrow\mathrm{alg}(\mathrm{GL}_{d-1}(M)),$$

and

$$\Phi^-:\mathrm{alg}(\mathbb{M}_d(M))\longrightarrow\mathrm{alg}(\mathbb{M}_{d-1}(M)),$$

defined by $\Psi^-=r_{\mathbb{V}_d,1}$ (resp. $\Phi^-=r_{\mathbb{V}_d,\theta}$) the functor of $\mathbb{V}_d$ coinvariants (resp. $(\mathbb{V}_d,\theta)$-coinvariants), cf. [**BZ77**] 1.8. For $\tau\in\mathrm{alg}(\mathbb{M}_d(M))$, the representation

$$\tau^{(k)}:=\Psi^-\circ(\Phi^-)^{k-1}(\tau)$$

is called the k-th derivative of τ. If $\tau^{(k)}\neq 0$ and $\tau^{(m)}=0$ for all $m>k$, then $\tau^{(k)}$ is called the highest derivative of τ. In the particular case where $k=d$, there is an unique irreducible representation τ_{nd} of $\mathbb{M}_d(M)$ with derivative of order d.

Definition 2.1.5. — An irreducible representation π of $\mathrm{GL}_d(M)$ is said generic, if its restriction to the mirabolic subgroup admits τ_{nd} as a subquotient.

Let π be an irreducible generic $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_d(M)$ and consider any stable lattice which gives us by modulo l reduction a $\overline{\mathbb{F}}_l$- representation uniquely determined up to semi-simplification. Then this modulo l reduction admits an unique generic irreducible constituant.

Lemma 2.1.6. — *Let $1\leq t\leq s$ and consider*

- $\underline{s}=(l_1\geq\cdots\geq l_r)$ *a partition of s with $l_1<t$;*
- $\pi_1,\cdots,\pi_r$ *irreducible cuspidal representations whose modulo l reductions are isomorphic to ϱ.*

There exists an irreducible subquotient τ of $\mathrm{st}_{l_1}(\pi_1)\times\cdots\times\mathrm{st}_{l_r}(\pi_r)$ such that

- *whatever is $\underline{s}'=(l'_1\geq\cdots\geq l'_{r'})$ with $l'_1\geq t$,*
- *and $\pi'_1,\cdots,\pi'_{r'}$ irreducible cuspidal representations with modulo l reduction isomorphic to ϱ,*

then τ is not a subquotient of the modulo l reduction of $\mathrm{st}_{l'_1}(\pi'_1)\times\cdots\times\mathrm{st}_{l'_r}(\pi'_{r'})$.

Proof. — Note that $\mathrm{st}_{l_1}(\pi_1)\times\cdots\times\mathrm{st}_{l_r}(\pi_r)$ has the same modulo l reduction as

$$(2)\quad \mathrm{st}_{l_1}(\pi_1\{\frac{l_1-s}{2}\})\times\mathrm{st}_{l_2}(\pi_2\{\frac{l_2-s}{2}+l_1-1\}\times\cdots\times\mathrm{st}_{l_r}(\pi_1\{\frac{l_r-s}{2}+l_1+\cdots+l_{r-1}\})$$

where the shifts are chosen so that $\mathrm{st}_{l_1}(\pi\{\frac{l_1-s}{2}\})\times\cdots\times\mathrm{st}_{l_r}(\pi\{\frac{s-l_r}{2}\})$ is the subquotient of (1) associated to

$$(\overbrace{1,\cdots,1}^{l_1-1},2,\overbrace{1,\cdots,1}^{l_1-1},2,\cdots,\overbrace{1,\cdots,1}^{l_r-1})\in\mathrm{Dec}(s).$$

Note that this irreducible constituant of (1) is the less non degenerate subquotient and we denote by τ is its modulo l reduction which remains irreducible.

The property stated in the lemma then follows from the fact that whatever is an irreducible subquotient of the modulo l reduction of $\mathrm{st}_{l'_1}(\pi'_1)\times\cdots\times\mathrm{st}_{l'_r}(\pi'_{r'})$, it has a non zero derivative of order l'_1, and so a non zero derivative of order t, while the derivative of order t of τ is zero.

□

Definition 2.1.7. — Elements τ constructed in the above lemma will be said to be (ϱ,t)-small.

Consider any non degenerate irreducible representation $\Pi:=\mathrm{st}_{l_1}(\pi_1)\times\cdots\times\mathrm{st}_{l_r}(\pi_r)$ where the modulo l reduction of $\pi_1,\cdots,\pi_r$ is isomorphic to ϱ. As the modulo l reduction of Π contains an unique irreducible non degenerate subquotient, there exists then an unique stable lattice such that its modulo l reduction has an irreducible generic socle τ_{gen}, cf. [**EGS15**] lemma 4.1.1.

Notation 2.1.8. — We denote by $\Gamma(\Pi)^{gen}$ the lattice of Π for which the socle of its modulo l reduction is generic.

2.2. Weil–Deligne inertial types. — Recall that a Weil-Deligne representation of W_M is a pair (r,N) where

- $r:W_M\longrightarrow\mathrm{GL}(V)$ is a smooth(8) representation on a finite dimensional $\overline{\mathbb{Q}}_l$-vector space V; and
- $N\in\mathrm{End}(V)$ is nilpotent such that
$$r(g)Nr(g)^{-1}=||g||N,$$
where $||\bullet||:W_M\to W_M/I_M\twoheadrightarrow q^{\mathbb{Z}}$ takes an arithmetic Frobenius element to q.

Remark. To a continuous(9) representation on a finite dimensional $\overline{\mathbb{Q}}_l$-vector space V, $\rho:W_M\longrightarrow\mathrm{GL}(V)$ is attached a Weil-Deligne representation denoted by $\mathrm{WD}(\rho)$. A Weil representation of W_M is also said of Galois type if it comes from a representation of G_M.

(8) i.e. continuous for the discrete topology on V

(9) relatively to the l-adic topology on V

Main example: let $\rho : W_M \longrightarrow \mathrm{GL}(V)$ be a smooth irreducible representation on a finite dimensional vector space V. For $k \geq 1$ an integer, we then define a Weil-Deligne representation
$$\mathrm{Sp}(\rho, k) := \big(V \oplus V(1) \oplus \cdots \oplus V(k-1), N\big),$$
where for $0 \leq i \leq k-2$, the isomorphism $N : V(i) \cong V(i+1)$ is induced by some choice of a basis of $\overline{L}(1)$ and $N_{|V(k-1)}$ is zero. Then every Frobenius semi-simple Weil-Deligne representation of W_M is isomorphic to some $\bigoplus_{i=1}^r \mathrm{Sp}(\rho_i, k_i)$, for smooth irreducible representations $\rho_i : W_M \longrightarrow \mathrm{GL}(V_i)$ and integers $k_i \geq 1$. Up to obvious reorderings, such a writing is unique.

Let now ρ be a continuous representation of W_M, or its Weil-Deligne representation $\mathrm{WD}(\rho)$, and consider its restriction to I_M, $\tau := \rho_{|I_M}$. Such an isomophism class of a finite dimensional continuous representation of I_M is then called *an inertial type*.

Notation 2.2.1. — Let $\mathcal{I}_0$ the set of inertial types that extend to a continuous irreducible representation of G_M.

Remark. $\tau \in \mathcal{I}_0$ might not be irreducible.

Let Part be the set of decreasing sequences of positive integers
$$\underline{d} = (\underline{d}(1) \geq \underline{d}(2) \geq \cdots)$$
viewed as a partition of $\sum \underline{d} := \sum_i \underline{d}(i)$. We also denote by $\mathrm{Part}(s)$ the set of partition of s: $\mathrm{Part} = \coprod_{s \geq 1} \mathrm{Part}(s)$.

Notation 2.2.2. — Let $f : \mathcal{I}_0 \longrightarrow \mathrm{Part}$ with finite support. We then denote by τ_f the restriction to I_M of
$$\bigoplus_{\tau_0 \in \mathcal{I}_0} \bigoplus_i \mathrm{Sp}(\rho_{\tau_0}, f(\tau_0)(i)),$$
where ρ_{τ_0} is a fixed extension of τ_0 to W_M.

Remark. By lemma 3.3 of [**MS21**] the isomorphism class of τ_f is independent of the choices of the ρ_{τ_0}.

The map from $\{f : \mathcal{I}_0 \longrightarrow \mathrm{Part}\}$ to the set of inertial types given by $f \mapsto \tau_f$, is a bijection. The dominance order $\preceq$ on Part induces a partial order on the set of inertial types.

We let rec_M denote the local reciprocity map of [**HT01**, Theorem A]. Fix an isomorphism $\imath : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. We normalize the local reciprocity map rec of [**HT01**, Theorem A], defined on isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(M)$ over $\mathbb{C}$ as follows: if π is the isomorphism class of an irreducible smooth representation of $\mathrm{GL}_n(M)$ over $\overline{\mathbb{Q}}_\ell$, then
$$\rho_\ell(\pi) \stackrel{\mathrm{def}}{=} \imath^{-1} \circ \mathrm{rec}_M \circ \imath(\pi \otimes_{\overline{\mathbb{Q}}_\ell} |\det|^{(1-n)/2}).$$

Then $\rho_\ell(\pi)$ is the isomorphism class of an n-dimensional, Frobenius semisimple Weil–Deligne representation of W_M over $\overline{\mathbb{Q}}_\ell$, independent of the choice of $\imath$. Moreover, if ρ is an isomorphism class of an n-dimensional, Frobenius semisimple Weil–Deligne representation of W_M over M, then $\rho_\ell^{-1}(\rho)$ is defined over M (cf. [**CEG**$^+$**16**, §1.8]).

Recall the following compatibility of the Langlands correspondence.

Lemma 2.2.3. — *If π and π' are irreducible generic representations of $\mathrm{GL}_d(M)$ such that $\rho_\ell(\pi)|I_M \cong \rho_\ell(\pi')|I_M$ then $\pi_{|\,\mathrm{GL}_d(\mathcal{O}_M)} \cong \pi'_{|\,\mathrm{GL}_d(\mathcal{O}_M)}$.*

Theorem 2.2.4. — *(cf.* [**BC09**] *proposition 6.3.3 or* [**Sho18**] *theorem 3.7)*
Let $G = \mathrm{GL}_n$. Let τ be a inertial type for M. Then there is a smooth irreducible $\mathrm{GL}_n(\mathcal{O}_M)$-representation $\sigma(\tau)$ over E such that for any irreducible admissible representation π of $\mathrm{GL}_n(M)$, one has:

(i) *if $\pi|_{\mathrm{GL}_n(\mathcal{O}_M)}$ contains $\sigma(\tau)$ then $\rho_\ell(\pi)|_{I_M} \preceq \tau$;*
(ii) *if $\rho_\ell(\pi)|_{I_M} = \tau$, then $\pi|_{\mathrm{GL}_n(\mathcal{O}_M)}$ contains $\sigma(\tau)$ with multiplicity one;*
(iii) *if $\rho_\ell(\pi)|_{I_M} \preceq \tau$ and π is generic, then $\pi|_{\mathrm{GL}_n(\mathcal{O}_M)}$ contains $\sigma(\tau)$ and the multiplicity is one if furthermore τ is maximal with respect to $\preceq$.*

Remark. For example take τ_0 the trivial representation and consider the following partitions $\underline{d} = (1 \geq 1 \geq \cdots \geq 1)$ (resp. (d)) denoted also by (1^d). Denote then by τ the associated inertial type of notation 2.2.2. Then $\sigma(\tau)$ is the trivial representation (resp. is inflated from the Steinberg representation of $\mathrm{GL}_d(\kappa)$). Note also that π contains $\sigma(\tau)$ if and only if π is unramified (resp. it implies that $r_\ell(\pi)|I_M$ is unipotent).

We need more details about the construction of $\sigma(\tau)$ which rests on the notion of SZ-stratum of [**SZ99**]. We first recall quickly the basic notions of type theory of Bushnell and Kutzko. Let $\mathbb{K}$ denote $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$. Let V be a vector space over F and let $G = \mathrm{Aut}_F(V)$ and $A = \mathrm{End}_F(V)$.

- An $\mathcal{O}_F$-lattice chain in V is a sequence $\mathcal{L} = (\Lambda_i)_{i\in\mathbb{Z}}$ of $\mathcal{O}_F$-lattices in$V$ such that $\Lambda_{i+1} \subseteq \Lambda_i$ for all $i \in \mathbb{Z}$ and such that there exists an period $e \geq 1$ with $\Lambda_{i+i} = \mathfrak{P}_M \Lambda_i$ for all $i \in \mathbb{Z}$.

- The hereditary $\mathcal{O}_M$-orders in A are those orders $\mathfrak{U}$ that arise as the stabiliser of some $\mathcal{O}_F$-lattice chain; it is maximal if and only if it stabilises a lattice chain of period $e = 1$.

- A hereditary order $\mathfrak{U} \subseteq A$ has a unique two-sided maximal ideal $\mathfrak{P} = \{x \in \mathfrak{U} : x\Lambda_i \subseteq \Lambda_{i+1}$ for all $i \in \mathbb{Z}\}$. We write $U(\mathfrak{U})$ for the group of units in $\mathfrak{U}$ and $U^1(\mathfrak{U}) := 1 + \mathfrak{P}$.

Definition 2.2.5. — A simple strata in A is $[\mathfrak{U}, m, 0, \beta]$ where

- $\mathfrak{U}$ is a hereditary $\mathcal{O}_K$-order in A;
- $m > 0$ is an integer;
- $\beta \in \mathfrak{P}^{-m} \setminus \mathfrak{P}^{1-m}$ is such that $E := L[\beta]$ is a field and $E^\times$ is contained in the normalized of $U(\mathfrak{U})$;

– the number $k_0(\beta, \mathfrak{U}(E))$ defined in [**BK93**] §1.4 is strictly negative.

The chain lattice defining $\mathfrak{U}$ can be seen as an $\mathcal{O}_E$-lattice chain and we denote $\mathfrak{B} := \mathfrak{U} \cap \mathrm{End}_E(V)$ its stabiliser. We define the groups $U(\mathfrak{B})$ and $U^1(\mathfrak{B})$ as for $\mathfrak{U}$. In [**BK93**], the authors associate to the simple stratum $[\mathfrak{U}, m, 0, \beta]$, compact open subgroups $J = J(\beta, \mathfrak{U})$, $J^1 = J^1(\beta, \mathfrak{U})$ and $H = H^1(\beta, \mathfrak{U})$ of $U(\mathfrak{U})$ such that:

– J^1 is a normal prop-p- subgroup of J;
– H^1 is a normal subgroup of J^1;
– $U(\mathfrak{B}) \subseteq J$ and $U^1(\mathfrak{B}) \subseteq J^1$ and the induced map

$$U(\mathfrak{B})/U^1(\mathfrak{B}) \longrightarrow J/J^1$$

is a isomorphism.

By [**BK93**] §3.2, there is a set $\mathcal{C}(\mathfrak{U}, 0, \beta)$ of simple characters of $H^1(\beta, \mathfrak{U})$. For $\theta \in \mathcal{C}(\mathfrak{U}, 0, \beta)$ there is a unique irreducible representation η of $J^1(\beta, \mathfrak{U})$ whose restriction to $H^1(\beta, \mathfrak{U})$ contains θ. There is then a distinguished class of β-extensions κ of η to $J(\beta, \mathfrak{U})$.

It is the main result of [**BK99**] that every Berstein component of $\mathrm{Rep}_{\mathbb{K}}(G)$ has a type and give an explicit construction of them. When Ω is supercuspidal then they construct a type (J, λ) such that $J = J(\beta, \mathfrak{U})$ for a simple stratum $[\mathfrak{U}, m, 0, \beta]$ in A in which $\mathfrak{B}$ is a maximal $\mathcal{O}_E$-order and $\lambda = \kappa \otimes \nu$ where κ is a β-extension of an irreducible representation η containing a simple character $\theta \in \mathcal{C}(\mathfrak{U}, 0, \beta)$ and ν is a cuspidal representation of $J/J^1 \cong \mathrm{GL}_{n/[E:L]}(k_E)$. This type is called maximal.

From [**BK99**] §4, the character θ determines a ps-character $(\Theta, 0, \beta)$ viewed as a function Θ on the set of simple strata $[\mathfrak{U}, m, 0, \beta]$ taking such a stratum to an element $\Theta(\mathfrak{U}) \in \mathcal{C}(\mathfrak{U}, 0, \beta)$. By [**BK99**] §4.5, the endo-class of this ps-character is determine by Ω.

If $[\mathbb{M}, \pi]$ is an inertial equivalence class of supercuspidal pair corresponding to a Berstein component Ω of $\mathrm{Rep}_{\mathbb{K}}(G)$, then let $(J_{\mathbb{M}}, \lambda_{\mathbb{M}})$ be a maximal type for the supercuspidal Berstein component $\Omega_{\mathbb{M}}$ of $\mathrm{Rep}_M(\mathbb{M})$ containing π. By the results of [**BK99**], there is, what is called a G-cover, (J, λ) of $(J_{\mathbb{M}}, \lambda_{\mathbb{M}})$ which is a type for Ω.

Definition 2.2.6. — (cf. [**Sho18**] definition 6.11)
A SZ-datum over $\mathbb{K}$, is a set

$$\mathfrak{S} = \Big\{ (E_i, \beta_i, V_i, \mathfrak{B}_i, \lambda_i), \ i = 1, \cdots, r \Big\}$$

where r is a positive integer and for each $i = 1, \cdots, r$ we have

– E_i/F is a finite extension generated by an element $\beta_i \in E_i$,
– V_i is an E_i-vector space of finite dimension N_i,
– $\mathfrak{B}_i \subseteq \mathrm{End}_{E_i}(V_i)$ is a maximal hereditary $\mathcal{O}_{E_i}$-order,

– let denote $\mathfrak{U}_i$ the associated $\mathcal{O}_M$-order in $A_i := \mathrm{End}_M(V_i)$ and let $m_i := -v_{E_i}(\beta_i)$. Then $[\mathfrak{U}_i, m_i, 0, \beta_i]$ is a simple stratum and λ_i is an $\mathbb{K}$-representation of $J_i := J(\beta_i, \mathfrak{U}_i)$ of the form $\kappa_i \otimes \nu_i$ where κ_i is a β_i-extension of the representation η_i of $J_i^1 := J^1(\beta_i, \mathfrak{U}_i)$ containing some simple character $\theta_i \in \mathcal{C}(\mathfrak{U}_i, 0, \beta_i)$ of $H_i^1 = H^1(\beta_i, \mathfrak{U}_i)$ and ν_i is an irreducible representation of $U(\mathfrak{B}_i)/U^1(\mathfrak{B}_i) \cong \mathrm{GL}_{N_i}(k_{E_i})$ over $\mathbb{K}$;
– no two of the θ_i are endo-equivalent in the sense of [**BK99**] §4.

Let $V = \bigoplus_{i=1}^r V_i$, $A = \mathrm{End}_M(V)$ and $G = \mathrm{Aut}_M(V)$. The Levi subgroup $\mathbb{M} = \prod_{i=1}^r \mathrm{Aut}_M(A_i) \subseteq G$ has compact open subgroups $J_{\mathbb{M}}^1 \lhd J_{\mathbb{M}}$ where $J_{\mathbb{M}}^1 = \prod_{i=1}^r J_i^1$ and similarly for $J_{\mathbb{M}}$. Let denote by $\eta_{\mathbb{M}} = \bigotimes_{i=1}^r \eta_i$, a representation of $J_{\mathbb{M}}^1$ and similarly for the representations $\kappa_{\mathbb{M}}$ and $\lambda_{\mathbb{M}}$ of $J_{\mathbb{M}}$. Note that $\eta_{\mathbb{M}}$ and $\kappa_{\mathbb{M}}$ are clearly irreducible and $\lambda_{\mathbb{M}}$ is irreducible by [**Sho18**] propositions 6.12.

Since no two of the θ_i are endo-equivalent, by [**BK99**] §8, see also [**MS14**] proposition 2.28, we have compact open subgroups J and J^1 of G and representations η of J^1, κ of J and λ of J such that (J^1, η) (resp. (J, κ), resp. (J, λ)) is a G-cover of $(J_{\mathbb{M}}^1, \eta_{\mathbb{M}})$ (resp. $(J_{\mathbb{M}}, \kappa_{\mathbb{M}})$, resp. $(J_{\mathbb{M}}, \lambda_{\mathbb{M}})$) and $J/J^1 = J_{\mathbb{M}}/J_{\mathbb{M}}^1$ with $\lambda = \kappa \otimes (\bigotimes_{i=1}^r \nu_i)$ under this identification.

Definition 2.2.7. — For $\mathfrak{S}$ an SZ-stratum and J and λ as above. Let K be a maximal compact subgroup of G such that $J_{\mathbb{M}} \subseteq K \cap \mathbb{M}$. We then denote by

$$\sigma(\mathfrak{S}) := \mathrm{ind}_J^K(\lambda),$$

which by [**Sho18**] theorem 6.16 is irreducible.

Start now from $\mathcal{P} \in \mathcal{I} = \{f : \mathcal{I}_0 \longrightarrow \mathrm{Part}\}$ and let $n = \sum_{\tau_0 \in \mathcal{I}_0} \sum f(\tau_0) \dim \tau_0$. Let V be a L-vector space of dimension n and let $G = \mathrm{Aut}_M(V)$. Let $(\mathbb{M}^0, \pi)$ be a supercuspidal pair in the inertial equivalence class associated to the Bernstein component attached to $\mathcal{P}$. Write $\mathbb{M}^0 = \prod_{i=1}^t \mathbb{M}_i^0$ where each $\mathbb{M}_i^0$ is the stabiliser of some n_i-dimensional subspace V_i^0 of V. Write $\pi = \bigotimes_{i=1}^t \pi_i$ and let Ω_i be the supercuspidal Berstein component containing π_i. For each Ω_i there is an associated endo-class of ps-character $\Theta_i^0 = (\Theta_i^0, 0, \beta_i^0)$. Let $\mathbb{M} = \prod_{i=1}^r \mathbb{M}_i^0$ the Levi subgroup of G obtaining from $\mathbb{M}$ by gathering $\mathbb{M}_j^0$ with $\mathbb{M}_k^0$ if and only if $\Theta_j^0 = \Theta_k^0$: we then write simply $(\Theta_i, 0, \beta_i)$ for the common value.

We attach to $\mathcal{P}$ a SZ-datum

$$\mathfrak{S}_{\mathcal{P}} = \Big\{ (E_i, \beta_i, V_i, \mathfrak{B}_i, \kappa_i \otimes \nu_i) : i = 1, \cdots, r \Big\}$$

as follows. Suppose first that $r = 1$ and write $(\Theta, 0, \beta)$ for the common value of $(\Theta_i^0, 0, \beta_i^0)$.

– $E = L[\beta]$;
– let (J_i^0, λ_i^0) be the maximal simple type for the supercuspidal Berstein component Ω_i^0 with $J_i^0 = J(\beta, UF_i^0)$ for a simple stratum $[\mathfrak{U}_i^0, m_i^0, 0, \beta_i^0]$ and λ_i^0 contains

$\theta_i^0 := \Theta(\mathfrak{U}_i^0, 0, \beta_i)$. As above we then have compact open subgroups $J^1 \subseteq J$ of G and a representation η of J^1 containing the simple character $\Theta(\mathfrak{U}, 0, \beta)$ where $\mathfrak{U}$ is a hereditary $\mathcal{O}_M$-order in A and $\mathcal{U}_i \cap B = \mathfrak{B}$ is a maximal hereditary $\mathcal{O}_E$-order.

- We choose compatible β-extensions, in the sense of [**Sho18**] §6.6, κ_i^0 of η_i^0 coming from a β-extension κ of η, and decompose each λ_i^0 as $\kappa_i^0 \otimes \nu_i^0$ where ν_i^0 is a cuspidal representation of $J_i^0/J_i^{1,0} = U(\mathfrak{B}_i^0)/U^1(\mathfrak{B}_i^0) \cong \mathrm{GL}_{n_i^0/[E:L]}(k_E)$ for an integer n_i^0. Then $J/J^1 \cong \mathrm{GL}_{n/[E:L]}(k_E)$ so that we can view each ν_i^0 as an element of $\overline{\mathcal{I}}_0$ and define an element $\overline{\mathcal{P}} \in \overline{\mathcal{I}}$ by $\overline{\mathcal{P}}(\nu_i^0) = \mathcal{P}(\tau_i)$ where $\tau_i \in \mathcal{I}_0$ corresponds to Ω_i.
- Write $\mathrm{st}(\overline{\mathcal{P}})$ for the irreducible representation of $\mathbb{M}_{\overline{\mathcal{P}}}(\kappa_E)$ whose tensor factors are the $\mathrm{st}(\sigma, \overline{\mathcal{P}}(\sigma)(i))$ for each (σ, i). We then denote by $\pi_{\overline{\mathcal{P}}} := \mathrm{ind}_{\mathbb{L}_{\overline{\mathcal{P}}}(\kappa_E)}^{\mathrm{GL}_{n/[E:L]}(\kappa_E)} \mathrm{st}(\overline{\mathcal{P}})$ where $\mathbb{L}_{\overline{\mathcal{P}}}$ is any parabolic subgroup with Levi factor $\mathbb{M}_{\overline{\mathcal{P}}}$. Let $\sigma_{\overline{\mathcal{P}}}$ be the unique irreducible representation of $\pi_{\overline{\mathcal{P}}}$ that is not contained in $\pi_{\overline{\mathcal{P}}'}$ for any $\overline{\mathcal{P}} \preceq \overline{\mathcal{P}}'$. We then write $\nu = \sigma_{\overline{\mathcal{P}}}$ a representation of $\mathrm{GL}_{n/[E:L]}(k_E)$ and see it as a representation of J/J^1.
- We repeat this construction for every $i = 1, \cdots, r$.

Notation 2.2.8. — For $\tau = \tau_{\mathcal{P}}$ we write $\sigma(\tau) := \sigma(\mathfrak{S}_{\mathcal{P}})$.

Theorem 2.2.9. — *(*[**Sho18**] *6.20)*
Let $\mathcal{P}' \in \mathcal{I}$ with degree n and let (M', π') be any discrete pair in the inertial equivalence class associated to $\mathcal{P}'$. For any parabolic subgroup Q' of G with Levi M', we have

$$\dim \mathrm{Hom}_K(\sigma(\mathfrak{S}_{\mathcal{P}}), \mathrm{ind}_{Q'}^G \pi') = \prod_{\tau_0 \in \mathcal{I}_0} m(\mathcal{P}(\tau_0), \mathcal{P}'(\tau_0)),$$

where for any two partition $\underline{\lambda}, \underline{\mu}$ of n, $m(\underline{\lambda}, \underline{\mu})$ is the usual Kostka number which counts the number of possible ways to fill the Young tableau with lines of respective size $\lambda_1, \cdots$ with μ_1 one, μ_2 two and so on, with the following properties: the sequence of labels in each columns is strictly increasing, while it is increasing in each lines.

Note that

- if $\underline{\lambda} < \underline{\mu}$ then $m(\underline{\lambda}, \underline{\mu}) = 0$;
- $m(\underline{\lambda}, \underline{\lambda}) = 1$;
- $m((n), \underline{\mu}) = 1$.

In particular if $\mathcal{P}$ is maximal that is if for every $\tau_0 \in \mathcal{I}_0$ we have $\mathcal{P}(\tau_0) = (\sum \mathcal{P}(\tau_0))$, then the multiplicity of $\sigma(\mathfrak{S}_{\mathcal{P}})$ in $\mathrm{ind}_{Q'}^G \pi'$ is 1 if $\sum \mathcal{P}'(\tau_0) = \sum \mathcal{P}(\tau_0)$ for every $\tau_0 \in \mathcal{I}_0$, otherwise it is 0.

Let $\mathfrak{S} = \Big\{ (E_i, \beta_i, V_i, \mathfrak{B}_i, \lambda_i); i = 1, \cdots, r \Big\}$ be an SZ-datum. Write $\lambda_i = \kappa_i \otimes \nu_i$ where the ν_i are irreducible representations of J_i/J_i^1. Note that the modulo l reduction $\overline{\kappa}_i$ of κ_i remains irreducible. We then decompose the semisimplification of the modulo

l reduction of ν_i:

$$\overline{\nu}_i^{ss} = \bigoplus_{j \in S_i} \mu_{i,j} \nu_{i,j}$$

where S_i is some finite indexing set, $\nu_{i,j}$ are distinct irreducible $\overline{\mathbb{F}}_l$-representations and $\mu_{i,j} \in \mathbb{N}$. For $\underline{j} = (j_1, \cdots, j_r) \in S_1 \times \cdots \times S_r$, define an SZ-datum $\mathfrak{S}_{\underline{j}}$ over $\overline{\mathbb{F}}_l$ by

$$\mathfrak{S}_{\underline{j}} = \Big\{ (E_i, \beta_i, V_i, \mathfrak{B}_i, \overline{\kappa}_i \otimes \nu_{i,j_i}); i = 1, \cdots, r \Big\}.$$

Theorem 2.2.10. — *(cf* [**Sho18**] *6.23)*
The semisimplified modulo l reduction of $\sigma(\mathfrak{S})$ is

$$\bigoplus_{\underline{j} \in S_1 \times \cdots \times S_r} \mu_{\underline{j}} \sigma(\mathfrak{S}_{\underline{j}}),$$

where $\mu_{\underline{j}} := \prod_{i=1}^r \mu_{i,j_i}$.

2.3. Kottwiz–Harris–Taylor Shimura varieties. — Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is a quadratic imaginary extension and $F^+/\mathbb{Q}$ is totally real. We fix a real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we will denote by F_v the completion of F at v, $\mathcal{O}_v$ its ring of integers with uniformizer ϖ_v and residue field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$ of cardinal q_v.

Let B be a division algebra with center F, of dimension d^2 such that at every place v of F, either B_v is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta \in B^{*=-1}$, denote by $\sharp_\beta$ the involution $v \mapsto v^{\sharp_\beta} = \beta v^* \beta^{-1}$ and let $G/\mathbb{Q}$ be the group of similitudes, denoted by G_τ in [**HT01**], defined for every $\mathbb{Q}$-algebra R by

$$G(R) \cong \{ (\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda \}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$ split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \cong (B_y^{op})^\times \times \mathbb{Q}_x^\times \cong \mathbb{Q}_x^\times \times \prod_{v_i^+} (B_{v_i^+}^{op})^\times, \tag{3}$$

where $x = \prod_i v_i^+$ in F^+ and we identify places of F^+ over x with places of F over y.

Convention 2.3.1. — For $x = yy^c$ a place of $\mathbb{Q}$ split in M and v a place of F over y, we shall make throughout the text the following abuse of notation: we denote $G(F_v)$ the factor $(B_{v_{|F^+}}^{op})^\times$ in the formula (3) so that

$$G(\mathbb{A}_{\mathbb{Q}}^{\infty,v}) := G(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \times \Big(\mathbb{Q}_p^\times \times \prod_{v_i^+ \neq v|F^+} (B_{v_i^+}^{op})^\times \Big).$$

In [**HT01**], the authors justify the existence of some G like before such that

- if x is a place of $\mathbb{Q}$ non split in M then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [**HT01**, page 90], a compact open subgroup K of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is said to be *sufficiently small* if there exists a place x of $\mathbb{Q}$ such that the projection from K^x to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

Notation 2.3.2. — Denote by $\mathcal{K}$ the set of sufficiently small compact open subgroups of $G(\mathbb{A}^{\infty})$. For $K \in \mathcal{K}$, write $\mathrm{Sh}_{K,\eta} \longrightarrow \mathrm{Spec}\, F$ for the associated Shimura variety of Kottwitz-Harris-Taylor type.

Definition 2.3.3. — Denote by Spl the set of places w of F such that $p_w := w_{|\mathbb{Q}} \neq l$ is split in E and $B_w^{\times} \cong \mathrm{GL}_d(F_w)$. For each $K \in \mathcal{K}$, we write $\mathrm{Spl}(K)$ for the subset of Spl of places such that K_v is the standard maximal compact of $\mathrm{GL}_d(F_v)$.

In the sequel, we fix a place v of F in Spl. The scheme $\mathrm{Sh}_{K,\eta}$ has a projective model $\mathrm{Sh}_{K,v}$ over $\mathrm{Spec}\, \mathcal{O}_v$ with special geometric fiber $\mathrm{Sh}_{K,\bar{s}_v}$. We have a projective system $(\mathrm{Sh}_{K,\bar{s}_v})_{K \in \mathcal{K}}$ which is naturally equipped with an action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathbb{Z}$ such that any $w_v \in W_{F_v}$ acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}_{F_v}^{-1}$ and $\mathrm{Art}_{F_v} : F_v^{\times} \xrightarrow{\sim} W_{F_v}^{ab}$.

Notation 2.3.4. — For $K \in \mathcal{K}$, the Newton stratification of the geometric special fiber $\mathrm{Sh}_{K,\bar{s}_v}$ is denoted by

$$\mathrm{Sh}_{K,\bar{s}_v} =: \mathrm{Sh}_{K,\bar{s}_v}^{\geq 1} \supset \mathrm{Sh}_{K,\bar{s}_v}^{\geq 2} \supset \cdots \supset \mathrm{Sh}_{K,\bar{s}_v}^{\geq d}$$

where $\mathrm{Sh}_{K,\bar{s}_v}^{=h} := \mathrm{Sh}_{K,\bar{s}_v}^{\geq h} - \mathrm{Sh}_{K,\bar{s}_v}^{\geq h+1}$ is an affine scheme, which is smooth and pure of dimension $d-h$. It is built up by the geometric points such that the connected part of the associated Barsotti–Tate group has rank h For each $1 \leq h < d$, write

$$i_h : \mathrm{Sh}_{K,\bar{s}_v}^{\geq h} \hookrightarrow \mathrm{Sh}_{K,\bar{s}_v}^{\geq 1}, \quad j^{\geq h} : \mathrm{Sh}_{K,\bar{s}_v}^{=h} \hookrightarrow \mathrm{Sh}_{K,\bar{s}_v}^{\geq h},$$

and $j^{=h} = i_h \circ j^{\geq h}$.

For $n \geq 1$, with our previous abuse of notation, consider $K^v(n) := K^v K_v(n)$ where

$$K_v(n) := \ker(\mathrm{GL}_d(\mathcal{O}_v) \twoheadrightarrow \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)).$$

Recall that $\mathrm{Sh}_{I^v(n),\bar{s}_v}^{=h}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely this means there exists a closed subscheme $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h}$ stabilized by the Hecke action of $P_{h,d}(F_v)$ and such that

$$\mathrm{Sh}_{K^v(n),\bar{s}_v}^{=h} = \mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h} \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n), \tag{4}$$

meaning that $\mathrm{Sh}_{K^v(n),\bar{s}_v}^{=h}$ is the disjoint union of copies of $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h}$ indexed by $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ and exchanged by the action of $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)$. We will denote by $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{\geq h}$ the closure of $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h}$ inside $\mathrm{Sh}_{K^v(n),\bar{s}_v}$.

Notation 2.3.5. — Let $1 \leq g \leq d$ and π_v be an irreducible cuspidal representation of $\mathrm{GL}_g(F_v)$. For $1 \leq t \leq s := \lfloor d/g \rfloor$, let Π_t any representation of $\mathrm{GL}_{tg}(F_v)$. We denote by

$$\widetilde{HT}_1(\pi_v, \Pi_t) := \mathcal{L}(\pi_v, t)_1 \otimes \Pi_t^{K_v(n)} \otimes \Xi^{\frac{t-s}{2}}$$

the Harris-Taylor local system on the Newton stratum $\mathrm{Sh}^{=tg}_{K^v(n),\bar{s}_v,1}$ where

- $\mathcal{L}(\pi_v,t)_1$ is the $\overline{\mathbb{Z}}_l$-local system given by Igusa varieties of [**HT01**] and associated to the representation $\pi_v[t]_D$ of the division algebra $D_{v,tg}/F_v$ with invariant $1/tg$, corresponding through Jacquet-Langlands correspondance to $\mathrm{st}_t(\pi_v^\vee)$: cf. [**Boy09**] §1.4 for more details;
- $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ is defined by $\Xi(\frac{1}{2}) = q^{1/2}$.

We also introduce the induced version

$$\widetilde{HT}(\pi_v,\Pi_t) := \Big(\mathcal{L}(\pi_v,t)_1 \otimes \Pi_t^{K_v(n)} \otimes \Xi^{\frac{t-s}{2}}\Big) \times_{P_{tg,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n),$$

where the unipotent radical of $P_{tg,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d}(\mathcal{O}_v/\mathcal{M}_v^n) \times W_v$$

is given

- by the action of g_v^c on $\Pi_t^{K_v(n)}$ and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{t-s}{2}}$, and
- the action of $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg\sigma_v) \in G(\mathbb{A}^{\infty,v}) \times \mathrm{GL}_{d-tg}(\mathcal{O}_v/\mathcal{M}_v^n) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v)_1 \otimes \Xi^{\frac{t-s}{2}}$.

We also introduce

$$HT(\pi_v,\Pi_t)_1 := \widetilde{HT}(\pi_v,\Pi_t)_1[d-tg],$$

and the perverse sheaf

$$P(t,\pi_v)_1 := {}^p j^{=tg}_{1,!*} HT(\pi_v, \mathrm{St}_t(\pi_v))_1 \otimes L_g(\pi_v)^\vee,$$

and their induced version, $HT(\pi_v,\Pi_t)$ and $P(t,\pi_v)$, where $L_g(\pi_v)^\vee$ is the contragredient of the representation of dimension g of $\mathrm{Gal}(\overline{F}_v/F_v)$ associated to π_v by the Langlands correspondence L_g.

Important property: over $\overline{\mathbb{Z}}_l$, there are at least two notions of intermediate extension associated to the two classical t-structures p and $p+$. By proposition 2.4.1 of [**Boy23b**], in the limit case where all $\overline{\mathbb{F}}_l$-cuspidal representations are supercuspidal, as recalled after proposition 2.1.4, all the p and $p+$ intermediate extensions of Harris-Taylor local systems coincide. The arguments in loc. cit. are rather difficult but if one accepts to restrict to the case where the irreducible constituants of $\overline{\rho}_{\mathfrak{m}}$ are all characters, then the proof of this fact is easy. Indeed as $\mathrm{Sh}^{\geq h}_{K^v(n),\bar{s}_v,1}$ is smooth over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, then $HT(\chi_v,\Pi_h)_1$ is perverse for the two t-structures with

$$i_1^{h\leq +1,*} HT(\chi_v,\Pi_h)_1 \in {}^p\mathcal{D}^{<0} \text{ and } i_1^{h\leq +1,!} HT(\chi_v,\Pi_h)_1 \in {}^{p+}\mathcal{D}^{\geq 1}.$$

Let now denote by

$$\Psi_{K,v} := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])(\frac{d-1}{2})$$

the nearby cycles autodual free perverse sheaf on $\mathrm{Sh}_{K,\bar{s}_v}$. Recall, cf. [**Boy23b**] proposition 3.1.3, that

$$\Psi_{K,v} \cong \bigoplus_{1\leq g\leq d} \bigoplus_{\varrho\in \mathrm{Scusp}(g)} \Psi_{K,\varrho}, \tag{5}$$

where

- $\mathrm{Scusp}(g)$ is the set of equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations of $\mathrm{GL}_g(F_v)$.
- The irreducible sub-quotients of $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are the Harris-Taylor perverse sheaves of $\Psi_{K,\overline{\mathbb{Q}}_l}$ associated to irreducible cuspidal representations π_v with modulo l reduction having supercuspidal support a Zelevinsky segment associated to ϱ.

In the limit case when $q_v \equiv 1 \mod l$ and $l > d$, recall that we do not have to bother about cuspidal $\overline{\mathbb{F}}_l$-representation which are not supercuspidal. In particular in the previous formula we can

- replace $\mathrm{Scusp}(g)$ by the set $\mathrm{Cusp}(g)$ of equivalence classes of cuspidal representations,
- and the Harris-Taylor perverse sheaves of $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are those associated to π_v such that its modulo l reduction is isomorphic to ϱ.

3. Nearby cycles and filtrations

3.1. Filtrations of stratification of Ψ_ϱ. — We now fix an irreducible $\overline{\mathbb{F}}_l$-cuspidal representation ϱ of $\mathrm{GL}_g(F_v)$ for some $1 \leq g \leq d$. We also introduce $s = \lfloor d/g \rfloor$.

Using the Newton stratification and following the constructions of [**Boy14**], we can define a $\overline{\mathbb{Z}}_l$-filtration

$$\mathrm{Fil}^0_!(\Psi_{K,\varrho}) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^s_!(\Psi_{K,\varrho}) = \Psi_{K,\varrho}$$

where $\mathrm{Fil}^t_!(\Psi_{K,\varrho})$ is the saturated image of $j^{=tg}_! j^{=tg,*}\Psi_{K,\varrho} \longrightarrow \Psi_{K,\varrho}$. We also denote by $\mathrm{coFil}^t_!(\Psi_\varrho) := \Psi_\varrho / \mathrm{Fil}^t_!(\Psi_\varrho)$. Dually we can define a cofiltration

$$\Psi_{K,\varrho} = \mathrm{coFil}^s_*(\Psi_{K,\varrho}) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}^1_*(\Psi_{K,\varrho})$$

where $\mathrm{coFil}^t_*(\Psi_{K,\varrho})$ is the saturated image of $\Psi_{K,\varrho} \longrightarrow j^{=tg}_* j^{=tg,*}\Psi_{K,\varrho}$: cf. figure 2 for an illustration. We denote by $\mathrm{Fil}^t_*(\Psi_\varrho) := \ker(\Psi_\varrho \twoheadrightarrow \mathrm{coFil}^t_*(\Psi_\varrho))$.

Over $\overline{\mathbb{Q}}_l$, the filtration $\mathrm{Fil}^\bullet_!(\Psi_{K,\varrho})$ coincides with the iterated kernel of N_v, i.e. $\mathrm{Fil}^t_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \cong \ker(N^t_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. Dually the cofiltration $\mathrm{coFil}^\bullet_!(\Psi_{K,\varrho})$ coincides with the iterated image of N_v, i.e. the kernel of $\Psi_{K,\varrho} \twoheadrightarrow \mathrm{coFil}^t_*(\Psi_{K,\varrho})$ is the image of N^t_v. Note that by Grothendieck-Verdier duality, we have $D(\mathrm{Fil}^t_!(\Psi_{K,\varrho})) \cong \mathrm{coFil}^t_*(\Psi_{K,\varrho})$.

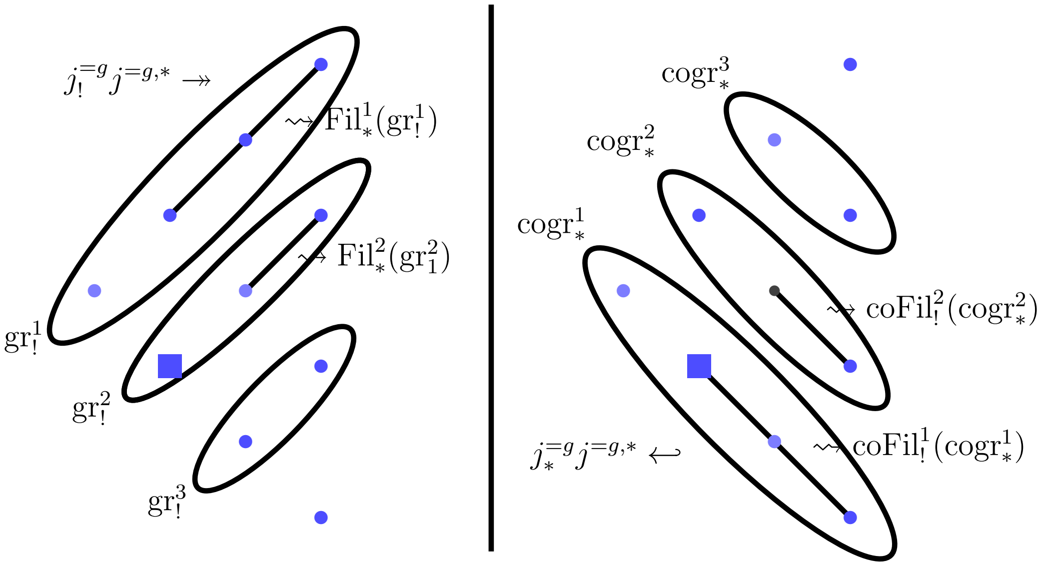

FIGURE 2. Filtrations of stratification of $\Psi_{K,\varrho}$

The graded parts $\mathrm{gr}^t_!(\Psi_{K,\varrho})$ are, by construction, free and admit a strict[10] filtration, cf. [**Boy14**] corollary 3.4.5

$$\mathrm{Fil}^{s-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho})) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^{k-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho})) = \mathrm{gr}^k_!(\Psi_{K,\varrho})$$

with

$$\mathrm{gr}^{i-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} P(i,\pi_v)(\frac{i+1-2k}{2}),$$

where $\mathrm{Cusp}(\varrho)$ is the set of equivalence classes of irreducible cuspidal representations with modulo l reduction isomorphic to ϱ.

Dually, $\mathrm{cogr}^k_*(\Psi_{K,\varrho})$ has a cofiltration

$$\mathrm{cogr}^k_*(\Psi_{K,\varrho}) = \mathrm{coFil}^{k-1}_!(\mathrm{cogr}^k_*(\Psi_{K,\varrho})) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}^{s-1}_!(\mathrm{cogr}^k_*(\Psi_{K,\varrho})),$$

with

$$\mathrm{cogr}^{i-1}_!(\mathrm{cogr}^k_*(\Psi_{K,\varrho})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} P(i,\pi_v)(\frac{2k-i-1}{2}).$$

Concerning the $\overline{\mathbb{Z}}_l$-structures, cf. the third global result of the introduction of [**Boy23b**] , for every $1 \leq k \leq i \leq s$, we have strict epimorphisms[11]

$$j_!^{=ig} j^{=ig,*} \,\mathrm{Fil}^{i-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho})) \twoheadrightarrow \mathrm{Fil}^{i-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho}))$$

(10) meaning the graded parts are free

(11) strict means that the cokernel is torsion free

as well as strict monomorphisms

$$\mathrm{coFil}_!^{i-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})) \hookrightarrow j_*^{=ig} j^{=ig,*}\, \mathrm{coFil}^{i-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})).$$

Exchange basic step: to go from one filtration to another, one can repeat the following process to exchange the order of appearance of two consecutive subquotient:

$$\begin{tikzcd}
 & P_1' \arrow[d, hook] \arrow[dr, hook] & & \\
P_2 \arrow[r, hook] \arrow[dr, hook] & X \arrow[r, two heads] \arrow[d] & P_1 \arrow[dr] & \\
 & P_2' \arrow[dr] & & T \arrow[dl, equal] \\
 & & T, &
\end{tikzcd}$$

where

- P_1 and P_2 are two consecutive subquotient in a given filtration and X is the subquotient gathering them as a subquotient of this filtration.
- Over $\overline{\mathbb{Q}}_l$, the extension $X \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is split, so that on can write X as an extension of P_2' by P_1' with $P_1' \hookrightarrow P_1$ and $P_2 \hookrightarrow P_2'$ have the same cokernel T, a perverse sheaf of torsion.

Remark. In the particular case when P_1 and P_2 are intermediate extensions of local systems living on respective strata of index h_1 and h_2 with $h_1 \neq h_2$, such that the two associated intermediate extensions for the p and $p+$ t-structure are isomorphic, then T is necessary zero and X is then split over $\overline{\mathbb{Z}}_l$. Indeed if T was not zero, then seen as a quotient of P_1 (resp. P_2') it has to be supported on the $\mathrm{Sh}_{K,\bar{s}_v}^{\geq h_1}$ (resp. $\mathrm{Sh}_{K,\bar{s}_v}^{\geq h_1}$) with $j^{=h_1,*}T \neq 0$ (resp. $j^{=h_2,*}T \neq 0$): the two conditions are then incompatible.

3.2. The abutment filtration of $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ is strict.

— We have spectral sequences

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}, \mathrm{gr}_*^{-p}(\mathrm{gr}_!^k(\Psi_{K,\varrho}))) \Rightarrow H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}\, \mathrm{gr}_!^k(\Psi_{K,\varrho})), \tag{6}$$

and

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}, \mathrm{gr}_!^{-p}(\Psi_{K,\varrho})) \Rightarrow H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K,\varrho}). \tag{7}$$

Definition 3.2.1. — For a finite set S of places of $\mathbb{Q}$ containing the places where G is ramified, denote by $\mathbb{T}_{abs}^S := \prod'_{x \notin S} \mathbb{T}_{x,abs}$ the abstract unramified Hecke algebra where $\mathbb{T}_{x,abs} \cong \mathbb{Z}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.

Example. For $w \in \mathrm{Spl}$, we have

$$\mathbb{T}_{w|_{\mathbb{Q}},abs} = \mathbb{Z}_l\big[T_{w',i}: \ i=1,\cdots,d, \ w'|(w|_{\mathbb{Q}})\big]$$

where $T_{w',i}$ is the characteristic function of

$$\mathrm{GL}_d(\mathcal{O}_{w'})\,\mathrm{diag}(\overbrace{\varpi_{w'},\cdots,\varpi_{w'}}^{i},\overbrace{1,\cdots,1}^{d-i})\,\mathrm{GL}_d(\mathcal{O}_{w'}) \subseteq \mathrm{GL}_d(F_{w'}).$$

Recall that $\mathbb{T}^S_{abs}$ acts through correspondances on each of the $H^i(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Z}}_l)$ where $K \in \mathcal{K}$ is maximal at each places outside S.

Notation 3.2.2. — For K unramified outside S, we denote by $\mathbb{T}(K)$ the image of $\mathbb{T}^S_{abs}$ inside $\mathrm{End}_{\overline{\mathbb{Z}}_l}(H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Z}}_l))$.

We also denote by

$$H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar\eta},\overline{\mathbb{Z}}_l) := \varinjlim_{K_v} H^{d-1}(\mathrm{Sh}_{K^vK_v,\bar\eta},\overline{\mathbb{Z}}_l), \tag{8}$$

where K_v describe the set of open compact subgroup of $\mathrm{GL}_d(\mathcal{O}_v)$. We also use similar notation for others cohomology groups.

Theorem 3.2.3. — *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}(K^v(\infty))$ such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible, cf. §5.2.*[12] *Then*

(i) $H^i(\mathrm{Sh}_{K^v(\infty),\bar\eta},\overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ is zero if $i \neq d-1$ and otherwise torsion free.
(ii) Moreover the spectral sequences (6) and (7), localized at $\mathfrak{m}$, degenerate at E_1 and the $E^{p,q}_{1,\mathfrak{m}}$ are zero for $p+q \neq 0$ and otherwise torsion free.

Proof. — (i) It is the main theorem of [**Boy23a**].

(ii) We follow closely the arguments of [**Boy23a**] dealing with all irreducible cuspidal representations instead of only characters in loc. cit. using crucially that in the limit case, the p and $p+$ intermediate extensions coincide exactly as it was the case for characters in loc. cit.

From (5) we are led to study the initial terms of the spectral sequence given by the filtration of $\Psi_{K^v(\infty),\varrho}$ for ϱ a irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation associated through local Langlands correspondance to an irreducible constituant of $\overline{\rho}_{\mathfrak{m},v}$. Recall also, as we are in the limit case, that

– as there do not exist irreducible $\overline{\mathbb{Q}}_l$-cuspidal representation of $\mathrm{GL}_g(F_v)$ for $g \leq d$ with modulo l reduction being not supercuspidal, the irreducible constituants of $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are the Harris-Taylor perverse sheaves $P(t,\pi_v)(\frac{t-1-2k}{2})$ where the modulo l reduction of π_v is isomorphic to ϱ and $0 \leq k < t$.
– Over $\overline{\mathbb{Z}}_l$, we do not have to worry about the difference between p and $p+$ intermediate extensions.

(12) Recall also that we suppose $q_v \equiv 1 \mod l$ and $l > d$.

From [**Boy23b**] §2.3, consider the following equivariant resolution

$$\begin{aligned}(9)\quad 0 \to j_!^{=sg} HT(\pi_v, \Pi_t\{\frac{t-s}{2}\} \times \mathrm{Speh}_{s-t}(\pi_v\{t/2\})) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \cdots \\ \longrightarrow j_!^{=(t+1)g} HT(\pi_v, \Pi_t\{-1/2\} \times \pi_v\{t/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow \\ j_!^{=tg} HT(\pi_v, \Pi_t) \longrightarrow {}^p j_{!*}^{=tg} HT(\pi_v, \Pi_t) \to 0,\end{aligned}$$

where Π_t is any representation of $\mathrm{GL}_{tg}(F_v)$, also called the infinitesimal part of the perverse sheaf ${}^p j_{!*}^{=tg} HT(\pi_v, \Pi_h)$.[13]

By adjunction property, for $1 \leq \delta \leq s-t$, the map

$$\begin{aligned}(10)\quad j_!^{=(t+\delta)g} HT(\pi_v, \Pi_t\{\frac{-\delta}{2}\} \times \mathrm{Speh}_\delta(\pi_v\{t/2\})) \otimes \Xi^{\delta/2} \\ \longrightarrow j_!^{=(t+\delta-1)g} HT(\pi_v, \Pi_t\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\pi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\end{aligned}$$

is given by

$$\begin{aligned}(11)\quad HT(\pi_v, \Pi_t\{\frac{-\delta}{2}\} \times \mathrm{Speh}_\delta(\pi_v\{t/2\})) \otimes \Xi^{\delta/2} \longrightarrow \\ j^{=(t+\delta)g,*}({}^p i^{(t+\delta)g,!}(j_!^{=(t+\delta-1)g} HT(\pi_v, \Pi_t\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}))\end{aligned}$$

To compute this last term we use the resolution (9) for $t+\delta-1$. Precisely denote by $\mathcal{H} := HT(\pi_v, \mathrm{st}_t(\pi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$, and write the previous resolution for $t+\delta-1$ as follows

$$0 \to K \longrightarrow j_!^{=(t+\delta)g} \mathcal{H}' \longrightarrow Q \to 0,$$
$$0 \to Q \longrightarrow j_!^{=(t+\delta-1)g} \mathcal{H} \longrightarrow {}^p j_{!*}^{=(t+\delta-1)g} \mathcal{H} \to 0,$$

with

$$\mathcal{H}' := HT\Big(\pi_v, \Pi_t\{\frac{1-\delta}{2}\} \times \big(\mathrm{Speh}_{\delta-1}(\pi_v\{-1/2\}) \times \pi_v\{\frac{\delta-1}{2}\}\big)\{t/2\}\Big) \otimes \Xi^{\delta/2}.$$

As the support of K is contained in $\mathrm{Sh}_{I,\bar{s}_v}^{\geq (t+\delta+1)g}$ then ${}^p i^{(t+\delta)g,!} K = K$ and $j^{=(t+\delta)g,*}({}^p i^{(t+\delta)g,!} K)$ is zero. Moreover ${}^p i^{(t+\delta)g,!}({}^p j_{!*}^{=(t+\delta-1)g} \mathcal{H})$ is zero by construction of the intermediate extension. We then deduce that

$$\begin{aligned}(12)\quad j^{=(t+\delta)g,*}({}^p i^{(t+\delta)g,!}(j_!^{=(t+\delta-1)g} HT(\pi_v, \Pi_t\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\pi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}})) \\ \cong HT\Big(\pi_v, \Pi_t\{\frac{1-\delta}{2}\} \\ \times \big(\mathrm{Speh}_{\delta-1}(\pi_v\{-1/2\}) \times \pi_v\{\frac{\delta-1}{2}\}\big)\{t/2\}\Big) \otimes \Xi^{\delta/2}\end{aligned}$$

In particular, up to homothety, the map (11), and so (10), is unique. Finally as the maps of (9) are strict, the given maps (10) are uniquely determined, that is, if we

[13] In $P(t, \pi_v)$ the infinitesimal part Π_t is $\mathrm{st}_t(\pi_v)$.

forget the infinitesimal parts, these maps are independent of the chosen t in (9), i.e. only depends on $t+\delta$.

For every $1 \leq t \leq s$, let denote by $i(t)$ the smallest index i such that $H^i(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=tg} HT(\pi_v,\Pi_t))_{\mathfrak{m}}$ has non trivial torsion: if it does not exist then we set $i(t)=+\infty$ and note that it does not depend on the choice of the infinitesimal part Π_t. By duality, as ${}^p j_{!*} = {}^{p+} j_{!*}$ for ours Harris-Taylor local systems, note that when $i(t)$ is finite then $i(t) \leq 0$. Suppose by absurdity there exists t with $i(t)$ finite and denote t_0 the biggest such t.

Lemma 3.2.4. — *For $1 \leq t \leq t_0$ then $i(t) = t - t_0$.*

Proof. — a) We first prove that for every $t_0 < t \leq s$, the cohomology groups of $j_!^{=tg} HT(\pi_v,\Pi_t)$ are torsion free. Consider the following strict filtration in the category of free perverse sheaves

$$(13)\quad (0) = \mathrm{Fil}^{-1-s}(\pi_v,h) \lhook\joinrel\relbar\joinrel\mapsto \mathrm{Fil}^{-s}(\pi_v,h) \lhook\joinrel\relbar\joinrel\mapsto \cdots \lhook\joinrel\relbar\joinrel\mapsto \mathrm{Fil}^{-t}(\pi_v,t) = j_!^{=tg} HT(\pi_v,\Pi_t)$$

where the symbol $\lhook\joinrel\relbar\joinrel\mapsto$ means a strict[14] monomorphism, with graded parts

$$\mathrm{gr}^{-k}(\pi_v,t) \cong {}^p j_{!*}^{=kg} HT(\pi_v, \Pi_t\{\frac{t-k}{2}\} \otimes \mathrm{st}_{k-t}(\pi_v\{t/2\}))(\frac{t-k}{2}).$$

Over $\overline{\mathbb{Q}}_l$, the result is proved in [**Boy09**] §4.3. Over $\overline{\mathbb{Z}}_l$, the result follows from the general constructions of [**Boy14**] and the fact that the p and $p+$ intermediate extensions are isomorphic for Harris-Taylor perverse sheaves associated to characters. The associated spectral sequence localized at $\mathfrak{m}$, is then concentrated in middle degree and torsion free which gives the claim.

b) Before watching the cases $t \leq t_0$, note that the spectral sequence associated to (9) for $t = t_0+1$, has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the maps

$$(14)\quad H^0\big(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=(t+\delta)g} HT_\xi(\pi_v, \mathrm{st}_t(\pi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\pi_v\{t/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=(t+\delta-1)g} HT_\xi(\pi_v, \mathrm{st}_t(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{t/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}}$$

are saturated, i.e. their cokernel are free $\overline{\mathbb{Z}}_l$-modules. Then from the previous fact stressed after (12), this property remains true when we consider the associated spectral sequence for $1 \leq t' \leq t_0$.

[14] i.e. the cokernel is free

c) Consider now $t=t_0$ and the spectral sequence associated to (9) where

$$(15)\quad E_2^{p,q}=H^{p+2q}(\mathrm{Sh}_{K^v(\infty),\bar s_v},j_!^{=(t+q)g} \\ HT_\xi(\pi_v,\mathrm{st}_t(\pi_v(-q/2))\times \mathrm{Speh}_q(\pi_v\{t/2\}))\otimes \Xi^{\frac{q}{2}})_{\mathfrak m}$$

By definition of t_0, we know that some of the $E_\infty^{p,-p}$ should have a non trivial torsion subspace. We saw that

- the contributions from the deeper strata are torsion free and
- $H^i(\mathrm{Sh}_{K^v(\infty),\bar s_v},j_!^{=t_0g}HT_\xi(\pi_v,\Pi_{t_0}))_{\mathfrak m}$ are zero for $i<0$ and is torsion free for $i=0$, whatever is Π_{t_0}.
- Then there should exist a non strict map $d_1^{p,q}$. But, we have just seen that it can not be maps between deeper strata.
- Finally, using the previous points, the only possibility is that the cokernel of

$$(16)\quad H^0\big(\mathrm{Sh}_{K^v(\infty),\bar s_v},j_!^{=(t_0+1)g}HT_\xi(\pi_v,\mathrm{st}_{t_0}(\pi_v\{\frac{-1}{2}\})\times\pi_v\{t_0/2\}))\otimes\Xi^{1/2}\big)_{\mathfrak m} \\ \longrightarrow \\ H^0\big(\mathrm{Sh}_{K^v(\infty),\bar s_v},j_!^{=t_0g}HT_\xi(\pi_v,\mathrm{st}_{t_0}(\pi_v))\big)_{\mathfrak m}$$

 has a non trivial torsion subspace.

In particular we have $i(t_0)=0$.

d) Finally using the fact 2.18 and the previous points, for any $1\leq t\leq t_0$, in the spectral sequence (15)

- by point a), $E_2^{p,q}$ is torsion free for $q\geq t_0-t+1$ and so it is zero if $p+2q\neq 0$;
- by affiness of the open strata, cf. [**Boy19**] theorem 1.8, $E_2^{p,q}$ is zero for $p+2q<0$ and torsion free for $p+2q=0$;
- by point b), the maps $d_2^{p,q}$ are saturated for $q\geq t_0-t+2$;
- by point c), $d_2^{-2(t_0-t+1),t_0-t+1}$ has a cokernel with a non trivial torsion subspace.
- Moreover, over $\overline{\mathbb Q}_l$, the spectral sequence degenerates at E_3 and $E_3^{p,q}=0$ if $(p,q)\neq(0,0)$.

We then deduce that $H^i(\mathrm{Sh}_{K^v(\infty),\bar s_v},{}^pj_{!*}^{=tg}HT_\xi(\pi_v,\Pi_t))_{\mathfrak m}$ is zero for $i<t-t_0$ and for $i=t-t_0$ it has a non trivial torsion subspace. □

Consider now the filtration of stratification of $\Psi_\varrho:=\Psi_{K^v(\infty),\varrho}$[15] constructed using the adjunction morphisms $j_!^{=h}j^{=h,*}$ as in [**Boy14**]

$$(17)\qquad \mathrm{Fil}^1_!(\Psi_\varrho)\lhook\joinrel\longrightarrow \mathrm{Fil}^2_!(\Psi_\varrho)\lhook\joinrel\longrightarrow\cdots\lhook\joinrel\longrightarrow \mathrm{Fil}^s_!(\Psi_\varrho)$$

where $\mathrm{Fil}^t_!(\Psi_\varrho)$ is the saturated image of $j_!^{=tg}j^{=tg,*}\Psi_\varrho\longrightarrow\Psi_\varrho$. For a fixed π_v, let denote by $\mathrm{Fil}^1_{!,\pi_v}(\Psi)\lhook\joinrel\longrightarrow \mathrm{Fil}^1_!(\Psi_\varrho)$ such that $\mathrm{Fil}^1_{!,\pi_v}(\Psi)\otimes_{\overline{\mathbb Z}_l}\overline{\mathbb Q}_l\cong \mathrm{Fil}^1_!(\Psi_{\pi_v})$ where Ψ_{π_v}

[15] i.e. with infinite level at v

is the direct factor of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ associated to π_v, cf. [**Boy14**]. From [**Boy23b**] 3.3.5, we have the following resolution of $\mathrm{gr}^t_{!,\pi_v}(\Psi_\varrho)$

$$(18) \quad 0 \to j_!^{=sg} HT(\chi_v, LT_{t,s}(\pi_v)) \otimes \pi_v^\vee(\frac{s-t}{2}) \longrightarrow \\ j_!^{=(s-1)g} HT(\pi_v, LT_{t,s-1}(\pi_v)) \otimes \pi_v^\vee(\frac{s-t-1}{2}) \longrightarrow \\ \cdots \longrightarrow j_!^{=tg} HT(\pi_v, \mathrm{st}_t(\pi_v)) \otimes \pi_v^\vee \longrightarrow \mathrm{gr}^t_{!,\pi_v}(\Psi_\varrho) \to 0,$$

where $LT_{t,t+\delta}(\pi_v) \hookrightarrow \mathrm{st}_t(\pi_v\{-\delta/2\}) \times \mathrm{Speh}_\delta(\pi_v\{t/2\})$, is the only irreducible subspace of this induced representation,

We can then apply the previous arguments a)-d) above: for $t \leq t_0$ (resp. $t > t_0$) the torsion of $H^i(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \mathrm{gr}^t_{!,\pi_v}(\Psi_{v,\xi}))_{\mathfrak{m}}$ is trivial for any $i \leq t - t_0$ (resp. for all i) and the free parts are concentrated for $i = 0$. Using the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ would have non trivial torsion which is false as $\mathfrak{m}$ is supposed to be KHT-free. $\square$

In particular the previous spectral sequence gives us a filtration of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ whose graded parts are

$$H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \mathrm{gr}^{-p}(\mathrm{gr}^k_!(\Psi_{K,\varrho})))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l,$$

for ϱ describing the equivalence classes of irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation of $\mathrm{GL}_g(F_v)$ with $1 \leq g \leq d$, and then $1 \leq k \leq p \leq \lfloor \frac{d}{g} \rfloor$.

3.3. Local and global monodromy. — Consider a fixed $\overline{\mathbb{F}}_l$-character ϱ and denote by Ψ_ϱ the direct factor of $\Psi_{K^v(\infty),v}$ associated to ϱ.

Over $\overline{\mathbb{Q}}_l$, the monodromy operator define a nilpotent morphism $N_{\varrho,\overline{\mathbb{Q}}_l} : \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \longrightarrow \Psi_\varrho(1) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ compatible with the filtration $\mathrm{Fil}^\bullet_!(\Psi_\varrho)$ in the sense that $\mathrm{Fil}^t_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ coincides with the kernel of $N^t_{\varrho,\overline{\mathbb{Q}}_l}$. The aim of this section is to construct a $\overline{\mathbb{Z}}_l$-version N_ϱ of $N_{\varrho,\overline{\mathbb{Q}}_l}$ such that $\mathrm{Fil}^t_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ coincides with the kernel of $N^t_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

First step: consider

$$0 \to \mathrm{Fil}^1_!(\Psi_\varrho) \longrightarrow \Psi_\varrho \longrightarrow \mathrm{coFil}^1_!(\Psi_\varrho) \to 0,$$

and the following long exact sequence

$$0 \to \hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \hom(\Psi_\varrho, \Psi_\varrho(1)) \longrightarrow \hom(\mathrm{Fil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \cdots$$

where hom is taken in the category of equivariant Hecke perverse sheaves with an action of $\mathrm{Gal}(\overline{F}_v/F_v)$. As $\mathrm{Fil}^1_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ coincides with the kernel of $N_{\varrho,\overline{\mathbb{Q}}_l}$, then $N_{\varrho,\overline{\mathbb{Q}}_l} \in \hom(\Psi_\varrho, \Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ comes from $\hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, so that we focus on $\hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1))$. From

$$0 \to \mathrm{gr}^2_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^1_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^2_!(\Psi_\varrho) \to 0,$$

we obtain

$$0 \to \hom(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \\ \hom(\mathrm{gr}^2_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho(1))) \longrightarrow \cdots$$

The socle of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ being contained in $\mathrm{Fil}^1_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, any map $\mathrm{coFil}^2_!(\Psi_\varrho) \longrightarrow \Psi_\varrho(1))$ can not be equivariant for the Galois action, so that we are led to look at

$$\hom(\mathrm{gr}^2_!(\Psi_\varrho), \Psi_\varrho(1)) \cong \hom(\mathrm{gr}^2_!(\Psi_\varrho), \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho(1))))$$

where

$$0 \to \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \longrightarrow \mathrm{Fil}^1_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^1_*(\mathrm{Fil}^1_!(\Psi_\varrho) \to 0.$$

Note that $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho(1))) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ and their $\overline{\mathbb{Z}}_l$-structure is obtained, cf. the introduction of [**Boy23b**] or equation (9), through the strict $\overline{\mathbb{Z}}_l$-epimorphisms

$$j^{=2g}_! j^{=2g,*} \mathrm{gr}^2_!(\Psi_\varrho) \twoheadrightarrow \mathrm{gr}^2_!(\Psi_\varrho), \quad \text{and} \quad j^{=2g}_! j^{=2g,*} \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \twoheadrightarrow \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)),$$

cf. figure 2 and the notations of the beginning of §3.1.

In particular to prove that $\mathrm{gr}^2_!(\Psi_\varrho)$ is isomorphic to $\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho(1)))$, it suffices to prove that the two local systems $j^{=2g,*} \mathrm{gr}^2_!(\Psi_\varrho)$ and $j^{=2g,*} \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho(1)))$ are isomorphic. In this case we can take[(16)] $N_v \in \hom(\Psi_\varrho, \Psi_\varrho(1)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ so that, over $\overline{\mathbb{Z}}_l$ we have $\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho(1))) = N_v(\mathrm{Fil}^2_!(\Psi_\varrho))$.

More generally to prove that the two perverse sheaves $\mathrm{gr}^{h+1}_!(\Psi_\varrho)$ and $\mathrm{Fil}^1_*(\mathrm{gr}^h_!(\Psi_\varrho(1)))$ are isomorphic, it suffices to prove that the two local systems $j^{=(h+1)g,*} \mathrm{gr}^{h+1}_!(\Psi_\varrho)$ and $j^{=(h+1)g,*} \mathrm{Fil}^1_*(\mathrm{gr}^h_!(\Psi_\varrho(1)))$ are isomorphic.

Second step: we want to prove that the local systems $j^{=2g,*} \mathrm{gr}^2_!(\Psi_\varrho)$ and $j^{=2g,*} \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ are isomorphic. Consider first the following situation: let $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$ be $\overline{\mathbb{Z}}_l$-local systems on a scheme X such that:

- $\mathcal{L}_k \hookrightarrow \mathcal{L}_{k+1}$ where the cokernel gr_{k+1} is torsion free;
- $\mathcal{L}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong (\mathcal{L}_k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l) \oplus (\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$ where $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is supposed to be irreducible;
- we introduce

$$\begin{tikzcd} \mathrm{gr}'_{k+1} \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \mathrm{gr}_{k+1,\overline{\mathbb{Q}}_l} \arrow[d, hook] \\ \mathcal{L}_{k+1} \arrow[r, hook] & \mathcal{L}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{tikzcd}$$

  We moreover suppose that $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is also irreducible so the various stable $\overline{\mathbb{Z}}_l$-lattices of gr_{k+1} are homothetic.

(16) As it is not clear that $\mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho(1)))$ is torsion free, we can not claim at this stage that $N_v \in \hom(\Psi_\varrho, \Psi_\varrho(1))$.

We then have

$$0 \to \mathcal{L}_k \oplus \mathrm{gr}'_{k+1} \longrightarrow \mathcal{L}_{k+1} \longrightarrow T \to 0,$$

where T is torsion and can be viewed as a quotient

$$\mathcal{L}_k \hookrightarrow \mathcal{L}'_k \twoheadrightarrow T, \quad \mathrm{gr}'_{k+1} \hookrightarrow \mathrm{gr}_{k+1} \twoheadrightarrow T,$$

with

$$\mathcal{L}_k \hookrightarrow \mathcal{L}_{k+1} \twoheadrightarrow \mathrm{gr}_{k+1}, \qquad \mathrm{gr}'_{k+1} \hookrightarrow \mathcal{L}_{k+1} \twoheadrightarrow \mathcal{L}'_k.$$

As $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is irreducible, then $\mathrm{gr}'_{k+1} \hookrightarrow \mathrm{gr}_{k+1}$ is given by multiplication by l^{δ_k} and, as the stable lattices of $\mathrm{gr}_k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are all isomorphic, the extension is characterized by this δ_k.

Consider then the $\overline{\mathbb{Z}}_l$-local system $\mathcal{L} := j^{=g,*}\Psi_\varrho$ and recall that

$$\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{i=1}^r HT_{\overline{\mathbb{Q}}_l}(\pi_{v,i}, \pi_{v,i}),$$

where we fix any numbering of $\mathrm{Cusp}(\varrho) = \{\pi_{v,1}, \cdots, \pi_{v,r}\}$. For $k = 1, \cdots, r$, we introduce

$$\begin{tikzcd} \mathcal{L}_k \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \bigoplus_{i=1}^k HT(\pi_{v,i}, \pi_{v,i}) \arrow[d, hook] \\ \mathcal{L} \arrow[r, hook] & \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{tikzcd}$$

Let denote by T_{k+1} the torsion local system such that

$$0 \to \mathcal{L}_k \oplus \mathrm{gr}_{k+1} \longrightarrow \mathcal{L}_{k+1} \longrightarrow T_{k+1} \to 0,$$

where $\mathrm{gr}_{k+1} := \mathcal{L}_{k+1}/\mathcal{L}_k$, as above. We can apply the previous remark and denote by δ_k the power of l which define the homothety $\mathrm{gr}'_{k+1} \hookrightarrow \mathrm{gr}_{k+1} \twoheadrightarrow T_{k+1}$. The set $\{\delta_k : k = 1, \cdots, r\}$ is then a numerical data to characterize $\mathcal{L}$ inside $j^{=1,*}\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$.

(i) To control $j^{=2g,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$, we use the general description above with

- local systems $\mathcal{L}^+_k$ for $k = 1, \cdots, r$ so that $\mathcal{L}^+_k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{i=1}^k HT_{\overline{\mathbb{Q}}_l}(\pi_{v,i}, \mathrm{st}_2(\pi_{v,i})(-1/2)$;
- with $\mathrm{gr}^{+,'}_{k+1}$ defined, as before, with

$$0 \to \mathcal{L}^+_k \oplus \mathrm{gr}^{+,'}_{k+1} \longrightarrow \mathcal{L}^+_{k+1} \longrightarrow T_{k+1} \to 0,$$

where T_{k+1} is killed by $l^{\delta^+_{k+1}}$.

We want to prove that $\delta^+_k = \delta_k$ for every $k = 1, \cdots, r$ where $\{\delta_k : k = 1, \cdots, r\}$ is the numerical data associated to $j^{=1,*}\Psi_\varrho$.

Let denote by

$$j^{=1}_{\neq 1} : \mathrm{Sh}_{K,\bar{s}_v} \setminus \mathrm{Sh}^{\geq 1}_{K,\bar{s}_v,1} \hookrightarrow \mathrm{Sh}_{K,\bar{s}_v}, \qquad i^1_1 : \mathrm{Sh}^{\geq 1}_{K,\bar{s}_v,1} \hookrightarrow \mathrm{Sh}^{\geq 1}_{K,\bar{s}_v} = \mathrm{Sh}_{K,\bar{s}_v}.$$

From [**Boy23b**] lemma B.3.2, $j^{=2g,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho))$ is obtained as follows. Let
$$P := {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} j^{=1,*}_{\neq 1} \Psi_\varrho$$
so that
$$0 \to P \longrightarrow j^{=1}_{\neq 1,!} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow {}^p j^{=1}_{\neq 1,!*} j^{=1,*}_{\neq 1} \Psi_\varrho \to 0.$$
Then P is the cosocle of $i_1^{1,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho))$ so that
$$j^{=2g,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho)) \cong j^{=2g,*} P \times_{P_{1,d}(F_v)} \mathrm{GL}_d(F_v),$$
where induction has the same meaning as in (4).

Note then that the numerical data associated to $j^{=2g,*}P$ are also given by $\{\delta_k^+ : k = 1, \cdots, r\}$. With the previous notations, consider the data associated to $\mathcal{L} := j^{=g,*}\Psi_\varrho$, i.e. a filtration
$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots \subseteq \mathcal{L}_r = \mathcal{L}$$
with graded parts gr^k and $\operatorname{gr}'_k \hookrightarrow \operatorname{gr}_k$ is given by multiplication by l^{δ_k}. We then have a strict filtration
$${}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_1 \subseteq \cdots \subseteq {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_r = P,$$
with graded parts ${}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*}\operatorname{gr}_k$. Indeed we have
$$\begin{aligned} {}^p h^{-2} i_1^{1,*} j^{=1}_{\neq 1,*}\operatorname{gr}_{k+1} = 0 \longrightarrow {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*}\mathcal{L}_k \longrightarrow& \\ {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*}\mathcal{L}_k \longrightarrow {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*}\operatorname{gr}_{k+1} &\longrightarrow {}^p h^{0} i_1^{1,*} j^{=1}_{\neq 1,*}\mathcal{L}_k \end{aligned}$$
where the free quotient of ${}^p h^0 i_1^{1,*} j^{=1}_{\neq 1,*}\mathcal{L}_k$ is zero. Moreover it is torsion free because its torsion corresponds to the difference between p and $p+$ intermediate extensions which are equal here from the main result of [**Boy23b**]. We then apply the exact functor $j^{=2g,*}$ and we induce from $P_{1,d}(F_v)$ to $\mathrm{GL}_d(F_v)$ to obtain the filtration $\mathcal{L}^+_\bullet$ of $j^{=2g,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho))$ where $\operatorname{gr}_k^{+,'} \hookrightarrow \operatorname{gr}_k^+$ is given by multiplication by l^{δ_k}.

(ii) Dually the same arguments applied to
$$0 \to {}^p j^{=1}_{\neq 1,!*} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow {}^p j^{=1}_{\neq 1,*} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow Q \to 0,$$
give us that $j^{=2g,*}Q$ is characterized by the data $\{\delta_k : k = 1, \cdots, r\}$. After inducing from $P_{1,d}(F_v)$ to $\mathrm{GL}_d(F_v)$, we obtain the description of the local system $\mathcal{A} := j^{=2g,*}A$ where A is defined as follows:
$$0 \to {}^p j^{=g}_{!*} j^{=g,*}\operatorname{coFil}^1_*(\Psi_\varrho) \longrightarrow \operatorname{coFil}^1_*(\Psi_\varrho) \longrightarrow A \to 0.$$
Concretely this means that ${}^p j^{=2g}_{!*}\mathcal{A}$ is the socle A_1 of A, which corresponds to the square dot in the right side of the figure 2.

We are interested by the local system associated to $j^{=2g,*}$ of the cosocle of $\operatorname{Fil}^2_!(\Psi_\varrho)$, which corresponds to the square dot in the left side of the figure 2. As explained in §3.1, we have to use basic exchange steps as many times as needed to move A_1 until it appears as the cosocle of $\operatorname{Fil}^2_!(\Psi_\varrho) \hookrightarrow \Psi_\varrho$.

Note then that all the perverse sheaves which are exchanged with A_1 during this process, are lattice of $j_{!*}^{=tg} HT_{\overline{\mathbb{Q}}_l}(\pi_v, \mathrm{st}_t(\pi_v))(\frac{1-t+\delta}{2})$ with $t \geq 3$, cf. figure 2. As explained in the remark after the definition of the exchange basic step, as ${}^p j_{!*}^{=2g} HT(\pi_v, \mathrm{st}_2(\pi_v)) \cong {}^{p+} j_{!*}^{=2g} HT(\pi_v, \mathrm{st}_2(\pi_v))$, for all these exchange, we have $T = 0$ and A_1 remains unchanged during all the basic exchange steps.

Third step: at this stage we constructed a $\overline{\mathbb{Q}}_l$-monodromy operator N_v such $\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho(1))) = N_v(\mathrm{Fil}^2_!(\Psi_\varrho))$. Recall that this monodromy operator induces

$$\alpha : \mathrm{coFil}^t_!(\mathrm{cogr}^t_*(\Psi_\varrho)) \longrightarrow \mathrm{cogr}^{t+1}_*(\Psi_\varrho(1))$$

such that $j^{=(t+1)g,*} \circ \alpha$ is then an isomorphism over $\overline{\mathbb{Z}}_l$. We say that α is an isomorphism. Indeed consider

$$0 \to {}^p j_{!*}^{=tg} j^{=tg,*} \mathrm{cogr}^t_*(\Psi_\varrho) \longrightarrow \mathrm{cogr}^t_*(\Psi_\varrho) \longrightarrow \mathrm{coFil}^t_!(\mathrm{cogr}^t_*(\Psi_\varrho)) \to 0,$$

with the following two strict monomorphisms

$$\alpha_1 : \mathrm{cogr}^{t+1}_*(\Psi_\varrho) \hookrightarrow j_*^{=(t+1)g} j^{=(t+1)g,*} \mathrm{cogr}^{t+1}_*(\Psi_\varrho) \tag{19}$$

and

$$\alpha_2 : \mathrm{coFil}^t_!(\mathrm{cogr}^t_*(\Psi_\varrho(1))) \hookrightarrow j_*^{=(t+1)g} j^{=(t+1)g,*} \mathrm{coFil}^t_!(\mathrm{cogr}^t_*(\Psi_\varrho(1))). \tag{20}$$

By composing α with α_2 in (20), we obtain

$$\begin{aligned} \alpha_1, \alpha_2 \circ \alpha \in \hom\Big(&\mathrm{cogr}^{t+1}_*(\Psi_\varrho), j_*^{=(t+1)g} j^{=(t+1)g,*} \mathrm{cogr}^{t+1}_*(\Psi_\varrho)\Big) \\ &\cong \hom\Big(j^{=(t+1)g,*} \mathrm{cogr}^{t+1}_*(\Psi_\varrho), j^{=(t+1)g,*} \mathrm{cogr}^{t+1}_*(\Psi_\varrho)\Big), \end{aligned} \tag{21}$$

by adjunction. By hypothesis α_1 and $\alpha_2 \circ \alpha$ coincides in this last space, so they are equal and α is then an isomorphism.

Notation 3.3.1. — Under the hypothesis of theorem 3.2.3 on $\mathfrak{m}$, the action of N_ϱ on Ψ_ϱ defined above for every $\overline{\mathbb{F}}_l$-character ϱ, induces a nilpotent monodromy operator $N^{coho}_{\mathfrak{m},v}$ on $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$. We also denote by $\overline{N}^{coho}_{\mathfrak{m},v} := N^{coho}_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ acting on $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$

4. Uniformity of automorphic sub-lattices

4.1. Infinite level at v. — For $K \in \mathcal{K}$ a compact open subgroup of $G(\mathbb{A}^\infty_{\mathbb{Q}})$ as before, we consider the category $\mathcal{C}(K)$ of finitely generated $\mathcal{O}_L$-modules with a continuous K-action and let $\mathcal{C}'(K)$ be a Serre subcategory of $\mathcal{C}(K)$. Let $\mathcal{C}(K)_Z$ be the subcategory of $\mathcal{C}(K)$ consisting of those $\sigma \in \mathcal{C}$ possessing a central character which lifts $\prod_{v \in \Sigma^+(K)} (\det \overline{\rho}_{|I_v} \bar{\varepsilon}) \circ \mathrm{Art}$ and let $\mathcal{C}'(K)_Z$ be a Serre subcategory of $\mathcal{C}(K)_Z$.

For each $v \in \Sigma^+(K)$, we fix

– a lift of a geometric Frobenius element $\mathrm{Frob}_v \in G_v := \mathrm{Gal}(\overline{F}_v/F_v)$,

- an element $\alpha_v \in \overline{\mathbb{Z}}_l$ lifting $\det \overline{\rho}_v(\mathrm{Frob}_v)$,
- a character $\psi_{\sigma,v} : G_v \longrightarrow \overline{\mathbb{Z}}_l^\times$ such that $\psi_{\sigma,v}(\mathrm{Frob}_v) = \alpha_v$ and the composite with Artin map has a restriction to I_v equal to the central character σ.

For $\sigma \in \mathcal{C}'(K)_Z$, we define

$$M_K(\sigma) := H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_v \otimes \mathcal{L}_{\sigma^\vee})_{\mathfrak{m}}^\vee,$$

where $\mathcal{L}_{\sigma^\vee}$ is the sheaf associated to $\sigma^\vee$. When we consider only the direct factor Ψ_ϱ of Ψ_v, we then denote by $M_{K,\varrho}(\sigma)$ the associated space.

Remark. As the $\mathfrak{m}$-localized cohomology groups are concentrated in middle degree, note that the functor $\sigma \mapsto M_{\infty,K,\varrho}(\sigma)$ is exact.

Let σ^v be a continuous finitely generated representation of K^v and let $\sigma_0(v)$ be the representation of $K = K^v K_v$ given by twisting the action of K^v on σ^v by the character $\psi \circ \mathrm{Art}_{F_v} \circ \det$ of K_v. We assume that $\sigma \in \mathcal{C}(K)_Z$. We then define

$$S_{K^v}(\sigma^v) := \lim_{\rightarrow n} H^{d-1}(\mathrm{Sh}_{K^v(n),\bar{s}_v}, \mathcal{L}^\vee_{\sigma_0(v)^\vee})_{\mathfrak{m}}.$$

We let $\mathbb{T}(\sigma^v)$ denote the image of $\mathbb{T}^S$ in $\mathrm{End}_{\mathcal{O}}(S_{K^v}(\sigma^v))$ and we write $\rho(\sigma^v)_{\mathfrak{m}}$ for the composite $\mathrm{Gal}_{F,s} \stackrel{\rho^{univ}}{\longrightarrow} \mathrm{GL}_d(R_S^{univ}) \longrightarrow \mathrm{GL}_d(\mathbb{T}(\sigma^v)_{\mathfrak{m}})$. We set

$$M_{K^v}(\sigma^v) \stackrel{\mathrm{def}}{=} \Big(\hom_{\mathbb{T}(\sigma^v)_{\mathfrak{m}}[\mathrm{Gal}_{F,S}]}(\rho(\sigma^v)_{\mathfrak{m}}, S_{K^v}(\sigma_v)_{\mathfrak{m}})\big)^*. \tag{22}$$

For a fixed K_v and a representation σ_v of K_v such that $\sigma := \sigma_v \otimes_{\mathcal{O}} \sigma^v$ is an element of $\mathcal{C}(K)_Z$, then we have a natural isomorphism

$$M_K(\sigma) \cong \hom_{K_v}(M_{K^v}(\sigma^v), \sigma_v^*)^*. \tag{23}$$

4.2. Typicity. — As explained in [**HT01**], the $\overline{\mathbb{Q}}_l$-cohomology of $\mathrm{Sh}_{K,\bar{\eta}}$ can be written as

$$H^{d-1}(\mathrm{Sh}_{K,\bar{\eta}}, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \cong \bigoplus_{\pi \in \mathcal{A}_K(\mathfrak{m})} (\pi^\infty)^K \otimes \sigma(\pi^\infty),$$

where

- $\mathcal{A}_K(\mathfrak{m})$ is the set of equivalence classes of automorphic representations of $G(\mathbb{A})$ with non trivial K-invariants and such that its modulo l Satake's parameters, outside the set S of places dividing the level K, are prescribed by $\mathfrak{m}$,
- and $\sigma(\pi^\infty)$ is a representation of $\mathrm{Gal}_{F,S}$.

As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible, then as explained in chapter VI of [**HT01**], if $\sigma(\pi^\infty)$ is non zero, then π is a weak transfer of a cohomological automorphic representation (Π, ψ) of $\mathrm{GL}_d(\mathbb{A}_F) \times \mathbb{A}_F^\times$ with $\Pi^\vee \cong \Pi^c$ where c is the complex conjugation. Attached to such a Π is a global Galois representation $\rho_{\Pi,l} : \mathrm{Gal}_{F,S} \longrightarrow \mathrm{GL}_d(\overline{\mathbb{Q}}_l)$ which is irreducible.

Theorem 4.2.1. — *(cf.* [**NF19**] *theorem 2.20)*
If $\rho_{\Pi,l}$ is strongly irreducible, meaning it remains irreducible when it is restricted to any finite index subgroup, then $\sigma(\pi^\infty)$ is a semi-simple representation of $\mathrm{Gal}_{F,S}$.

Remark. The Tate conjecture predicts that $\sigma(\pi^\infty)$ is always semi-simple.

Definition 4.2.2. — (cf. [**Sch18**] §5) We say that $\mathfrak{m}$ is KHT-typic for K if, as a $\mathbb{T}(K)_\mathfrak{m}[\mathrm{Gal}_{F,S}]$-module,

$$H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Z}}_l)_\mathfrak{m}\cong\sigma_{\mathfrak{m},K}\otimes_{\mathbb{T}(K)_\mathfrak{m}}\rho_{\mathfrak{m},K},$$

for some $\mathbb{T}(K)_\mathfrak{m}$-module $\sigma_{\mathfrak{m},K}$ on which $\mathrm{Gal}_{F,S}$ acts trivially and

$$\rho_{\mathfrak{m},K}:\mathrm{Gal}_{F,S}\longrightarrow\mathrm{GL}_d(\mathbb{T}(K)_\mathfrak{m})$$

is the stable lattice of $\bigoplus_{\widetilde{\mathfrak{m}}\subseteq\mathfrak{m}}\rho_{\widetilde{\mathfrak{m}}}$ introduced in the introduction.

Proposition 4.2.3. — *We suppose that for all $\pi\in\mathcal{A}_K(\mathfrak{m})$, the Galois representation $\sigma(\pi^\infty)$ is semi-simple. Then $\mathfrak{m}$ is KHT-typic for K.*

Proof. — By proposition 5.4 of [**Sch18**] it suffices to deal with $\overline{\mathbb{Q}}_l$-coefficients. From [**HT01**] proposition VII.1.8 and the semi-simplicity hypothesis, then $\sigma(\pi^\infty)\cong\widetilde{R}(\pi)^{\bigoplus n(\pi)}$ where $\widetilde{R}(\pi)$ is of dimension d. We then write

$$(\pi^\infty)^K\otimes_{\overline{\mathbb{Q}}_l}R(\pi)\cong(\pi^\infty)^K\otimes_{\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l}}(\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

and $(\pi^\infty)^K\otimes_{\overline{\mathbb{Q}}_l}\sigma(\pi^\infty)\cong((\pi^\infty)^K)^{\bigoplus n(\pi)}\otimes_{\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l}}(\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l})^d$ and finally

$$H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Q}}_l)_\mathfrak{m}\cong\sigma_{\mathfrak{m},K,\overline{\mathbb{Q}}_l}\otimes_{\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l}}(\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

with $\sigma_{\mathfrak{m},K,\overline{\mathbb{Q}}_l}\cong\bigoplus_{\pi\in\mathcal{A}_K(\mathfrak{m})}((\pi^\infty)^I)^{\bigoplus n(\pi)}$. The result then follows from [**HT01**] theorem VII.1.9 which insures that $R(\pi)\cong\rho_{\widetilde{\mathfrak{m}}}$, if $\widetilde{\mathfrak{m}}$ is the prime ideal associated to π, □

In particular with the notations of §4.1, we have an isomorphism

$$M_{K^v}(\sigma^v)\otimes_{\mathbb{T}(\sigma_v)_\mathfrak{m}}\rho(\sigma^v)_\mathfrak{m}\longrightarrow S_{K^v}(\sigma^v)_\mathfrak{m}.$$

4.3. Lattices. — For each place $w\in S\backslash\{v\}$ choose an inertial type τ_w and a lattice $\sigma^0(\tau_w)$ in $\sigma_L(\tau_w)$ and let $\sigma^v:=\otimes_{w\in S\backslash\{v\}}\sigma^0(\tau_w)$ the corresponding representation of $K^v=\prod_{w\in S\backslash\{v\}}K_w$.

Recall that $\mathcal{O}_L$ is the ring of integers of a large enough finite extension L of $\mathbb{Q}_l$. Let $\lambda:\mathbb{T}_\mathfrak{m}\longrightarrow\mathcal{O}_L$ be a system(17) of Hecke eigenvalues corresponding to some minimal prime ideal of $\mathbb{T}_\mathfrak{m}:=\mathbb{T}(\sigma^v\sigma^0(\tau_v))_\mathfrak{m}$ for some lattice $\sigma^0(\tau_v)$ of $\sigma_L(\tau_v)$ define just after: note that $\mathbb{T}_\mathfrak{m}$ does not depend on this choice. Let $\Pi_v:=(M_{K^v}(\sigma^v)^*[1/l])[\lambda]$ which by strong multiplicity one is an irreducible representation of $\mathrm{GL}_d(F_v)$. We suppose

(17)Note that $\lambda \mod \varpi_L$ is given by $\mathfrak{m}$, i.e. the modulo l reduction of λ is fixed.

that $\sigma_L(\tau_v)$ appears with multiplicity one in $(\Pi_v)_{|K_v}$. The eigenspace $M_{K^v}(\sigma^v)^*[\lambda]$ is a lattice in Π_v and

$$\sigma_\lambda(\tau_v) := \sigma_L(\tau_v) \cap M_{K^v}(\sigma^v)^*[\lambda]$$

is K_v-stable lattice in $\sigma_L(\tau_v)$.

For any lattice $\sigma^1(\tau_v)$ of $\sigma_L(\tau_v)$, it follows from (23) that there is a natural isomorphism

$$\hom_{K_v}(\sigma^1(\tau_v), \sigma_\lambda(\tau_v))^* \cong M_{K^v}(\sigma^v\sigma^1(\tau_v))/\lambda. \tag{24}$$

Consider some injection

$$\iota : \sigma_1(\tau_v) \hookrightarrow \sigma_2(\tau_v)$$

between two lattices of $\sigma_L(\tau_v)$. Then ι induces a injective map

$$M_{K^v}(\sigma^v\sigma_1(\tau_v)) \hookrightarrow M_{K^v}(\sigma^v\sigma_2(\tau_v)).$$

Lemma 4.3.1. — *Suppose there exists λ_0 so that the natural map induces by ι*

$$M_{K^v}(\sigma^v\sigma_1(\tau_v))/\lambda_0 \hookrightarrow M_{K^v}(\sigma^v\sigma_2(\tau_v))/\lambda_0 \tag{25}$$

is an isomorphism, then this is true for every $\lambda \equiv \lambda_0 \mod \varpi_L$.

Proof. — For any λ the natural inclusion

$$\mathcal{O}_L \cong M_{K^v}(\sigma^v\sigma_1(\tau_v))/\lambda \hookrightarrow M_{K^v}(\sigma^v\sigma_2(\tau_v))/\lambda \cong \mathcal{O}_L$$

is either an isomorphism or multiplication by some power of ϖ_L and so zero modulo ϖ_L. By hypothesis modulo ϖ_L, the natural map ι induces an isomorphism

$$M_{K^v}(\sigma^v\sigma_1(\tau_v))/(\lambda_0, \varpi_L) \cong M_{K^v}(\sigma^v\sigma_2(\tau_v))/(\lambda_0, \varpi_L).$$

As the following diagram is commutative

$$\begin{tikzcd} R_\infty^{\tau_v} \arrow[r, "\lambda_0"] \arrow[dr, "\lambda"'] & \mathbb{Z}_L \arrow[r, two heads] & \mathbb{F}_L \\ & \mathbb{Z}_L \arrow[ur] & \end{tikzcd}$$

with common kernel $\mathfrak{m}$, we then deduce that ι induces an isomorphism

$$M_{K^v}(\sigma^v\sigma_1(\tau_v))/(\lambda, \varpi_L) \cong M_{K^v}(\sigma^v\sigma_2(\tau_v))/(\lambda, \varpi_L),$$

so that the natural map induces by ι

$$M_{K^v}(\sigma^v\sigma_1(\tau_v))/\lambda \hookrightarrow M_{K^v}(\sigma^v\sigma_2(\tau_v))/\lambda$$

is also an isomorphism. □

Proposition 4.3.2. — *Consider two automorphic irreducible representations Π and Π' associated to respectively two systems*[18] *of Hecke eigenvalues $\lambda, \lambda' : \mathbb{T}_\mathfrak{m} \longrightarrow \mathcal{O}_L$. We denote by*

$$\Pi_v := (M_{K^v}(\sigma^v)^*[1/l])[\lambda] \quad \textit{and} \quad \Pi'_v := (M_{K^v}(\sigma^v)^*[1/l])[\lambda'].$$

[18] with modulo l reduction fixed by $\mathfrak{m}$

Consider a K-type τ_v such that $\sigma_L(\tau_v)$ appears with multiplicity one in $(\Pi_v)_{|K_v}$ (resp. in $(\Pi'_v)_{|K_v}$) and let $\sigma_\lambda(\tau_v)$ (resp. $\sigma_{\lambda'}(\tau_v)$) be the stable lattice in $\sigma_L(\tau_v)$ defined by

$$\sigma_\lambda(\tau_v) := \sigma_L(\tau_v) \cap M_{K^v}(\sigma^v)^*[\lambda], \quad \textit{resp. } \sigma_{\lambda'}(\tau_v) := \sigma_L(\tau_v) \cap M_{K^v}(\sigma^v)^*[\lambda'].$$

Then these two lattices are homothetic.

Proof. — Up to homothety any stable lattice σ_+ is such that

$$\sigma_0 := \sigma_\lambda(\tau_v) \subsetneq \sigma_+ \subsetneq \varpi_L^{-\delta}\sigma_\lambda(\tau_v),$$

where δ is minimal and $\sigma_0 \not\subseteq \varpi_L\sigma_+$. The natural morphism

$$\mathcal{O}_L \cong \hom_{\mathrm{GL}_d(\mathcal{O}_v)}(\sigma_+, \sigma_\lambda(\tau_v)) \longrightarrow \hom_{\mathrm{GL}_d(\mathcal{O}_v)}(\sigma_0, \sigma_\lambda(\tau_v)) \cong \mathcal{O}_L$$

is given by multiplication by ϖ_L^δ, while

$$\hom_{\mathrm{GL}_d(\mathcal{O}_v)}(\sigma_0, \sigma_\lambda(\tau_v)) \longrightarrow \hom_{\mathrm{GL}_d(\mathcal{O}_v)}(\sigma_-, \sigma_\lambda(\tau_v))$$

is a isomorphism. From the isomorphism (24), ι induces

$$M_{K^v}(\sigma^v\sigma_-)/\lambda = M_{K^v}(\sigma^v\sigma_0)/\lambda = \varpi_L^\delta M_{K^v}(\sigma^v\sigma_+)/\lambda. \tag{26}$$

If $\sigma_{\lambda'}(\tau_v)$ were equal to σ_+ then the morphism induced by ι

$$\hom_{\mathrm{GL}_d(\mathcal{O}_v)}(\sigma_+, \sigma_{\lambda'}(\tau_v)) \hookrightarrow \hom_{\mathrm{GL}_d(\mathcal{O}_v)}(\sigma_0, \sigma_{\lambda'}(\tau_v))$$

would be an isomorphism and by (24), the natural injection would give the equality

$$M_{K^v}(\sigma^v\sigma_0)/\lambda' = M_{K^v}(\sigma^v\sigma_+)/\lambda',$$

which, by the previous lemma, is not compatible with (26). □

5. Proof of Ihara's lemma

5.1. Supersingular locus as a zero dimensional Shimura variety. — As explained in the introduction, we follow the strategy of [**Boy22**] which consists to transfer the genericity property of Ihara's lemma concerning $\overline{G}$ to the genericity of the cohomology of KHT-Shimura varieties.

Let $\overline{G}$ be a similitude group as in the introduction such that moreover there exists a prime number p_0 split in E and v_0^+ a place of F^+ above p_0, identified as before to a place v_0 of F, such that $\overline{B}_{v_0}$ is a division algebra: in particular $v_0 \neq v$. Consider then, with the usual abuse of notation, $G/\mathbb{Q}$ such that $G(\mathbb{A}_\mathbb{Q}^{\infty,v_0}) \cong \overline{G}(\mathbb{A}_\mathbb{Q}^{\infty,v_0})$ with $G(F_{v_0}) \cong \mathrm{GL}_d(F_{v_0})$ and $G(\mathbb{R})$ of signatures $(1, n-1), (0, n)^r$. The KHT Shimura variety $\mathrm{Sh}_{K,v_0} \to \operatorname{spec} \mathcal{O}_{v_0}$ associated to G with level K, has a Newton stratification of its special fiber with supersingular locus

$$\mathrm{Sh}^{=d}_{K,\bar{s}_{v_0}} = \coprod_{i \in \ker^1(\mathbb{Q},G)} \mathrm{Sh}^{=d}_{K,\bar{s}_{v_0},i}.$$

For a equivariant sheaf $\mathcal{F}_{K,i}$ on $\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0},i}$ seen as acompatible system over $\mathrm{Sh}^{=d}_{K^vK_v,\bar{s}_{v_0},i}$ for K_v describing the set of open compact subgroups of $\mathrm{GL}_d(\mathcal{O}_v)$, its fiber at a compatible system $z_{K^v(\infty),i}$ of supersingular point $z_{K^vK_v,i}$, has an action of $\overline{G}(\mathbb{A}_{\mathbb{Q}}^\infty) \times \mathrm{GL}_d(F_v)^0$ where $\mathrm{GL}_d(F_v)^0$ is the kernel of the valuation of the determinant so that, cf. [**Boy09**] proposition 5.1.1, as a $\mathrm{GL}_d(F_v)$-module, we have

$$H^0(\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0},i}, \mathcal{F}_{K^v(\infty),i}) \cong \Big(\mathrm{ind}_{\overline{G}(\mathbb{Q})}^{\overline{G}(\mathbb{A}^{\infty,v})\times\mathbb{Z}}\, z^*_{K^{v_0}(\infty),i}\mathcal{F}_{K^{v_0}(\infty),i}\Big)^{K^v},$$

with $\delta \in \overline{G}(\mathbb{Q}) \mapsto (\delta^{\infty,v_0}, \mathrm{val}\circ\mathrm{rn}(\delta_{v_0})) \in \overline{G}(\mathbb{A}^{\infty,v_0,v}) \times \mathbb{Z}$ and where the action of $g_{v_0} \in \mathrm{GL}_d(F_{v_0})$ is given by those of $(g_0^{-\mathrm{val}\det g_{v_0}} g_{v_0}, \mathrm{val}\det g_{v_0}) \in \mathrm{GL}_d(F_{v_0})^0 \times \mathbb{Z}$ where $g_0 \in \mathrm{GL}_d(F_{v_0})$ is any fixed element with $\mathrm{val}\det g_0 = 1$. Moreover, cf. [**Boy09**] corollaire 5.1.2, if $z^*_{K^{v_0}(\infty),i}\mathcal{F}_{K^{v_0}(\infty),i}$ is provided with an action of the kernel $(D^\times_{v_0,d})^0$ of the valuation of the reduced norm, action compatible with those of $\overline{G}(\mathbb{Q}) \hookrightarrow D^\times_{v_0,d}$, then as a $G(\mathbb{A}^\infty)$-module, we have

$$H^0(\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0},i}, \mathcal{F}_{K^v(\infty),i}) \cong \Big(\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty), \Lambda) \otimes_{D^\times_{v_0,d}} \mathrm{ind}_{(D^\times_{v_0,d})^0}^{D^\times_{v_0,d}}\, z_i^*\mathcal{F}_{\mathcal{I},i}\Big)^{K^v} \tag{27}$$

In particular, cf. lemma 2.3.1 of [**Boy22**], let $\overline{\pi}$ be an irreducible sub-$\overline{\mathbb{F}}_l$-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ for $\mathfrak{m}$ such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible. Write its local component $\bar{\pi}_{v_0} \cong \pi_{v_0}[s]_D$ with π_{v_0} an irreducible cuspidal representation of $\mathrm{GL}_g(F_{v_0})$ with $d = sg$. Then $(\overline{\pi}^{v_0})^{K^v}$ is a sub-representation of $H^0(\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0}}, HT(\pi^\vee_{v_0}, s))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ and, cf. proposition 2.3.2 of [**Boy22**], a sub-$\overline{\mathbb{F}}_l$-representation of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. Indeed, cf. theorem 3.2.3,

- by the main result of [**Boy23a**], as $l > d \geq 2$ and $\overline{\rho}_{\mathfrak{m}}$ is irreducible, then $\mathfrak{m}$ is KHT free so that hypothesis (H1) of [**Boy22**] is fulfilled.
- Theorem 3.2.3 gives us that the filtration of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_{v_0}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ induced by the filtration of the nearby cycles at v_0, is strict.(19)

Finally if the analog of Ihara's lemma for $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ is true for the action of $\mathrm{GL}_d(F_v)$, then this is also the case for $\overline{G}$. We now focus on the genericity of irreducible sub-$\mathrm{GL}_d(F_v)$-modules of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{\eta}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ using the nearby cycles at the place v.

5.2. Level raising. — To a cohomological minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}(K)$, which corresponds to a maximal ideal of $\mathbb{T}(K)[\frac{1}{l}]$, is associated both a near equivalence class of $\overline{\mathbb{Q}}_l$-automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ and a Galois representation

$$\rho_{\widetilde{\mathfrak{m}}} : G_F := \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{GL}_d(\overline{\mathbb{Q}}_l)$$

such that the eigenvalues of the Frobenius morphism at an unramified place w are given by the Satake parameters of the local component $\Pi_{\widetilde{\mathfrak{m}},w}$ of $\Pi_{\widetilde{\mathfrak{m}}}$. The semi-simple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of the maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S_K$ containing $\widetilde{\mathfrak{m}}$.

(19) In [**Boy22**] hypothesis (H3) was introduced for this property to be true.

We now allow infinite level at v and we denote by $\mathbb{T}(K^v(\infty))$ the associated Hecke algebra. We fix a maximal ideal $\mathfrak{m}$ in $\mathbb{T}(K^v(\infty))$ such that the associated Galois representation $\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_d(\mathbb{F})$ is irreducible.

Remark. For every minimal prime $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$, note that $\Pi_{\widetilde{\mathfrak{m}},v}$ looks like $\mathrm{st}_{s_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{s_r}(\pi_{v,r})$ where $\pi_{v,i}$ is an irreducible cuspidal representation of $\mathrm{GL}_{g_i}(F_v)$ and $s_1 g_1 + \cdots + s_r g_r = d$.

Let $\mathcal{S}_v(\mathfrak{m})$ be the supercuspidal support of the modulo l reduction of any $\Pi_{\widetilde{m},v}$ in the near equivalence class associated to a minimal prime ideal $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$. Recall that $\mathcal{S}_v(\mathfrak{m})$ is a multi-set, i.e. a set with multiplicities which only depends on $\mathfrak{m}$. We decompose it according to the set of Zelevinsky lines: as we supposed $q_v \equiv 1 \mod l$ then every Zelevinsky line is reduced to a single equivalence class of an irreducible (super)cuspidal $\overline{\mathbb{F}}_l$-representations ϱ of some $\mathrm{GL}_{g(\varrho)}(F_v)$ with $1 \leq g(\varrho) \leq d$.

$$\mathcal{S}_v(\mathfrak{m}) = \coprod_{1 \leq g \leq d} \coprod_{\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}(g,v)} \mathcal{S}_\varrho(\mathfrak{m}),$$

where $\mathrm{Cusp}_{\overline{\mathbb{F}}_l}(g,v)$ is the set of irreducible cuspidal $\overline{\mathbb{F}}_l$-representations of $\mathrm{GL}_g(F_v)$.

Notation 5.2.1. — We denote by $l_\varrho(\mathfrak{m})$ the multiplicity of $\mathcal{S}_\varrho(\mathfrak{m})$.

For $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$, the local component $\Pi_{\widetilde{\mathfrak{m}},v}$ of $\Pi_{\widetilde{\mathfrak{m}}}$ can then be written as a full induced representation $\bigtimes_{1 \leq g \leq d} \bigtimes_{\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}(g,v)} \Pi_{\widetilde{\mathfrak{m}},\varrho}$ where each $\Pi_{\widetilde{\mathfrak{m}},\varrho}$ is also a full induced representation

$$\Pi_{\widetilde{\mathfrak{m}},\varrho} \cong \bigtimes_{i=1}^{r_\varrho(\widetilde{\mathfrak{m}})} \mathrm{St}_{l_{\varrho,i}(\widetilde{\mathfrak{m}})}(\pi_{v,i})$$

where $r_l(\pi_{v,i}) \cong \varrho$, $l_{\varrho,1}(\widetilde{\mathfrak{m}}) \geq \cdots \geq l_{\varrho,r_\varrho(\widetilde{\mathfrak{m}})}(\widetilde{\mathfrak{m}})$ and $\sum_{i=1}^r l_{\varrho,i}(\widetilde{\mathfrak{m}}) = l_\varrho(\mathfrak{m})$.

Suppose now that there exists $1 \leq g \leq d$ and $\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}(g,v)$ such that $\min_{\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}}\{r_\varrho(\widetilde{\mathfrak{m}})\} \geq 2$ and let $l_{\varrho,1} := \max_{\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}}\{l_{\varrho,1}(\widetilde{\mathfrak{m}})\}$ which is then strictly less than $l_\varrho(\mathfrak{m})$.

Fact from [**Boy10**] **§3**: for an irreducible cuspidal representation π_v such that its modulo l reduction is isomorphic to ϱ, $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, P(t,\pi_v))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is the sum of the contributions of $\Pi_{\widetilde{\mathfrak{m}}}$ with $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}}}$ is of the following shape: $\mathrm{st}_t(\pi'_v) \times \psi$ where π'_v is an unramified twist of π_v and ψ is any representation of $\mathrm{GL}_{d-tg}(F_v)$ whose cuspidal support is not linked to those of $\mathrm{st}_t(\pi'_v)$.

In particular for every $t > l_{\varrho,1}$, $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, P(t,\pi_v))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is zero, so that, as everything is torsion free,

$$H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \mathrm{gr}_*^{l_{\varrho,1}(\varrho)-1}(\mathrm{gr}_!^1(\Psi_\varrho)))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \hookrightarrow H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi_{K^v(\infty),v}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

Moreover this subspace, as a $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_d(F_v)$, has a subspace of the following shape $\mathrm{st}_{l_1(\varrho)}(\varrho) \times \tau$ where the supercuspidal support of τ contains ϱ. In particular as $q_v \equiv 1 \mod l$ and $l > d$, this induced representation has both a generic and a non generic subspace.

We can then conclude that for the genericity property to be true for KHT Shimura varieties, one needs a level raising property as in proposition 3.3.1 of [**Boy22**]. Hopefully such statements exist under some rather mild hypothesis as for example the following result of T. Gee.

Theorem 5.2.2. — *([**Gee11**] theorem 5.1.5) Let $F = F^+E$ be a CM field where F^+ is totally real and E is imaginary quadratic. Let $d > 1$ and $l > d$ be a prime which is unramified in F^+ and split in E. Suppose that*

$$\overline{\rho} : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation which is unramified at all places of F lying above primes which do not split in E and which satisfies the following properties.

- *$\overline{\rho}$ is automorphic of weight $\underline{a}$, where we assume that for all $\tau \in (\mathbb{Z}^d)^{\mathrm{hom}(F,\mathbb{C})}$ we have either*[20]

$$l - 1 - d \geq a_{\tau,1} \geq \cdots \geq a_{\tau,d} \geq 0 \text{ or } l - 1 - d \geq a_{c\tau,1} \geq \cdots \geq a_{c\tau,d} \geq 0.$$

- *$\overline{F}^{\ker \mathrm{ad}\, \overline{\rho}}$ does not contain $F(\zeta_l)$.*
- *$\overline{\rho}(G_{F^+(\zeta_l)})$ is big.*

Let u be a finite place of F^+ which split in F and not dividing l. Choose an inertial type τ_v and a place v of F above u. Assume that $\overline{\rho}_{|G_{F_v}}$ has a lift to characteristic zero of type τ_v.

Then there is an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\underline{a}$ and level prime to l such that

- *$\overline{r}_{l,\iota}(\pi) \cong \overline{\rho}$.*
- *$r_{l,\iota}(\pi)_{|G_{F_v}}$ has type τ_v.*
- *π is unramified at all places $w \neq v$ of F at which $\overline{\rho}$ is unramified.*

Remark. In this text we focus only on the trivial coefficients $\overline{\mathbb{Z}}_l$, i.e. to the case $a_{\tau,1} = \cdots = a_{\tau,d} = a_{c\tau,1} = \cdots = a_{c\tau,d} = 0$, but we could also deals with others weights as in the previous theorem.

5.3. Various filtrations.

— Recall that

$$H^{d-1}(\mathrm{Sh}_{K,\bar{\eta}_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \cong \overline{\sigma}_K \otimes_{\overline{R}_K} \overline{\rho}_K,$$

with $\overline{\sigma}_K/\overline{\mathfrak{m}} \cong \overline{\rho}_{\mathfrak{m}}$. It can be decomposed as follows.

[20] Note that these conditions imply $\overline{\rho}^c \cong \overline{\rho}^\vee \epsilon^{1-d}$.

– It is the direct sum
$$\bigoplus_{1\leq g\leq d}\bigoplus_{\varrho\in\mathrm{Cusp}_{\overline{\mathbb{F}}_l}(g,v)} H^0(\mathrm{Sh}_{K,\bar{s}_v},\Psi_\varrho)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l.$$
– For some fixed $\overline{\mathbb{F}}_l$-supercuspidal representation ϱ of $\mathrm{GL}_g(F_v)$, the maximal ideal $\mathfrak{m}$ gives us at the place v an element $\overline{\mathcal{P}}\in\overline{\mathcal{I}}$ and in particular numbers s_ϱ for every irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation ϱ.
– For $t>s_\varrho$, and π_v irreducible cuspidal with modulo l reduction isomorphic to ϱ, then $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\mathcal{P}(t,\pi_v))_{\mathfrak{m}}=(0)$.

We then have a filtration of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\Psi_\varrho)_{\mathfrak{m}}$ coming from the previous filtration of Ψ_ϱ:
$$(0)=\mathrm{Fil}^0_\varrho\subseteq\mathrm{Fil}^1_\varrho\subseteq\cdots\subseteq\mathrm{Fil}_\varrho^{\frac{s_\varrho(s_\varrho+1)}{2}},$$
where for $k=s_\varrho+(s_\varrho-1)+\cdots+(s_\varrho-t)-\delta$ with $0\leq t\leq s_\varrho-1$ and $0\leq\delta<s_\varrho-t-1$ we have
$$\mathrm{gr}^k_\varrho\cong H^0(\mathrm{Sh}_{K,\bar{s}_v},\mathcal{P}(\varrho,s_\varrho-\delta))(\frac{1-s_\varrho+2t}{2})_{\mathfrak{m}}.$$
The modulo l monodromy operator $\overline{N}_\varrho$ induces an isomorphism
$$\mathrm{gr}_\varrho^{s_\varrho+(s_\varrho-1)+\cdots+(s_\varrho-t)-\delta}\cong\mathrm{gr}_\varrho^{s_\varrho+(s_\varrho-1)+\cdots+(s_\varrho-t-1)-\delta}.$$

We then consider partitions $(l_1(\varrho)\geq l_2(\varrho),\cdots)$ of s_ϱ. The idea of the proof is to pass, step by step, from the previous filtration, coming from a sheaf filtration, to another one
$$(0)=\mathrm{Fil}^{fin}_{-1-s_\varrho}\subseteq\mathrm{Fil}^{fin}_{-s_\varrho}\subseteq\cdots\subseteq\mathrm{Fil}^{fin}_{-1}=H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\Psi_\varrho)_{\mathfrak{m}}$$
where for $r=1,\cdots,s_\varrho$, the graded part gr^{fin}_{-r} is a lattice of
$$\bigoplus_{\underline{s}_\varrho\in\mathrm{Part}_r(s_\varrho)}\bigoplus_{\Pi\in\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)}(\Pi^\infty)^{K^v}\otimes\sigma_\varrho(\Pi),\tag{28}$$
where

– $\mathrm{Part}_r(s_\varrho)$ is the subset of partition $\underline{s}_\varrho=(l_1\geq\cdots\geq l_k)$ of s_ϱ such that $l_1=r$;
– for $\underline{s}_\varrho=(l_1\geq\cdots\geq l_k)$ a partition of s_ϱ, $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)$ is the set of isomorphism classes of irreducible automorphic with $\mathfrak{m}$-K^v non trivial invariants and such that
$$\Pi_v\cong\mathrm{st}_{l_1}(\pi_{v,1})\times\cdots\times\mathrm{st}_{l_k}(\pi_{v,k})\times\psi$$
where the modulo l reduction of $\pi_{v,1},\cdots,\pi_{v,k}$ are isomorphic to ϱ and the supercuspidal support of the modulo l reduction of ψ does not contain ϱ.
– $\sigma_\varrho(\Pi)$ is the ϱ-part of $\sigma(\Pi)$ i.e. with the above notations,
$$\sigma_\varrho(\Pi)\cong\mathrm{Sp}_{l_1}(\rho_{v,1})\oplus\cdots\oplus\mathrm{Sp}_{l_k}(\rho_{v,k})$$
where for $i=1,\cdots,k$, the Galois representation $\rho_{v,i}$ is the contragredient of the representation associated to $\pi_{v,i}$ by the local Langlands correspondence.

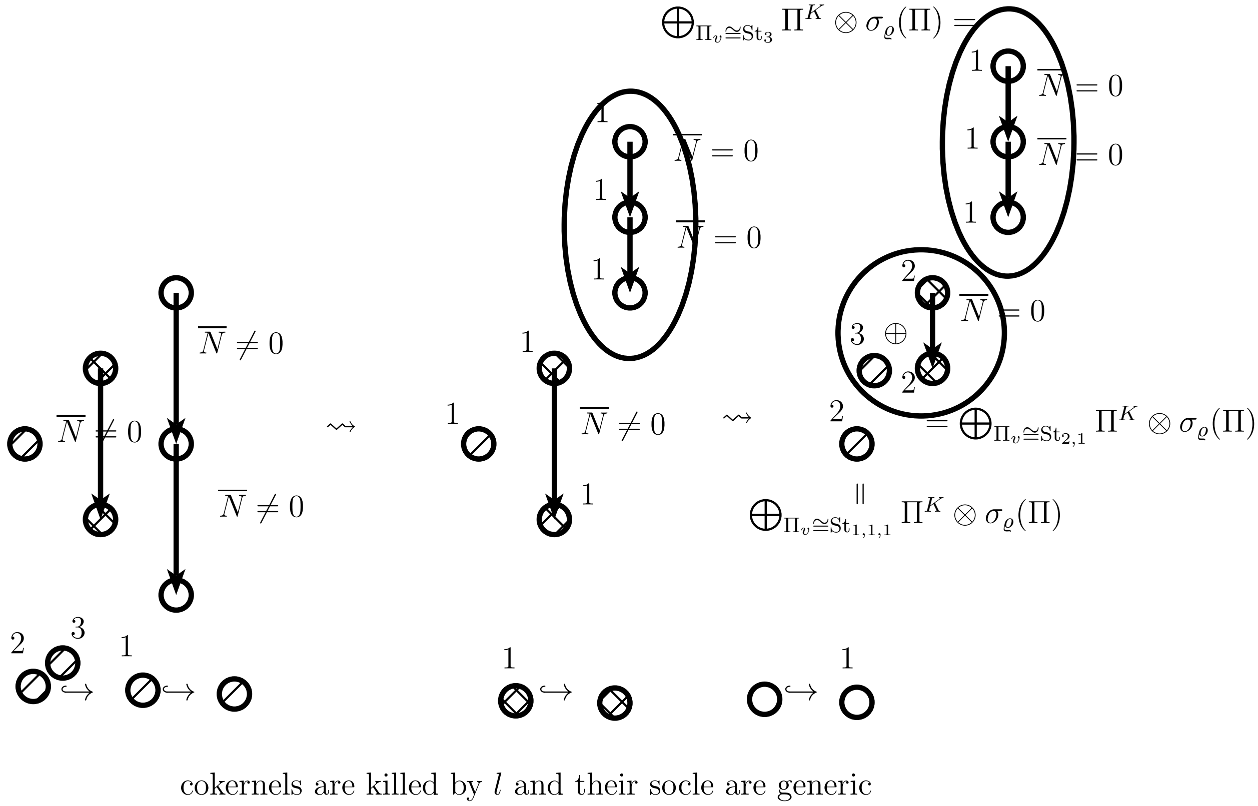

FIGURE 3. Three filtrations of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\Psi_\varrho)_{\mathfrak{m}}$ when $s_\varrho=3$

At each step r we prove the ϱ-part of Ihara's lemma for representations $\Pi\in\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)$ with $\underline{s}_\varrho\in\mathrm{Part}_r(s_\varrho)$ in the following sense:

– for such a irreducible automorphic representation Π, let $\Gamma(\Pi)$ be the lattice of Π_v induced by the $\mathcal{O}_L$-cohomology through

$$(\Pi^{\infty,v})^{K^v}\otimes\Pi_v\otimes\sigma(\Pi)\hookrightarrow H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\Psi_\varrho)_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l;$$

– then $\Gamma(\Pi)\otimes_{\mathcal{O}_L}\mathcal{O}_L/\varpi_L\mathcal{O}_L$, as a $\mathbb{F}_L$-representation of $\mathrm{GL}_d(F_v)$, has a ϱ-generic socle, i.e representations of the form $\mathrm{st}_{l_\varrho(\mathfrak{m})}(\varrho)\times\psi$ where ϱ does not belong to the supercuspidal support of ψ.

5.3.1. Case where $s_\varrho=3$. — Before considering the general case, we first want to explain the case where $s_\varrho=3$ and $d=gs_\varrho$: we advise the reader to follow the details of the proof with the help of figure 3. We first describe it.

– It illustrates three filtrations separated by the symbols $\rightsquigarrow$; the graded parts are represented by circles corresponding to the cohomology of Harris-Taylor perverse sheaves: compare the filtration on the left with those in the figure 2. In particular each circle can be viewed as some lattice of

$$\bigoplus_{\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s}_\varrho)} (\Pi^\infty)^{K^v} \otimes \sigma'_\varrho(\Pi)$$

for some partition $\underline{s}_\varrho$ of $s_\varrho = 3$ with the following precisions.

- When the circle is filled with diagonal lines with slope 1, it corresponds to $\underline{s}_\varrho = (1,1,1)$ or to the contribution of 1 in the partition $\underline{s}_\varrho = (2,1)$ in the sense that $\sigma'_\varrho(\Pi)$ is the Galois representation $\rho_l(\pi_{v,2})^\vee$ associated to $\pi_{v,2}$ in $\Pi_v \cong \mathrm{st}_2(\pi_{v,1}) \times \pi_{v,2}$, i.e. the contragredient of the Galois representation associated to $\pi_{v,2}$ by the local Langlands correspondence.
- When the circle is filled with diagonal lines with slope ± 1, it corresponds to the contribution of 2 in the partition $\underline{s}_\varrho = (2,1)$, in the sense that $\sigma'_\varrho(\Pi)$ is the Galois representation associated either to $\pi_{v,1}\{1/2\}$ or $\pi_{v,1}\{-1/2\}$ in $\Pi_v \cong \mathrm{st}_2(\pi_{v,1}) \times \pi_{v,2}$: the sign corresponds to the weight of the perverse Harris-Taylor sheaf whose cohomology gives the circle.
- When the circle is empty, it then corresponds to $\underline{s}_\varrho = (3)$: then $\sigma'_\varrho(\Pi)$ correspond to the Galois representation associated to $\pi_v\{k\}$ in $\Pi_v \cong \mathrm{st}_3(\pi_v)$ where $k \in \{-1,0,1\}$ corresponds to the weight of the perverse Harris-Taylor sheaf whose cohomology gives the circle.

– For each of these three filtrations illustrated in figure 3, subspaces appears from bottom to top.
– Arrows correspond to the nilpotent monodromy operator and we explicit if its modulo l reduction $\overline{N}$ is zero or non zero, see after for more details.
– In the second filtration we gather the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(3))$ so that in the large ellipse we obtain as a quotient the contribution of $\bigoplus_{\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(3))} (\Pi^\infty)^{K^v} \otimes \sigma_\varrho(\Pi)$ for all of $\sigma_\varrho(\Pi)$. The indices 1 above the circles means that the lattices are modified which is materialize in the bottom of the figure with the precision that cokernels are killed by ϖ_L with generic cosocle as $\mathbb{F}_L$-representations.
– In the last filtration on the right of the figure, we gather in the large circle the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1)$ with another modifications of the lattices: note that one have to separate the contributions of $(2,1)$ with $(1,1,1)$ in the circle filled with lines with slope 1.

The automorphic filtration

For each of these circles viewed as the $\overline{\mathbb{Z}}_l$-cohomology group of Harris-Taylor perverse sheaves, we will also consider filtrations as follows.

– For $n=1$ we fix any numbering of elements in $\mathcal{A}_{K^v,\bar{s}_v}(\varrho, \underline{s}_\varrho)$ which appears in $\mathcal{A}_{K^v(1),\bar{s}_v}(\varrho, \underline{s}_\varrho)$, i.e. having non trivial vectors invariants under $K_v(1)$. We then obtain successive subspaces of the $\overline{\mathbb{Q}}_l$-contribution of this circle: the intersection with the $\overline{\mathbb{Z}}_l$ cohomology gives us successive lattices $\Gamma(\Pi)$ of $(\Pi^\infty)^{K^v} \otimes \sigma'_\varrho(\Pi)$ for theses elements $\Pi \in \mathcal{A}_{K^v,\bar{s}_v}(\varrho, \underline{s}_\varrho)$ having non zero vectors invariant under $K_v(1)$.

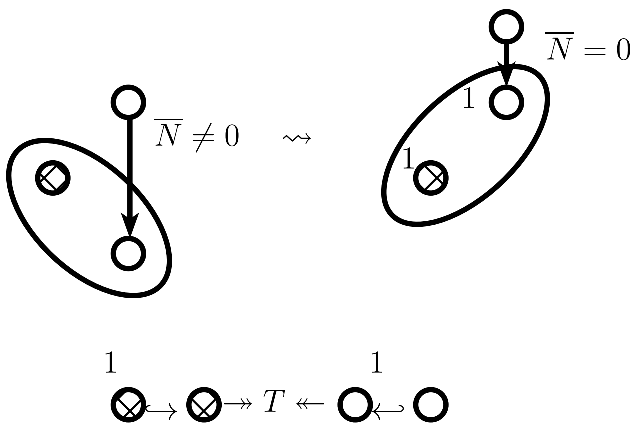

Figure 4. First step

- We then take $n = 2$, and consider elements of $\mathcal{A}_{K^v(2),\bar{s}_v}(\varrho, \underline{s}_\varrho)$ which do not appears with $\mathcal{A}_{K^v(1),\bar{s}_v}(\varrho, \underline{s}_\varrho)$: we fix a numbering of them and obtain other lattices of their contribution.
- We keep on this construction for any n and speak about *the automorphic filtration* of this circle.

Remark. Note that the lattice $\Gamma(\Pi)$ might depends on the ordering of any of the $\mathcal{A}_{K^v(n),\mathfrak{m}}(\varrho, \underline{s}_\varrho)$. To deal with finite number of graded parts, in the following we will argue for some fixed n: this also allows us to have everything defined over some finite extension L of $\mathbb{Q}_l$ as before.

A- We first explain how to pass from the first filtration to the second one.

Step 1: the exchange is non trivial
Consider as illustrated in the figure 4, the three first quotients of the first filtration, which correspond to the three circles in the bottom on the left part of the figure 3. We then exchange the first two as explained in figure 1. We want first to explain why this exchange is non trivial, i.e. the first extension is non split or equivalently T is non zero. For this we examine more precisely $\overline{N}$ on this space denoted by V.

a) On the left side of the figure 4 we know that the arrow is a isomorphism: we just write $\overline{N} \neq 0$ in the figure. We can then read the dimension of the image of $\overline{N}$ from what happens over $\overline{\mathbb{Q}}_l$. More precisely, over $\overline{\mathbb{Q}}_l$, the rank of the monodromy operator is equal to the cardinal of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho, (3))$ meaning for each finite level K_v, $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, (3))$ contributes to $g \dim_{\overline{\mathbb{Q}}_l}(\Pi^\infty)^{K^v K_v}$.

Remark. It is also possible to argue with $K^v(\infty)$, i.e. with infinite dimension at v: to be able to count something with finite dimension, one can look at the contribution of a K_v-type for example.

b) We then look at the right side of the figure 4. Looking at the last two graded parts, over $\overline{\mathbb{F}}_l$, as explained in the introduction, the modulo l reduction of the nilpotent monodromy operator is zero on any $\sigma_\varrho(\Pi)$ so when considering a lattice of a direct sum of $\sigma_\varrho(\Pi)$, the rank of $\overline{N}$ has to be strictly less than those of N over $\overline{\mathbb{Q}}_l$. In the above picture we simplify this observation by simply writing $\overline{N}=0$.

So the ellipse of the figure 4 cannot be split, i.e. the exchange is non trivial and the T appearing in figure 4 is non zero.

- As a quotient of the empty circle on the right of the last line, corresponding to the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(3))$, we deduce that T has to be extensions of $\overline{\mathbb{F}}_l$-generic representations of $\mathrm{GL}_d(F_v)$. To see this, consider an automorphic filtration of the empty circle and view T as a the limit of T_n where T_n corresponds to the exchange of the automorphic representation for $K^v(n)$. Then every T_n is a extension of ϱ-generic representations.
  Remark. in view of the next steps we just remember that T has a ϱ-generic socle;
- We then look at T as a quotient of left part of the last line. Consider an automorphic filtration of the circle filled with diagonal lines corresponding to automorphic representations in $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$. Let denote by $\Gamma(\Pi)$ be the lattice given by the initial circle and $\Gamma^1(\Pi)$ after the exchange, i.e. for the circle indexed with a 1. If $\Gamma(\Pi)$ is modified, for the first time it becomes $\Gamma(\Pi)^{gen}$ the lattice with a ϱ-generic socle. Then it is not possible to modify $\Gamma(\Pi)^{gen}$ to obtain a new lattice $\Gamma(\Pi)^2$ such that $\Gamma(\Pi)^{gen}/\Gamma(\Pi)^2$ is ϱ-generic: indeed the modulo l reduction of $\Gamma(\Pi)^{gen}$ contains an unique ϱ-generic constituant and it has to be in the image of

$$\Gamma(\Pi)^2\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l\longrightarrow\Gamma(\Pi)^{gen}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l.$$

Finally after the exchange, the subspace of the new filtration illustrated by a circle filled with lines and indexed by 1 in the figure 4, provided with its automorphic filtration, is such that the lattices are either the same as the initial one or it is the one with a ϱ-generic socle.

There is then at least one $\Pi_0\in\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$ such that its lattice is $\Gamma(\Pi_0)^{gen}$ and is not the one before the exchange. Take Π_0 as the first one. Note that modulo l we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{st}_3(\varrho) \arrow[r, hook] & \overline{X} \arrow[r, two heads] \arrow[d, equal] & \overline{\Gamma}(\Pi_0) \\
\overline{\Gamma}^{gen}(\Pi_0) \arrow[r, hook] & \overline{X} \arrow[r, two heads] & \mathrm{st}_3(\varrho)
\end{tikzcd}$$

where before the exchange, $\mathrm{st}_3(\varrho) = \ker(\overline{X} \twoheadrightarrow \overline{\Gamma}(\Pi_0))$ belongs to the image of $\overline{N}_\varrho$. Then as

$$\mathrm{st}_3(\varrho) = \ker(\overline{\Gamma}^{gen}(\Pi_0) \longrightarrow \overline{\Gamma}(\Pi_0))$$

we then deduce that, after the exchange, modulo l, the subspace $\mathrm{st}_3(\varrho)$ of $\overline{\Gamma}^{gen}(\Pi_0)$ maps to $\mathrm{st}_3(\varrho) = \ker(\overline{X} \twoheadrightarrow \overline{\Gamma}(\Pi_0)$ and then belongs to the image of $\overline{N}_\varrho$.

Step 2: the subspace lattice of Π_0 *has also a* ϱ*-generic socle*
Maybe Π_0 does not appears first in the automorphic filtration. Let denote by k the index of the graded part corresponding to Π_0 in the automorphic filtration of the circle on the bottom in the right side of the figure 4, i.e. the one indexed by 1 and filled with diagonal lines: we denote by W the cohomology represented by this circle.

We may moreover suppose that k is minimal among all modified lattices and we want to prove that $k = 1$. We reason by absurd and assume $k \geq 2$. Let then denote by $\Gamma(\Pi)$ the lattice which appears just before $\Gamma(\Pi_0)^{gen}$, i.e. the graded part of index $k - 1$. We then exchange these two graded parts so that before the exchange the successive graded parts are

$$W = \Gamma_-, \quad \Gamma(\Pi), \quad \Gamma(\Pi_0)^{gen}, \quad \Gamma_+$$

and after there are

$$W = \Gamma_-, \quad \Gamma(\Pi_0), \quad \Gamma(\Pi)', \quad \Gamma_+$$

and

$$\begin{tikzcd} \Gamma(\Pi_0) \arrow[r, hook] & \Gamma(\Pi_0)^{gen} \arrow[r, two heads] & T \arrow[d, equal] \\ \Gamma(\Pi) \arrow[r, hook] & \Gamma(\Pi)' \arrow[r, two heads] & T, \end{tikzcd} \tag{29}$$

where T is l-torsion, non zero and not ϱ-generic. As k is minimal, then $\Gamma_-, \Gamma(\Pi_0)$ is induced by the circle filled with diagonal lines in the left part of the figure 4, i.e. before any exchange, so that T is non zero as Π_0 was chosen so that $\Gamma(\Pi_0)^{gen}$ is not the same lattice as before the exchange. The modulo l reduction of these two filtrations gives a filtration of the modulo l reduction $\overline{W}$ of W with successive graded parts

$$\overline{W} = \overline{\Gamma}_-, \quad \overline{\Gamma}(\Pi_0), \quad \overline{\Gamma}(\Pi)', \quad \overline{\Gamma}_+ \quad = \quad \overline{\Gamma}_-, \quad \overline{\Gamma}(\Pi), \quad \overline{\Gamma}(\Pi_0)^{gen}, \quad \overline{\Gamma}_+.$$

We then focus on the ϱ-generic constituant of $\overline{\Gamma}(\Pi_0)$:

– viewed inside $\overline{\Gamma}(\Pi_0)$, it is not in the image of $\overline{N}$,
– but viewed in the second filtration, i.e. as a subsapce of $\overline{\Gamma}(\Pi_0)^{gen}$, we have seen it is in the image of $\overline{N}$;
– but as the modulo l reduction of T in (29) is not generic, these two subquotient are equal: contradiction.

Remark. Note that here ϱ-generic is the same as generic.

Step 3: every subspace lattice has a ϱ-generic socle
Consider now $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$ and the two previous lattices of Π_v with

$$0 \to \Gamma(\Pi)^{gen} \longrightarrow \Gamma(\Pi) \longrightarrow \mathrm{st}_3(\varrho) \to 0. \tag{30}$$

We want to distinguish these two lattices through their restriction to $K_v = \mathrm{GL}_d(\mathcal{O}_v)$. Note that $\mathrm{st}_3(\varrho)_{|K_v}$ contains an unique K_v-type $\overline{\sigma}_{\max}$ which is moreover maximal. From theorem 2.2.9 $\overline{\sigma}_{\max}$ appears with multiplicity one in the modulo l reduction of $\Gamma(\Pi)_{|K_v}$. Write $\Pi_v \cong \mathrm{st}_2(\pi_{v,1}) \times \pi_{v,2}$. With the notations of §2.2, to these two cuspidal representations $\pi_{v,1}$ and $\pi_{v,2}$, is associated a SZ-datum and in particular cuspidal representations ν_1 and ν_2 of a finite linear group.
(a) We first suppose that ν_2 and ν_1 are not isomorphic. Then Π_v contains an unique K_v-type σ_L whose modulo l reduction contains two K_v-types, $\overline{\sigma}_{\max}$ and another one $\overline{\sigma}_{\min}$. We denote by σ (resp. σ^{gen}) the lattice of σ_L obtained from $\Gamma(\Pi)$ (resp. $\Gamma(\Pi)^{gen}$) through $\sigma_L \hookrightarrow (\Pi_v)_{|K_v}$. By the multiplicity one property, the short exact sequence (30) gives

$$0 \to \sigma^{gen} \longrightarrow \sigma \longrightarrow T \to 0,$$

where T is non zero as it contains at least $\overline{\sigma}_{\max}$ as a sub-quotient. Consider any $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$, we know that its subspace lattice induced by the cohomology is either $\Gamma(\Pi)$ or $\Gamma(\Pi)^{gen}$. Proposition 4.3.2 tells us that it has to be $\Gamma(\Pi)^{gen}$.

(b) Consider now the case where the two cuspidal representations ν_1 and ν_2 are isomorphic. We consider another cuspidal representation ν_2' isomorphic to ν_2 modulo ϖ_L, which is not isomorphic to ν_1 and we denote by $\pi'_{v,2}$ the corresponding cuspidal representation constructed as in §2.2 using the same data for $\pi_{v,2}$ and replacing ν_2 by ν_2'.

By [**Gee11**] theorem 5.1.5, cf. theorem 5.2.2, there exists another system of Hecke eigenvalues λ' so that $\lambda' \equiv \lambda \mod \varpi_L$ and the associated local component Π'_v is isomorphic to $\mathrm{st}_2(\pi_{v,1}) \times \pi'_{v,2}$: note that λ' now verifies the property as in the previous case (a) so that Π'_v has only one K_v-type σ'_L. This type is constructed as explained in §2.2 so that we can copy the construction for Π_v to obtain a representation σ_L: note that, with the notations of the paragraph after definition 2.2.7, $\overline{\sigma}_{\overline{\mathcal{P}}}$ is not equal to $\pi_{\overline{\mathcal{P}}}$ and so σ_L is not a K_v-type.

We then consider the induced lattices Γ_{ind} (resp. Γ'_{ind}) of Π_v (resp. Π'_v) and the associated lattices σ_{ind} (resp. σ'_{ind}) of σ_L (resp. σ'_L). Note that modulo ϖ_L, we have

$$\overline{\Gamma}_{ind} \cong \overline{\Gamma}'_{ind} \qquad \text{and} \qquad \overline{\sigma}_{ind} \cong \overline{\sigma}'_{ind}.$$

Let then denote by Γ_{gen} and Γ'_{gen} the lattices such the socle of

$$\Gamma_{gen}/\varpi_L\Gamma_{gen} \cong \Gamma'_{gen}/\varpi_L\Gamma'_{gen}$$

is generic. We then also introduce the pullbacks

$$\begin{tikzcd} \Gamma_{gen} \arrow[r, hook] & \Gamma_{ind} \\ \sigma_{gen} \arrow[u, hook, dashed] \arrow[r, hook, dashed] & \sigma_{ind} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \Gamma'_{gen} \arrow[r, hook] & \Gamma'_{ind} \\ \sigma'_{gen} \arrow[u, hook, dashed] \arrow[r, hook, dashed] & \sigma'_{ind} \arrow[u, hook] \end{tikzcd}$$

The semi-simplification of $\sigma_{gen}/\varpi_L\sigma_{gen}$ is, cf. theorem 2.2.10, the sum of the only two K_v-types of $\overline{\Gamma}_{gen}$ and so is equal to the semi-simplification of $\sigma'_{gen}/\varpi_L\sigma'_{gen}$ and the image of

$$\sigma_{gen}/\varpi_L\sigma_{gen} \hookrightarrow \overline{\Gamma}_{gen} \cong \overline{\Gamma}'_{gen}$$

is $\sigma'_{gen}/\varpi_L\sigma'_{gen}$ so that

$$\sigma_{gen}/\varpi_L\sigma_{gen} \cong \sigma'_{gen}/\varpi_L\sigma'_{gen},$$

and

$$\begin{tikzcd} \sigma_{gen}/\varpi_L\sigma_{gen} \arrow[r] \arrow[d, "\cong"] & \sigma_{ind}/\varpi_L\sigma_{ind} \arrow[d, "\cong"] \\ \sigma'_{gen}/\varpi_L\sigma'_{gen} \arrow[r] & \sigma'_{ind}/\varpi_L\sigma'_{ind} \end{tikzcd}$$

is commutative.

Case (a) gives us that the natural morphism

$$M(\sigma^v\sigma'_{gen})/(\lambda',\varpi_L) \longrightarrow M(\sigma^v\sigma'_{ind})/(\lambda',\varpi_L)$$

induced by $\iota : \sigma'_{gen} \hookrightarrow \sigma'_{ind}$ is zero so that it is the same for

$$M(\sigma^v\sigma_{gen})/(\lambda,\varpi_L) \longrightarrow M(\sigma^v\sigma_{ind})/(\lambda,\varpi_L). \tag{31}$$

But if the lattice induced by λ were Γ_{ind}, then

$$\hom_{K_v}(\sigma_{ind}/\varpi_L, \sigma_{ind}/\varpi_L) \longrightarrow \hom_{K_v}(\sigma_{gen}/\varpi_L, \sigma_{ind}/\varpi_L)$$

is non zero as the image of identity is $\bar{\iota} \neq 0$ and so (31) is non zero.

Remark. In case (b) if we do not modify one of the cuspidal representations, the problem is that the K_v-type whose modulo l reduction contains $\overline{\sigma}_{\max}$, remains irreducible modulo ϖ_L and so have, up to homothety, only one stable lattice. We are then not able to detect the right $\mathrm{GL}_{3s_\varrho}(F_v)$-lattice. In the general case we might also have to modify one of the cuspidal representation in the cuspidal support to avoid such a situation.

Remark. The case $s_\varrho = 3$ is more easy than the general one because we are concerned with only one partition $(2,1)$ so that when we consider any other automorphic contribution Π it shares the same partition $(2,1)$ with Π_0 and the modulo l reduction of the types of Π and Π_0 coincide. In the general case we will need to manage the existence of various partitions contributing to the considered circle.

Step 4: Ihara's lemma for $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$
By now concerning the new filtration, those illustrated on the right side of the figure

4, we denote again by W the contribution of the circle filled with diagonal lines and indexed by 1:

$$0 \to W \longrightarrow V \longrightarrow W' \to 0.$$

We focus on an automorphic filtration of W where the graded parts are lattices of the contribution of some $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1)$: we have seen that the first graded part, i.e. the subspace one, is such that its modulo l reduction is a non trivial extension of the irreducible non ϱ-generic constituant by the ϱ-generic one.

Consider any graded part $\Gamma(\Pi)$ and suppose that its modulo l reduction has a non ϱ-generic subspace: we state that it is not a subspace of W. Indeed we have seen that subspace lattice of the contribution of Π is $\Gamma(\Pi)^{gen}$ and

$$0 \to \Gamma(\Pi)^{gen} \longrightarrow \Gamma(\Pi) \longrightarrow T \to 0,$$

where T is l-torsion and non zero as by hypothesis $\Gamma(\Pi)$ is supposed to be non isomorphic to $\Gamma(\Pi)^{gen}$. Tensoring with $\overline{\mathbb{F}}_l$, we then obtain that the non ϱ-generic constituant of $\Gamma(\Pi)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$, which is supposed to be a subspace of W and which belongs to the image of $\Gamma(\Pi)^{gen} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. But we know that this non ϱ-generic constituant is not a subspace of $\Gamma(\Pi)^{gen} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$: contradiction.

Step 5: final exchanges to arrive at the second filtration
We now exchange W' with the two circles of the first filtration of the figure 3: those filled with diagonal lines with slope 1 and those with slopes ± 1. We then arrive at the second filtration of this figure. To be in position to pass from the second filtration to the third one, we need to give more informations on the arrow $\overline{N} \neq 0$ of the second filtration of the figure 3.

Let consider the subspace represented by the first three graded parts, i.e. we remove the quotient given by the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(3))$. Before all the exchanges, the image of the considered $\overline{N}$ was equal to the contribution of the circle pointed by this arrow. In particular all the $(\varrho,2)$-small subquotients were in the image of $\overline{N}$. But as the exchanges made are only related to $\mathrm{st}_3(\varrho)$, we then deduce that after all the exchanges, all the $(\varrho,2)$-small subquotients of the circle pointed by the arrow are still in the image of $\overline{N}$.

B- We now explain how to pass from the second filtration to the third one.

The situation is very similar, we just focus on the differences. Let denote again by V the quotient gathering the three first graded parts of this second filtration. Note that

- the modulo l reduction of the first graded part, i.e. the circle filled with lines with slopes ± 1, has a ϱ-generic socle;
- the rank of $\overline{N}$, concerning the $(\varrho,2)$-small subquotients, can be directly compared to its $\overline{\mathbb{Q}}_l$-version as explained above.

Step 1: the exchange is non trivial
As step 1 of A, the exchange between the first two graded parts is necessarily non trivial because after the exchange the $\overline{\mathbb{F}}_l$-contribution of $\overline{N}$, concerning $(\varrho, 2)$-small subquotients, in the first two graded parts is strictly less than its $\overline{\mathbb{Q}}_l$-contribution.

Remark. The easiest way maybe to count things, is to look at the contributions of K_v-type for the partition $(2,1)$: note that $\mathrm{st}_3(\pi_v)$ never contributes to these contributions.

Concerning the torsion module T produced by the various exchanges,

(i) on one side as a quotient of the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$, it follows from A step 3, one of them has a ϱ-generic socle but compared to A we don't know whether it is always a non trivial extension of $\mathrm{st}_3(\varrho)$ by $LT_\varrho(1,1)$ or it is $\mathrm{st}_3(\varrho)$ alone with $LT_\varrho(1,1)$ appearing alone after;
(ii) on the other side, note that modulo ϖ_L, the representation $\pi_{v,1} \times \pi_{v,2} \times \pi_{v,3}$ has a subquotient which is not a constituant of any $\mathrm{st}_2(\pi_{v,1}) \times \pi_{v,2}$, where all the $\pi_{v,i}$ are supposed to be isomorphic to ϱ modulo ϖ_L. As before we then deduce that T is killed by ϖ_L.

We then deduce that after the exchange, concerning the circle filled with line of slope 1 and indexed with a 2, its automorphic filtration separating the contributions of $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(1,1,1))$, has a graded part $\Gamma(\Pi_0)$ such that its modulo ϖ_L reduction, as a $\mathbb{F}_L$-representation of $\mathrm{GL}_d(F_v)$, has a ϱ-generic socle.

Step 2-3-4: lattices for elements of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(1,1,1))$
The arguments of A-step 2 applies and gives us that the lattice associated to Π_0 viewed as a subspace of V has also a ϱ-generic socle. Repeating the arguments of A-step 3, thanks to proposition 4.3.2, we then deduce that whatever is $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(1,1,1))$, its lattice induced by V as a subspace, has also a ϱ-generic socle. In particular in (i) above, we deduce that all l-torsion modules appearing in the exchanges are necessary a non trivial extension of $LT_\varrho(1,1)$ by $\mathrm{st}_3(\varrho)$ and the subspace lattice of any $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(1,1,1))$, has a socle filtration with at least three graded parts where the two first ones are $\mathrm{st}_3(\varrho)$ and $LT_\varrho(1,1)$.

Finally the previous arguments of A-step 4 show that no non ϱ-generic subquotient of the contributions of elements of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(1,1,1))$ can give an irreducible subspace of the modulo l cohomology.

We then arrive at our final filtration where we succeeded to gather the contributions of the various elements of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)$ such that no non ϱ-generic irreducible constituant of the various graded parts could be a subspace of the modulo l cohomology: Ihara's lemma is then proved in this case.

5.3.2. The general case. — We now consider the general case and we follow closely the arguments explained when $s_\varrho = 3$. We first have to define the different filtrations and then explain how to pass from one to its following.

For some fixed r between 1 and s_ϱ we suppose that we have already proved the following property:

- consider an automorphic representation Π which contributes to $H^0(\mathrm{Sh}_{K,\bar{s}_v},\Psi_\varrho)_\mathfrak{m}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ with $\Pi_v\cong \mathrm{st}_{l_1(\varrho)}\pi_{v,1}\times\cdots \mathrm{st}_{l_k(\varrho)}\pi_{v,k}\times\psi$ where $(l_1(\varrho)\geq l_2(\varrho),\cdots,l_k(\varrho))$ is a partition of s_ϱ with $r_l(\pi_{v,i})\cong\varrho$ for $i=1,\cdots,k$ and the supercuspidal support of the modulo l reduction of ψ does not contain ϱ;
- suppose that $l_1(\varrho)\geq r$, then the $\mathrm{GL}_d(F_v)$-lattice induced by
$$(\Pi^{\infty,v})^{K^v}\otimes\Pi_v\otimes\rho_l(\pi_v)^\vee\hookrightarrow H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\Psi_\varrho)_\mathfrak{m}$$
is such that the only irreducible subspace of its modulo l reduction is $\mathrm{st}_{s_\varrho}(\varrho)\times\psi$ where ψ is some irreducible $\overline{\mathbb{F}}_l$-representation whose supercuspidal support does not contain ϱ.

For the same fixed r between 1 and s_ϱ we consider the following filtration:
$$(0)=\mathrm{Fil}_\varrho^{-r(r+1)/2}(r)\subseteq\cdots\subseteq\mathrm{Fil}^0_\varrho(r)\subseteq\mathrm{Fil}^1_\varrho(r)=H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\Psi_\varrho)_\mathfrak{m}$$
such that

- $\mathrm{gr}^1_\varrho(r)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ is the direct sum
$$\bigoplus_{\underline{s}_\varrho>(r)}\ \bigoplus_{\Pi\in\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)}(\Pi^\infty)^{K^v(\infty)}\otimes\sigma_\varrho(\Pi),$$
where $(l_1\geq l_2\geq\cdots)>(r)$ means $l_1>r$;
- for $k=1+2+\cdots+(t+1)-\delta$ with $0\leq t\leq r-1$ and $0\leq\delta\leq t$, the graded part $\mathrm{gr}_\varrho^{-r(r+1)/2+k}(r)$ is a lattice $\Gamma(t,\delta)^{init}$ of[21]
$$\bigoplus_{\underline{s}_\varrho=(r\geq\cdots)}\ \bigoplus_{\Pi\in\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)}(\Pi^\infty)^{K^v(\infty)}\otimes\rho_l(\Pi_v)(\frac{r-1-2t+\delta}{2})$$
$$\hookrightarrow\bigoplus_{\pi_v/r_l(\pi_v)\cong\varrho}H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\mathcal{P}(r-\delta)(\frac{r-1-2t+\delta}{2}))_\mathfrak{m}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l.$$
Moreover with the same notations for $0\leq\delta\in\{t-1,t\}$, if we denote by $\Gamma(t,\delta)^0$ the lattice induced by
$$\bigoplus_{\pi_v/r_l(\pi_v)\cong\varrho}H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\mathcal{P}(t+1+\delta,\pi_v)(\frac{t-\delta}{2}))_\mathfrak{m}$$
on the previous subspace, then we have
$$\Gamma(t,\delta)^{init}\hookrightarrow\Gamma(t,\delta)^0$$
where the cokernel does not contain any (ϱ,r)-small subquotient.

(21) The contribution of the automorphic representations $\Pi\in\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,\underline{s}_\varrho)$ for $\underline{s}_\varrho=(l_1\geq\cdots)$ with $l_1>r$ was already put in $\mathrm{gr}^0_\varrho(r)$ while the others which contributes to the cohomology of $\mathcal{P}(t+1+\delta)(\frac{t-\delta}{2})$ remains in this graded parts.

Remark. In the case $s_\varrho = 3$,

- in the second filtration of the figure 3, $\mathrm{gr}^1_\varrho(2)$ is the first ellipse gathering the contributions of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(3))$. The image of the circle pointed by (resp. at the origin of) the vector corresponds to $(t,\delta)=(0,0)$ (resp. $(1,0)$) and the last one is indexed by $(1,1)$.
- In the third filtration of this figure, $\mathrm{gr}^1_\varrho(1)$ gathers both the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(3))$ and those of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$ in the large circle. There is then only remaining graded part for $(t,\delta)=(0,0)$ which corresponds to the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(1,1,1))$ which corresponds to the remaining part of the cohomology of $\bigoplus_{\pi_v/r_l(\pi_v)\cong\varrho}\mathcal{P}(1,\pi_v)(0)$ when the contribution of $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(2,1))$ has been removed.

– For every $0\leq t\leq r-1$, the modulo ϖ_L monodromy operator $\overline{N}_\varrho$ induces

$$\mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t+1)-(t-1))}(r)\longrightarrow \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+t-(t-1))}(r),$$

which is surjective relatively to all the (ϱ,r)-small subquotients.

– if $\Gamma_{K^v,\mathfrak{m}}(r,t,\delta)$ is the *initial* lattice of $\mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2\cdots+(t+1)-\delta)}(r)$ induced by $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v},\mathcal{P}(r-\delta)(-\frac{r-1-2t+\delta}{2}))_\mathfrak{m}$ then

$$\mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t+1)-\delta)}(r)\hookrightarrow\Gamma_{K^v,\mathfrak{m}}(r,t,\delta)$$

where the cokernel does not contain any (ϱ,r)-small subquotient, and the kernel of the modulo ϖ_L reduction of this injection has a ϱ-generic socle.

5.3.3. Proof of the induction. — Note first that the case $r=s_\varrho$ is clearly satisfied. We now suppose that the result is true for r and we prove it is then true for $r-1\geq 1$. We follow closely the arguments of the case $s_\varrho=3$.

Step 1: the exchange is non trivial
Consider the three first graded parts $V:=\mathrm{Fil}_\varrho^{-r(r+1)/2+2}(r)$ where the nilpotent monodromy operator induces $\mathrm{gr}_\varrho^{-r(r+1)/2+2}(r)\longrightarrow\mathrm{gr}_\varrho^{-r(r+1)/2}(r)$ which remains an isomorphism modulo ϖ_L. We then exchange the two first graded parts and then denote the new graded parts as

$$\widetilde{\mathrm{gr}}_\varrho^{-r(r+1)/2+1}(r,\underline{s}_\varrho(r,\max)),\qquad \widetilde{\mathrm{gr}}_\varrho^{-r(r+1)/2}(r,\underline{s}_\varrho(r,\max)),\qquad \mathrm{gr}_\varrho^{-r(r+1)/2+2}(r)$$

with

$$0\to\widetilde{\mathrm{gr}}_\varrho^{-r(r+1)/2+1}(r)\longrightarrow\mathrm{gr}_\varrho^{-r(r+1)/2+1}(r)\longrightarrow T\to 0,$$

and

$$0\to\mathrm{gr}_\varrho^{-r(r+1)/2}(r)\longrightarrow\widetilde{\mathrm{gr}}_\varrho^{-r(r+1)/2}(r)\longrightarrow T\to 0,$$

where T is torsion. The situation is exactly similar as step 1 for the case $s_\varrho=3$, so that by considering the modulo ϖ_L reduction of the nilpotent monodromy operator on V, we deduce that the exchange is non trivial, i.e. $T\neq 0$. Considering the modulo ϖ_L reduction of the second short exact sequence, the induction hypothesis tells us that the socle of $T[\varpi_L]$ is ϱ-generic.

Consider an automorphic filtration of $\mathrm{gr}_\varrho^{-r(r+1)/2+1}(r)$ whose graded parts $\mathrm{grr}_\varrho^k(r)$ correspond to the contribution of some $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s}_\varrho)$ for some partition $\underline{s}_\varrho = (r-1 \geq \cdots)$ of s_ϱ starting with $r-1$. We infer a similar filtration of $\widetilde{\mathrm{gr}}_\varrho^{-r(r+1)/2+1}(r)$ with graded parts

$$0 \to \widetilde{\mathrm{grr}}_\varrho^k(r) \longrightarrow \mathrm{grr}_\varrho^k(r) \longrightarrow T_\varrho^k(r) \to 0.$$

As T is also a quotient of $\widetilde{\mathrm{gr}}_\varrho^{r(r+1)/2-1}(r)$, we then deduce that there exists k such that $T_\varrho^k(r)[\varpi_L]$ has a non zero socle which is ϱ-generic.

Step 2: the subspace lattice of Π_0 *has also a* ϱ*-generic socle*
The same arguments of the case $s_\varrho = 3$ apply without any change.

Step 3: every subspace lattice attached to any $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s}_\varrho)$ *for any partition* $\underline{s}_\varrho = (r-1 \geq \cdots)$*, has also a* ϱ*-generic socle*
The main ingredient is the same as in the case $s_\varrho = 3$, that is proposition 4.3.2, but now we have to struggle with the fact that usually there are many partitions $\underline{s}_\varrho = (r-1 \geq \cdots)$ starting with $r-1$: the problem is that proposition 4.3.2 needs the K_v-type to have multiplicity one in $(\Pi_v)_{|K_v}$.

(i) From some $\underline{s}_\varrho^0$ *to* $(r-1, s_\varrho - r + 1)$
By now we proved the existence of some $\Pi_0 \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s}_\varrho^0)$ for which its subspace lattice Γ_0^{gen} induced by the cohomology is such that its modulo l reduction has a ϱ-generic socle. The local component of Π_0 at v looks like

$$\Pi_{0,v} \cong \mathrm{st}_{r-1}(\pi_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,k}) \times \psi^\varrho$$

where

- $\underline{s}_\varrho^0 = (r-1 \geq l_1 \geq \cdots \geq l_k)$,
- the modulo l reduction of $\pi_{v,i}$ is isomorphic to ϱ for $i = 0, \cdots, k$
- and ϱ does not belong to the supercuspidal support of the modulo l reduction of ψ^ϱ.

Consider now

$$\Pi'_v \cong \mathrm{st}_{r-1}(\pi_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}\{\delta_1/2\}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,1}\{\delta_k/2\}) \times \psi^\varrho$$

where $\delta_1 = s_\varrho - r$, $\delta_2 = \delta_1 - 2l_1$... and $\delta_k = \delta_1 - 2l_1 - \cdots - 2l_{k-1} = r - s_\varrho$.

Remark. The δ_i are chosen so that

$$\mathrm{st}_{s_\varrho - r+1}(\pi_{v,1}) \hookrightarrow \mathrm{st}_{l_1}(\pi_{v,1}\{\delta_1/2\}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,1}\{\delta_k/2\}). \tag{32}$$

Lemma 5.3.1. — *To any stable lattice* Γ *of*

$$\mathrm{st}_{r-1}(\pi_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,k})$$

is associated a stable lattice Γ' *of*

$$\mathrm{st}_{r-1}(\pi_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}\{\delta_1/2\}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,1}\{\delta_k/2\})$$

such that $\overline{\Gamma} := \Gamma/\varpi_L\Gamma \cong \Gamma'/\varpi_L\Gamma' =: \overline{\Gamma}'$.

Proof. — Let denote by Γ_{ind} (resp. Γ'_{ind}) the induced stable lattice of the first (resp. second) induced representation of the lemma, and note that their modulo l reduction is semi-simple and so isomorphic. Consider then a stable lattice Γ contained in Γ_{ind} such that $\tau := \Gamma_{ind}/\Gamma$ is l-torsion and let Γ' be defined by the following pullback:

$$\begin{tikzcd}
 & \tau \arrow[r, equal] & \tau \\
l\Gamma'_{ind} \arrow[r, hook] \arrow[d, equal] & \Gamma'_{ind} \arrow[r, two heads] \arrow[u, two heads] & \tau \oplus \tau' \arrow[u, two heads] \\
l\Gamma'_{ind} \arrow[r, hook] & \Gamma' \arrow[r, dashed] \arrow[u, hook, dashed] & \tau'. \arrow[u, hook]
\end{tikzcd}$$

We then have $\overline{\Gamma}' \cong \overline{\Gamma}$. We can then repeat the argument with this (Γ, Γ') instead of $(\Gamma_{ind}, \Gamma'_{ind})$ so that, step by step, we are finally able to cover all the cases. $\square$

Recall that for each of the $\pi_{v,i}$ is associated, cf. notation 2.2.8, a SZ datum and in particular a cuspidal representation σ_i of a finite linear group.

- If all these σ_i for $i = 0, \cdots, k$ are pairwise non isomorphic, we then consider σ_L the associated K_v-type of $\mathrm{st}_{r-1}(\pi_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,k})$: it is obtained by inducing the K_v-types of the Steinberg representations.(22) Note that from theorem 2.2.10, the modulo ϖ_L-reduction of σ_L is reducible.
- Otherwise, as in A step 3 for $s_\varrho = 3$, to avoid to consider a K_v-type with an irreducible modulo ϖ_L reduction, we modify the σ_i to $\widetilde{\sigma}_i$ such that they are then pairwise distinct.
  Remark. Note that l divides $q-1$ and $l > d$ such that there exists, up to increase L, at least d distinct characters of $\mathbb{F}_q^\times \longrightarrow \mathcal{O}_L^\times$ all to congruent to the trivial one modulo ϖ_L.
  Let denote by $\widetilde{\pi}_{v,i}$ the cuspidal representation associated to the K_v-type of $\pi_{v,i}$ where we replace σ_i by $\widetilde{\sigma}_i$. Consider then the K_v-type $\widetilde{\sigma}_L$ of $\mathrm{st}_{r-1}(\widetilde{\pi}_{v,0}) \times \mathrm{st}_{l_1}(\widetilde{\pi}_{v,1}) \times \cdots \times \mathrm{st}_{l_k}(\widetilde{\pi}_{v,k})$, which also have a reducible modulo ϖ_L reduction. (23) We then copy the construction of $\widetilde{\sigma}_L$ replacing $\widetilde{\sigma}_i$ by σ_i, to obtain σ_L which is now reducible. Note also that $\widetilde{\sigma}_L$ is both irreducible and appears with multiplicity one in $\mathrm{st}_{r-1}(\widetilde{\pi}_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,k})$, which allows to apply proposition 4.3.2.

Replacing the $\pi_{v,i}$ for $i = 1, \cdots, k$, by $\widetilde{\pi}_{v,1}$, and $\pi_{v,0}$ by $\widetilde{\pi}_{v,0}$, in the construction of σ_L we obtain a subspace σ'_L of $\mathrm{st}_{r-1}(\widetilde{\pi}_{v,0}) \times \mathrm{st}_{l_1}(\widetilde{\pi}_{v,1}\{\delta_1/2\}) \times \cdots \times \mathrm{st}_{l_k}(\widetilde{\pi}_{v,1}\{\delta_k/2\})$. By

(22) With the notations of loc. cit., we have $\sigma_{\overline{\mathcal{P}}} = \pi_{\overline{\mathcal{P}}}$.

(23) Note that for $\widetilde{\sigma}_L$, with the notations of the paragraph after definition 2.2.7, $\overline{\sigma}_{\overline{\mathcal{P}}}$ is not equal to $\pi_{\overline{\mathcal{P}}}$ and so $\widetilde{\sigma}_L$ is not a K_v-type.

construction in notation 2.2.8, the K_v-type $\sigma_{st,L}$ of $\mathrm{st}_{r-1}(\widetilde{\pi}_{v,0}) \times \mathrm{st}_{s_\varrho - r+1}(\widetilde{\pi}_{v,1})$ appears with multiplicity one in σ'_L and has a reducible modulo ϖ_L reduction.

For a stable lattice Γ we obtain lattices Γ' and $\widetilde{\Gamma}$ as explained in the previous lemma and we denote by Γ^{st} the lattice of $\mathrm{st}_{r-1}(\widetilde{\pi}_{v,0}) \times \mathrm{st}_{s_\varrho - r+1}(\widetilde{\pi}_{v,1})$ given by Γ' through the injective map (32). We then obtain a lattice σ of σ_L and, using the previous lemma, lattices σ' of σ'_L and $\widetilde{\sigma}$ of $\widetilde{\sigma}_L$. By construction for the induced lattices, note that

$$\overline{\sigma}^{',ind} := \sigma^{',ind}/\varpi_L \sigma^{',ind} \cong \sigma^{ind}/\varpi_L \sigma^{ind} =: \overline{\sigma}^{ind} \cong \widetilde{\sigma}^{ind}/\varpi_L \widetilde{\sigma}^{ind}$$

as a $\mathbb{F}_L$-representation of $K_v = \mathrm{GL}_d(\mathcal{O}_v)$. We also obtained a lattice σ_{st}^{ind} of $\sigma_{st,L}$ with

$$\overline{\sigma}_{st}^{ind} := \sigma_{st}^{ind}/\varpi_L \sigma_{st}^{ind} \hookrightarrow \overline{\sigma}^{',ind} \cong \overline{\sigma}^{ind}.$$

Starting with Γ^{gen} we denote the associated lattices by σ^{gen}, $\widetilde{\sigma}^{gen}$, $\sigma^{gen,'}$ and σ_{st}^{gen}. Reasoning as in step 3 for case $s_\varrho = 3$, in the diagram

$$\begin{tikzcd} \widetilde{\sigma}^{gen}/\varpi_L \arrow[r, hook] & \widetilde{\Gamma}/\varpi_L \arrow[d, "\cong"] \\ \sigma^{gen}/\varpi_L \arrow[r, hook] & \Gamma/\varpi_L \end{tikzcd}$$

the image of $\widetilde{\sigma}^{gen}/\varpi_L$ in Γ/ϖ_L is σ^{gen}/ϖ_L: indeed the semi-simplification of $\widetilde{\sigma}^{gen}/\varpi_L$ (resp. σ^{gen}/ϖ_L) is, cf. theorem 2.2.10, the sum of all the K_v-types of $\widetilde{\Gamma}/\varpi_L$ (resp. Γ/ϖ_L). In the same way we also have $\sigma^{gen}/\varpi_L \cong \sigma^{gen,'}/\varpi_L$.

As the modulo ϖ_L reduction of Γ_{st}^{gen} has two irreducible subquotients, $\Gamma_{st,L}$ has a unique stable lattice $\Gamma_{st,+}$ such that

$$\Gamma_{st}^{gen} \subsetneq \Gamma_{st,+} \subsetneq \varpi_L^{-1} \Gamma_{st}^{gen}.$$

Note that the cokernel $\Gamma_{st,+}/\Gamma_{st}^{gen}$ contains exactly one K_v-type, the maximal one $\overline{\sigma}_{\max}$ of $\mathrm{st}_{s_\varrho}(\varrho)$. We then deduce that the cokernel of $\sigma_{st}^{gen} \hookrightarrow \sigma_{st,+}$ is non zero as it contains at least $\overline{\sigma}_{\max}$. We then construct the lattice Γ'_+ by pushout

$$\begin{tikzcd} \Gamma_{st}^{gen} \arrow[r, hook] \arrow[d, hook] & \Gamma_{st,+} \arrow[d, hook, dashed] \\ \Gamma^{gen,'} \arrow[r, hook, dashed] & \Gamma'_+, \end{tikzcd}$$

giving also rise to a lattice Γ_+. We then obtained lattices $\sigma_{st,+}$, σ'_+, σ_+ and $\widetilde{\sigma}_+$.

Recall that ψ^ϱ is a representation of $\mathrm{GL}_{d-s_\varrho g}(F_v)$ such that ϱ does not belong to the supercuspidal support of its ϖ_L-modulo reduction. Consider a representation ψ_ϱ such that, up to multiplicity, the supercuspidal support of its modulo ϖ_L reduction is equal to ϱ. Any stable lattice of $\psi_\varrho \times \psi^\varrho$ is then induced form stable lattices Γ_ϱ and Γ^ϱ of respectively ψ_ϱ and ψ^ϱ. Consider then any stable lattice Γ^ϱ of ψ^ϱ that we induce with the previous lattices σ^{gen}, $\sigma' \cdots$ to obtain lattices denoted σ_ψ^{gen}, $\sigma'_\psi \cdots$.

With the notations of §4.3, consider then the following commutative diagram
(33)
$$\begin{tikzcd}
& \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}^{gen}_{st,\psi}) \arrow[r] \arrow[d, hook] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{st,+,\psi}) \arrow[d, hook] & \\
\overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}^{gen}_{\psi}) \arrow[r, equal] \arrow[d, equal] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}^{gen,'}_{\psi}) \arrow[r] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}'_{+,\psi}) \arrow[r, equal] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{+,\psi}) \arrow[d, equal] \\
\overline{M}_{K^v}(\overline{\sigma}^v\widetilde{\sigma}^{gen}_{\psi}/\varpi_L) & & & \overline{M}_{K^v}(\overline{\sigma}^v\widetilde{\sigma}_{+,\psi}/\varpi_L)
\end{tikzcd}$$

By [**Gee11**] theorem 5.1.5, cf. theorem 5.2.2, there exists another system of Hecke eigenvalues $\widetilde{\lambda}$ so that $\widetilde{\lambda} \equiv \lambda_0 \mod \varpi_L$ and the associated local component Π'_v is isomorphic to
$$\mathrm{st}_{r-1}(\widetilde{\pi}_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}) \times \cdots\cdots \mathrm{st}_{l_k}(\pi_{v,k}) \times \psi^{\varrho}.$$
Proposition 4.3.2 tells us that
$$M_{K^v}(\sigma^v\widetilde{\sigma}^{gen}_{\psi})/(\varpi_L,\widetilde{\lambda}) \longrightarrow M_{K^v}(\sigma^v\widetilde{\sigma}_{+,\psi})/(\varpi_L,\widetilde{\lambda})$$
is either zero or an isomorphism, and it is independent on the choice of $\widetilde{\lambda}$. However for λ_0, as the lattice induced by the cohomology on σ_L is σ^{gen}, then
$$\overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}^{gen}_{\psi})/\lambda_0 \longrightarrow \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{+,\psi})/\lambda_0$$
can not be an isomorphism, so it is zero.

The commutativity of the diagram imposes that the top horizontal map is also zero. Consider then a system of Hecke eigenvalues λ_1 associated to a automorphic representation $\Pi_1 \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho,(r-1,s_\varrho-r+1))$ with
$$\Pi_{1,v} \cong \mathrm{st}_{r-1}(\pi^1_{v,0}) \times \mathrm{st}_{s_\varrho-r+1}(\pi^1_{v,1}) \times \psi_1^{\varrho},$$
where the modulo ϖ_L reduction of $\pi^1_{v,0}$ and $\pi^1_{v,1}$ is isomorphic to ϱ while those of ψ_1^ϱ does not contain ϱ in its supercuspidal support. As above, the construction of $\sigma_{st,L}$ can be used, replacing $\widetilde{\pi}_{v,0}$ (resp. $\pi_{v,1}$) by $\pi^1_{v,0}$ (resp. $\pi^1_{v,1}$) and ψ^ϱ by ψ^ϱ_1, to obtain the subspace σ^1_{st,L,ψ_1} of $(\Pi_{1,v})_{K_v}$. The lattices $\sigma^{gen}_{st,\psi}$ and $\sigma_{st,+,\psi}$ then gives us lattices $\sigma^{gen,1}_{st,\psi_1}$ and $\sigma^1_{st,+,\psi_1}$ such that
$$\sigma^1_{st,+,\psi_1}/\varpi_L \cong \sigma_{st,+,\psi}/\varpi_L \qquad \text{and} \qquad \sigma^{gen,1}_{st,\psi_1}/\varpi_L \cong \sigma^{gen}_{st,\psi}/\varpi_L.$$
We have seen that
$$\overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}^{gen}_{st,\psi}) \longrightarrow \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{st,+,\psi})$$
is zero modulo λ_0 and so it is also zero modulo λ_1. In particular we see that $\Gamma^1_{st,+}$ can not be the subspace lattice of Π^1. Finally we are able to eliminate stable lattice on the right of Γ^{gen}_{st} which are not isomorphic to it.

To deal with lattices $\Gamma_{st,-}$ contained in Γ^{gen}_{st}
$$\varpi_L\Gamma^{gen}_{st} \subsetneq \Gamma_{st,-} \subsetneq \Gamma^{gen}_{st},$$

i.e. to go on the left, we repeat the same arguments but now playing with

$$\Pi_{v,-} \cong \mathrm{st}_{r-1}(\pi_{v,0}) \times \mathrm{st}_{l_1}(\pi_{v,1}\{\delta_1/2\}) \times \cdots \mathrm{st}_{l_k}(\pi_{v,1}\{\delta_k/2\}) \times \psi$$

where $\delta_1 = r - s_\varrho$, $\delta_2 = \delta_1 + 2l_1$... and $\delta_k = \delta_1 + 2l_1 + \cdots + 2l_{k-1} = s_\varrho - r$ so that

$$\mathrm{st}_{l_1}(\pi_{v,1}\{\delta_1/2\}) \times \cdots \times \mathrm{st}_{l_k}(\pi_{v,1}\{\delta_k/2\}) \twoheadrightarrow \mathrm{st}_{s_\varrho - r+1}(\pi_{v,1}). \tag{34}$$

We construct Γ'_- by pullback

$$\begin{tikzcd} \Gamma_{st,-} \arrow[r, hook] & \Gamma_{st}^{gen} \\ \Gamma'_- \arrow[u, dashed, two heads] \arrow[r, hook, dashed] & \Gamma^{gen,'} . \arrow[u, two heads] \end{tikzcd}$$

With similar notations as above, we have a commutative diagram

(35)

$$\begin{tikzcd} & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{st,-,\psi}) \arrow[r] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{st,\psi}^{gen}) & \\ \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{-,\psi}) \arrow[r, equal] \arrow[d, equal] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}'_{-,\psi}) \arrow[r] \arrow[u, two heads] & \overline{M}_{K^v}(\overline{\sigma}^v\overline{\sigma}_{\psi}^{gen,'}), \arrow[r, equal] \arrow[u, two heads] & \overline{M}_{\infty}(\overline{\sigma}^v\overline{\sigma}_{\psi}^{gen}) \arrow[d, equal] \\ \overline{M}_{K^v}(\overline{\sigma}^v\widetilde{\sigma}_{-,\psi}/\varpi_L) & & & \overline{M}_{K^v}(\overline{\sigma}^v\widetilde{\sigma}_{-,\psi}/\varpi_L) \end{tikzcd}$$

where the bottom horizontal map has to be an isomorphism as it is modulo λ_0. The commutativity of the above diagram imposes the top horizontal map is also an isomorphism so that $\Gamma^1_{st,-}$ ca not be the subspace lattice of Π^1.

(ii) From $(r-1, s_\varrho - r + 1)$ to any $\underline{s}_\varrho$
We repeat the same arguments but starting now from lattices

$$\varpi_L \Gamma^{gen,'} \subsetneq \Gamma'_- \subsetneq \Gamma^{gen,'} \subsetneq \Gamma'_+ \subsetneq \varpi_L^{-1}\Gamma^{gen,'}.$$

Using pullback and pushout, we then construct lattices $\Gamma_{st,\pm}$ and we conclude through the commutative diagrams (33) and (35) through the fact that if the top horizontal map is non zero then the bottom horizontal map is also non zero, and so necessary an isomorphism as explained before.

Step 4: Ihara's lemma for $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho, (r-1 \geq \cdots))$
The arguments are exactly the same as in the case $s_\varrho = 3$. Precisely consider as before an automorphic filtration of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \overline{\mathbb{Z}}_l)_\mathfrak{m}$ such that the graded parts correspond to some automorphic representation. Consider such a graded part associated to $\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s}_\varrho)$ where $\underline{s}_\varrho = (r-1 \geq \cdots)$ is a partition of s_ϱ starting with $r-1$. Let denote by $\Gamma(\Pi)$ the associated lattice and suppose that its modulo l reduction has a non ϱ-generic subspace τ which is a subspace of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \overline{\mathbb{F}}_l)_\mathfrak{m}$:

we want to prove it is absurd. Let $\Gamma(\Pi)^{gen}$ be the subspace lattice associated to Π so that

$$0 \to \Gamma(\Pi)^{gen} \longrightarrow \Gamma(\Pi) \longrightarrow T \to 0,$$

where T is l-torsion. The modulo l reduction of $\Gamma(\Pi)$ is such that

$$0 \to W \longrightarrow \Gamma(\Pi) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \longrightarrow T \to 0,$$

where the socle of T is generic and where W is the image of

$$\Gamma(\Pi)^{gen} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \longrightarrow \Gamma(\Pi) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

and non zero as we have seen that the modulo l We then remark that τ is necessary a subspace of W but as τ is not a subspace of $\Gamma(\Pi)^{gen} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ it cannot be a subspace of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$.

Step 5: Final exchanges to conclude the induction
We then exchange every contribution coming from $\mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s})$ for any $\underline{s} = (r \geq \cdots)$ until we arrive at the filtration for $r-1$. We then just have to check the hypothesis on the modulo l monodromy operator. Before all the exchanges we know that for every $0 \leq t \leq r-2$, the modulo ϖ_L reduction of

$$N : \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+t-(t-1))}(r) \longrightarrow \mathrm{gr}_\varrho^{\frac{-r(r+1)}{2}+(1+2+\cdots+(t-1)-(t-1))}(r),$$

is surjective relatively to all the (ϱ, r)-small subquotients and also for the $(\varrho, r-1)$-small ones. After we have to check that the modulo ϖ_L reduction of

$$N : \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+t-(t-1))}(r-1) \longrightarrow \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-1)-(t-1))}(r-1), \tag{36}$$

remains surjective relatively to all the $(\varrho, r-1)$-small subquotients. By construction we have

$$\begin{array}{ccc}
\displaystyle\bigoplus_{\underline{s}_\varrho \leq (r-1)} \bigoplus_{\Pi \in \mathcal{A}_{K^v,\mathfrak{m}}(\varrho, \underline{s}_\varrho)} (\Pi^\infty)^{K^v} \otimes L(\Pi_v)(\ldots) & \hookrightarrow & \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+t-(t-1))}(r) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \\
& \searrow & \\
& & \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-1)-(t-1))}(r) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l
\end{array}$$

and if Γ_1 (resp. Γ_2) is the lattice induced on this subspace by $\mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+t-(t-1))}(r)$ (resp. by $\mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-1)-(t-1))}(r)$), then we have

$$0 \to \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-1)-(t-2))}(r-1) \longrightarrow \Gamma_1 \longrightarrow T_1 \to 0,$$

$$0 \to \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-2)-(t-2))}(r-1) \longrightarrow \Gamma_2 \longrightarrow T_2 \to 0,$$

where none of the irreducible constituants of T_1 and T_2 are $(\varrho, r-1)$-small in the sense of definition 2.1.7. By the induction hypothesis, the monodromy operator induces

$\Gamma_1 \longrightarrow \Gamma_2$ such that, after tensoring with $\mathbb{F}_L$, the cokernel does not have any $(\varrho, r-1)$-subquotients. We then deduce that the induced map

$$N : \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-1)-(t-2))}(r-1) \longrightarrow \mathrm{gr}_\varrho^{-\frac{r(r+1)}{2}+(1+2+\cdots+(t-2)-(t-2))}(r-1)$$

is such that the cokernel of its modulo ϖ_L reduction does not have any $(\varrho, r-1)$-subquotients.

5.4. Genericity for KHT-Shimura varieties. — Consider an irreducible subspace τ of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi)_{\mathfrak{m}}$. To prove that τ is generic, we are led to prove it is ϱ-generic for every ϱ in its supercuspidal support. Recall that

$$H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi)_{\mathfrak{m}} \cong \bigoplus_{\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}} H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi_\varrho)_{\mathfrak{m}}$$

and, from the typicity property, for ϱ belonging to the supercuspidal support of τ, τ is also a subspace of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi_\varrho)_{\mathfrak{m}}$. From step 4 of the previous section, we know that τ has to be ϱ-generic.

5.5. Breuil's lattice conjecture for $l \neq p$. — Consider an inertial type τ_v and its associated K_v-type $\sigma_L(\tau_v)$. Consider also a system of Hecke eigenvalues $\lambda : \mathbb{T}_m \longrightarrow \mathbb{Z}_L$ associated to some automorphic representation Π which appears in middle cohomology group of $\mathrm{Sh}_{K^v(\infty),\bar{\eta}_n}$ with coefficients in $\mathcal{L}^\vee_{\sigma_0(v)}$ for $\sigma_0(v) = \sigma^v \sigma_v$ where σ^v is a continuous finitely generated representation of K^v. When $\sigma_L(\tau_v)$ appears with multiplicty one in Π_v, we can define

$$\sigma_\lambda(\tau_v) := M_{K^v}(\sigma^v)^*[\lambda] \cap \sigma_L(\tau_v),$$

which is a stable lattice of $\sigma_L(\tau_v)$: from theorem 2.2.4 (iii), we can apply this construction to the maximal inertia type. Proposition 4.3.2 tells us that this lattice only depends on the modulo l reduction of λ. One possible translation of Breuil's lattice conjecture to our situation could be the following.

Proposition 5.5.1. — *The lattice $\sigma_\lambda(\tau_v)$ depends only on the local datum Π_v.*

Proof. — Consider Π_1 and Π_2 associated to two systems of Hecke eigenvalues λ_1 and λ_2 as above: we moreover suppose that $\Pi_{v,1} \cong \Pi_{v,2}$. From Ihara's lemma we know that the lattice of $\Pi_{v,1}$ is the one such that the socle of its modulo l reduction is irreducible and generic: this lattice is then isomorphic to those of $\Pi_{v,2}$ which proves the statement. □

However note that the previous proposition is not so interesting as there are very few stable lattices compare to the case where $l = p$.

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Pascal Boyer • *E-mail :* `boyer@math.univ-paris13.fr`, Université Paris 13, Sorbonne Paris Nord, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse (France), Coloss: ANR-19-PRC