IHARA'S LEMMA IN HIGHER DIMENSION: THE LIMIT CASE

by

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Abstract. — Clozel, Harris and Taylor proposed in [**CHT08**] conjectural generalizations of the classical Ihara's lemma for GL_2, to higher dimensional similitude groups. We prove these conjectures in the so called *limit case*, which after base change is the essential one, under some mild hypothesis coming from a level raising theorem of Gee in [**Gee11**].

Contents

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1. Introduction

1.1. Ihara's original Lemma: origin and proofs. — In the Taylor-Wiles method Ihara's lemma is the key ingredient to extend a $R = T$ property from the minimal case to a non minimal one. It is usually formulated by the injectivity of some map as follows.

Let $\Gamma = \Gamma_0(N)$ be the usual congruence subgroup of $SL_2(\mathbb{Z})$ for some $N > 1$, and for a prime p not dividing N let $\Gamma' := \Gamma \cap \Gamma_0(p)$. We then have two degeneracy maps

$$\pi_1, \pi_2 : X_{\Gamma'} \longrightarrow X_\Gamma$$

between the compactified modular curves of levels Γ' and Γ respectively, induced by the inclusion

$$\Gamma' \hookrightarrow \Gamma \text{ and } \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma' \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \hookrightarrow \Gamma.$$

For $l \neq p$, we then have a map

$$\pi^* := \pi_1^* + \pi_2^* : H^1(X_\Gamma, \mathbb{F}_l)^2 \longrightarrow H^1(X_{\Gamma'}, \mathbb{F}_l).$$

Theorem 1.1.1. — *Let $\mathfrak{m}$ be a maximal ideal of the Hecke algebra acting on these cohomology groups which is non Eisenstein, i.e. that corresponds to an irreducible Galois representation. Then after localizing at $\mathfrak{m}$, the map π^* is injective.*

Diamond and Taylor in [**DT94**] proved an analogue of Ihara's lemma for Shimura curves over $\mathbb{Q}$. For a general totally real number field F with ring of integers $\mathcal{O}_F$, Manning and Shotton in [**MS**] succeeded to prove it under some large image hypothesis. Their strategy is entirely different from those of [**DT94**]but rather consists roughly

- to carry Ihara's lemma for a compact Shimura curve $Y_{\bar{K}}$ associated to a definite quaternion algebra $\overline{D}$ ramified at some auxiliary place v of F, in level $\bar{K} = \bar{K}^v \bar{K}_v$ an open compact subgroup of $D \otimes \mathbb{A}_{F,f}$ unramified at v,
- to the indefinite situation X_K relatively to a quaternion division algebra D ramified at all but one infinite place of F, and isomorphic to $\bar{D}$ at all finite places of F different to v, and with level K agreing with $\bar{K}^v$ away from v.

Indeed in the definite case Ihara's statement is formulated by the injectivity of

$$\pi^* = \pi_1^* + \pi_2^* : H^0(Y_{\bar{K}}, \mathbb{F}_l)_\mathfrak{m} \oplus H^0(Y_{\bar{K}}, \mathbb{F}_l)_\mathfrak{m} \longrightarrow H^0(Y_{\bar{K}_0(w)}, \mathbb{F}_l)_\mathfrak{m}$$

where both $\overline{D}$ and $\bar{K}$ are unramified at the place w and $\bar{K}_0(w)_w$ is the subgroup of $\mathrm{GL}_2(F_w)$ of elements which are upper triangular modulo p.

The proof goes like this, cf. [**MS**] theorem 6.8. Suppose $(f, g) \in \ker \pi^*$. Regarding f and g as K^v-invariant function on $\overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})$, then $f(x) = -g(x\omega)$ where $\omega = \begin{pmatrix} \varpi_w & 0 \\ 0 & 1 \end{pmatrix}$, ϖ_v being an uniformizer for F_w and $\overline{G}$ being the algebraic group over $\mathcal{O}_F$ associated to $\mathcal{O}_{\overline{D}}^\times$ the inversible group of the maximal order $\mathcal{O}_{\overline{D}}$ of $\overline{D}$: note that $\overline{G}(F_w) \cong \mathrm{GL}_2(F_w)$. Then f is invariant under K^v and $\omega^{-1}K^v\omega$ so that, using the strong approximation theorem for the subgroup of $\overline{G}$ of elements of reduced norm 1, then f factors through the reduced norm map, and so is supported on Eisenstein maximal ideals.

The link between X_K and Y_{K^v} is given by the geometry of the integral model of the Shimura curve $X_{K_0(v)}$ with $\Gamma_0(v)$-level structure. The main new ingredient of [**MS**] to carry this geometric link to Ihara's lemma goes through the patching technology which allows to obtain maximal Cohen-Macaulay modules over deformation rings. Using a flatness property and Nakayama's lemma, there are then able to extend a surjective property, dual to the injectivity in the Ihara's lemma, from the maximal unipotent locus on the deformation space to the whole space, and recover the Ihara's statement reducing by the maximal ideal of the deformation ring.

Recently Caraiani and Tamiozzo following closely [**MS**] also obtained Ihara's lemma for Hilbert varieties essentially because Galois deformations rings are the same and so regular which is not the case beyond GL_2.

1.2. Generalisations of Ihara's Lemma. — To generalize the classical Ihara's lemma in higher dimension, there are essentially two approaches.

- The first natural one developed by Clozel, Harris and Taylor in their first proof of Sato-Tate theorem [**CHT08**], focuses on the H^0 with coefficients in $\mathbb{F}_l$ of a zero dimensional Shimura variety associated to higher dimensional definite division algebras. More precisely consider a totally real field F^+ and a imaginary quadratic extension $E/\mathbb{Q}$ and define $F = F^+E$. We then consider $\overline{G}/\mathbb{Q}$ an unitary group with $\overline{G}(\mathbb{Q})$ compact so that $\overline{G}$ becomes an inner form of GL_d over F. This means, cf. §2.3, we have fixed a division algebra $\overline{B}$ with center F, of dimension d^2, provided with an involution of the second kind such that its restriction to F is the complex conjugation. We moreover suppose that at every place w of F, either $\overline{B}_w$ is split or a local division algebra.

Let v be a place of F above a prime number p split in E and such that $\overline{B}_v^\times \cong \mathrm{GL}_d(F_v)$ where F_v is the associated local field with ring of integers $\mathcal{O}_v$ and residue field $\kappa(v)$.

Notation 1.2.1. — Let q_v be the order of the residue field $\kappa(v)$.

Consider then an open compact subgroup $\overline{K}^v$ infinite at v in the following sense: $\overline{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{v_i^+} \overline{B}_{v_i^+}^{op,\times}$ where $p = \prod_i v_i^+$ in F^+ and we identify places of F^+ over $p = uu^c \in E$ with places of F over u. We then ask $\overline{K}_p^v = \mathbb{Z}_p^\times \times \prod_{w|u} \overline{K}_w$ to be such that $\overline{K}_v$ is restricted to the identity element.

The associated Shimura variety with level $\overline{K} = \overline{K}^v \overline{K}_v$ for some finite level $\overline{K}_v$ at v, denoted by $\overline{\mathrm{Sh}}_{\overline{K}}$, is then such that its $\mathbb{C}$-points are $\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}_{\mathbb{Q}}^\infty)/\overline{K}$ and for l a prime not divisible by v, its H^0 with coefficients in $\overline{\mathbb{F}}_l$ is then identified with the space

$$S_{\overline{G}}(\overline{K},\overline{\mathbb{F}}_l) = \{f : \overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}_{\mathbb{Q}}^\infty)/\overline{K} \longrightarrow \overline{\mathbb{F}}_l \text{ locally constant}\}.$$

Replacing $\overline{K}$ by $\overline{K}^v$, we then obtain an admissible smooth representation of $\mathrm{GL}_d(F_v)$ equipped with an action of the Hecke algebra $\mathbb{T}(\overline{K}^v)$ defined as the image of the abstract unramified Hecke algebra, cf. definition 3.2.1, inside $\mathrm{End}\Big(S_{\overline{G}}(\overline{K}^v,\overline{\mathbb{F}}_l)\Big)$.

To a maximal ideal $\mathfrak{m}$ of $\mathbb{T}(\overline{K}^v)$ is associated a Galois $\overline{\mathbb{F}}_l$-representation $\overline{\rho}_{\mathfrak{m}}$, cf. §4.2. We consider the case where this representation is irreducible. Note in particular that such an $\mathfrak{m}$ is then not pseudo-Eisenstein in the usual terminology.

Conjecture 1.2.2. — *(cf. conjecture B in* [**CHT08**]*)*
Any irreducible $\mathrm{GL}_d(F_v)$*-submodule of* $S_{\overline{G}}(\overline{K}^v,\overline{\mathbb{F}}_l)_{\mathfrak{m}}$ *is generic.*

- For rank 2 unitary groups, we recover the previous statement as the characters are exactly those representations which do not have a Whittaker model, i.e. are the non generic ones.

- For $d \geq 2$, over $\overline{\mathbb{Q}}_l$, the generic representations of $\mathrm{GL}_d(F_v)$ are the irreducible parabolically induced representations $\mathrm{st}_{t_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{t_r}(\pi_{v,r})$ where for $i = 1, \cdots, r$,

– $\pi_{v,i}$ is an irreducible cuspidal representation of $\mathrm{GL}_{g_i}(F_v)$,
– $\mathrm{st}_{t_i}(\pi_{v,i})$ is a Steinberg representations, cf. definition 2.1.2,
– $\sum_{i=1}^r t_i g_i = d$ where the Zelevinsky segments $[\pi_{v,i}\{\frac{1-t_i}{2}\}, \pi_{v,i}\{\frac{t_i-1}{2}\}]$are not linked in the sense of [**Zel80**].

- Over $\overline{\mathbb{F}}_l$ every irreducible generic representation is obtained as the unique generic subquotient of the modulo l reduction of a generic representation. It can also be characterized intrinsically using representation of the mirabolic subgroup, cf. §2.1.

Here we will be mainly interested in the following weak form of Ihara's lemma, except that we will allow ramified characters, see the main theorem below.

Definition 1.2.3. — (cf. definition of [**CHT08**] 5.1.9)
An admissible smooth $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module M is said to have the weak Ihara property if for every $m \in M^{\mathrm{GL}_d(\mathcal{O}_v)}$ which is an eigenvector of $\overline{\mathbb{F}}_l[\mathrm{GL}_d(\mathcal{O}_v)\backslash \mathrm{GL}_d(F_v)/ \mathrm{GL}_d(\mathcal{O}_v)]$, every irreducible submodule of the $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module generated by m, is generic.

Remark. In particular if we ask about $S_{\overline{G}}(\overline{K}^v,\overline{\mathbb{F}}_l)_{\mathfrak{m}}$ having the weak Ihara property, then $S_{\overline{G}}(\overline{K}^v,\overline{\mathbb{F}}_l)_{\mathfrak{m}}$ should have non trivial unramified vectors so that the supercuspidal support of the restriction $\overline{\rho}_{\mathfrak{m},v}$ of $\overline{\rho}_{\mathfrak{m}}$ to the decomposition subgroup at v, is made of unramified characters.

- The second approach is to find a map playing the same role as $\pi^* = \pi_1^* + \pi_2^*$. It is explained in section 5.1 of [**CHT08**] with the help of the element

$$\theta_v \in \mathbb{Z}_l[K_1(v^n)\backslash \mathrm{GL}_d(F_v)/\mathrm{GL}_d(\mathcal{O}_{F_v})]$$

constructed by Russ Mann, cf. proposition 5.1.7 of [**CHT08**], where F_v is here a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_v$.

Definition 1.2.4. — An admissible smooth $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module M is said to have the almost Ihara property if $\theta_v : M^{\mathrm{GL}_d(\mathcal{O}_v)} \longrightarrow M$ is injective.

Recall that l is called quasi-banal for $\mathrm{GL}_d(F_v)$ if either $l \nmid \sharp \mathrm{GL}_d(\kappa_v)$ (the banal case) or $l > d$ and $q_v \equiv 1 \mod l$ (the limit case).

Proposition 1.2.5. — *(cf.* [**CHT08**] *lemma 5.1.10)*
Suppose that l is quasi-banal and M is a $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$-module verifying the Ihara property. If $\ker(\theta_v : M^{\mathrm{GL}_d(\mathcal{O}_v)} \longrightarrow M)$ is a $\mathbb{F}_l[\mathrm{GL}_d(\mathcal{O}_{F_v})\backslash \mathrm{GL}_d(F_v)/\mathrm{GL}_d(\mathcal{O}_{F_v})]$-module, then M has the almost Ihara property.

Applications: the generalizations of the classical Ihara's lemma were introduced in [**CHT08**] to prove a non minimal $R = \mathbb{T}$ theorem. The weaker statement $R^{red} = \mathbb{T}$ where R^{red} is the reduced quotient of R, was later obtained unconditionally using Taylor's *Ihara avoidance* method, cf. [**Tay08**] which was enough to prove the Sato-Tate conjecture. However, the full $R = \mathbb{T}$ theorem would have applications to special values of the adjoint L-function and would imply that R is a complete intersection. It should also be useful for generalizing the local-global compatibility results of [**Eme**].

In [**Mos21**], the author also proved that Ihara's property in the quasi-banal case is equivalent to the following result.

Proposition 1.2.6. — *(cf.* [**Mos21**] *corollary 9.5)*
Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathbb{T}^S$ and $f \in S_{\overline{G}}(\overline{K}^v \mathrm{GL}_d(\mathcal{O}_v), \overline{\mathbb{F}}_l)$. Let K_v be the Iwahori subgroup of $\mathrm{GL}_d(\mathcal{O}_v)$, then the $\overline{\mathbb{F}}_l[\mathrm{K}_v \backslash \mathrm{GL}_d(F_v)/\mathrm{GL}_d(\mathcal{O}_v)]$-submodule of $S_{\overline{G}}(\overline{K}^v \mathrm{K}_v, \overline{\mathbb{F}}_l)$ generated by f is of dimension $d!$.

1.3. Main result. — With the previous notations, let q_v be the order of the residue field of F_v. We fix some prime number l unramified in F^+ and split in E and we place ourself in the limit case where $q_v \equiv 1 \mod l$ with $l > d$, which is, after by base change, the crucial case to consider.

Definition 1.3.1. — As in definition 2.5.1 of [**CHT08**], we say that a subgroup $H \subseteq \mathrm{GL}_d(\overline{\mathbb{F}}_l)$ is big if :

- H has no l-power order quotients;
- $H^i(H, \mathfrak{g}_d^0(\overline{\mathbb{F}}_l)) = (0)$ for $i = 0, 1$ and where $\mathfrak{g}_d := \mathrm{Lie}\,\mathrm{GL}_d$ and $\mathfrak{g}_d^0$ is the trace zero subspace of $\mathfrak{g}_d$;

– for all irreducible $\overline{\mathbb{F}}_l[H]$-submodules W of $\mathfrak{g}_d(\overline{\mathbb{F}}_l)$, we can find $h \in H$ and $\alpha \in \overline{\mathbb{F}}_l$ satisfying the following properties.
  - The α-generalized eigenspace $V(h,\alpha)$ of h on $\overline{\mathbb{F}}_l^d$ is one dimensional.
  - Let $\pi_{h,\alpha} : \overline{\mathbb{F}}_l^d \twoheadrightarrow V(h,\alpha)$ be the h-equivariant projection of $\overline{\mathbb{F}}_l^d$ to $V(h,\alpha)$ and let $i_{h,\alpha} : V(h,\alpha) \hookrightarrow \overline{\mathbb{F}}_l^d$ be the h-equivariant injection of $V(h,\alpha)$ into $\overline{\mathbb{F}}_l^d$. Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.

Theorem 1.3.2. — *In the limit case, suppose moreover that there exists a prime $p_0 = u_0 \bar{u}_0$ split in E with a place $v_0|u_0$ of F such that $\overline{B}_{v_0}$ is a division algebra. Consider $\mathfrak{m}$ such that*

$$\overline{\rho}_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_d(\overline{\mathbb{F}}_l)$$

is an irreducible representation which is unramified at all places of F lying above primes which do not split in E and which satisfies the following hypothesis:

– *after semi-simplification $\overline{\rho}_{\mathfrak{m},v}$ is a direct sum of characters;*
– *$\overline{F}^{\ker \mathrm{ad}\, \overline{\rho}}$ does not contain $F(\zeta_l)$ where ζ_l is any primitive l-root of* 1*;*
– *$\overline{\rho}(G_{F^+(\zeta_l)})$ is big.*

Then Ihara's lemma of the conjecture 1.2.2 is true, i.e. every irreducible $\mathrm{GL}_d(F_v)$*-submodule of $S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ is generic.*

– The first hypothesis, if you moreover suppose that the characters are unramified, corresponds to the weak form of Ihara's lemma of definition 1.2.3. Using the main result of [**Boy23b**] proving that p and $p+$ intermediate extensions of Harris-Taylor local systems, in the limit case, coincide, it should be possible to remove this restriction: however we found the exposition much more easy dealing only with characters especially as for applications, by base change, we can reduce to the case of characters.
– The last two hypothesis come from theorem 5.1.5 of [**Gee11**] which is some level raising and lowering statement, cf. theorem 4.2.2. Any other similar statement, for example theorem 4.4.1 of [**BLGGT14**], with different hypothesis can then be used to formulate another statement.

In [**Boy20**] we essentially proved conjecture 1.2.2 in the banal case under some restrictive hypothesis. The basic idea is to introduce geometry and move from the Shimura variety associated to $\overline{G}$ which is of dimension zero, to another Shimura variety Sh_K associated to some reductive group G and level K, of strictly positive dimension, so that $S_{\overline{G}}(\overline{K}, \overline{\mathbb{F}}_l)$ appears in a certain cohomology group of some sheaf over Sh_K. The strategy is then to construct filtration of this cohomology group coming from geometry so that the graded parts, which are expected to be more easy to handle with, also verify the genericity property of there irreducible sub-spaces.

More explicitly we study the middle degree cohomology group of the KHT Shimura variety $\mathrm{Sh}_{K^v(\infty)}$ associated to some similitude group $G/\mathbb{Q}$ such that $G(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \cong$

$\overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$, cf. §2.3 for more details, and with level $K^v(\infty) := \overline{K}^v$ meaning finite level outside v and infinite level at v.

The localization at $\mathfrak{m}$ of the cohomology groups of $\mathrm{Sh}_{K^v(\infty)}$ can be computed as the cohomology of the geometric special fiber $\mathrm{Sh}_{K^v(\infty),\bar{s}_v}$ of $\mathrm{Sh}_{K^v(\infty)}$, with coefficient in the complex of nearby cycles $\Psi_{K^v(\infty),v}$.

The Newton stratification of $\mathrm{Sh}_{K^v(\infty),\bar{s}_v}$ gives us a filtration of $\Psi_{K^v(\infty),v}$, cf. [**Boy19**], and so a filtration $\mathrm{Fil}^\bullet(K^v(\infty))$ of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ and the main point of [**Boy20**] is to prove that the modulo l reduction of each graded part of this filtration verifies the Ihara property, i.e. each of their irreducible sub-space are generic. To realize this strategy we need first the cohomology groups of $\mathrm{Sh}_{K^v(\infty)}$ to be torsion free: this point is now essentially settled by the main result of [**Boy23a**]. More crucially the previous filtration $\mathrm{Fil}^\bullet(K^v(\infty))$ should be strict, i.e. its graded parts have to be torsion free, cf. theorem 3.2.3.

It appears that the graded parts of $\mathrm{Fil}^\bullet(K^v(\infty))$ are parabolically induced and in the limit case when the order q_v of the residue field is such that $q_v \equiv 1 \mod l$, the socle of the modulo l reduction of these parabolic induced representations are no more irreducible and do not fulfill the Ihara property, i.e. some of their subspaces are not generic. It then appears that we have at least

- to verify that the modulo l reduction of the first non trivial graded part of $\mathrm{Fil}^\bullet(K^v(\infty))$ verifies the genericity property of its irreducible submodule. For this we need a level raising statement as theorem 5.1.5 in [**Gee11**], cf. theorem 4.2.2, or theorem 4.4.1 of [**BLGGT14**].
- Then we have to understand that the extensions between the graded parts of $\mathrm{Fil}^\bullet(K^v(\infty)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ are non split.

One problem about this last point is that the $\overline{\mathbb{Q}}_l$-cohomology is split. For any irreducible automorphic representation Π of $G(\mathbb{A})$ cohomological for, say, the trivial coefficients, the $\overline{\mathbb{Z}}_l$-cohomology defines a lattice $\Gamma(\Pi)$ of $(\Pi^\infty)^{K^v(\infty)} \otimes \sigma(\Pi)_v$ whose modulo l reduction gives a subspace of the $\overline{\mathbb{F}}_l$-cohomology: Ihara's lemma predicts that the socle of this subspace is still generic, i.e. it gives informations about the lattice $\Gamma(\Pi)$. We then see that non splitness of $\mathrm{Fil}^\bullet(K^v(\infty)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ should be understood in a very flexible point of view.

One possible strategy is, using the fact that the $\overline{\mathbb{Q}}_l$-cohomology is split, to start from the filtration $\mathrm{Fil}^\bullet(K^v(\infty))$ and modify it in order to arrive to another one where all the modulo l reduction of the graded parts fulfill the Ihara property i.e. their irreducible subspaces are generic. The main ingredient to construct modifications of filtrations is to consider following situations:

- a filtration $\mathrm{Fil}^\bullet$ of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_v}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ whose graded parts $\mathrm{gr}^\bullet$ are torsion free;
- let k and $X := \mathrm{Fil}^k / \mathrm{Fil}^{k-2}$ such that $X \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong (gr^{k-1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l) \oplus (\mathrm{gr}^k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$.

– We can then define $\widetilde{\mathrm{gr}^{k-1}}$ and $\widetilde{\mathrm{gr}^k}$ with

$$\begin{tikzcd}
 & \widetilde{\mathrm{gr}^k} \arrow[d, hook] \arrow[dr, hook] & & \\
\mathrm{gr}^{k-1} \arrow[r, hook] \arrow[dr, hook] & X \arrow[r, two heads] \arrow[d] & \mathrm{gr}^k \arrow[dr] & \\
 & \widetilde{\mathrm{gr}^{k-1}} \arrow[dr] & & T \arrow[dl, equal] \\
 & & T &
\end{tikzcd}$$

obtained by taking $\widetilde{\mathrm{gr}^k} := X \cap (\mathrm{gr}^k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$, and where T is torsion. Passing modulo l, we then obtained a priori two distinct filtrations.

Let us first explain why something interesting should happen during this process.

– We can define a $\overline{\mathbb{F}}_l$-monodromy operator for the Galois action at the place v.[(1)] We are looking for a geometric monodromy operator N_v^{geo} which then exists whatever are the coefficients, $\overline{\mathbb{Q}}_l$, $\overline{\mathbb{Z}}_l$ and $\overline{\mathbb{F}}_l$, compatible with tensor products. One classical construction is known in the semi-stable reduction case, cf. [**Ill94**] §3, which corresponds to the case where the level at v of our Shimura variety is of Iwahori type.[(2)] Using our knowledge of the $\overline{\mathbb{Z}}_l$-nearby cycles described completely in [**Boy23b**], we can construct such a geometric nilpotent monodromy operator which generalizes the semi-stable case by allowing ramified characters, cf. §3.3.
– Taking this geometric monodromy operator, we then obtain a cohomological monodromy operator $N_{v,\mathfrak{m}}^{coho}$ acting on $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$ as soon as the irreducible constituants of the restriction $\overline{\rho}_{\mathfrak{m},v}$ of $\overline{\rho}_{\mathfrak{m}}$ to the decomposition group at v, are characters. One of the main point, cf theorem 3.2.3, is that the graded parts of the filtration of $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$ induced by the Newton filtration on the nearby cycles spectral sequence, are all torsion free, so that in particular we are in position to understand quite enough the action of $\overline{N}_{v,\mathfrak{m}}^{coho} := N_{v,\mathfrak{m}}^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ on $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, and prove that its nilpotency order is as large as possible.
– Note that as $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, then the modulo l reduction of the monodromy operator acting on $\rho_{\widetilde{\mathfrak{m}}}$ does not depend on the choice of the prime ideal $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$ so that it is usually trivial.

(1) Note that over $\overline{\mathbb{F}}_l$ the usual arithmetic approach for defining the nilpotent monodromy operator, is hopeless because, up to consider a finite extension of F_v, such a $\overline{\mathbb{F}}_l$-representation has a trivial action of the inertia group.
(2) This corresponds to automorphic representations Π such that the cuspidal support of Π_v is made of unramified characters, and so with the weak form of Ihara's lemma of definition 1.2.3.

Finally, as $N_v^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is far from being trivial, there should be non split extensions between the graded parts of $\mathrm{Fil}^\bullet(K^v(\infty))$.

However this strategy seems difficult to implement directly because of counting problems: to deal with finite number of representations you need to work with a finite level at the place v and then pass to the limit. It seems first difficult to count liftings of a fixed representation and secondly when increasing the level, it should be not easy to glue back things together. Our approach in some sense consists to consider all the liftings together using typicness of the cohomology, cf. §4.3. The proof then goes into three main steps:

- we first prove, cf. theorem 3.2.3, that the filtration of the middle cohomology groups constructed from the filtration of stratification of the nearby cycles perverse sheaf, has torsion free graded parts, otherwise the all cohomology would have non trivial torsion classes which is not the case by [**Boy23a**];
- secondly, §3.3, using results from [**Boy23b**] about various filtrations of stratification of the nearby cycle perverse sheaf over $\overline{\mathbb{Z}}_l$, we define an integral monodromy operator on the nearby cycles perverse sheaf;
- this integral geometric monodromy operator gives us a monodromy operator on $\rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, a representation with coefficients in an artinian local ring defined as the modulo l reduction of the image of some Hecke algebra acting on the middle cohomology group with finite level. The index of nilpotency of this monodromy operator is then as large as possible. Working at various finite level and using Matlis duality for artinian ring, we finally prove, §4.3, the genericity of the socle of the middle cohomology group at infinite level at v, for the action of $\mathrm{GL}_d(F_v)$.

To conclude this long introduction, note that Ihara's lemma in Clozel-Harris-Taylor formulation, was stated in order to be able to do level raising. In our proof we use level raising statements, proved thanks to Taylor's Ihara avoidance in [**Tay08**], in order to prove Ihara's lemma.

2. Preliminaries

2.1. Representations of $\mathrm{GL}_d(L)$. — Consider a finite extension $L/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We denote by $|-|$ its absolute value. For a representation π of $\mathrm{GL}_d(L)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \mathrm{val} \circ \det}.$$

Notation 2.1.1. — For π_1 and π_2 representations of respectively $\mathrm{GL}_{n_1}(L)$ and $\mathrm{GL}_{n_2}(L)$, we will denote by

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(L)}^{\mathrm{GL}_{n_1+n_2}(L)} \pi_1\{\frac{n_2}{2}\} \otimes \pi_2\{-\frac{n_1}{2}\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d

with Levi
$$\mathrm{GL}_{r_1}\times\mathrm{GL}_{r_2-r_1}\times\cdots\times\mathrm{GL}_{r_k-r_{k-1}}.$$

Recall that a representation ϱ of $\mathrm{GL}_d(L)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero, these two notions coincides, but this is no more true over $\overline{\mathbb{F}}_l$.

Definition 2.1.2. — (see [**Zel80**] §9 and [**Boy10**] §1.4) Let g be a divisor of $d=sg$ and π an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_g(L)$. The induced representation

$$\pi\{\frac{1-s}{2}\}\times\pi\{\frac{3-s}{2}\}\times\cdots\times\pi\{\frac{s-1}{2}\} \tag{1}$$

holds an unique irreducible quotient (resp. subspace) denoted $\mathrm{st}_s(\pi)$ (resp. $\mathrm{Speh}_s(\pi)$); it is a generalized Steinberg (resp. Speh) representation. Their cuspidal support is the Zelevinsky segment
$$[\pi\{\frac{1-s}{2}\},\pi\{\frac{s-1}{2}\}]:=\Big\{\pi\{\frac{1-s}{2}\},\pi\{\frac{3-s}{2}\},\cdots,\pi\{\frac{s-1}{2}\}\Big\}.$$

More generally the set of sub-quotients of the induced representation (1) is in bijection with the following set.
$$\mathrm{Dec}(s)=\{(t_1,\cdots,t_r),\text{ such that } t_i\geq 1\text{ and }\sum_{i=1}^r t_i=s\}.$$
For any $\underline{s}\in\mathrm{Dec}(s)$, we the denote by $\mathrm{st}_{\underline{s}}(\pi)$ the associated irreducible sub-quotient of (1). Following Zelevinsky, we fix this bijection such that $\mathrm{Speh}_s(\pi)$ corresponds to (s) and $\mathrm{st}_s(\pi)$ to $(1,\cdots,1)$. The Lubin-Tate representation $LT_{h,s}(\pi)$ will also appear in the following, it corresponds with $(\overbrace{1,\cdots,1}^{h},s-h)$.

Proposition 2.1.3. — *(cf. [**Vig96**] III.5.10) Let π be an irreducible cuspidal representation of $\mathrm{GL}_g(K)$ with a stable $\overline{\mathbb{Z}}$-lattice[(3)], then its modulo l reduction is irreducible and cuspidal (but not necessary supercuspidal).*

We now suppose as explained in the introduction that
$$q\equiv 1\mod l\quad\text{and}\quad l>d$$
so the following facts are verified (cf. [**Vig96**] §III):

- the modulo l reduction of every irreducible cuspidal representation of $\mathrm{GL}_g(L)$ for $g\leq d$, is supercuspidal.

[(3)] We say that π is integral.

– For a $\overline{\mathbb{F}}_l$-irreducible supercuspidal representation ϱ of $\mathrm{GL}_g(L)$, the parabolic induced representation $\varrho \times \cdots \times \varrho$, with s copies of ϱ, is semi-simple with irreducible constituants the modulo l reduction of the set of elements of $\{\mathrm{st}_{\underline{s}}(\pi)$ such that $\underline{s} \in \mathrm{Dec}(s)\}$, where π is any cuspidal representation whose modulo l reduction is isomorphic to ϱ.

Concerning the notion of genericity, consider the mirabolic subgroup $M_d(L)$ of $\mathrm{GL}_d(L)$ as the set of matrices with last row $(0, \cdots, 0, 1)$: we denote by

$$V_d(L) = \{(m_{i,j}) \in M_d(L) : \ m_{i,j} = \delta_{i,j} \text{ for } j < d\}.$$

its unipotent radical. We fix a non trivial character ψ of L and let θ be the character of $V_d(L)$ defined by $\theta((m_{i,j})) = \psi(m_{d-1,d})$. For $G = \mathrm{GL}_r(L)$ or $M_r(L)$, we denote by $\mathrm{alg}(G)$ the abelian category of smooth representations of G and, following [**BZ77**], we introduce

$$\Psi^- : \mathrm{alg}(M_d(L)) \longrightarrow \mathrm{alg}(\mathrm{GL}_{d-1}(L)),$$

and

$$\Phi^- : \mathrm{alg}(M_d(L)) \longrightarrow \mathrm{alg}(M_{d-1}(L)),$$

defined by $\Psi^- = r_{V_d,1}$ (resp. $\Phi^- = r_{V_d,\theta}$) the functor of V_d coinvariants (resp. (V_d, θ)-coinvariants), cf. [**BZ77**] 1.8. For $\tau \in \mathrm{alg}(M_d(L))$, the representation

$$\tau^{(k)} := \Psi^- \circ (\Phi^-)^{k-1}(\tau)$$

is called the k-th derivative of τ. If $\tau^{(k)} \neq 0$ and $\tau^{(m)} = 0$ for all $m > k$, then $\tau^{(k)}$ is called the highest derivative of τ. In the particular case where $k = d$, there is an unique irreducible representation τ_{nd} of $M_d(L)$ with derivative of order d.

Definition 2.1.4. — An irreducible representation π of $\mathrm{GL}_d(L)$ is said generic, if its restriction to the mirabolic subgroup admits τ_{nd} as a subquotient.

Let π be an irreducible generic $\overline{\mathbb{Q}}_l$-representation of $\mathrm{GL}_d(L)$ and consider any stable lattice which gives us by modulo l reduction a $\overline{\mathbb{F}}_l$- representation uniquely determined up to semi-simplification. Then this modulo l reduction admits an unique generic irreducible constituant.

2.2. Weil–Deligne inertial types. — Recall that a Weil-Deligne representation of W_L is a pair (r, N) where

– $r : W_L \longrightarrow \mathrm{GL}(V)$ is a smooth[(4)] representation on a finite dimensional vector space V; and
– $N \in \mathrm{End}(V)$ is nilpotent such that

$$r(g)Nr(g)^{-1} = ||g||N,$$

where $|| \bullet || : W_L \longrightarrow W_L/I_L \twoheadrightarrow q^{\mathbb{Z}}$ takes an arithmetic Frobenius element to q.

(4) i.e. continuous for the discrete topology on V

Remark. To a continuous[5] representation on a finite dimensional $\mathbb{Q}_l$-vector space V, $\rho : W_L \longrightarrow \mathrm{GL}(V)$ is attached a Weil-Deligne representation denoted by $\mathrm{WD}(\rho)$. A Weil representation of W_L is also said of Galois type if it comes from a representation of G_L.

Main example: let $\rho : W_L \longrightarrow \mathrm{GL}(V)$ be a smooth irreducible representation on a finite dimensional vector space V. For $k \geq 1$ an integer, we then define a Weil-Deligne representation

$$\mathrm{Sp}(\rho, k) := \big(V \oplus V(1) \oplus \cdots \oplus V(k-1), N\big),$$

where for $0 \leq i \leq k-2$, the isomorphism $N : V(i) \cong V(i+1)$ is induced by some choice of a basis of $\overline{L}(1)$ and $N_{|V(k-1)}$ is zero. Then every Frobenius semi-simple Weil-Deligne representation of W_L is isomorphic to some $\bigoplus_{i=1}^r \mathrm{Sp}(\rho_i, k_i)$, for smooth irreducible representations $\rho_i : W_L \longrightarrow \mathrm{GL}(V_i)$ and integers $k_i \geq 1$. Up to obvious reorderings, such a writing is unique.

Let now ρ be a continuous representation of W_L, or its Weil-Deligne representation $\mathrm{WD}(\rho)$, and consider its restriction to I_L, $\tau := \rho_{|I_L}$. Such an isomophism class of a finite dimensional continuous representation of I_L is then called *an inertial type*.

Notation 2.2.1. — Let $\mathcal{I}_0$ the set of inertial types that extend to a continuous irreducible representation of G_L.

Remark. $\tau \in \mathcal{I}_0$ might not be irreducible.

Let Part be the set of decreasing sequences of positive integers $\underline{d} = (\underline{d}(1) \geq \underline{d}(2) \geq \cdots)$ viewed as a partition of $\sum \underline{d} := \sum_i \underline{d}(i)$.

Notation 2.2.2. — Let $f : \mathcal{I}_0 \longrightarrow \mathrm{Part}$ with finite support. We then denote by τ_f the restriction to I_L of

$$\bigoplus_{\tau_0 \in \mathcal{I}_0} \bigoplus_i \mathrm{Sp}(\rho_{\tau_0}, f(\tau_0)(i)),$$

where ρ_{τ_0} is a fixed extension of τ_0 to W_L.

Remark. By lemma 3.3 of [**MS**] the isomorphism class of τ_f is independent of the choices of the ρ_{τ_0}.

The map from $\{f : \mathcal{I}_0 \longrightarrow \mathrm{Part}\}$ to the set of inertial types given by $f \mapsto \tau_f$, is a bijection. The dominance order $\preceq$ on Part induces a partial order on the set of inertial types.

We let rec_L denote the local reciprocity map of [**HT01**, Theorem A]. Fix an isomorphism $\imath \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$. We normalize the local reciprocity map rec of [**HT01**, Theorem A], defined on isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(L)$ over

[5] relatively to the l-adic topology on V

$\mathbb{C}$ as follows: if π is the isomorphism class of an irreducible smooth representation of $\mathrm{GL}_n(L)$ over $\overline{\mathbb{Q}}_\ell$, then

$$\rho_\ell(\pi) \stackrel{\text{def}}{=} \imath^{-1} \circ \mathrm{rec}_L \circ \imath(\pi \otimes_{\overline{\mathbb{Q}}_\ell} |\det|^{(1-n)/2}).$$

Then $\rho_\ell(\pi)$ is the isomorphism class of an n-dimensional, Frobenius semisimple Weil–Deligne representation of W_L over $\overline{\mathbb{Q}}_\ell$, independent of the choice of $\imath$. Moreover, if ρ is an isomorphism class of an n-dimensional, Frobenius semisimple Weil–Deligne representation of W_L over M, then $\rho_\ell^{-1}(\rho)$ is defined over M (cf. [**CEG$^+$16**, §1.8]).

Recall the following compatibility of the Langlands correspondence.

Lemma 2.2.3. — *If π and π' are irreducible generic representations of $\mathrm{GL}_d(L)$ such that $\rho_\ell(\pi)|I_L \cong \rho_\ell(\pi')|I_L$ then $\pi_{|\mathrm{GL}_d(\mathcal{O}_L)} \cong \pi'_{|\mathrm{GL}_d(\mathcal{O}_L)}$.*

2.3. Kottwiz–Harris–Taylor Shimura varieties. — Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is a quadratic imaginary extension and $F^+/\mathbb{Q}$ is totally real. We fix a real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we will denote by F_v the completion of F at v, $\mathcal{O}_v$ its ring of integers with uniformizer ϖ_v and residue field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$ of cardinal q_v.

Let B be a division algebra with center F, of dimension d^2 such that at every place v of F, either B_v is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta \in B^{*=-1}$, denote by $\sharp_\beta$ the involution $v \mapsto v^{\sharp_\beta} = \beta v^* \beta^{-1}$ and let $G/\mathbb{Q}$ be the group of similitudes, denoted by G_τ in [**HT01**], defined for every $\mathbb{Q}$-algebra R by

$$G(R) \cong \{(\lambda, g) \in R^\times \times (B^{op} \otimes_\mathbb{Q} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$ split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \cong (B_y^{op})^\times \times \mathbb{Q}_x^\times \cong \mathbb{Q}_x^\times \times \prod_{v_i^+} (B^{op}_{v_i^+})^\times, \tag{2}$$

where $x = \prod_i v_i^+$ in F^+ and we identify places of F^+ over x with places of F over y.

Convention 2.3.1. — For $x = yy^c$ a place of $\mathbb{Q}$ split in M and v a place of F over y, we shall make throughout the text the following abuse of notation: we denote $G(F_v)$ the factor $(B^{op}_{v|_{F^+}})^\times$ in the formula (2) so that

$$G(\mathbb{A}_\mathbb{Q}^{\infty,v}) := G(\mathbb{A}_\mathbb{Q}^{\infty,p}) \times \Big(\mathbb{Q}_p^\times \times \prod_{v_i^+ \neq v|F^+} (B^{op}_{v_i^+})^\times \Big).$$

In [**HT01**], the authors justify the existence of some G like before such that

- if x is a place of $\mathbb{Q}$ non split in M then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [**HT01**, page 90], a compact open subgroup K of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is said to be *sufficiently small* if there exists a place x of $\mathbb{Q}$ such that the projection from K^x to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

Notation 2.3.2. — Denote by $\mathcal{K}$ the set of sufficiently small compact open subgroups of $G(\mathbb{A}^{\infty})$. For $K \in \mathcal{K}$, write $\mathrm{Sh}_{K,\eta} \longrightarrow \operatorname{Spec} F$ for the associated Shimura variety of Kottwitz-Harris-Taylor type.

Definition 2.3.3. — Denote by Spl the set of places w of F such that $p_w := w_{|\mathbb{Q}} \neq l$ is split in E and $B_w^{\times} \cong \mathrm{GL}_d(F_w)$. For each $K \in \mathcal{K}$, we write $\mathrm{Spl}(K)$ for the subset of Spl of places such that K_v is the standard maximal compact of $\mathrm{GL}_d(F_v)$.

In the sequel, we fix a place v of F in Spl. The scheme $\mathrm{Sh}_{K,\eta}$ has a projective model $\mathrm{Sh}_{K,v}$ over $\operatorname{Spec} \mathcal{O}_v$ with special geometric fiber $\mathrm{Sh}_{K,\bar{s}_v}$. We have a projective system $(\mathrm{Sh}_{K,\bar{s}_v})_{K\in\mathcal{K}}$ which is naturally equipped with an action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathbb{Z}$ such that any $w_v \in W_{F_v}$ acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}_{F_v}^{-1}$ and $\mathrm{Art}_{F_v} : F_v^{\times} \xrightarrow{\sim} W_{F_v}^{ab}$.

Notation 2.3.4. — For $K \in \mathcal{K}$, the Newton stratification of the geometric special fiber $\mathrm{Sh}_{K,\bar{s}_v}$ is denoted by

$$\mathrm{Sh}_{K,\bar{s}_v} =: \mathrm{Sh}_{K,\bar{s}_v}^{\geq 1} \supset \mathrm{Sh}_{K,\bar{s}_v}^{\geq 2} \supset \cdots \supset \mathrm{Sh}_{K,\bar{s}_v}^{\geq d}$$

where $\mathrm{Sh}_{K,\bar{s}_v}^{=h} := \mathrm{Sh}_{K,\bar{s}_v}^{\geq h} - \mathrm{Sh}_{K,\bar{s}_v}^{\geq h+1}$ is an affine scheme, which is smooth and pure of dimension $d-h$. It is built up by the geometric points such that the connected part of the associated Barsotti–Tate group has rank h For each $1 \leq h < d$, write

$$i_h : \mathrm{Sh}_{K,\bar{s}_v}^{\geq h} \hookrightarrow \mathrm{Sh}_{K,\bar{s}_v}^{\geq 1}, \quad j^{\geq h} : \mathrm{Sh}_{K,\bar{s}_v}^{=h} \hookrightarrow \mathrm{Sh}_{K,\bar{s}_v}^{\geq h},$$

and $j^{=h} = i_h \circ j^{\geq h}$.

For $n \geq 1$, with our previous abuse of notation, consider $K^v(n) := K^v K_v(n)$ where

$$K_v(n) := \ker(\mathrm{GL}_d(\mathcal{O}_v) \twoheadrightarrow \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)).$$

Recall that $\mathrm{Sh}_{I^v(n),\bar{s}_v}^{=h}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely this means there exists a closed subscheme $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h}$ stabilized by the Hecke action of $P_{h,d}(F_v)$ and such that

$$\mathrm{Sh}_{K^v(n),\bar{s}_v}^{=h} = \mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h} \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n), \tag{3}$$

meaning that $\mathrm{Sh}_{K^v(n),\bar{s}_v}^{=h}$ is the disjoint union of copies of $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h}$ indexed by $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ and exchanged by the action of $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)$. We will denote by $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{\geq h}$ the closure of $\mathrm{Sh}_{K^v(n),\bar{s}_v,1}^{=h}$ inside $\mathrm{Sh}_{K^v(n),\bar{s}_v}$.

Notation 2.3.5. — Let $1 \leq h \leq d$ and Π_h any representation of $\mathrm{GL}_h(F_v)$. For χ_v a character of $F_v^{\times}$, we then denote by

$$\widetilde{HT}_1(\chi_v, \Pi_h) := \mathcal{L}(\chi_v, t)_1 \otimes \Pi_h^{K_v(n)} \otimes \Xi^{\frac{h-d}{2}}$$

the Harris-Taylor local system on the Newton stratum $\mathrm{Sh}^{=h}_{K^v(n),\bar{s}_v,1}$ where

- $\mathcal{L}(\chi_v,t)_1$ is the constant sheaf $\overline{\mathbb{Z}}_l$ where the fundamental group acts through
$$\pi_1 \twoheadrightarrow \mathcal{D}_{v,h}^\times \xrightarrow{\chi_v} \mathcal{O}_v^\times$$
where $\mathcal{D}_{v,h}$ is the maximal order of the division algebra $D_{v,h}/F_v$ with invariant $1/h$, and the first surjection is given by the Igusa varieties of [**HT01**];
- $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ is defined by $\Xi(\frac{1}{2}) = q^{1/2}$.

We also introduce the induced version
$$\widetilde{HT}(\chi_v,\Pi_h) := \Big(\mathcal{L}(\chi_v,t)_1 \otimes \Pi_h^{K_v(n)} \otimes \Xi^{\frac{h-d}{2}}\Big) \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n),$$
where the unipotent radical of $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ acts trivially and the action of
$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n) \times W_v$$
is given

- by the action of g_v^c on $\Pi_h^{K_v(n)}$ and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{h-d}{2}}$, and
- the action of $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg\sigma_v) \in G(\mathbb{A}^{\infty,v}) \times \mathrm{GL}_{d-h}(\mathcal{O}_v/\mathcal{M}_v^n) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\chi_v)_1 \otimes \Xi^{\frac{h-d}{2}}$.

We also introduce
$$HT(\chi_v,\Pi_h)_1 := \widetilde{HT}(\chi_v,\Pi_h)_1[d-tg],$$
and the perverse sheaf
$$P(h,\chi_v)_1 := {}^p j^{=h}_{1,!*} HT(\chi_v, \mathrm{St}_h(\chi_v))_1 \otimes \chi_v^{-1},$$
and their induced version, $HT(\chi_v,\Pi_h)$ and $P(h,\chi_v)$.

Note that over $\overline{\mathbb{Z}}_l$, there are at least two notions of intermediate extension associated to the two classical t-structures p and $p+$. However in our situation they all coincide. Indeed as $\mathrm{Sh}^{\geq h}_{K^v(n),\bar{s}_v,1}$ is smooth over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, then $HT(\chi_v,\Pi_h)_1$ is perverse for the two t-structures with
$$i_1^{h\leq +1,*} HT(\chi_v,\Pi_h)_1 \in {}^p\mathcal{D}^{<0} \text{ and } i_1^{h\leq+1,!} HT(\chi_v,\Pi_h)_1 \in {}^{p+}\mathcal{D}^{\geq 1}.$$

Let now denote by
$$\Psi_{K,v} := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])(\frac{d-1}{2})$$
the nearby cycles autodual free perverse sheaf on $\mathrm{Sh}_{K,\bar{s}_v}$. Recall, cf. [**Boy23b**] proposition 3.1.3, that

$$\Psi_{K,v} \cong \bigoplus_{1\leq g\leq d} \bigoplus_{\varrho\in \mathrm{Scusp}(g)} \Psi_{K,\varrho}, \tag{4}$$

where

– $\mathrm{Scusp}(g)$ is the set of equivalence classes of irreducible supercuspidal representations of $\mathrm{GL}_g(F_v)$.
– The irreducible sub-quotients of $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are the Harris-Taylor perverse sheaves of $\Psi_{K,\overline{\mathbb{Q}}_l}$ associated to irreducible cuspidal representations π_v with modulo l reduction having supercuspidal support a Zelevinsky segment associated to ϱ.

Remark. In the limit case when $q_v \equiv 1 \mod l$ and $l > d$, recall that we do not have to bother about cuspidal $\overline{\mathbb{F}}_l$-representation which are not supercuspidal. In particular in the previous formula we can

– replace $\mathrm{Scusp}(g)$ by the set $\mathrm{Cusp}(g)$ of equivalence classes of cuspidal representations,
– and the Harris-Taylor perverse sheaves of $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are those associated to π_v such that its modulo l reduction is isomorphic to ϱ.

Moreover regarding the main statement about Ihara's lemma, we will only be concerned by $\Psi_{K,\varrho}$ for ϱ a character.

3. Nearby cycles and filtrations

3.1. Filtrations of stratification of Ψ_ϱ. — Using the Newton stratification and following the constructions of [**Boy14**], we can define a $\overline{\mathbb{Z}}_l$-filtration

$$\mathrm{Fil}^0_!(\Psi_{K,\varrho}) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^d_!(\Psi_{K,\varrho}) = \Psi_{K,\varrho}$$

where $\mathrm{Fil}^h_!(\Psi_{K,\varrho})$ is the saturated image of $j^{=h}_! j^{=h,*}\Psi_{K,\varrho} \longrightarrow \Psi_{K,\varrho}$. We also denote by $\mathrm{coFil}^h_!(\Psi_\varrho) := \Psi_\varrho / \mathrm{Fil}^h_!(\Psi_\varrho)$. Dually we can define a cofiltration

$$\Psi_{K,\varrho} = \mathrm{coFil}^d_*(\Psi_{K,\varrho}) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}^1_*(\Psi_{K,\varrho})$$

where $\mathrm{coFil}^h_*(\Psi_{K,\varrho})$ is the saturated image of $\Psi_{K,\varrho} \longrightarrow j^{=h}_* j^{=h,*}\Psi_{K,\varrho}$: cf. figure 3.3 for an illustration. We denote by $\mathrm{Fil}^h_*(\Psi_\varrho) := \ker(\Psi_\varrho \twoheadrightarrow \mathrm{coFil}^h_*(\Psi_\varrho))$.

Over $\overline{\mathbb{Q}}_l$, the filtration $\mathrm{Fil}^\bullet_!(\Psi_{K,\varrho})$ coincides with the iterated kernel of N_v, i.e. $\mathrm{Fil}^k_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \cong \ker(N_v^k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. Dually the cofiltration $\mathrm{coFil}^\bullet_!(\Psi_{K,\varrho})$ coincides with the iterated image of N_v, i.e. the kernel of $\Psi_{K,\varrho} \twoheadrightarrow \mathrm{coFil}^h_*(\Psi_{K,\varrho})$ is the image of N_v^h. Note that by Grothendieck-Verdier duality, we have $D(\mathrm{Fil}^h_!(\Psi_{K,\varrho})) \cong \mathrm{coFil}^h_*(\Psi_{K,\varrho})$.

The graded parts $\mathrm{gr}^k_!(\Psi_{K,\varrho})$ are, by construction, free and admit a strict[(6)] filtration, cf. [**Boy14**] corollary 3.4.5

$$\mathrm{Fil}^{d-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho})) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^{k-1}_*(\mathrm{gr}^k_!(\Psi_{K,\varrho})) = \mathrm{gr}^k_!(\Psi_{K,\varrho})$$

(6) meaning the graded parts are free

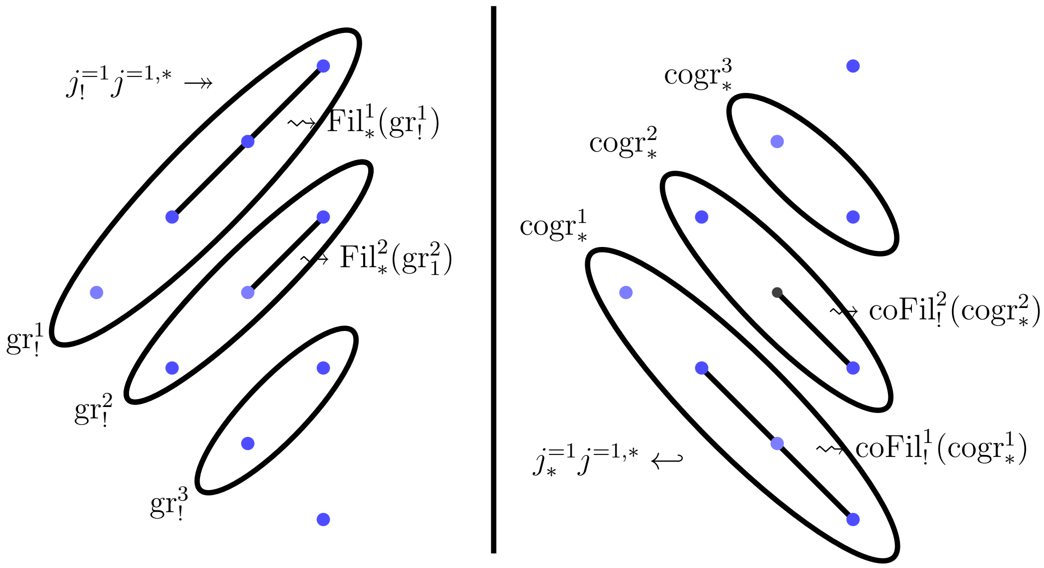

FIGURE 1. Filtrations of stratification of $\Psi_{K,\varrho}$

with

$$\mathrm{gr}_*^{i-1}(\mathrm{gr}_!^k(\Psi_{K,\varrho})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} P(i,\pi_v)(\frac{i+1-2k}{2}),$$

where $\mathrm{Cusp}(\varrho)$ is the set of equivalence classes of irreducible cuspidal representations with modulo l reduction isomorphic to ϱ.

Dually, $\mathrm{cogr}_*^k(\Psi_{K,\varrho})$ has a cofiltration

$$\mathrm{cogr}_*^k(\Psi_{K,\varrho}) = \mathrm{coFil}_!^{k-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}_!^{d-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})),$$

with

$$\mathrm{cogr}_!^{i-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} P(i,\pi_v)(\frac{2k-i-1}{2}).$$

Concerning the $\overline{\mathbb{Z}}_l$-structures, cf. the third global result of the introduction of [**Boy23b**] , for every $1 \leq k \leq i \leq d$, we have strict epimorphisms(7)

$$j_!^{=i} j^{=i,*}\, \mathrm{Fil}_*^{i-1}(\mathrm{gr}_!^k(\Psi_{K,\varrho})) \twoheadrightarrow \mathrm{Fil}_*^{i-1}(\mathrm{gr}_!^k(\Psi_{K,\varrho}))$$

as well as strict monomorphisms

$$\mathrm{coFil}_!^{i-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})) \hookrightarrow j_*^{=i} j^{=i,*}\, \mathrm{coFil}^{i-1}(\mathrm{cogr}_*^k(\Psi_{K,\varrho})).$$

(7) strict means that the cokernel is torsion free

Exchange basic step: to go from filtration to another, one can repeat the following process to exchange the order of appearance of two consecutive subquotient:

$$\begin{tikzcd}
 & P'_1 \arrow[d, hook] \arrow[dr, hook] & & \\
P_2 \arrow[r, hook] \arrow[dr, hook] & X \arrow[r, two heads] \arrow[d] & P_1 \arrow[dr] & \\
 & P'_2 \arrow[dr] & & T \arrow[dl, equal] \\
 & & T, &
\end{tikzcd}$$

where

- P_1 and P_2 are two consecutive subquotient in a given filtration and X is the subquotient gathering them as a subquotient of this filtration.
- Over $\overline{\mathbb{Q}}_l$, the extension $X \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is split, so that on can write X as an extension of P'_2 by P'_1 with $P'_1 \hookrightarrow P_1$ and $P_2 \hookrightarrow P'_2$ have the same cokernel T, a perverse sheaf of torsion.

Remark. In the particular case when P_1 and P_2 are intermediate extensions of local systems living on different strata such that the two associated intermediate extensions for the p and $p+$ t-structure are isomorphic, then T is necessary zero and X is then split over $\overline{\mathbb{Z}}_l$.

3.2. The canonical filtration of $H^0(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ is strict. — We have spectral sequences

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}, \mathrm{gr}_*^{-p}(\mathrm{gr}_!^k(\Psi_{K,\varrho}))) \Rightarrow H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}\, \mathrm{gr}_!^k(\Psi_{K,\varrho})), \tag{5}$$

and

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}, \mathrm{gr}_!^{-p}(\Psi_{K,\varrho})) \Rightarrow H^{p+q}(\mathrm{Sh}_{K,\bar{s}_v}, \Psi_{K,\varrho}). \tag{6}$$

Definition 3.2.1. — For a finite set S of places of $\mathbb{Q}$ containing the places where G is ramified, denote by $\mathbb{T}^S_{abs} := \prod'_{x \notin S} \mathbb{T}_{x,abs}$ the abstract unramified Hecke algebra where $\mathbb{T}_{x,abs} \cong \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.

Example. For $w \in \mathrm{Spl}$, we have

$$\mathbb{T}_{v|_{\mathbb{Q}},abs} = \overline{\mathbb{Z}}_l\big[T_{v',i} : \ i = 1, \cdots, d, \ v'|(v|_{\mathbb{Q}})\big]$$

where $T_{v',i}$ is the characteristic function of

$$\mathrm{GL}_d(\mathcal{O}_{v'})\, \mathrm{diag}(\overbrace{\varpi_{v'}, \cdots, \varpi_{v'}}^{i}, \overbrace{1, \cdots, 1}^{d-i})\, \mathrm{GL}_d(\mathcal{O}_{v'}) \subseteq \mathrm{GL}_d(F_{v'}).$$

Recall that $\mathbb{T}^S_{abs}$ acts through correspondances on each of the $H^i(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Z}}_l)$ where $K\in\mathcal{K}$ is maximal at each places outside S.

Notation 3.2.2. — For K unramified outside S, we denote by $\mathbb{T}(K)$ the image of $\mathbb{T}^S_{abs}$ inside $\mathrm{End}_{\overline{\mathbb{Z}}_l}(H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Z}}_l))$.

We also denote by

$$H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar\eta},\overline{\mathbb{Z}}_l):=\varinjlim_{K_v} H^{d-1}(\mathrm{Sh}_{K^vK_v,\bar\eta},\overline{\mathbb{Z}}_l),$$

where K_v describe the set of open compact subgroup of $\mathrm{GL}_d(\mathcal{O}_v)$. We also use similar notation for others cohomology groups.

Theorem 3.2.3. — *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}(K^v(\infty))$ such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible, cf. §4.2, and the irreducible constituants of its restriction to the decomposition group at the place v are characters.*[8] *Then*

(i) $H^i(\mathrm{Sh}_{K^v(\infty),\bar\eta},\overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ is zero if $i\neq d-1$ and otherwise torsion free.
(ii) Moreover the spectral sequences (5) and (6), localized at $\mathfrak{m}$, degenerate at E_1 and the $E^{p,q}_{1,\mathfrak{m}}$ are zero for $p+q\neq 0$ and otherwise torsion free.

Proof. — (i) It is the main theorem of [**Boy23a**].

(ii) From (4) we are led to study the initial terms of the spectral sequence given by the filtration of $\Psi_{K^v(\infty),\varrho}$ for ϱ a character which is a constituant of $\overline{\rho}_{\mathfrak{m},v}$. Recall also, as we are in the limit case, that

– as there do not exist irreducible $\overline{\mathbb{Q}}_l$-cuspidal representation of $\mathrm{GL}_g(F_v)$ for $g\leq d$ with modulo l reduction being not supercuspidal, the irreducible constituants of $\Psi_{K,\varrho}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ are the Harris-Taylor perverse sheaves $P(h,\chi_v)(\frac{h-1-2k}{2})$ where the modulo l reduction of χ_v is isomorphic to ϱ and $0\leq k<h$.
– Over $\overline{\mathbb{Z}}_l$, we do not have to worry about the difference between p and $p+$ intermediate extensions.

From [**Boy23b**] §2.3, consider the following equivariant resolution

$$(7)\quad 0\to j_!^{=d}HT(\chi_v,\Pi_h\{\frac{h-d}{2}\}\times\mathrm{Speh}_{d-h}(\chi_v\{h/2\}))\otimes\Xi^{\frac{d-h}{2}}\longrightarrow\cdots$$
$$\longrightarrow j_!^{=h+1}HT(\chi_v,\Pi_h\{-1/2\}\times\chi_v\{h/2\})\otimes\Xi^{\frac12}\longrightarrow$$
$$j_!^{=h}HT(\chi_v,\Pi_h)\longrightarrow {}^pj_{!*}^{=h}HT(\chi_v,\Pi_h)\to 0,$$

where Π_h is any representation of $\mathrm{GL}_h(F_v)$, also called the infinitesimal part of the perverse sheaf ${}^pj_{!*}^{=h}HT(\chi_v,\Pi_h)$.[9]

[8] Recall also that we suppose $q_v\equiv 1 \mod l$ and $l>d$.
[9] In $P(h,\chi_v)$ the infinitesimal part Π_h is $\mathrm{st}_h(\chi_v)$.

By adjunction property, for $1 \leq \delta \leq d-h$, the map

$$(8)\quad j_!^{=h+\delta} HT(\chi_v, \Pi_h\{\frac{-\delta}{2}\} \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \\ \longrightarrow j_!^{=h+\delta-1} HT(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$$

is given by

$$(9)\quad HT(\chi_v, \Pi_h\{\frac{-\delta}{2}\} \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \longrightarrow \\ j^{=h+\delta,*}({}^p i^{h+\delta,!}(j_!^{=h+\delta-1} HT(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}))$$

To compute this last term we use the resolution (7) for $h+\delta-1$. Precisely denote by $\mathcal{H} := HT(\chi_v, \mathrm{st}_h(\chi\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$, and write the previous resolution for $h+\delta-1$ as follows

$$0 \to K \longrightarrow j_!^{=h+\delta}\mathcal{H}' \longrightarrow Q \to 0,$$

$$0 \to Q \longrightarrow j_!^{=h+\delta-1}\mathcal{H} \longrightarrow {}^p j_{!*}^{=h+\delta-1}\mathcal{H} \to 0,$$

with

$$\mathcal{H}' := HT\Big(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2}.$$

As the support of K is contained in $\mathrm{Sh}_{I,\bar{s}_v}^{\geq h+\delta+1}$ then ${}^p i^{h+\delta,!}K = K$ and $j^{=h+\delta,*}({}^p i^{h+\delta,!}K)$ is zero. Moreover ${}^p i^{h+\delta,!}({}^p j_{!*}^{=h+\delta-1}\mathcal{H})$ is zero by construction of the intermediate extension. We then deduce that

$$(10)\quad j^{=h+\delta,*}({}^p i^{h+\delta,!}(j_!^{=h+\delta-1} HT(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}})) \\ \cong HT\Big(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \\ \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2}$$

In particular, up to homothety, the map (9), and so (8), is unique. Finally as the maps of (7) are strict, the given maps (8) are uniquely determined, that is, if we forget the infinitesimal parts, these maps are independent of the chosen h in (7), i.e. only depends on $h+\delta$.

For every $1 \leq h \leq d$, let denote by $i(h)$ the smallest index i such that $H^i(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=h} HT(\chi_v, \Pi_h)_{\mathfrak{m}}$ has non trivial torsion: if it does not exist then we set $i(h) = +\infty$ and note that it does not depend on the choice of the infinitesimal part Π_h. By duality, as ${}^p j_{!*} = {}^{p+} j_{!*}$ for Harris-Taylor local systems associated to characters, note that when $i(h)$ is finite then $i(h) \leq 0$. Suppose by absurdity there exists h with $i(h)$ finite and denote h_0 the biggest such h.

Lemma 3.2.4. — *For $1 \leq h \leq h_0$ then $i(h) = h - h_0$.*

Note that a similar result is proved in [**Boy17**] when the level is maximal at v.

Proof. — a) We first prove that for every $h_0 < h \leq d$, the cohomology groups of $j_!^{=h} HT(\chi_v, \Pi_h)$ are torsion free. Consider the following strict filtration in the category of free perverse sheaves

$$(0) = \mathrm{Fil}^{-1-d}(\chi_v, h) \hookrightarrow\!\!\!\!\!+\!\!\!\rightarrow \mathrm{Fil}^{-d}(\chi_v, h) \hookrightarrow\!\!\!\!\!+\!\!\!\rightarrow \cdots \hookrightarrow\!\!\!\!\!+\!\!\!\rightarrow \mathrm{Fil}^{-h}(\chi_v, h) = j_!^{=h} HT(\chi_v, \Pi_h) \tag{11}$$

where the symbol $\hookrightarrow\!\!\!\!\!+\!\!\!\rightarrow$ means a strict[10] monomorphism, with graded parts

$$\mathrm{gr}^{-k}(\chi_v, h) \cong {}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{st}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}).$$

Over $\overline{\mathbb{Q}}_l$, the result is proved in [**Boy09**] §4.3. Over $\overline{\mathbb{Z}}_l$, the result follows from the general constructions of [**Boy14**] and the fact that the p and $p+$ intermediate extensions are isomorphic for Harris-Taylor perverse sheaves associated to characters. The associated spectral sequence localized at $\mathfrak{m}$, is then concentrated in middle degree and torsion free which gives the claim.

b) Before watching the cases $h \leq h_0$, note that the spectral sequence associated to (7) for $h = h_0 + 1$, has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the maps

$$\begin{aligned} H^0\big(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=h+\delta} HT_\xi(\chi_v, \mathrm{st}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}} \\ \longrightarrow \\ H^0\big(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=h+\delta-1} HT_\xi(\chi_v, \mathrm{st}_h(\chi_v\{\frac{1-\delta}{2}\}) \\ \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}} \end{aligned} \tag{12}$$

are saturated, i.e. their cokernel are free $\overline{\mathbb{Z}}_l$-modules. Then from the previous fact stressed after (10), this property remains true when we consider the associated spectral sequence for $1 \leq h' \leq h_0$.

c) Consider now $h = h_0$ and the spectral sequence associated to (7) where

$$E_2^{p,q} = H^{p+2q}(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=h+q} HT_\xi(\chi_v, \mathrm{st}_h(\chi_v(-q/2)) \times \mathrm{Speh}_q(\chi_v\{h/2\})) \otimes \Xi^{\frac{q}{2}})_{\mathfrak{m}} \tag{13}$$

By definition of h_0, we know that some of the $E_\infty^{p,-p}$ should have a non trivial torsion subspace. We saw that

– the contributions from the deeper strata are torsion free and

[10] i.e. the cokernel is free

- $H^i(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v,\Pi_{h_0}))_{\mathfrak{m}}$ are zero for $i<0$ and is torsion free for $i=0$, whatever is Π_{h_0}.
- Then there should exist a non strict map $d_1^{p,q}$. But, we have just seen that it can not be maps between deeper strata.
- Finally, using the previous points, the only possibility is that the cokernel of

$$H^0\big(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=h_0+1} HT_\xi(\chi_v,\mathrm{st}_{h_0}(\chi_v\{\frac{-1}{2}\})\times\chi_v\{h_0/2\}))\otimes\Xi^{1/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v,\mathrm{st}_{h_0}(\chi_v))\big)_{\mathfrak{m}} \tag{14}$$

  has a non trivial torsion subspace.

In particular we have $i(h_0)=0$.

d) Finally using the fact 2.18 and the previous points, for any $1\leq h\leq h_0$, in the spectral sequence (13)

- by point a), $E_2^{p,q}$ is torsion free for $q\geq h_0-h+1$ and so it is zero if $p+2q\neq 0$;
- by affiness of the open strata, cf. [**Boy17**] theorem 1.8, $E_2^{p,q}$ is zero for $p+2q<0$ and torsion free for $p+2q=0$;
- by point b), the maps $d_2^{p,q}$ are saturated for $q\geq h_0-h+2$;
- by point c), $d_2^{-2(h_0-h+1),h_0-h+1}$ has a cokernel with a non trivial torsion subspace.
- Moreover, over $\overline{\mathbb{Q}}_l$, the spectral sequence degenerates at E_3 and $E_3^{p,q}=0$ if $(p,q)\neq(0,0)$.

We then deduce that $H^i(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v,\Pi_h))_{\mathfrak{m}}$ is zero for $i<h-h_0$ and for $i=h-h_0$ it has a non trivial torsion subspace. $\square$

Consider now the filtration of stratification of $\Psi_\varrho:=\Psi_{K^v(\infty),\varrho}$[11] constructed using the adjunction morphisms $j_!^{=h}j^{=h,*}$ as in [**Boy14**]

$$\mathrm{Fil}_!^1(\Psi_\varrho)\lhook\joinrel\longrightarrow\mathrm{Fil}_!^2(\Psi_\varrho)\lhook\joinrel\longrightarrow\cdots\lhook\joinrel\longrightarrow\mathrm{Fil}_!^d(\Psi_\varrho) \tag{15}$$

where $\mathrm{Fil}_!^h(\Psi_\varrho)$ is the saturated image of $j_!^{=h}j^{=h,*}\Psi_\varrho\longrightarrow\Psi_\varrho$. For our fixed χ_v, let denote by $\mathrm{Fil}^1_{!,\chi_v}(\Psi)\lhook\joinrel\longrightarrow\mathrm{Fil}_!^1(\Psi_\varrho)$ such that $\mathrm{Fil}^1_{!,\chi_v}(\Psi)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l\cong\mathrm{Fil}_!^1(\Psi_{\chi_v})$ where Ψ_{χ_v} is the direct factor of $\Psi_\varrho\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ associated to χ_v, cf. [**Boy14**]. From [**Boy23b**] 3.3.5, we have the following resolution of $\mathrm{gr}^h_{!,\chi_v}(\Psi_\varrho)$

$$0\to j_!^{=d}HT(\chi_v,LT_{h,d}(\chi_v))\otimes\chi_v^{-1}(\frac{d-h}{2})\longrightarrow j_!^{=d-1}HT(\chi_v,LT_{h,d-1}(\chi_v))\otimes\chi_v^{-1}(\frac{d-h-1}{2})\longrightarrow \cdots\longrightarrow j_!^{=h}HT(\chi_v,\mathrm{st}_h(\chi_v))\otimes\chi_v^{-1}\longrightarrow\mathrm{gr}^h_{!,\chi_v}(\Psi_\varrho)\to 0, \tag{16}$$

[11] i.e. with infinite level at v

where $LT_{h,h+\delta}(\chi_v) \hookrightarrow \mathrm{st}_h(\chi_v\{-\delta/2\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})$, is the only irreducible subspace of this induced representation,

We can then apply the previous arguments a)-d) above: for $h \leq h_0$ (resp. $h > h_0$) the torsion of $H^i(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \mathrm{gr}^h_{!,\chi_v}(\Psi_{v,\xi}))_{\mathfrak{m}}$ is trivial for any $i \leq h - h_0$ (resp. for all i) and the free parts are concentrated for $i = 0$. Using the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ would have non trivial torsion which is false as $\mathfrak{m}$ is supposed to be KHT-free. $\square$

In particular the previous spectral sequence gives us a filtration of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ whose graded parts are

$$H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \mathrm{gr}^{-p}(\mathrm{gr}^k_!(\Psi_{K,\varrho})))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l,$$

for ϱ describing the equivalence classes of irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation of $\mathrm{GL}_g(F_v)$ with $1 \leq g \leq d$, and then $1 \leq k \leq p \leq \lfloor \frac{d}{g} \rfloor$.

3.3. Local and global monodromy. — Consider a fixed $\overline{\mathbb{F}}_l$-character ϱ and denote by Ψ_ϱ the direct factor of $\Psi_{K^v(\infty),v}$ associated to ϱ.

Over $\overline{\mathbb{Q}}_l$, the monodromy operator define a nilpotent morphism $N_{\varrho,\overline{\mathbb{Q}}_l} : \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \longrightarrow \Psi_\varrho(1) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ compatible with the filtration $\mathrm{Fil}^\bullet_!(\Psi_\varrho)$ in the sense that $\mathrm{Fil}^h_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ coincides with the kernel of $N^h_{\varrho,\overline{\mathbb{Q}}_l}$. The aim of this section is to construct a $\overline{\mathbb{Z}}_l$-version N_ϱ of $N_{\varrho,\overline{\mathbb{Q}}_l}$ such that $\mathrm{Fil}^h_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ coincides with the kernel of $N^h_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

First step: consider

$$0 \to \mathrm{Fil}^1_!(\Psi_\varrho) \longrightarrow \Psi_\varrho \longrightarrow \mathrm{coFil}^1_!(\Psi_\varrho) \to 0,$$

and the following long exact sequence

$$0 \to \hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \hom(\Psi_\varrho, \Psi_\varrho(1)) \longrightarrow \hom(\mathrm{Fil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \cdots$$

where hom is taken in the category of equivariant Hecke perverse sheaves with an action of $\mathrm{Gal}(\overline{F}_v/F_v)$. As $\mathrm{Fil}^1_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ coincides with the kernel of $N_{\varrho,\overline{\mathbb{Q}}_l}$, then $N_{\varrho,\overline{\mathbb{Q}}_l} \in \hom(\Psi_\varrho, \Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ comes from $\hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, so that we focus on $\hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1))$. From

$$0 \to \mathrm{gr}^2_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^1_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^2_!(\Psi_\varrho) \to 0,$$

we obtain

$$\begin{aligned} 0 \to \hom(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \hom(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \\ \hom(\mathrm{gr}^2_!(\Psi_\varrho), \Psi_\varrho(1)) \longrightarrow \mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho(1))) \longrightarrow \cdots \end{aligned}$$

The socle of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ being contained in $\mathrm{Fil}^1_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, any map $\mathrm{coFil}^2_!(\Psi_\varrho) \longrightarrow \Psi_\varrho(1))$ can not be equivariant for the Galois action, so that we are led to look at

$$\hom(\mathrm{gr}^2_!(\Psi_\varrho), \Psi_\varrho(1)) \cong \hom(\mathrm{gr}^2_!(\Psi_\varrho), \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho(1))))$$

where

$$0 \to \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \longrightarrow \mathrm{Fil}^1_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^1_*(\mathrm{Fil}^1_!(\Psi_\varrho) \to 0.$$

Note that $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho(1))) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ and their $\overline{\mathbb{Z}}_l$-structure is obtained, cf. the introduction of [**Boy23b**] or equation (7) for a character alone, through the strict $\overline{\mathbb{Z}}_l$-epimorphisms

$$j^{=2}_! j^{=2,*} \mathrm{gr}^2_!(\Psi_\varrho) \twoheadrightarrow \mathrm{gr}^2_!(\Psi_\varrho), \quad \text{and} \quad j^{=2}_! j^{=2,*} \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \twoheadrightarrow \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)),$$

cf. figure and the notations of the beginning of §3.1.

In particular to prove that $\mathrm{gr}^2_!(\Psi_\varrho)$ is isomorphic to $\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho(1)))$, it suffices to prove that the two local systems $j^{=2,*} \mathrm{gr}^2_!(\Psi_\varrho)$ and $j^{=2,*} \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho(1)))$ are isomorphic. In this case we can take[12] $N_v \in \hom(\Psi_\varrho, \Psi_\varrho(1)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ so that, over $\overline{\mathbb{Z}}_l$ we have $\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho(1))) = N_v(\mathrm{Fil}^2_!(\Psi_\varrho))$.

More generally to prove that the two perverse sheaves $\mathrm{gr}^{h+1}_!(\Psi_\varrho)$ and $\mathrm{Fil}^1_*(\mathrm{gr}^h_!(\Psi_\varrho(1)))$ are isomorphic, it suffices to prove that the two local systems $j^{=h+1,*} \mathrm{gr}^{h+1}_!(\Psi_\varrho)$ and $j^{=h+1,*} \mathrm{Fil}^1_*(\mathrm{gr}^h_!(\Psi_\varrho(1)))$ are isomorphic.

Second step: we want to prove that the local systems $j^{=2,*} \mathrm{gr}^2_!(\Psi_\varrho)$ and $j^{=2,*} \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ are isomorphic. Consider first the following situation: let $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$ be $\overline{\mathbb{Z}}_l$-local systems on a scheme X such that:

- $\mathcal{L}_k \hookrightarrow \mathcal{L}_{k+1}$ where the cokernel gr_{k+1} is torsion free;
- $\mathcal{L}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong (\mathcal{L}_k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l) \oplus (\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)$ where $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is supposed to be irreducible;
- we introduce

$$\begin{tikzcd} \mathrm{gr}'_{k+1} \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \mathrm{gr}_{k+1,\overline{\mathbb{Q}}_l} \arrow[d, hook] \\ \mathcal{L}_{k+1} \arrow[r, hook] & \mathcal{L}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{tikzcd}$$

  We moreover suppose that $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is also irreducible so the various stable $\overline{\mathbb{Z}}_l$-lattices of gr_{k+1} are homothetic.

We then have

$$0 \to \mathcal{L}_k \oplus \mathrm{gr}'_{k+1} \longrightarrow \mathcal{L}_{k+1} \longrightarrow T \to 0,$$

where T is torsion and can be viewed as a quotient

$$\mathcal{L}_k \hookrightarrow \mathcal{L}'_k \twoheadrightarrow T, \quad \mathrm{gr}'_{k+1} \hookrightarrow \mathrm{gr}_{k+1} \twoheadrightarrow T,$$

with

$$\mathcal{L}_k \hookrightarrow \mathcal{L}_{k+1} \twoheadrightarrow \mathrm{gr}_{k+1}, \qquad \mathrm{gr}'_{k+1} \hookrightarrow \mathcal{L}_{k+1} \twoheadrightarrow \mathcal{L}'_k.$$

(12) As it is not clear that $\mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho(1)))$ is torsion free, we can not claim at this stage that $N_v \in \hom(\Psi_\varrho, \Psi_\varrho(1))$.

As $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is irreducible, then $\mathrm{gr}'_{k+1} \hookrightarrow \mathrm{gr}_{k+1}$ is given by multiplication by l^{δ_k} and, as the stable lattices of $\mathrm{gr}_k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are all isomorphic, the extension is characterized by this δ_k.

Consider then the $\overline{\mathbb{Z}}_l$-local system $\mathcal{L} := j^{=1,*}\Psi_\varrho$ and recall that

$$\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{i=1}^{r} HT_{\overline{\mathbb{Q}}_l}(\chi_{v,i}, \chi_{v,i}),$$

where we fix any numbering of $\mathrm{Cusp}(\varrho) = \{\chi_{v,1}, \cdots, \chi_{v,r}\}$. For $k = 1, \cdots, r$, we introduce

$$\begin{tikzcd} \mathcal{L}_k \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \bigoplus_{i=1}^{k} HT(\chi_{v,i}, \chi_{v,i}) \arrow[d, hook] \\ \mathcal{L} \arrow[r, hook] & \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{tikzcd}$$

Let denote by T_{k+1} the torsion local system such that

$$0 \to \mathcal{L}_k \oplus \mathrm{gr}_{k+1} \longrightarrow \mathcal{L}_{k+1} \longrightarrow T_{k+1} \to 0,$$

where $\mathrm{gr}_{k+1} := \mathcal{L}_{k+1}/\mathcal{L}_k$, as above. We can apply the previous remark and denote by δ_k the power of l which define the homothety $\mathrm{gr}'_{k+1} \hookrightarrow \mathrm{gr}_{k+1} \twoheadrightarrow T_{k+1}$. The set $\{\delta_k : k = 1, \cdots, r\}$ is then a numerical data to characterize $\mathcal{L}$ inside $j^{=1,*}\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$.

(i) To control $j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$, we use the general description above with

- local systems $\mathcal{L}_k^+$ for $k = 1, \cdots, r$ so that $\mathcal{L}_k^+ \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{i=1}^k HT_{\overline{\mathbb{Q}}_l}(\chi_{v,i}, \mathrm{st}_2(\chi_{v,i})(-1/2)$;
- with $\mathrm{gr}^{+,'}_{k+1}$ defined, as before, with

$$0 \to \mathcal{L}_k^+ \oplus \mathrm{gr}^{+,'}_{k+1} \longrightarrow \mathcal{L}^+_{k+1} \longrightarrow T_{k+1} \to 0,$$

where T_{k+1} is killed by $l^{\delta^+_{k+1}}$.

We want to prove that $\delta_k^+ = \delta_k$ for every $k = 1, \cdots, r$ where $\{\delta_k : k = 1, \cdots, r\}$ is the numerical data associated to $j^{=1,*}\Psi_\varrho$.

Let denote by

$$j^{=1}_{\neq 1} : \mathrm{Sh}_{K,\bar{s}_v} \setminus \mathrm{Sh}^{\geq 1}_{K,\bar{s}_v,1} \hookrightarrow \mathrm{Sh}_{K,\bar{s}_v}, \qquad i^1_1 : \mathrm{Sh}^{\geq 1}_{K,\bar{s}_v,1} \hookrightarrow \mathrm{Sh}^{\geq 1}_{K,\bar{s}_v} = \mathrm{Sh}_{K,\bar{s}_v}.$$

From [**Boy23b**] lemma B.3.2, $j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ is obtained as follows. Let

$$P := {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} j^{=1,*}_{\neq 1} \Psi_\varrho$$

so that

$$0 \to P \longrightarrow j^{=1}_{\neq 1,!} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow {}^p j^{=1}_{\neq 1,!*} j^{=1,*}_{\neq 1} \Psi_\varrho \to 0.$$

Then P is the cosocle of $i_1^{1,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ so that

$$j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \cong j^{=2,*} P \times_{P_{1,d}(F_v)} \mathrm{GL}_d(F_v),$$

where induction has the same meaning as in (3).

Note then that the numerical data associated to $j^{=2,*}P$ are also given by $\{\delta_k^+ : k = 1, \cdots, r\}$. With the previous notations, consider the data associated to $\mathcal{L} := j^{=1,*}\Psi_\varrho$, i.e. a filtration

$$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots \subseteq \mathcal{L}_r = \mathcal{L}$$

with graded parts gr^k and $\mathrm{gr}'_k \hookrightarrow \mathrm{gr}_k$ is given by multiplication by l^{δ_k}. We then have a strict filtration

$${}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_1 \subseteq \cdots \subseteq {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_r = P,$$

with graded parts ${}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathrm{gr}_k$. Indeed we have

$$\begin{aligned} {}^p h^{-2} i_1^{1,*} j^{=1}_{\neq 1,*} \mathrm{gr}_{k+1} = 0 \longrightarrow {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_k \longrightarrow \\ {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_k \longrightarrow {}^p h^{-1} i_1^{1,*} j^{=1}_{\neq 1,*} \mathrm{gr}_{k+1} \longrightarrow {}^p h^{0} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_k \end{aligned}$$

where the free quotient of ${}^p h^{0} i_1^{1,*} j^{=1}_{\neq 1,*} \mathcal{L}_k$ is zero. Moreover it is torsion free because its torsion corresponds to the difference between p and $p+$ intermediate extensions which are equal here from the main result of [**Boy23b**]. We then apply the exact functor $j^{=2,*}$ and we induce from $P_{1,d}(F_v)$ to $\mathrm{GL}_d(F_v)$ to obtain the filtration $\mathcal{L}^+_\bullet$ of $j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ where $\mathrm{gr}_k^{+,'} \hookrightarrow \mathrm{gr}_k^+$ is given by multiplication by l^{δ_k}.

(ii) Dually the same arguments applied to

$$0 \to {}^p j^{=1}_{\neq 1,!*} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow {}^p j^{=1}_{\neq 1,*} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow Q \to 0,$$

give us that $j^{=2,*}Q$ is characterized by the data $\{\delta_k : k = 1, \cdots, r\}$. After inducing from $P_{1,d}(F_v)$ to $\mathrm{GL}_d(F_v)$, we obtain the description of the local system $\mathcal{A} := j^{=2,*}A$ where A is defined as follows:

$$0 \to {}^p j^{=1}_{!*} j^{=1,*}\,\mathrm{coFil}^1_*(\Psi_\varrho) \longrightarrow \mathrm{coFil}^1_*(\Psi_\varrho) \longrightarrow A \to 0.$$

Concretely this means that ${}^p j^{=2}_{!*}\mathcal{A}$ is the socle A_1 of A but as we are interested by the local system associated to $j^{=2,*}$ of the cosocle of $\mathrm{Fil}^2_!(\Psi_\varrho)$, as explained in §3.1, we have to use basic exchange steps as many times as needed to move A_1 until it appears as the cosocle of $\mathrm{Fil}^2_!(\Psi_\varrho) \hookrightarrow \Psi_\varrho$.

Note then that all the perverse sheaves which are exchanged with A_1 during this process, are lattice of $j^{=h}_{!*} HT_{\overline{\mathbb{Q}}_l}(\chi_v, \mathrm{st}_h(\chi_v))(\frac{1-h+\delta}{2})$ with $h \geq 3$. As explained in the remark after the definition of the exchange basic step, as ${}^p j^{=2}_{!*} HT(\chi_v, \mathrm{st}_2(\chi_v)) \cong {}^{p+} j^{=2}_{!*} HT(\chi_v, \mathrm{st}_2(\chi_v))$, for all these exchange, we have $T = 0$ and A_1 remains unchanged during all the basic exchange steps.

Third step: at this stage we constructed a $\overline{\mathbb{Q}}_l$-monodromy operator N_v such $\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho(1))) = N_v(\mathrm{Fil}^2_!(\Psi_\varrho))$. Recall that this monodromy operator induces

$$\alpha : \mathrm{coFil}^h_!(\mathrm{cogr}^h_*(\Psi_\varrho)) \longrightarrow \mathrm{cogr}^{h+1}_*(\Psi_\varrho(1))$$

such that $j^{=h+1,*}\circ\alpha$ is then an isomorphism over $\overline{\mathbb{Z}}_l$. We say that α is an isomorphism. Indeed consider

$$0 \to {}^p j_{!*}^{=h} j^{=h,*} \operatorname{cogr}_*^h(\Psi_\varrho) \longrightarrow \operatorname{cogr}_*^h(\Psi_\varrho) \longrightarrow \operatorname{coFil}_!^h(\operatorname{cogr}_*^h(\Psi_\varrho)) \to 0,$$

with the following two strict monomorphisms

$$\alpha_1 : \operatorname{cogr}_*^{h+1}(\Psi_\varrho) \hookrightarrow j_*^{=h+1} j^{=h+1,*} \operatorname{cogr}_*^{h+1}(\Psi_\varrho) \tag{17}$$

and

$$\alpha_2 : \operatorname{coFil}_!^h(\operatorname{cogr}_*^h(\Psi_\varrho(1))) \hookrightarrow j_*^{=h+1} j^{=h+1,*} \operatorname{coFil}_!^h(\operatorname{cogr}_*^h(\Psi_\varrho(1))). \tag{18}$$

By composing α with α_2 in (18), we obtain

$$\begin{aligned}\alpha_1, \alpha_2\circ\alpha \in \hom\Big(\operatorname{cogr}_*^{h+1}(\Psi_\varrho), j_*^{=h+1} j^{=h+1,*} \operatorname{cogr}_*^{h+1}(\Psi_\varrho)\Big)\\ \cong \hom\Big(j^{=h+1,*} \operatorname{cogr}_*^{h+1}(\Psi_\varrho), j^{=h+1,*} \operatorname{cogr}_*^{h+1}(\Psi_\varrho)\Big),\end{aligned} \tag{19}$$

by adjunction. By hypothesis α_1 and $\alpha_2\circ\alpha$ coincides in this last space, so they are equal and α is then an isomorphism.

Notation 3.3.1. — Under the hypothesis of theorem 3.2.3 on $\mathfrak{m}$, the action of N_ϱ on Ψ_ϱ defined above for every $\overline{\mathbb{F}}_l$-character ϱ, induces a nilpotent monodromy operator $N_{\mathfrak{m},v}^{coho}$ on $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$. We also denote by $\overline{N}_{\mathfrak{m},v}^{coho} := N_{\mathfrak{m},v}^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ acting on $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$

4. Proof of the Ihara's lemma

4.1. Supersingular locus as a zero dimensional Shimura variety. — As explained in the introduction, we follow the strategy of [**Boy20**] which consists to transfer the genericity property of Ihara's lemma concerning $\overline{G}$ to the genericity of the cohomology of KHT-Shimura varieties.

Let $\overline{G}$ be a similitude group as in the introduction such that moreover there exists a prime number p_0 split in E and v_0^+ a place of F^+ above p_0, identified as before to a place v_0 of F, such that $\overline{B}_{v_0}$ is a division algebra: in particular $v_0 \neq v$. Consider then, with the usual abuse of notation, $G/\mathbb{Q}$ such that $G(\mathbb{A}_{\mathbb{Q}}^{\infty,v_0}) \cong \overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,v_0})$ with $G(F_{v_0}) \cong \mathrm{GL}_d(F_{v_0})$ and $G(\mathbb{R})$ of signatures $(1,n-1),(0,n)^r$. The KHT Shimura variety $\mathrm{Sh}_{K,v_0} \to \operatorname{spec}\mathcal{O}_{v_0}$ associated to G with level K, has a Newton stratification of its special fiber with supersingular locus

$$\mathrm{Sh}_{K,\bar{s}_{v_0}}^{=d} = \coprod_{i\in\ker^1(\mathbb{Q},G)} \mathrm{Sh}_{K,\bar{s}_{v_0},i}^{=d}.$$

For a equivariant sheaf $\mathcal{F}_{K,i}$ on $\mathrm{Sh}_{K^v(\infty),\bar{s}_{v_0},i}^{=d}$ seen as acompatible system over $\mathrm{Sh}_{K^vK_v,\bar{s}_{v_0},i}^{=d}$ for K_v describing the set of open compact subgroups of $\mathrm{GL}_d(\mathcal{O}_v)$, its fiber at a compatible system $z_{K^v(\infty),i}$ of supersingular point $z_{K^vK_v,i}$, has an action of

$\overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathrm{GL}_d(F_v)^0$ where $\mathrm{GL}_d(F_v)^0$ is the kernel of the valuation of the determinant so that, cf. [**Boy09**] proposition 5.1.1, as a $\mathrm{GL}_d(F_v)$-module, we have

$$H^0(\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0},i}, \mathcal{F}_{K^v(\infty),i}) \cong \Big(\mathrm{ind}_{\overline{G}(\mathbb{Q})}^{\overline{G}(\mathbb{A}^{\infty,v})\times\mathbb{Z}}\, z^*_{K^{v_0}(\infty),i}\mathcal{F}_{K^{v_0}(\infty),i}\Big)^{K^v},$$

with $\delta \in \overline{G}(\mathbb{Q}) \mapsto (\delta^{\infty,v_0}, \mathrm{val}\circ\mathrm{rn}(\delta_{v_0})) \in \overline{G}(\mathbb{A}^{\infty,v_0,v}) \times \mathbb{Z}$ and where the action of $g_{v_0} \in \mathrm{GL}_d(F_{v_0})$ is given by those of $(g_0^{-\mathrm{val}\det g_{v_0}} g_{v_0}, \mathrm{val}\det g_{v_0}) \in \mathrm{GL}_d(F_{v_0})^0 \times \mathbb{Z}$ where $g_0 \in \mathrm{GL}_d(F_{v_0})$ is any fixed element with $\mathrm{val}\det g_0 = 1$. Moreover, cf. [**Boy09**] corollaire 5.1.2, if $z^*_{K^{v_0}(\infty),i}\mathcal{F}_{K^{v_0}(\infty),i}$ is provided with an action of the kernel $(D^\times_{v_0,d})^0$ of the valuation of the reduced norm, action compatible with those of $\overline{G}(\mathbb{Q}) \hookrightarrow D^\times_{v_0,d}$, then as a $G(\mathbb{A}^\infty)$-module, we have

$$H^0(\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0},i}, \mathcal{F}_{K^v(\infty),i}) \cong \Big(\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A}^\infty),\Lambda) \otimes_{D^\times_{v_0,d}} \mathrm{ind}_{(D^\times_{v_0,d})^0}^{D^\times_{v_0,d}}\, z_i^*\mathcal{F}_{\mathcal{I},i}\Big)^{K^v} \tag{20}$$

In particular, cf. lemma 2.3.1 of [**Boy20**], let $\overline{\pi}$ be an irreducible sub-$\overline{\mathbb{F}}_l$-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/K^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ for $\mathfrak{m}$ such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible. Write its local component $\bar{\pi}_{v_0} \cong \pi_{v_0}[s]_D$ with π_{v_0} an irreducible cuspidal representation of $\mathrm{GL}_g(F_{v_0})$ with $d = sg$. Then $(\overline{\pi}^{v_0})^{K^v}$ is a sub-representation of $H^0(\mathrm{Sh}^{=d}_{K^v(\infty),\bar{s}_{v_0}}, HT(\pi^\vee_{v_0}, s))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ and, cf. proposition 2.3.2 of [**Boy20**], a sub-$\overline{\mathbb{F}}_l$-representation of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. Indeed, cf. theorem 3.2.3,

- by the main result of [**Boy23a**], as $l > d \geq 2$ and $\overline{\rho}_{\mathfrak{m}}$ is irreducible, then $\mathfrak{m}$ is KHT free so that hypothesis (H1) of [**Boy20**] is fulfilled.
- Theorem 3.2.3 gives us that the filtration of $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}_{v_0}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ induced by the filtration of the nearby cycles at v_0, is strict.(13)

Finally if the analog of Ihara's lemma for $H^{d-1}(\mathrm{Sh}_{K^v(\infty),\bar{\eta}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ is true for the action of $\mathrm{GL}_d(F_v)$, then this is also the case for $\overline{G}$. We now focus on the genericity of irreducible sub-$\mathrm{GL}_d(F_v)$-modules of $H^0(\mathrm{Sh}_{K^v(\infty),\bar{\eta}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ using the nearby cycles at the place v.

4.2. Level raising. — To a cohomological minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}(K)$, which corresponds to a maximal ideal of $\mathbb{T}(K)[\frac{1}{l}]$, is associated both a near equivalence class of $\overline{\mathbb{Q}}_l$-automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ and a Galois representation

$$\rho_{\widetilde{\mathfrak{m}}} : G_F := \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{GL}_d(\overline{\mathbb{Q}}_l)$$

such that the eigenvalues of the Frobenius morphism at an unramified place w are given by the Satake parameters of the local component $\Pi_{\widetilde{\mathfrak{m}},w}$ of $\Pi_{\widetilde{\mathfrak{m}}}$. The semi-simple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of the maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S_K$ containing $\widetilde{\mathfrak{m}}$.

We now allow infinite level at v and we denote by $\mathbb{T}(K^v(\infty))$ the associated Hecke algebra. We fix a maximal ideal $\mathfrak{m}$ in $\mathbb{T}(K^v(\infty))$ such that

(13) In [**Boy20**] hypothesis (H3) was introduced for this property to be true.

– the associated Galois representation $\overline{\rho}_{\mathfrak{m}} : G_F \to \mathrm{GL}_d(\mathbb{F})$ is irreducible;
– $\overline{\rho}_{\mathfrak{m}}|W_{F_v}$, after semi-simplification, is a direct sum of characters.

Remark. For every minimal prime $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$, note that $\Pi_{\widetilde{\mathfrak{m}},v}$ looks like $\mathrm{st}_{s_1}(\chi_{v,1}) \times \cdots \times \mathrm{st}_{s_r}(\chi_{v,r})$ with $s_1 + \cdots + s_r = d$.

Let $\mathcal{S}_v(\mathfrak{m})$ be the supercuspidal support of the modulo l reduction of any $\Pi_{\widetilde{m},v}$ in the near equivalence class associated to a minimal prime ideal $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$. Recall that $\mathcal{S}_v(\mathfrak{m})$ is a multi-set, i.e. a set with multiplicities which only depends on $\mathfrak{m}$. We decompose it according to the set of Zelevinsky lines: as we supposed $q_v \equiv 1 \mod l$ then every Zelevinsky line is reduced to a single equivalence class of an irreducible (super)cuspidal $\overline{\mathbb{F}}_l$-representations ϱ of some $\mathrm{GL}_{g(\varrho)}(F_v)$ with $1 \leq g(\varrho) \leq d$. Moreover our second hypothesis tells us that we are only concerned with ϱ being a character:

$$\mathcal{S}_v(\mathfrak{m}) = \coprod_{\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}(1,v)} \mathcal{S}_\varrho(\mathfrak{m}),$$

where $\mathrm{Cusp}_{\overline{\mathbb{F}}_l}(1,v)$ is the set of $\overline{\mathbb{F}}_l$-characters of $F_v^\times$.

Notation 4.2.1. — We denote by $l_\varrho(\mathfrak{m})$ the multiplicity of $\mathcal{S}_\varrho(\mathfrak{m})$.

For $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$, the local component $\Pi_{\widetilde{\mathfrak{m}},v}$ of $\Pi_{\widetilde{\mathfrak{m}}}$ can then be written as a full induced representation $\bigtimes_{\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}(1,v)} \Pi_{\widetilde{\mathfrak{m}},\varrho}$ where each $\Pi_{\widetilde{\mathfrak{m}},\varrho}$ is also a full induced representation

$$\Pi_{\widetilde{\mathfrak{m}},\varrho} \cong \bigtimes_{i=1}^{r_\varrho(\widetilde{\mathfrak{m}})} \mathrm{St}_{l_{\varrho,i}(\widetilde{\mathfrak{m}})}(\pi_{v,i})$$

where $r_l(\pi_{v,i}) \cong \varrho$, $l_{\varrho,1}(\widetilde{\mathfrak{m}}) \geq \cdots \geq l_{\varrho,r_\varrho(\widetilde{\mathfrak{m}})}(\widetilde{\mathfrak{m}})$ and $\sum_{i=1}^r l_{\varrho,i}(\widetilde{\mathfrak{m}}) = l_\varrho(\mathfrak{m})$.

Suppose now that there exists $\varrho \in \mathrm{Cusp}_{\overline{\mathbb{F}}_l}(1,v)$ such that $\min_{\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}}\{r_\varrho(\widetilde{\mathfrak{m}})\} \geq 2$ and let $l_{\varrho,1} := \max_{\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}}\{l_{\varrho,1}(\widetilde{\mathfrak{m}})\}$ which is then strictly less than $l_\varrho(\mathfrak{m})$.

Recall that for a character χ_v such that its modulo l reduction is isomorphic to ϱ, $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, P(h,\chi_v))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ as computed in [**Boy10**], is the sum of the contributions of $\Pi_{\widetilde{\mathfrak{m}}}$ with $\widetilde{\mathfrak{m}} \subseteq \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}}}$ is of the following shape: $\mathrm{st}_h(\chi'_v) \times ?$ where χ'_v/χ_v is unramified and ? is any representation of $\mathrm{GL}_{d-h}(F_v)$ whose cuspidal support is not linked to those of $\mathrm{st}_h(\chi'_v)$.

In particular for every $h > l_{\varrho,1}$, $H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, P(h,\chi_v))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is zero, so that, as everything is torsion free,

$$H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \mathrm{gr}_*^{l_{\varrho,1}(\varrho)-1}(\mathrm{gr}_!^1(\Psi_\varrho)))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \hookrightarrow H^0(\mathrm{Sh}_{K^v(\infty),\bar{s}_v}, \Psi_{K^v(\infty),v}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

Moreover this subspace, as a $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_d(F_v)$, has a subspace of the following shape $\mathrm{st}_{l_1(\varrho)}(\varrho) \times \tau$ where the supercuspidal support of τ contains ϱ. In particular as $q_v \equiv 1 \mod l$ and $l > d$, this induced representation has both a generic and a non generic subspace.

We can then conclude that for the genericity property to be true for KHT Shimura varieties, one needs a level raising property as in proposition 3.3.1 of [**Boy20**]. Hopefully such statements exist under some rather mild hypothesis as for example the following result of T. Gee.

Theorem 4.2.2. — *(*[**Gee11**] *theorem 5.1.5) Let $F = F^+E$ be a CM field where F^+ is totally real and E is imaginary quadratic. Let $d > 1$ and $l > d$ be a prime which is unramified in F^+ and split in E. Suppose that*

$$\overline{\rho} : G_F \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_l)$$

is an irreducible representation which is unramified at all places of F lying above primes which do not split in E and which satisfies the following properties.

- *$\overline{\rho}$ is automorphic of weight $\underline{a}$, where we assume that for all $\tau \in (\mathbb{Z}^d)^{\mathrm{hom}(F,\mathbb{C})}$ we have either*[14]

$$l - 1 - d \geq a_{\tau,1} \geq \cdots \geq a_{\tau,d} \geq 0 \text{ or } l - 1 - d \geq a_{c\tau,1} \geq \cdots \geq a_{c\tau,d} \geq 0.$$

- *$\overline{F}^{\ker \mathrm{ad}\,\overline{\rho}}$ does not contain $F(\zeta_l)$.*
- *$\overline{\rho}(G_{F^+(\zeta_l)})$ is big.*

Let u be a finite place of F^+ which split in F and not dividing l. Choose an inertial type τ_v and a place v of F above u. Assume that $\overline{\rho}_{|G_{F_v}}$ has a lift to characteristic zero of type τ_v.

Then there is an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\underline{a}$ and level prime to l such that

- *$\overline{r}_{l,\iota}(\pi) \cong \overline{\rho}$.*
- *$r_{l,\iota}(\pi)_{|G_{F_v}}$ has type τ_v.*
- *π is unramified at all places $w \neq v$ of F at which $\overline{\rho}$ is unramified.*

Remark. In this text we focus only on the trivial coefficients $\overline{\mathbb{Z}}_l$, i.e. to the case $a_{\tau,1} = \cdots = a_{\tau,d} = a_{c\tau,1} = \cdots = a_{c\tau,d} = 0$, but we could also deals with others weights as in the previous theorem.

4.3. Genericity for KHT-Shimura varieties. — As explained in [**HT01**], the $\overline{\mathbb{Q}}_l$-cohomology of $\mathrm{Sh}_{K,\bar{\eta}}$ can be written as

$$H^{d-1}(\mathrm{Sh}_{K,\bar{\eta}}, \overline{\mathbb{Q}}_l)_{\mathfrak{m}} \cong \bigoplus_{\pi \in \mathcal{A}_{\xi,K}(\mathfrak{m})} (\pi^\infty)^K \otimes V(\pi^\infty),$$

where

[14] Note that these conditions imply $\overline{\rho}^c \cong \overline{\rho}^\vee \epsilon^{1-d}$.

– $\mathcal{A}_K(\mathfrak{m})$ is the set of equivalence classes of automorphic representations of $G(\mathbb{A})$ with non trivial K-invariants and such that its modulo l Satake's parameters outside S are prescribed by $\mathfrak{m}$,
– and $V(\pi^\infty)$ is a representation of $\mathrm{Gal}_{F,S}$.

As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible, then as explained in chapter VI of [**HT01**], if $V(\pi^\infty)$ is non zero, then π is a weak transfer of a cohomological automorphic representation (Π,ψ) of $\mathrm{GL}_d(\mathbb{A}_F)\times\mathbb{A}_F^\times$ with $\Pi^\vee\cong\Pi^c$ where c is the complex conjugation. Attached to such a Π is a global Galois representation $\rho_{\Pi,l}:\mathrm{Gal}_{F,S}\longrightarrow\mathrm{GL}_d(\overline{\mathbb{Q}}_l)$ which is irreducible.

Theorem 4.3.1. — *(cf.* [**NF19**] *theorem 2.20)*
If $\rho_{\Pi,l}$ is strongly irreducible, meaning it remains irreducible when it is restricted to any finite index subgroup, then $V(\pi^\infty)$ is a semi-simple representation of $\mathrm{Gal}_{F,S}$.

Remark. The Tate conjecture predicts that $V(\pi^\infty)$ is always semi-simple.

Definition 4.3.2. — (cf. [**Sch18**] §5) We say that $\mathfrak{m}$ is KHT-typic for K if, as a $\mathbb{T}(K)_{\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,

$$H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Z}}_l)_{\mathfrak{m}}\cong\sigma_{\mathfrak{m},K}\otimes_{\mathbb{T}(K)_{\mathfrak{m}}}\rho_{\mathfrak{m},K},$$

for some $\mathbb{T}(K)_{\mathfrak{m}}$-module $\sigma_{\mathfrak{m},K}$ on which $\mathrm{Gal}_{F,S}$ acts trivially and

$$\rho_{\mathfrak{m},K}:\mathrm{Gal}_{F,S}\longrightarrow\mathrm{GL}_d(\mathbb{T}(K)_{\mathfrak{m}})$$

is the stable lattice of $\bigoplus_{\widetilde{\mathfrak{m}}\subseteq\mathfrak{m}}\rho_{\widetilde{\mathfrak{m}}}$ introduced in the introduction.

Proposition 4.3.3. — *We suppose that for all $\pi\in\mathcal{A}_K(\mathfrak{m})$, the Galois representation $V(\pi^\infty)$ is semi-simple. Then $\mathfrak{m}$ is KHT-typic for K.*

Proof. — By proposition 5.4 of [**Sch18**] it suffices to deal with $\overline{\mathbb{Q}}_l$-coefficients. From [**HT01**] proposition VII.1.8 and the semi-simplicity hypothesis, then $V(\pi^\infty)\cong\widetilde{R}(\pi)^{\bigoplus n(\pi)}$ where $\widetilde{R}(\pi)$ is of dimension d. We then write

$$(\pi^\infty)^K\otimes_{\overline{\mathbb{Q}}_l}R(\pi)\cong(\pi^\infty)^K\otimes_{\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l}}(\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

and $(\pi^\infty)^K\otimes_{\overline{\mathbb{Q}}_l}V(\pi^\infty)\cong((\pi^\infty)^K)^{\bigoplus n(\pi)}\otimes_{\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l}}(\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l})^d$ and finally

$$H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Q}}_l)_{\mathfrak{m}}\cong\sigma_{\mathfrak{m},K,\overline{\mathbb{Q}}_l}\otimes_{\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l}}(\mathbb{T}(K)_{\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

with $\sigma_{\mathfrak{m},K,\overline{\mathbb{Q}}_l}\cong\bigoplus_{\pi\in\mathcal{A}_K(\mathfrak{m})}((\pi^\infty)^I)^{\bigoplus n(\pi)}$. The result then follows from [**HT01**] theorem VII.1.9 which insures that $R(\pi)\cong\rho_{\widetilde{\mathfrak{m}}}$, if $\widetilde{\mathfrak{m}}$ is the prime ideal associated to π, □

Let ϱ be a $\overline{\mathbb{F}}_l$-character with $l_\varrho(\mathfrak{m})>0$. Then $H^0(\mathrm{Sh}_{K,\bar\eta_v},\Psi_{K,\varrho})_{\mathfrak{m}}$ as a direct factor of $H^{d-1}(\mathrm{Sh}_{K,\bar\eta},\overline{\mathbb{Q}}_l)_{\mathfrak{m}}$, is also typic, i.e.

$$H^0(\mathrm{Sh}_{K,\bar\eta_v},\Psi_{K,\varrho})_{\mathfrak{m}}\cong\sigma_{\mathfrak{m},K,\varrho}\otimes_{\mathbb{T}(K)_{\mathfrak{m}}}\rho_{\mathfrak{m},K,\varrho}.$$

The monodromy operator $N^{coho}_{\mathfrak{m},\varrho}$ acting on $H^0(\mathrm{Sh}_{K,\bar\eta_v},\Psi_{K,\varrho})_{\mathfrak{m}}$ is such that

$$N^{coho}_{\mathfrak{m},\varrho}\otimes_{\mathbb{Z}_l}\overline{\mathbb{Q}}_l\cong \mathrm{Id}\otimes N_{\mathfrak{m},\varrho,\overline{\mathbb{Q}}_l},$$

i.e. it acts trivially on the first factor $\sigma_{\mathfrak{m},K,\varrho}$. We then deduce that $N^{coho}_{\mathfrak{m},\varrho}$ induces a nilpotent operator $N_{\mathfrak{m},\varrho}$ (resp. $\overline{N}_{\mathfrak{m},\varrho}$) on $\rho_{\mathfrak{m},K,\varrho}$ (resp. $\overline{\rho}_{\mathfrak{m},K,\varrho}:=\rho_{\mathfrak{m},K,\varrho}\otimes_{\mathbb{Z}_l}\overline{\mathbb{F}}_l$).

In the following, we will work with the following levels at v.

Notation 4.3.4. — For $\alpha\in\mathbb{N}\setminus\{0\}$, we will denote by

$$\mathrm{K}_v(\alpha):=\ker\Big(\mathrm{GL}_d(\mathcal{O}_v)\longrightarrow \mathrm{GL}_d(\mathcal{O}_v/\varpi_v^{\alpha})\Big),$$

the open compact subgroup of $\mathrm{GL}_d(\mathcal{O}_v)$. For a fixed level K^v outside v, we denote by $R_{K,\varrho}(\alpha)$ the image of $\mathbb{T}(K^v)_{\mathfrak{m}}$ acting on the direct factor $H^0(\mathrm{Sh}_{K^v\mathrm{K}_v(\alpha),\bar s_v},\Psi_\varrho)_{\mathfrak{m}}$ of $H^0(\mathrm{Sh}_{K^v\mathrm{K}_v(\alpha),\bar s_v},\Psi_v)_{\mathfrak{m}}$.

We now focus on the modulo l reduction of the previous objects

$$H^0(\mathrm{Sh}_{K^v\mathrm{K}_v(\alpha),\bar\eta_v},\Psi_\varrho)_{\mathfrak{m}}\otimes_{\mathbb{Z}_l}\overline{\mathbb{F}}_l\cong\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha),$$

where $\overline{\sigma}_{K,\varrho}(\alpha):=\sigma_{\mathfrak{m},K^v\mathrm{K}_v(\alpha),\varrho}\otimes_{\mathbb{Z}_l}\overline{\mathbb{F}}_l$, similarly $\overline{\rho}_{K,\varrho}(\alpha):=\rho_{\mathfrak{m},K^v\mathrm{K}_v(\alpha),\varrho}\otimes_{\mathbb{Z}_l}\overline{\mathbb{F}}_l$ and $\overline{R}_{K,\varrho}(\alpha):=R_{K,\varrho}(\alpha)\otimes_{\mathbb{Z}_l}\overline{\mathbb{F}}_l$.

Note that $\overline{R}_{K,\varrho}(\alpha)$ is an artinian local commutative ring with maximal ideal $\overline{\mathfrak{m}}_{K,\varrho}(\alpha)$ and with finite length $r_{K,\varrho}(\alpha)$. Let denote by

$$(0)=I_0\subseteq I_1\subseteq I_2\subseteq\cdots\subseteq I_{r_{K,\varrho}(\alpha)}=\overline{R}_{K,\varrho}(\alpha),$$

the ideals of $\overline{R}_{K,\varrho}(\alpha)$ such that for $k=1,\cdots,r$, the quotients I_k/I_{k-1} are simple and so isomorphic to $\overline{R}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\cong\overline{\mathbb{F}}_l$. For any irreducible cuspidal $\overline{\mathbb{F}}_l$-representation ϱ, we consider the filtration of $\overline{\sigma}_{K,\varrho}(\alpha)$:

$$(0)\subseteq I_1\overline{\sigma}_{K,\varrho}(\alpha)\subseteq\cdots\subseteq I_{r_{K,\varrho}(\alpha)-1}\overline{\sigma}_{K,\varrho}(\alpha)\subseteq\overline{\sigma}_{K,\varrho}(\alpha).$$

As $\overline{\rho}_{K,\varrho}(\alpha)$ is $\overline{R}_{K,\varrho}(\alpha)^d$ provided with a Galois action, it is flat as a $\overline{R}_{K,\varrho}(\alpha)$-module, so that we have the following filtration

$$(0)\subseteq I_1\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha)\subseteq\cdots\subseteq I_{r-1}\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha)\subseteq\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha),\tag{21}$$

with

$$\begin{array}{ccccc}
I_{k-1}\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha) & \hookrightarrow & I_k\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha) & \twoheadrightarrow & I_k\overline{\sigma}_{K,\varrho}(\alpha)/I_{k-1}\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho} \\
\| & & \| & & \\
\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}I_{k-1}\overline{\rho}_{K,\varrho}(\alpha) & \longrightarrow & \overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}I_k\overline{\rho}_{K,\varrho}(\alpha) & \twoheadrightarrow & \overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}I_{k-1}\overline{\rho}_{K,\varrho}(\alpha)/I_k\overline{\rho}
\end{array}$$

In other words, the graded parts are

$$\begin{aligned} I_k\overline{\sigma}_{K,\varrho}(\alpha)/I_{k-1}\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha) &\cong \overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)} I_k\overline{\rho}_{K,\varrho}(\alpha)/I_{k-1}\overline{\rho}_{K,\varrho}(\alpha) \\ &\cong \overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)} \big(\overline{\rho}_{K,\varrho}\otimes I_k/I_{k-1}\big) \\ &\cong \overline{\sigma}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\otimes_{\overline{\mathbb{F}}_l}\overline{\rho}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha). \end{aligned}$$

Remark. As expected all these graded parts are isomorphic as $\overline{R}_{K,\varrho}(\alpha)$-module. They are also equipped with an action of the modulo l local Hecke algebra $\overline{\mathbb{T}}_v(\alpha) = \overline{\mathbb{F}}_l[\mathrm{K}_v(\alpha)\backslash \mathrm{GL}_d(\mathcal{O}_v)/\mathrm{K}_v(\alpha)]$ at the place v with level $\mathrm{K}_v(\alpha)$. At this stage we do not know anything about the structure of these graded parts as $\overline{\mathbb{T}}_v(\alpha)$-modules. Note also that this filtration is not trivial: indeed otherwise as $N_{\mathfrak{m},\varrho}/\mathfrak{m}$ is trivial, then

$$\overline{N}^{coho}_{\mathfrak{m},\varrho} := N^{coho}_{\mathfrak{m},\varrho}\otimes_{R_{K,\varrho}(\alpha)}\overline{R}_{K,\varrho}(\alpha)$$

would be trivial but we know it is not the case as its order of nilpotency is $l_\varrho(\mathfrak{m})$ as soon as $K^v\,\mathrm{K}(\alpha)$ is small enough.

Notation 4.3.5. — Let $E_{K,\varrho}(\alpha)$ be the injective hull of $R_{K,\varrho}(\alpha)/\mathfrak{m}_{K,\varrho}(\alpha)$ and

$$D: M \longrightarrow D(M) = \hom_{R_{K,\varrho}(\alpha)}(M, E_{K,\varrho}(\alpha))$$

be the Maltis duality functor for the local artinian ring $\overline{R}_{K,\varrho}(\alpha)$ where M is any finitely generated $\overline{R}_{K,\varrho}(\alpha)$-module.

Recall the following properties:

(a) D is exact and $D\circ D(M)$ is canonically isomorphic to M;
(b) $\mathrm{length}_{\overline{R}_{K,\varrho}(\alpha)} D(M) = \mathrm{length}_{\overline{R}_{K,\varrho}(\alpha)} M$;
(c) for every ideal I of $\overline{R}_{K,\varrho}(\alpha)$, we have $D(M[I]) \cong D(M)/ID(M)$ and $D(M/IM)\cong D(M)[I]$;
(d) $D(\overline{R}_{K,\varrho}(\alpha)) = E_{K,\varrho}(\alpha)$ and $D(\overline{R}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)) \cong \overline{R}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)$;
(e) $M\otimes_{\overline{R}_{K,\varrho}(\alpha)} D(N)\cong D\big(\hom_{\overline{R}_{K,\varrho}(\alpha)}(M,N)\big)$.

Remark. as $\overline{\mathbb{F}}_l\hookrightarrow \overline{R}_{K,\varrho}(\alpha)$, then we have $E_{K,\varrho}(\alpha) = \hom_{\overline{\mathbb{F}}_l}(\overline{R}_{K,\varrho}(\alpha),\overline{\mathbb{F}}_l)$.

Lemma 4.3.6. — *With the previous notations, we have*

$$\begin{aligned} \Big(\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha)\Big)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)] &\cong \overline{\sigma}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\otimes_{\overline{\mathbb{F}}_l}\overline{\rho}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)] \\ &\cong \overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]\otimes_{\overline{\mathbb{F}}_l}\overline{\rho}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha). \end{aligned}$$

Proof. — From the previous properties $\Big(\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha)\Big)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]$ is isomorphic to

$$\begin{aligned}
&\cong \Big(D^2\big(\overline{\sigma}_{K,\varrho}(\alpha)\big)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha)\Big)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)] \cong D\Big(\hom_{\overline{R}_{K,\varrho}(\alpha)}(\overline{\rho}_{K,\varrho}(\alpha), D(\overline{\sigma}_{K,\varrho}(\alpha)))\Big)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]\\
&\cong D\Big(\hom_{\overline{R}_{K,\varrho}(\alpha)}(\overline{\rho}_{K,\varrho}(\alpha), D(\overline{\sigma}_{K,\varrho}(\alpha)))\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{R}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\Big)\\
&\cong D\Big(\hom_{\overline{R}_{K,\varrho}(\alpha)}\big(\overline{\rho}_{K,\varrho}(\alpha), D(\overline{\sigma}_{K,\varrho}(\alpha))/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\big)\Big)\\
&\cong D\Big(\hom_{\overline{R}_{K,\varrho}(\alpha)}\big((\overline{\rho}_{K,\varrho}(\alpha)), D(\overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)])\big)\Big)\\
&\cong \overline{\rho}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)} D^2(\overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)])\\
&\cong \overline{\rho}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)] \cong \overline{\rho}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}}\overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]
\end{aligned}$$

The second isomorphism is obtained by the symmetry of the proof. □

With the notations of §4.2, for a fixed ϱ, theorem 4.2.2 gives us the existence of $\widetilde{\mathfrak{m}}\subseteq\mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v}\cong \mathrm{st}_{l_\varrho(\mathfrak{m})}(\chi_v)\times ?$ where χ_v is a character lifting ϱ and ? is a representation of $\mathrm{GL}_{d-l_\varrho(\mathfrak{m}}(F_v)$ such that the supercuspidal support of its modulo l reduction does not contain ϱ. We consider any α_0 big enough such that $\Pi_{\widetilde{\mathfrak{m}},v}$ has non trivial vectors invariant under $\mathrm{K}_v(\alpha_0)$.

For $\alpha\geq\alpha_0$, let k be minimal such that $\overline{N}^{l_\varrho(\mathfrak{m})-1}_{\mathfrak{m},\varrho}$ is non zero on $\overline{\rho}_{K,\varrho}(\alpha)$, and so intersects $\overline{\rho}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]$. Consider then

$$0\neq z\in\overline{\rho}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]\cap\mathrm{Im}\big((\overline{N}^{l_\varrho(\mathfrak{m})-1}_{\mathfrak{m},\varrho})_{|I_k\rho_{K,\varrho}(\alpha)}\big).$$

Note that the map

$$\begin{aligned}
\overline{\sigma}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\otimes_{\overline{\mathbb{F}}_l}\overline{\rho}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha) &\cong \overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)} I_k\overline{\rho}_{K,\varrho}(\alpha)/I_{k-1}\overline{\rho}_{K,\varrho}(\alpha)\\
&\longrightarrow \overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{\mathbb{F}}_l}\overline{\rho}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]\cong(\overline{\sigma}_{K,\varrho}(\alpha)\otimes_{\overline{R}_{K,\varrho}(\alpha)}\overline{\rho}_{K,\varrho}(\alpha))[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]
\end{aligned}$$

induced by $\overline{N}^{l_\varrho(\mathfrak{m})-1}_{\mathfrak{m},\varrho}$ is then non zero and contains $\overline{\sigma}_{K,\varrho}(\alpha)/\overline{\mathfrak{m}}_{K,\varrho}(\alpha)\otimes_{\overline{\mathbb{F}}_l}\overline{\mathbb{F}}_l z$ which is then contained in $H^0(\mathrm{Sh}_{K^v\mathrm{K}_v(\alpha),\bar{s}_v},{}^pj_{!*}^{=l_\varrho(\mathfrak{m})}HT_{\overline{\mathbb{F}}_l}(\varrho,\mathrm{st}_{l_\varrho(\mathfrak{m})}(\varrho)))_{\mathfrak{m}}$ whose image in the Grothendieck group of $\overline{\mathbb{F}}_l[\mathbb{T}_v(\mathrm{K}_v(\alpha))]$-modules is equal to

$$\sum_{\widetilde{\mathfrak{m}}\subseteq\mathfrak{m}} n_{\widetilde{\mathfrak{m}}} r_l\Big(\Pi_{\widetilde{\mathfrak{m}}^\infty}\Big)^{K^v\mathrm{K}_v(\alpha)},$$

where

- $n_{\widetilde{\mathfrak{m}}}$ is a positive integer we do not need to precise,
- and the sum goes on the set of $\widetilde{\mathfrak{m}}\subseteq\mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v}$ is of the form $\mathrm{st}_{l_\varrho(\mathfrak{m})}(\chi_v)\times ?$ where χ_v is a lift of ϱ and ? is a representation of $\mathrm{GL}_{d-l_\varrho(\mathfrak{m})}(F_v)$ whose modulo l reduction does not contain ϱ in its supercuspidal support.

Using the second isomorphism of the lemma 4.3.6, we then deduce that the image of $\overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]$ in the Grothendieck group of $\overline{\mathbb{F}}_l[\overline{\mathbb{T}}_v(\mathrm{K}_v(\alpha))]$-modules has the same shape with multiplicities $n'_{\widetilde{\mathfrak{m}}} \leq n_{\widetilde{\mathfrak{m}}}$. We then conclude that as a $\overline{\mathbb{F}}_l$-representation of $\mathrm{GL}_d(F_v)$, every irreducible constituant of

$$\overline{\sigma}_{K,\varrho}[\overline{\mathfrak{m}}_K] = \lim_{\rightarrow \alpha} \overline{\sigma}_{K,\varrho}(\alpha)[\overline{\mathfrak{m}}_{K,\varrho}(\alpha)]$$

is of the form $\mathrm{st}_{l_\varrho(\mathfrak{m})}(\varrho) \times ?$ where ϱ does not belong to the supercuspidal support of ?. Varying ϱ we conclude that every irreducible constituant of $\overline{\sigma}_{K,\varrho}[\overline{\mathfrak{m}}_K]$ is isomorphic to $\bigtimes_\varrho \mathrm{st}_{l_\varrho(\mathfrak{m})}(\varrho)$, i.e. is generic.

Consider finally an irreducible $\overline{\mathbb{T}}(K^v)_{\mathfrak{m}} \times \mathrm{GL}_d(F_v)$-submodule V of σ_{K^v}: then $V = V[\overline{\mathfrak{m}}_K] \subseteq \sigma_{K^v}[\overline{\mathfrak{m}}_K]$ is necessary generic as a representation of $\mathrm{GL}_d(F_v)$, which finishes the proof.

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