FILTRATIONS OF THE PERVERSE SHEAF OF NEARBY CYCLES IN THE SEMI STABLE SITUATION

by

Pascal Boyer

Abstract. — In the strict semi stable reduction situation, we describe the various filtrations of the perverse sheaf of nearby cycles coming from the stratification of the special fiber and determine which extensions given by these filtrations, are split or not. Considering the similarity with the results of [**Boy23**], it could be a good introduction before reading loc. cit.

Contents

1. Introduction

Throughout the paper Λ will denote either $\overline{\mathbb{Q}}_l$, $\overline{\mathbb{Z}}_l$ or $\mathbb{F}_l$ where l is some prime number distinct from another prime number p equal to the residue characteristic κ of some discrete henselian valuation ring R.

2010 ***Mathematics Subject Classification.*** — 11F70, 11F80, 11F85, 11G18, 20C08.
Key words and phrases. — Ihara's lemma, Shimura varieties, torsion in the cohomology, Galois representations.

The computation of nearby cycles $R^i\Psi(\Lambda)$ for a scheme $X \longrightarrow \operatorname{spec} R$ with semi-stable reduction, is well understood, see [**Ill94**] theorem 3.2 for example. It is also well known that the complex $\Psi(\Lambda) := R\Psi(\Lambda)[d-1](\frac{d-1}{2})$ of nearby cycles, where $d = \dim X$, is perverse and autodual relative to the Grothendieck-Verdier duality. The first goal of this work is to describe, when $\Lambda = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, its irreducible constituants in the category of perverse sheaves on Y the geometric special fiber of X.

In [**Boy14**] we explain, for $\Lambda = \overline{\mathbb{Q}}_l, \overline{\mathbb{Z}}_l$ or $\overline{\mathbb{F}}_l$, given a stratification of Y, how to construct various filtrations of $\Psi(\Lambda)$. The second goal of this work is then to describe these filtrations and understand which extensions are split or unsplit. We finally explain how one can define the nilpotent monodromy operator from our computations.

The results are illustrated in figure 1 as explained after using the notations of the next sections

- Each circle represents a graded part isomorphic to some $i_*^{(h)}{}^p j_{!*}^{(h)}\Lambda^{(h)}(\frac{h-1-2(k-1)}{2})$ for $1 \leq h \leq r$ and $1 \leq k \leq h$, cf. notation 3.4.
- The first diagonal line is the first graded part $\operatorname{gr}^1_!(\Psi(\Lambda))$ of the filtration $\operatorname{Fil}^\bullet_!(\Psi(\Lambda))$ of stratification of $\Psi(\Lambda)$, cf. definition 4.7, constructed using the adjunction morphisms $j_!^{(h)} j^{(h)} \to \mathrm{Id}$. It also admits a filtration $\operatorname{Fill}^{-k}(\operatorname{gr}^1_!(\Psi(\Lambda))$ for $1 \leq k \leq r$, where the last graded part $\operatorname{grr}_{-1}(\operatorname{gr}^1_!(\Psi(\Lambda))$, i.e. the top quotient, corresponds to the circle on the left of this line and corresponds to $i_*^{(1)}{}^p j_{!*}^{(1)}\Lambda^{(1)}$, the previous one corresponds to $i_*^{(2)}{}^p j_{!*}^{(2)}\Lambda^{(2)}(\frac{1}{2})$ and so on until the first one $\operatorname{grr}^{-r}(\operatorname{gr}^1_!(\Psi(\Lambda))$, the socle subspace, is isomorphic to $i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-1}{2})$, cf. §5.
- The second diagonal represents $\operatorname{gr}^2_!(\Psi(\Lambda))$ having a filtration $\operatorname{Fill}^{-k}(\operatorname{gr}^2_!(\Psi(\Lambda))$ for $2 \leq k \leq r$, where the last graded parts $\operatorname{grr}^2(\operatorname{gr}^2_!(\Psi(\Lambda))$ appears on the left of this line and corresponds to $i_*^{(2)}{}^p j_{!*}^{(2)}\Lambda^{(2)}(\frac{-1}{2})$, the previous one corresponding to $i_*^{(3)}{}^p j_{!*}^{(3)}\Lambda^{(3)}$ and so on until the first one $\operatorname{grr}^{-r}(\operatorname{gr}^2_!(\Psi(\Lambda)) \cong i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-3}{2})$.
- The graded parts have to be read from the top to the bottom, in particular the socle (resp. top) appears in the top (resp. bottom) of the figure and corresponds to $i_*^{(r)}{}^p j^{(r)}\Lambda^{(r)}(\frac{r-1}{2})$ (resp. $i_*^{(r)}{}^p j^{(r)}\Lambda^{(r)}(-\frac{r-1}{2})$).
- The nilpotent monodromy operator, defined whatever are the coefficients ring, is represented by the arcs of circle starting from bottom to top, cf. §7
- Each segment represents some non trivial extensions in the sense that for a segment with slope 1 (resp. -1) and for $I \subseteq J \subseteq \{1, \cdots, r\}$ with $\sharp J \setminus I = 1$, $i_{J,*}{}^p j_{J,!*}\Lambda_J(\frac{\delta}{2})$ (resp. $i_{I,*}{}^p j_{I,!*}\Lambda_I(\frac{\delta}{2})$) being an irreducible constituant of the circle in the top of this segment and $i_{I,*}{}^p j_{I,!*}\Lambda_I(\frac{\delta-1}{2})$ (resp. $i_{J,*}{}^p j_{J,!*}\Lambda_J(\frac{\delta-1}{2})$) being an irreducible constituant of the circle of the bottom of this segment, then the first have to appears before the second one as a graded parts of any filtration of $\Psi(\Lambda)$: cf. lemmas 5.1 and 6.2 and their dual version.
- The adjunction morphisms gives natural epimorphisms $i_*^{(k)} j_!^{(k)}\Lambda^{(k)}(\frac{1-k}{2}) \twoheadrightarrow \operatorname{gr}^k_!(\Psi(\Lambda))$ where, when $\Lambda = \overline{\mathbb{Z}}_l$, we never need to saturate in the sense of definition 4.9: cf. the discussion after the proof of lemma 6.2.

- Dually we also have monomorphisms $\mathrm{cogr}_{*,k}(\Psi(\Lambda)) \hookrightarrow i_*^{(r-k+1)} j_*^{(r-k+1)} \Lambda^{(r-k+1)}(\frac{r-k}{2})$ induced by adjunction morphisms, where for $\Lambda = \overline{\mathbb{Z}}_l$, the cokernel are torsion free.
- For every $1 \leq h \leq r$ and for every $h \leq k \leq r$, the adjunction morphism gives a epimorphism, cf. 8, without taking saturation when $\Lambda = \overline{\mathbb{Z}}_l$
$$i_*^{(k)} j_!^{(k)} \Lambda^{(k)}(\frac{k-1-2(h-1)}{2}) \twoheadrightarrow \mathrm{Fill}^{-k}(\mathrm{gr}_!^h(\Psi(\Lambda)).$$
- Dually the adjunction morphism gives monomorphisms with torsion free cokernels, cf. 9
$$\mathrm{cogr}_{*,h}(\Psi(\Lambda))/\,\mathrm{Fill}^k(\mathrm{cogr}_{*,h}(\Psi(\Lambda)) \hookrightarrow i_*^{(k+1)} j_*^{(k+1)} \Lambda^{(k+1)}(\frac{1-k+2(h-1)}{2})$$
for $1 \leq h \leq r$ and $r-h \leq k \leq r-1$.

Combining compatibly the symbols i_I^*, $i_{I,*}$, $j_{I,*}$, j_I^*, the duality symbol D and the versions (h), on can compute their ${}^p h^\delta$ either in the Grothendieck group of perverse sheaves on Y but also their filtrations and decide if the various extensions are split or not. One example is given in the proposition 6.4 where we give a new proof of [**Dat12**] theorem 2.2 in the strict semi stable case, cf. proposition 6.4: in loc. cit. the author explains how to reduce from the general semi stable case to the strict one.

In a much more ramified situation given by the special fiber of Kottwitz-Harris-Taylor Shimura varieties at a split prime p, in [**Boy23**] we are also able to obtain the same results which at the end looks very similar to the semi stable reduction case but the arguments are much more complicated because first, contrary to the semi stable case, we do not know a priori the sheaf cohomology groups $R^i\Psi(\Lambda)$ and secondly the local systems which appears are not constant sheaves so that their intermediate extensions for p and $p+$ do not generally coincide.

2. Nearby cycles in the semi-stable case

Let denote by K a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field κ. Recall that a scheme $X \longrightarrow \operatorname{spec} \mathcal{O}_K$ of finite type is said strictly semi-stable purely of relative dimension $d-1$ if it is, Zariski locally on X, etale over $\operatorname{spec} \mathcal{O}_K[T_1, \cdots, T_d]/(T_1 \cdots T_r - \varpi_K)$ for a prime element ϖ_K in $\mathcal{O}_K$ and an integer $1 \leq r \leq d$. Note that in particular

- X is regular and flat over $\operatorname{spec} \mathcal{O}_K$,
- its generic fiber X_η is smooth of relative dimension $d-1$,
- its special fiber $Y := X_s$ is a divisor of X with simple normal crossings.

We now moreover suppose that Y is globally the sum of smooth divisors Y_i for $i = 1, \cdots, r$.

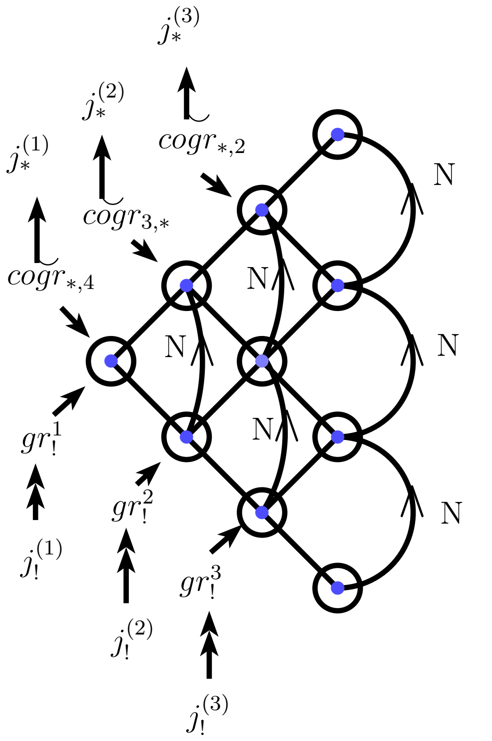

FIGURE 1. Filtrations of $\Psi(\Lambda)$ when $r=4$.

If $\overline{K}$ is a algebraic closure of K, and $X_{\overline{\eta}} := X_\eta \times_{\mathrm{spec}\, K} \mathrm{spec}\, \overline{K}$ with geometric special fiber $X_{\overline{s}}$, we have a diagram of cartesian squares

$$\begin{tikzcd} X_{\overline{\eta}} \arrow[r, hook, "\overline{j}"] \arrow[d] & \overline{X} \arrow[d] & X_{\overline{s}} \arrow[l, hook', "\overline{i}"'] \arrow[d] \\ \overline{\eta} \arrow[r, hook] & \overline{S} & \overline{s} \arrow[l, hook'] \end{tikzcd}$$

where $(\overline{S}, \overline{\eta}, \overline{s})$ is the strict henselian triple attached to (S, η, s) and the choice of $\overline{K}$.

Let $\mathcal{F}$ be a $\overline{\mathbb{Z}}_l$-locally constant free sheaf on X_η where $l \neq p$. The complexe of nearby cycles of $\mathcal{F}$

$$R\Psi(\mathcal{F}) := \overline{i}^* R\overline{j}_* \mathcal{F}_{\overline{\eta}}$$

is an object of the triangulated category of $\overline{\mathbb{Z}}_l$-sheaves on $X_{\overline{s}}$ with a continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$ compatible with its action on $X_{\overline{s}}$: moreover this action factors through the tame quotient $\mathrm{Gal}(\eta_t/\eta)$.

Let $Y_1, \cdots, Y_r$ be the irreducible components of $Y := X_{\overline{s}}$. For a non-empty subset $I \subseteq \{1, \cdots, r\}$, we put $Y_I := \bigcap_{i \in I} Y_i$ and let $i_I : Y_I \hookrightarrow Y$ be the associated immersion.

The scheme Y_I is smooth of dimension $d-k$ if $k = \sharp I$. We then put

$$Y^{(k)} := \bigcup_{\substack{I \subseteq \{1,\cdots,r\} \\ \sharp I = k}} Y_I,$$

and let $i^{(k)} : Y^{(k)} \hookrightarrow Y$ be the associated immersion. For every $I \subsetneq \{1, \cdots, r\}$, we put

$$Y_I^0 := Y_I \setminus \bigcup_{i \notin I} Y_{I \cup \{i\}},$$

and $j_I : Y_I^0 \hookrightarrow Y_I$ the open immersion. We also denote by

$$Y^{(k),0} := \coprod_{\substack{I \subseteq \{1,\cdots,r\} \\ \sharp I = k}} Y_I^0,$$

and $j^{(k)} : Y^{(k),0} \hookrightarrow Y^{(k)}$.

Theorem 2.1. — *(cf.* [**Ill94**] *theorem 3.2 (c))*
Let Λ be the constant sheaf $\overline{\mathbb{Z}}_l$ (or $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$). We then have

- $R^0\Psi(\Lambda) \cong \Lambda$,
- $R^1\Psi(\Lambda) \cong \big(\bigoplus_{i=1}^r \Lambda_{Y_i}\big)/\Lambda\ (-1)$ *where Λ injects diagonally in the sum,*
- $R^q\Psi(\Lambda) \cong \bigwedge^q R^1\Psi(\Lambda)$.

3. Image of $\Psi(\Lambda)$ in the Grothendieck group

We now denote by $\Psi(\Lambda) := R\Psi(\Lambda)[d-1](\frac{d-1}{2})$ which is then a perverse sheaf of weight 0. If D denote the Grothendieck-Verdier duality functor then $D\Psi(\Lambda) \cong \Psi(\Lambda)$. In particular when $\Lambda = \overline{\mathbb{Z}}_l$, then $\Psi(\Lambda)$ is a torsion free perverse sheaf.

The basic tool to determine the perverse irreducible constituants of some perverse sheaf whose you only know its sheaf cohomology groups is given by the following classical statements.

Proposition 3.1. — *Let $D_c^b(X, \mathbb{K})$ be the derived category of $\overline{\mathbb{Q}}_l$-constructible complexes on a $\mathbb{F}_q$-scheme X equipped with the autodual perversity t-structure (resp. the trivial t-structure) with core the category* $\mathrm{Perv}(X)$ *(resp.* $\mathrm{Const}(X)$*) of perverse sheaves (resp. constructible sheaves) on X. For any object $\mathcal{F}$ of $D_c^b(X, \overline{\mathbb{Q}}_l)$, its image in* $\mathrm{Groth}(\mathrm{Perv}(X))$ *is determined by its image in* $\mathrm{Groth}(\mathrm{Const}(X))$.

Proof. — Given a locally small triangulated category A, we consider its Grothendieck group $K(A)$ defined as the free group generated by the isomorphism classes of objects in A quotiented by the relations: $A = B + C$ for every distinguished triangle $B \longrightarrow A \longrightarrow C \xrightarrow{+1}$. The result then follows from the following classical lemma. □

Lemma 3.2. — *Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a derived category equipped with a non-degenerate t-structure: we denote by $\mathcal{C}$ its core, which is then an abelian category of Grothendieck group* Groth$(\mathcal{C})$*. The map that associates to an object $\mathcal{F}$ of $\mathcal{D}$*

$$\sum_i (-1)^i [{}^p h^i \mathcal{F}] \in \mathrm{Groth}(\mathcal{C})$$

induces an isomorphism of the Grothendieck group $K(\mathcal{D})$ of the triangulated category $\mathcal{D}$ onto Groth$(\mathcal{C})$*.*

Proof. — For any object $\mathcal{G}$ of $\mathcal{D}$ and for any $n \in \mathbb{Z}$, we have a distinguished triangle

$$\tau_{\leq n}\mathcal{G} \longrightarrow \mathcal{G} \longrightarrow \tau_{\geq n+1}\mathcal{G} \xrightarrow{+1}$$

such that if $\mathcal{G}$ is an object of $\mathcal{D}^{\geq n}$, we have $[\mathcal{G}] = (-1)^n [{}^p h^n \mathcal{G}] + [\tau_{n+1}\mathcal{G}]$ because $\tau_{\leq n}\mathcal{G} = \tau_{\leq n}\tau_{\geq n}\mathcal{G} = ({}^p h^n \mathcal{G})[n]$. Let a and b be such that $\mathcal{F}$ is an object of $\mathcal{D}^{[a,b]}$. Applying the above to $\tau_{\geq n}\mathcal{F}$ for n varying from a to b, we obtain the equality $[\mathcal{F}] = \sum_i (-1)^i [{}^p h^i \mathcal{F}]$, so that the mapping $\sum_i \alpha_i [P_i] \in \mathrm{Groth}(\mathcal{C}) \mapsto \sum_i \alpha_i [P_i] \in K(\mathcal{D})$ is the inverse of that in the statement. □

Proposition 3.3. — *For $\Lambda = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, and Y a scheme equipped with a stratification*

$$Y^{(r)} \subseteq Y^{(r-1 \leq r)} \subseteq \cdots \subseteq Y^{(1 \leq r)} = Y,$$

let $\mathcal{P}$ be a perverse Λ-sheaf on Y. Then the image of $\mathcal{P}$ in the Grothendieck group of perverse Λ-sheaves on Y is determined by those of $\sum_i (-1)^i [h^i(\mathcal{P})_{|Y^{(h)}}]$ in the Grothendieck group of locally constant sheaves on $Y^{(h)} := Y^{((h \geq h+1)} - X^{(h+1)}$ for all $1 \leq h \leq r$.

Proof. — It suffices to consider the case of an open immersion $j : U \hookrightarrow Y$ of the complement $i : Z \hookrightarrow Y$. From the distinguished triangle

$$j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \xrightarrow{+1},$$

we deduce that the image of an object $\mathcal{F}$ of $D^b_c(Y, \Lambda)$ in Groth(Const(Y)) is determined by that of $j^*\mathcal{F}$ in Groth(Const(U)) and that of $i^*\mathcal{F}$ in Groth(Const(Z)). □

We then want to describe the irreducible constituents of $\Psi(\Lambda)$ and its various filtrations by perverse sheaves.

Notation 3.4. — For every $I \subseteq \{1, \cdots, r\}$, we put

$$\Lambda_I := \Lambda_{Y_I^0}[d - \sharp I](\frac{d - \sharp I}{2}).$$

For $1 \leq h \leq d$, we put

$$\Lambda^{(h)} := \Lambda_{Y^{(h),0}}[d - h](\frac{d - h}{2}).$$

All these shifted local systems are then perverse sheaves of weight 0 on their support.

Lemma 3.5. — *For $\Lambda = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, and for $I \subsetneq \{1, \cdots, r\}$, we have the following equality in the Grothendieck group of perverse sheaves on Y:*

$$i_{I,*} j_{I,!} \Lambda_I = \sum_{k=0}^{r-\sharp I} \sum_{\substack{J \supset I \\ \sharp J \setminus I = k}} i_{J,*} {}^p j_{J,!*} \Lambda_J \Big(\frac{\sharp J - \sharp I}{2}\Big). \tag{1}$$

Remark. note that as the j_J are affine, then $j_{I,!}$ preserve the perverse t-structure. When $\Lambda = \overline{\mathbb{Z}}_l$, it preserves both t-structures p and $p+$ so that $j_{I,!}\Lambda_I$ are free perverse sheaves. Note also that, as Y_I is smooth then ${}^p j_{I,!*}\Lambda_I$ is equal to $\Lambda_{Y_I}[d - \sharp I]$ and, when $\Lambda = \overline{\mathbb{Z}}_l$, also equal to ${}^{p+} j_{I,!*}\Lambda_I$.

Proof. — We start from ${}^p j_{I,!*}\Lambda_I = \Lambda_{Y_I}[d - \sharp I]$ which by proposition 3.3 gives us the equality in the Grothendieck group

$$i_{I,*} {}^p j_{I,!*} \Lambda_I = \sum_{k=0}^{r-\sharp I} \sum_{\substack{J \supset I \\ \sharp J \setminus I = k}} (-1)^k i_{J,*} j_{J,!} \Lambda_J \Big(\frac{\sharp J - \sharp I}{2}\Big).$$

and so

$$i_{I,*} j_{I,!} \Lambda_I = i_{I,*} {}^p j_{I,!*} \Lambda_I + \sum_{k=1}^{r-\sharp I} \sum_{\substack{J \supset I \\ \sharp J \setminus I = k}} (-1)^{k-1} i_{J,*} j_{J,!} \Lambda_J \Big(\frac{\sharp J - \sharp I}{2}\Big). \tag{2}$$

We then argue by induction on $\sharp I$ by assuming the result true for every J such that $\sharp J > \sharp I$: note that the case $\sharp J = r$ is obvious as $Y^0_{\{1,\cdots,r\}} = Y_{\{1,\cdots,r\}}$. We then count the multiplicity of $i_{J,*} {}^p j_{J,!*} \Lambda_J (\frac{\sharp J - \sharp I}{2})$ in the right hand side of (2) with $J = I \coprod \{i_1, \cdots, i_\delta\}$. The contribution for a fixed k in (2), by the induction hypothesis, is $\binom{\delta}{k}(-1)^{k-1}$ and the result follows from the classical equality

$$1 = \sum_{k=1}^{\delta} (-1)^{k-1} \binom{\delta}{k}.$$

□

Considering all $I \subseteq \{1, \cdots, r\}$ with the same order, the previous lemma become

$$i^{(h)}_* {}^p j^{(h)}_{!*} \Lambda^{(h)} = \sum_{k=0}^{r-h} (-1)^k \binom{h+k}{h} i^{(h+k)}_* j^{(h+k)}_! \Lambda^{(h+k)} \Big(\frac{k}{2}\Big) \tag{3}$$

$$i^{(h)}_* j^{(h)}_! \Lambda^{(h)} = \sum_{k=0}^{r-h} \binom{h+k}{h} i^{(h+k)}_* {}^p j^{(h+k)}_{!*} \Lambda^{(h+k)} \Big(\frac{k}{2}\Big). \tag{4}$$

Lemma 3.6. — *Let $s \in \{1, \cdots, r\}$ be fixed and put for every $1 \leq h \subseteq \leq r$*

$$j^{(h)}_{\neq s} : Y^{(h)} \setminus (Y^{(h)} \cap Y_s) \hookrightarrow Y^{(h)} \hookleftarrow Y^{(h)} \cap Y_s : i^{(h)}_s.$$

For $P := \bigoplus_{\substack{\sharp I=h \\ s\not\in I}} {}^p j_{I,!*}\Lambda_I$, we have the following short exact sequence

$$0 \to i_*^{(h+1)}{}^p j_{s,!*}^{(h+1)}\Lambda_s^{(h+1)}(\frac{1}{2}) \longrightarrow i_*^{(h)} j_{\neq s,!}^{(h)} j_{\neq s}^{(h),*} P \longrightarrow P \to 0,$$

where $\Lambda_s^{(h+1)}$ is the sheaf Λ supported on $\bigcup_{\substack{\sharp I=h+1 \\ s\in I}} Y_I^0$. Moreover the extension does not split.

Proof. — By construction of intermediate extensions, note first that ${}^p j_{\neq s,!}^{(h)} j_{\neq s}^{(h),*} P \cong P$ so that ${}^p h^0 i_s^{(h)} P = 0$. As moreover the ${}^p h^\delta i_s^{(h)} P = 0$ for every $\delta < 0$ to compute the last one, i.e. when $\delta = -1$. we can simply first look at $\sum_\delta (-1)^\delta {}^p h^\delta i_s^{(h)} P$ in the Grothendieck group of perverse sheaves on Y. We start with the following equality

$$P = \sum_{k=0}^{r-h} \sum_{\substack{\sharp J=h+k \\ s\not\in J}} (-1)^k \binom{h+k}{h} i_{J,*}{}^p j_{J,!}\Lambda_J(\frac{k}{2}) + \sum_{k=1}^{r-h} \sum_{\substack{\sharp J=h+k \\ s\in J}} (-1)^k \binom{h+k-1}{h-1} i_{J,*}{}^p j_{J,!}\Lambda_J(\frac{k}{2}),$$

so that

$$\begin{aligned} -i_*^{(h)}{}^p h^{-1} i_s^{(h),*} P = \textstyle\sum_\delta (-1)^\delta i_*^{(h)}{}^p h^\delta i_s^{(h),*} P \quad &= \textstyle\sum_{k=1}^{r-h} \sum_{\substack{\sharp J=h+k \\ s\in J}} (-1)^k \binom{h+k-1}{h-1} i_{J,*}{}^p j_{J,!}\Lambda_J(\frac{k}{2}), \\ &= -i_*^{(h+1)}{}^p j_{s,!*}^{(h+1)}\Lambda_s^{(h+1)}(\tfrac{1}{2}). \end{aligned}$$

Put $j_s^{(h+1)} : \bigcup_{\substack{\sharp I=h+1 \\ s\in I}} Y_I^0$ and note that

$$j_s^{(h+1),*}\big({}^p h^{-1} i_s^{(h)}\big) \cong \Lambda_s^{(h+1)}(\frac{1}{2})$$

so that the image of the adjunction morphism

$$j_{s,*}^{(h+1)} j_s^{(h+1),*}\big({}^p h^{-1} i_s^{(h)}\big) \longrightarrow {}^p h^{-1} i_s^{(h)}$$

has ${}^p j_{s,!*}^{(h+1)}\Lambda_s^{(h+1)}(\frac{1}{2})$ as a quotient: the previous equality in the Grothendieck group then tells us that it is an isomorphism. □

Proposition 3.7. — *The image of $\Psi(\Lambda)$ in the Grothendieck group of perverse sheaves on X is equal to*

$$\sum_{h=1}^{r}\sum_{k=0}^{h-1} i_*^{(h)}{}^p j_{!*}^{(h)}\Lambda^{(h)}(\frac{h-1-2k}{2}) = \sum_{i=0}^{r-1}\sum_{k=i+1}^{r} i_*^{(k)}{}^p j_{!*}^{(k)}\Lambda^{(k)}(\frac{k-1-2i}{2}).$$

Proof. — We fix $I \subseteq \{1,\cdots,r\}$ and, for a closed geometric point z of Y_I^0, we look at the stalk $(h^i\Psi)_z$ of $h^i\Psi$ at z. The answer is given by the previous description of the $R^i\Psi(\Lambda)$ recalled above:

- for $i = 0$,we obtain $\Lambda[d-1](\frac{d-1}{2})$,
- for $i = 1$, we have $\Lambda^{\sharp I-1}[d-2](\frac{d-2}{2})(-\frac{1}{2})$,
- and for $i \geq 2$, the isomorphism $R^i\Psi(\Lambda) \cong \bigwedge^i R^1\Psi(\Lambda)$, gives

$$\Lambda^{\binom{\sharp I-1}{i}}[d-1-i](\frac{d-1-i}{2})(-\frac{i}{2}).$$

Now using (3), for a fixed $0 \leq i \leq r-1$, the multiplicity of $i_*^{(h)} j_!^{(h)} \Lambda^{(h)}(\frac{h-1-2i}{2})$ in $\sum_{k=i+1}^r i_*^{(k)} {}^p j_{!*}^{(k)} \Lambda^{(k)}(\frac{k-1-2i}{2})$ is equal to

$$\sum_{k=i+1}^{r} (-1)^{h-k} \binom{h}{k} = (-1)^{h-k-1} \binom{h-1}{i}.$$

Indeed starting with $\binom{h}{1} - \binom{h}{0} = \binom{h-1}{0}$, by an easy inductive argument we have

$$\binom{h}{k} - \binom{h}{k-1} + \cdots + (-1)^k \binom{h}{0} = \binom{h}{k} - \binom{h-1}{k-2} = \binom{h-1}{k-1}.$$

We then deduce that for every $1 \leq h \leq r$, the restriction to $Y^{(h),0}$ of $\sum_i (-1)^i h^i P$ for $P = \Psi(\Lambda)$ coincides with those for P equals to the formula given by the proposition. The result then follows from the proposition 3.3. $\square$

4. Filtrations of stratification from [Boy14]: generalities

Let S be the spectrum of a finite field and Y a finite-type scheme over S, then the usual t-structure on $\mathcal{D}(Y, \Lambda) := D_c^b(Y, \Lambda)$ is

$$A \in {}^p\mathcal{D}^{\leq 0}(Y, \Lambda) \Leftrightarrow \forall x \in Y,\ h^k i_x^* A = 0,\ \forall k > -\dim \overline{\{x\}}$$
$$A \in {}^p\mathcal{D}^{\geq 0}(Y, \Lambda) \Leftrightarrow \forall x \in Y,\ h^k i_x^! A = 0,\ \forall k < -\dim \overline{\{x\}}$$

where $i_x : \operatorname{spec} \kappa(x) \hookrightarrow Y$ and $h^k(K)$ is the k-th sheaf of cohomology of K.

Notation 4.1. — We denote by ${}^p\mathcal{C}(Y, \Lambda)$ the heart pf these t-structure: the cohomological functors are then denoted by ${}^p h^i$. For a functor T, we put ${}^p T := {}^p h i^0 \circ T$.

Remark. ${}^p\mathcal{C}(Y, \Lambda)$ is a Noetherian abelian category and Λ-linear. For Λ a field, this t-structure is autodual for Verdier duality. For $\Lambda = \overline{\mathbb{Z}}_l$, we can equip the $\overline{\mathbb{Z}}_l$-linear abelian category ${}^p\mathcal{C}(Y, \overline{\mathbb{Z}}_l)$ with a torsion theory $(\mathcal{T}, \mathcal{F})$ where $\mathcal{T}$ (resp. $\mathcal{F}$) is the full subcategory of objects of l^∞-torsion T (resp. l-free F) , i.e. such that $l^N 1_T$ is zero for N sufficiently large (resp. $l.1_F$ is a monomorphism).

Definition 4.2. — As in [**Jut09**], we define the dual t-structure

$${}^{p+}\mathcal{D}^{\leq 0}(Y, \overline{\mathbb{Z}}_l) := \{A \in {}^p\mathcal{D}^{\leq 1}(Y, \overline{\mathbb{Z}}_l) :\ {}^p h^1(A) \in \mathcal{T}\}$$
$${}^{p+}\mathcal{D}^{\geq 0}(Y, \overline{\mathbb{Z}}_l) := \{A \in {}^p\mathcal{D}^{\geq 0}(Y, \overline{\mathbb{Z}}_l) :\ {}^p h^0(A) \in \mathcal{F}\}$$

with heart ${}^{p+}\mathcal{C}(Y, \overline{\mathbb{Z}}_l)$ equipped with its torsion theory $(\mathcal{F}, \mathcal{T}[-1])$ dual of those of ${}^p\mathcal{C}(Y, \overline{\mathbb{Z}}_l)$.

Notation 4.3. — (cf. [**Boy14**] §1.3) Let

$$\mathcal{F}(Y, \Lambda) := {}^p\mathcal{C}(Y, \Lambda) \cap {}^{p+}\mathcal{C}(Y, \Lambda) = {}^p\mathcal{D}^{\leq 0}(Y, \Lambda) \cap {}^{p+}\mathcal{D}^{\geq 0}(Y, \Lambda)$$

be the quasi-abelian category of free perverse sheaves on X with coefficients in Λ.

Let $j : U \hookrightarrow Y$ an affine open immersion and $i : F := Y \setminus U \hookrightarrow Y$. Then $j_!$ is t-exact and, cf. [**Boy14**] proposition 1.3.14, $j_! = {}^pj_! = {}^{p+}j_!$ with $j_!(\mathcal{F}(U,\Lambda)) \subseteq \mathcal{F}(Y,\Lambda)$. Starting from the distinguished triangle

$$j_!j^*L \longrightarrow L \longrightarrow i_*i^*L \rightsquigarrow,$$

and using the perversity of L and $j_!j^*L$, the long exact sequence associated to it gives

$$0 \to i_*{}^ph^{-1}i^*L \longrightarrow {}^pj_!j^*L \longrightarrow L \longrightarrow i_*{}^ph^0i^*L \to 0,$$

where $i_*{}^ph^{-1}i^*L$ is free as ${}^pj_!j^*L = j_!j^*L$ is free.

Definition 4.4. — A monomorphism $f : L \hookrightarrow L'$ in ${}^p\mathcal{C}(Y,\Lambda)$ between free perverse sheaves, is said strict if, and only if, its cokernel in ${}^p\mathcal{C}(Y,\Lambda)$ is free. This is equivalent to ask f to be a monomorphism of ${}^{p+}\mathcal{C}(Y,\Lambda)$.

Definition 4.5. — A *bimorphism* of $\mathcal{F}(Y,\Lambda)$ is a monomorphism which is also a epimorphism, and we will denote by $L \hookrightarrow\!\!\!\!\!\twoheadrightarrow L'$ such a bimorphism between L and L'. If moreover the kernel in ${}^{p+}\mathcal{C}(Y,\Lambda)$ has its support of dimension strictly less that the dimension of the support of L, we will denote by $L \hookrightarrow\!\!\!\!\!\twoheadrightarrow_+ L'$.

For $L \in \mathcal{F}(Y,\Lambda)$, we consider the following diagram

$$\begin{tikzcd}
 & L \arrow[rrd, "\mathrm{can}_{*,L}"] & & \\
{}^{p+}j_!j^*L \arrow[ru, "\mathrm{can}_{!,L}"] \arrow[r, two heads, "+"] & {}^pj_{!*}j^*L \arrow[r, hook, two heads, "+"'] & {}^{p+}j_{!*}j^*L \arrow[r, hook, "+"] & {}^pj_*j^*L
\end{tikzcd}$$

where the bottom line is, cf. the remark following 1.3.12 in [**Boy14**], is the canonical factorization of ${}^{p+}j_!j^*L \longrightarrow {}^pj_*j^*L$ and where the maps $\mathrm{can}_{!,L}$ and $\mathrm{can}_{*,L}$ are given by adjunction.

Notation 4.6. — We put $\mathcal{P}_L := i_*{}^ph^{-1}_{libre}i^*j_*j^*L = \ker_{\mathcal{F}}\Big({}^{p+}j_!j^*L \twoheadrightarrow {}^pj_{!*}j^*L\Big)$. With the notations of the above diagram, we put

$$\mathrm{Fil}^0_{U,!}(L) = \mathrm{Im}_{\mathcal{F}}(\mathrm{can}_{!,L}) \quad \text{et} \quad \mathrm{Fil}^{-1}_{U,!}(L) = \mathrm{Im}_{\mathcal{F}}\Big((\mathrm{can}_{!,L})_{|\mathcal{P}_L}\Big).$$

We now suppose that Y is equipped with a stratification

$$Y^{(r)} \subseteq Y^{(r-1)} \subseteq \cdots \subseteq Y^{(1)}.$$

For $1 \leq h < r$, we put $Y^{(1\leq h)} := Y^{(1)} - Y^{(h+1)}$ and $j^{(1\leq h)} : Y^{(1\leq h)} \hookrightarrow Y^{(1)}$.

Definition 4.7. — For $P \in \mathcal{F}(Y,\Lambda)$ we define

$$\mathrm{Fil}^h_!(L) := \mathrm{Im}_{\mathcal{F}}\Big({}^pj_!^{(1\leq h)}j^{(1\leq h),*}P \longrightarrow P\Big).$$

Proposition 4.8. — *(cf. [**Boy14**] §2.2) The previous definition functionally equips every $L \in \mathcal{F}(Y,\Lambda)$ with a filtration of stratification.*

$$0 = \mathrm{Fil}^0_!(L) \subseteq \mathrm{Fil}^1_!(L) \subseteq \mathrm{Fil}^2_!(L) \cdots \subseteq \mathrm{Fil}^{r-1}_!(L) \subseteq \mathrm{Fil}^r_!(L) = L.$$

Definition 4.9. — For $L, L' \in \mathcal{F}(Y, \overline{\mathbb{Z}}_l)$ and $f : L \longrightarrow L'$ a morphism the usual cokernel $L/\operatorname{Im}_{\mathcal{C}} f$ where $\operatorname{Im}_{\mathcal{C}} f$ is the image of f in the category ${}^p\mathcal{C}(Y, \Lambda)$, might have non trivial torsion. However if we consider $L/\operatorname{Im}_{\mathcal{C}^+} f$ where $\operatorname{Im}_{\mathcal{C}^+} f$ is the image of f in ${}^{p+}\mathcal{C}(Y, \Lambda)$, then it is necessary free. We call this modification *the saturation process*.

Remark. When Y is a point and $f : L := \mathbb{Z}_l \longrightarrow L = \mathbb{Z}_l$ is the multiplication by l, then $L/\operatorname{Im}_{\mathcal{C}} f \cong \mathbb{Z}_l/l\mathbb{Z}_l$ while $L/\operatorname{Im}_{\mathcal{C}^+} f \cong \mathbb{Z}_l/\mathbb{Z}_l$. In other words $\operatorname{Im}_{\mathcal{C}^+} f = \mathbb{Z}_l$ is the usual saturation of $l\mathbb{Z}_l \subseteq \mathbb{Z}_l$.

The above filtrations of stratification are generally not fine enough. In [**Boy14**] proposition 2.3.3, using $\operatorname{Fil}^{-1}_{U,!}$, we construct in a functorial manner the exhaustive filtration of stratification of any object L of $\mathcal{F}(Y, \Lambda)$,

$$(5) \quad 0 = \operatorname{Fill}_!^{-2^{r-1}}(L) \subseteq \operatorname{Fill}_!^{-2^{r-1}+1}(L) \subseteq \cdots \subseteq \operatorname{Fill}_!^{0}(L) \subseteq \cdots \subseteq \operatorname{Fill}_!^{2^{r-1}-1}(L) = L,$$

such that all graduates grr^k are free and are, on $\overline{\mathbb{Q}}_l$, simple, i.e. verify ${}^p j^{(h)}_{!*} j^{(h),*} \operatorname{grr}^k \hookrightarrow\!\!\!\!\rightarrow_+ \operatorname{grr}^k$, where $Y^{(h)}$ is the support of grr^k.

Dually using $\operatorname{can}_{*,L}$ with

$$\operatorname{coFil}_{*,h}(L) := \operatorname{coIm}_{\mathcal{F}}\big(L \longrightarrow j^{(h)}_* j^{(h),*} L\big),$$

we also define a cofiltration

$$L = \operatorname{coFil}_{*,r}(L) \twoheadrightarrow \operatorname{coFil}_{*,r-1}(L) \twoheadrightarrow \cdots \twoheadrightarrow \operatorname{coFil}_{*,1}(L) \twoheadrightarrow \operatorname{coFil}_{*,0}(L) = 0,$$

and a filtration

$$0 = \operatorname{Fil}^{-r}_*(L) \hookrightarrow \operatorname{Fil}^{-r+1}_*(L) \hookrightarrow \cdots \hookrightarrow \operatorname{Fil}^{-1}_*(L) \hookrightarrow \operatorname{Fil}^{0}_*(L) = L$$

where $\operatorname{Fil}^{-h}_*(L) := \ker_{\mathcal{F}}\big(L \twoheadrightarrow \operatorname{coFil}_{*,h}(L)\big)$.

5. Filtrations of $j_{I,!}\Lambda_I$

For $\Lambda = \overline{\mathbb{Q}}_l$, the weight filtration of $j_{I,!}\Lambda_I$, thanks to the equality (1), can be written, up to a shift of $\sharp I$:

$$(0) = \operatorname{Fil}^W_{-r-1}(I,!) \subseteq \operatorname{Fil}^W_{-r}(I,!) \subseteq \cdots \subseteq \operatorname{Fil}^W_{\sharp I}(I,!) = j_{I,!}\Lambda_I$$

with graded parts $\operatorname{gr}^W_{-k}(I,!) = \sum_{\substack{J \supset I \\ \sharp J = k}} i_{J,*}{}^p j_{J,!*}\Lambda_J(\frac{\sharp J - \sharp I}{2})$. For $\Lambda = \overline{\mathbb{Z}}_l$, we also have a similar filtration. Moreover as the intermediate extensions for the two t-structures p and $p+$ are the same for the local systems $\Lambda_{Y_I^0}$ so that for $\Lambda = \overline{\mathbb{F}}_l$ we also have a filtration as above with graduate parts the $\sum_{\substack{J \subseteq I \\ \sharp J \setminus I = k}} i_{J,*}{}^p j_{J,!*}\Lambda_J(\frac{\sharp J - \sharp I}{2})$.

Lemma 5.1. — *For $\Lambda = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$ and for every $\sharp I < k < r-1$, the extension*

$$0 \to \operatorname{gr}^W_{-k}(I,!) \longrightarrow \operatorname{Fil}^W_{-k+1}(I,!)/\operatorname{Fil}^W_{-k-1}(I,!) \longrightarrow \operatorname{gr}^W_{-k+1}(I,!) \to 0,$$

does not split.

Proof. — Note first that this is obvious for $k = \sharp I + 1$ because $i_{I,*}{}^p j_{I,!*}\Lambda_{Y_I^0}[d-\sharp I]$ is the top of $i_{I,*}j_{I,!}\Lambda_{Y_I^0}[d-\sharp I]$. We now suppose $k > \sharp I + 1$ and we look at the spectral sequence

$$E_1^{p,q} = h^{p+q}\,\mathrm{gr}^W_{-p}(I,!) \Rightarrow h^{p+q} j_{I,!}\Lambda_{Y_I^0}[d-\sharp I],$$

computing the sheaf cohomology groups of the extension by zero of $\Lambda_{Y_I^0}[d-\sharp I]$. Consider any $J \supset I$ with $\sharp J = -k$ and a generic point z of Y_J^0. Concerning the stalks at z:

- $(E_\infty^n)_z$ is zero,
- $(h^{p+q}\,\mathrm{gr}^W_{-k}(I,!))_z$ is non zero if and only $p+q = d-\sharp J$,
- $(h^{p+q}\,\mathrm{gr}^W_{-k+1}(I,!))_z$ is non zero if and only $p+q = d-\sharp J - 1$,
- for every $\delta \notin \{-k,-k+1\}$, $(h^{p+q}\,\mathrm{gr}^W_\delta(I,!))_z = 0$ for $p+q \in \{d-\sharp J-1, d-\sharp J, d-\sharp J+1\}$.

Thus for any filtration of $j_{I,!}\Lambda_{Y_I^0}[d-\sharp I]$, the index of the graded part containing $\mathrm{gr}^W_{-k}(I,!))_z$ should be less than those containing $\mathrm{gr}^W_{-k+1}(I,!))_z$, which proves the statement. □

Remark. otherwise stated, up to obvious modifications such as refinement or forgetting some terms, this weigh filtration of $j_{I,!}\Lambda_I$ is unique. As a consequence we deduce that the adjunction morphism

$$(6) \qquad \bigoplus_{\substack{J\supset I\\ \sharp J = k}} j_{J,!}\Lambda_J\Big(\frac{\sharp J - \sharp I}{2}\Big) \cong j_!^{(k)} j^{(k),*}\,\mathrm{Fil}^W_{-k}(I,!) \longrightarrow \mathrm{Fil}^W_{-k}(I,!),$$

is surjective for $\Lambda = \overline{\mathbb{Q}}_l$ and $\overline{\mathbb{F}}_l$. In particular it is also surjective for $\Lambda = \overline{\mathbb{Z}}_l$ without saturating the image. Dually the weight filtration of $i_{I,*}j_{I,*}\Lambda_I$ up to a shift of $\sharp I$ is

$$(0) = \mathrm{Fil}^W_{r+1}(I,*) \subseteq \mathrm{Fil}^W_r(I,*) \subseteq \cdots \subseteq \mathrm{Fil}^W_{\sharp I}(I,*) = j_{I,*}\Lambda_I,$$

with graded parts $\mathrm{gr}^W_k(I,*) = \sum_{\substack{J\supset I\\ \sharp J\setminus I = r-k}} i_{J,*}{}^p j_{J,!*}\Lambda_J(\frac{\sharp I - \sharp J}{2})$. Moreover the adjunction morphism
(7)

$$j_{I,*}\Lambda_I/\,\mathrm{Fill}^W_{k+1}(I,*) \hookrightarrow j_!^{(r-k+\sharp I)} j^{(r-k+\sharp I),*}\Big(j_{I,*}\Lambda_I/\,\mathrm{Fill}^W_{k+1}(I,*)\Big) \cong \bigoplus_{\substack{J\supset I\\ \sharp J\setminus I = r-k}} j_{J,*}\Lambda_J\Big(\frac{\sharp I - \sharp J}{2}\Big),$$

is injective with, for $\Lambda = \overline{\mathbb{Z}}_l$, a torsion free cokernel.

Notation 5.2. — Let $P_I = i_{I,*}{}^p h^{-1} i_I^* j_{I,*}\Lambda_I)$ be the kernel of the canonical morphism

$$P_I = \ker\Big(j_{I,!}\Lambda_I \longrightarrow {}^p j_{I,!*}\Lambda_I\Big).$$

We denote by $Y_{I^+} := \bigcup_{J\supsetneq I} Y_J$ its support and as before $j_{I^+} : Y_{I^+}^0 \hookrightarrow Y_{I^+}$.

From the previous lemma, we deduce that, for $\Lambda \in \{\overline{\mathbb{Q}}_l, \overline{\mathbb{Z}}_l, \overline{\mathbb{F}}_l\}$, the adjunction morphism

$$j_{I^+,!}j_{I^+}^* P_I \twoheadrightarrow P_I,$$

is surjective. Indeed the cokernel has support in $Y_I \setminus Y_I^0$ and the previous lemma tells us that, over $\overline{\mathbb{Q}}_l$ and $\overline{\mathbb{F}}_l$, the top of P_I is ${}^p j_{I,!*} j_I^* P_I$ so that the cokernel is zero. In particular, for $\Lambda = \overline{\mathbb{Z}}_l$, there is no need to use a saturation process as in definition 4.9.

Lemma 5.3. — *For $J_0 \supset I$ with $\sharp J_0 \setminus I = 1$, we have a short exact sequence*

$$0 \to j_{I^+\setminus J_0,!}j_{I^+\setminus J_0}^* P_I \longrightarrow P_I \longrightarrow {}^p j_{J_0,!*}\Lambda_J(\frac{1}{2}) \to 0,$$

and a epimorphism

$$j_{I^+,!}j_{I^+}^* P_I \twoheadrightarrow P_I$$

without taking saturation.

Proof. — By construction P_I (resp. $j_{I^+\setminus J_0,!}j_{I^+\setminus J_0}^* P_I$) is the constant sheaf $\Lambda[d-\sharp I-1](\frac{d-\sharp I}{2})$ on Y_{I^+} (resp. on $Y_{I^+}\setminus Y_{J_0}$) while ${}^p j_{J_0,!*}\Lambda_J$ is the constant sheaf $\Lambda[d-\sharp J_0](\frac{d-\sharp J_0}{2})$ on Y_{J_0} with $\sharp J_0 = \sharp I + 1$.

The second statement is proved in the same way. □

Dually we obtain we obtain a cofiltration of $L := j_{I,*}\Lambda_I$:

$$i_{I,*}j_{I,*}\Lambda_I = \mathrm{coFil}_{*,r-\sharp I+1}(L) \twoheadrightarrow \mathrm{coFil}_{*,r-\sharp I}(L) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}_{*,1}(L) \twoheadrightarrow \mathrm{coFil}_{*,0}(L) = 0$$

with graded parts $\mathrm{cogr}_{*,k}(L) := \ker(\mathrm{coFil}_{*,k}(L) \twoheadrightarrow \mathrm{coFil}_{*,k-1}(L))$ having image in the Grothendieck group of perverse sheaves on X equal to

$$\sum_{J\supset I} i_{J,*}{}^p j_{J,!*}\Lambda_J(\frac{\sharp I - \sharp J}{2}),$$

the extension between two consecutive graded parts being moreover unsplit.

Proposition 5.4. — *For every $I \subseteq J$ we have*

$$i_{J,*}{}^p h^0 i_J^*\big(j_{I,*}\Lambda_I\big) \cong i_{I,*}j_{I,*}\Lambda_J(-\frac{1}{2}).$$

Proof. — By an obvious induction, we only need to deal with the case $\sharp J \setminus I = 1$. Put $j_{\overline{I}\setminus J} : Y_I \setminus Y_J \hookrightarrow Y_I$ and recall that $i_{J,*}{}^p h^0 i_J^*\big(j_{I,*}\Lambda_I\big)$ is the cokernel of the adjunction morphism

$$j_{\overline{I}\setminus J,!}j_{\overline{I}\setminus J}^* j_{I,*}\Lambda_I \longrightarrow j_{I,*}\Lambda_I,$$

so that, considering the constituant of $j_{I,*}\Lambda_I$ having support in Y_J, its image in the Grothendieck group of perverse sheaves on Y_I is less or equal to

$$\sum_{H\supset J} {}^p j_{H,!*}\Lambda_H(\frac{\sharp I - \sharp H}{2}).$$

This is in fact an equality as, (1) tells us that any of the ${}^p j_{H,!*}\Lambda_H(\frac{\sharp I-\sharp H}{2})$ is not a subquotient of $j_{\bar{I}\setminus J,!}j^*_{\bar{I}\setminus J}j_{I,*}\Lambda_I$.

In particular the cofiltration of stratification of $L := i_{J,*}{}^p h^0 i_J^*\big(j_{I,*}\Lambda_I\big)$

$$\mathrm{coFil}_{*,r-\sharp J+1}(L) \twoheadrightarrow \mathrm{coFil}_{*,r-\sharp J}(L) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}_{*,1}(L) \twoheadrightarrow \mathrm{coFil}_{*,0}(L) = 0$$

is such that its graded parts $\mathrm{cogr}_{*,k}(L)$ have image in the Grothendieck group of perverse sheaves on X equal to

$$\sum_{H\supset J} {}^p j_{H,!*}\Lambda_H(\frac{\sharp I-\sharp H}{2}),$$

where moreover the extension between two consecutive graded parts are unsplit. The same being true for the cofiltration of stratification of $j_{J,*}\Lambda_J(-\frac{1}{2})$ we then deduce that the adjunction morphism

$$L \longrightarrow j_{J,*}j_J^*L$$

is a isomorphism. $\square$

6. Filtrations of $\Psi(\Lambda)$

Let

$$\mathrm{Fil}^p_!(\Psi(\Lambda)) = \mathrm{Im}_{\mathcal{F}}\Big(j_!^{(1\leq p)}j^{(1\leq p),*}\Psi(\Lambda) \longrightarrow \Psi(\Lambda)\Big),$$

be the filtration of stratification of $\Psi(\Lambda)$

$$(0) = \mathrm{Fil}^0_!(\Psi) \subseteq \mathrm{Fil}^1_!(\Psi) \subseteq \cdots \subseteq \mathrm{Fil}^r_!(\Psi) = \Psi(\Lambda).$$

Proposition 6.1. — *For $1 \leq k \leq r$, the image of $\mathrm{gr}^k_!(\Psi(\Lambda)) \otimes_\Lambda \overline{\mathbb{Q}}_l$ in the Grothendieck group of perverse $\overline{\mathbb{Q}}_l$-sheaves on X is equal to that of $\sum_{i=k}^r i_*^{(i)}{}^p j_{!*}^{(i)}\Lambda^{(i)}(\frac{i-1-2(k-1)}{2})$. Moreover $\mathrm{gr}^k_!(\Psi(\Lambda))$ has a filtration*

$$(0) = \mathrm{Fill}^{-r-1}(\mathrm{gr}^k_!(\Psi(\Lambda))) \subseteq \mathrm{Fill}^{-r}(\mathrm{gr}^k_!(\Psi(\Lambda))) \subseteq \cdots \mathrm{Fill}^k(\mathrm{gr}^k_!(\Psi(\Lambda))) = \mathrm{gr}^k_!(\Psi(\Lambda))$$

with $\mathrm{grr}^h(\mathrm{gr}^k_!(\Psi(\Lambda))) \cong i_*^{(h)}{}^p j_{!*}^{(h)}\Lambda^{(h)}(\frac{h-1-2(k-1)}{2})$.

Remark. Note first that, considering the weight in the equality of proposition 3.7, the image of $\mathrm{gr}^k_!(\Psi(\Lambda))\otimes_\Lambda\overline{\mathbb{Q}}_l$ is necessary less than $\sum_{i=k}^r i_*^{(i)}{}^p j_{!*}^{(i)}\Lambda^{(k)}(\frac{i-1-2(k-1)}{2})$. Moreover as explained in the previous section, lemma 5.1 implies that the filtration $\mathrm{Fill}_\bullet$ of each $\mathrm{gr}^k_!(\Psi(\Lambda))$ is essentially unique.

Proof. — We argue as in the proof of lemma 5.1 by proving the following observation. Consider I with $\sharp I = i+1$ then there does not exists a filtration of $\Psi(\Lambda)$ such that there exists two graduate parts respectively isomorphic to ${}^p j_{I,!*}\Lambda^{(k)}(\frac{i-2(k-1)}{2})$ and ${}^p j_{!*}^{(i)}\Lambda^{(i)}(\frac{i-1-2(k-1)}{2})$ with respective index $r_1 < r_2$.

Indeed otherwise for z a generic point of Y_I^0 the stalks at z with weight $2(k-1)-i+1$ appears

– in degree $\sharp I-d$ for ${}^p j_{I,!*}\Lambda^{(k)}(\frac{i-2(k-1)}{2})$,
– in degree $\sharp I-d-1$ for ${}^p j_{!*}^{(i)}\Lambda^{(i)}(\frac{i-1-2(k-1)}{2})$
– in degree $<\sharp I-d-1$ otherwise.

So if $r_1<r_2$, we then see that $h^{-d+\sharp I}\Psi(\Lambda)$ would have a non zero stalks at z with weight $2(k-1)-i+1$ which is not the case. We then deduce that every irreducible constituant of $\sum_{i=k}^r i_*^{(i)}{}^p j_{!*}^{(i)}\Lambda^{(i)}(\frac{i-1-2(k-1)}{2})$ has to be a constituant of $\mathrm{Fil}_!^k(\Psi)$ which gives the result. □

As the p and $p+$ intermediate extensions of our local systems are isomorphic, we then obtain for $\Lambda=\overline{\mathbb{F}}_l$, a filtration like in the previous proposition with graded parts the ${}^p j_{!*}^{(h)}\Lambda^{(h)}(\frac{h-1-2(k-1)}{2})$. Consider then two consecutive graded parts in the filtration $\mathrm{Fill}^\bullet(\mathrm{gr}_!^k(\overline{\mathbb{F}}_l))$, say for example $P_I:={}^p j_{I,!*}\Lambda_I(\frac{\sharp I-1-2(k-1)}{2})$ and $P_-:={}^p j_{J,!*}\Lambda_J(\frac{\sharp J-1-2(k-1)}{2})$ with $\sharp I=\sharp J+1$. We choose $s\in I\setminus J$ and consider the extension X inside $\mathrm{gr}_!^k(\overline{\mathbb{F}}_l)$:

$$0\to P_I\longrightarrow X\longrightarrow P_-\to 0.$$

Lemma 6.2. — *The above extension X does not split.*

Proof. — We first start by proving the following statement which the analog of the lemma 5.3 for $\Psi(\Lambda)$.

Lemma 6.3. — *For $h\in\{1,\cdots,r\}$, we put $j_{\neq h}:Y\setminus Y_h\hookrightarrow Y$. Then the cokernel $L:=i_{h,*}{}^p h^0 i_h^*\Psi(\Lambda)$ of the adjunction morphism*

$$j_{\neq h,!}j_{\neq h}^*\Psi(\Lambda)\longrightarrow\Psi(\Lambda)$$

is isomorphic to $j_{h,*}\Lambda^{(h)}$.

Proof. — Recall the short exact sequence

$$0\to\Psi(\Lambda)\longrightarrow\overline{j}_!\Lambda[d-1](\frac{d-1}{2})\longrightarrow\overline{j}_*\Lambda[d-1](\frac{d-1}{2})\to 0,$$

with $\Psi(\Lambda)={}^p h^{-1}\overline{i}^*\overline{j}_*\Lambda[d-1](\frac{d-1}{2})$. Note also that for every $\delta\neq -1$, we have ${}^p h^\delta\overline{i}^*\overline{j}_*\Lambda[d-1]=(0)$. As $\overline{j}_{\neq h}:X_{\bar\eta}\setminus Y_h\hookrightarrow X_{\bar\eta}$ is affine, for $\overline{i}_h:Y_h\hookrightarrow X_{\bar\eta}$, the long exact sequence associated to the distinguished triangle

$$\overline{j}_{\neq h,!}\overline{j}_{\neq h}^*\Lambda[d-1]\longrightarrow\Lambda[d-1]\longrightarrow\overline{i}_{h,*}\overline{i}_h^*\Lambda[d-1]\rightsquigarrow^{+1},$$

as the first two terms are perverse, then ${}^p h^\delta\overline{i}_h^*\Lambda[d-1]=(0)$ for $\delta\neq -1$. Writting $\overline{i}_h=\overline{i}\circ i_h$ and considering the spectral sequence

$$E_2^{r,s}={}^p h^r i_h^*{}^p h^s\overline{i}^*\Lambda[d-1]\Rightarrow{}^p h^{r+s}\overline{i}_h^*\Lambda[d-1],$$

we then deduce that $i_h^*\Psi(\Lambda)$ is perverse and that

$$0 \to j_{\neq h,!}j_{\neq h}^*\Psi(\Lambda) \longrightarrow \Psi(\Lambda) \longrightarrow i_{h,*}{}^ph^0 i_h^*\Psi(\Lambda) \to 0.$$

We then observe that the stalks at geometric points of $\Psi(\Lambda)$ and that of

$$\bigoplus_{\substack{J\supset\{h\}\\ \sharp J=r-k+1}} {}^pj_{J,!*}\Lambda_J(-\frac{r-k}{2}),$$

are the same, so that the image of $i_{h,*}{}^ph^0i_h^*\Psi(\Lambda)$ in the Grothendieck group of perverse sheaves on Y is equal to that of this direct sum. Moreover $\Psi(\Lambda)$ has a quotient Q with the following cofiltration of stratification

$$Q = \mathrm{coFil}_{*,r}(Q) \twoheadrightarrow \mathrm{coFil}_{*,r-1}(Q) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}_{*,1}(Q) \twoheadrightarrow \mathrm{coFil}_{*,0}(Q) = 0,$$

with graded parts $\mathrm{cogr}_{k,*}(Q)$ having image in the Grothendieck group of perverse sheaves on X equal to

$$\sum_{\substack{J\supset\{h\}\\ \sharp J=r-k+1}} i_{J,*}{}^pj_{J,!*}\Lambda_J(-\frac{r-k}{2}),$$

where the extensions are unsplit. We then deduce that Q is isomorphic with L and that the adjunction morphism

$$L \longrightarrow j_{h,*}j_h^*L$$

is an isomorphism. □

We now go back to the extension X and we consider the short exact sequence

$$0 \to j_{\neq s,!}j_{\neq s}^*\Psi(\Lambda) \longrightarrow \Psi(\Lambda) \longrightarrow i_{h,*}{}^ph^0 i_s^*\Psi(\Lambda) \to 0,$$

so that our extension X lives inside the first term $j_{\neq s,!}j_{\neq s}^*\Psi(\Lambda)$ By t-exactness of $j_{\neq s,!}j_{\neq s}^*$ and using lemma 3.6 we see that $j_{\neq s,!}j_{\neq s}^*\Psi(\Lambda)$ can be written, up to a twist of weights, as successive unsplit extensions of the form

$$0 \to {}^pj_{s,!*}^{(h+1)}\Lambda_s^{(h+1)}(\frac{1}{2}) \longrightarrow j_{\neq s,!}^{(h)}j_{\neq s}^{(h),*}\Big(\bigoplus_{\substack{\sharp I=h\\ s\notin I}} {}^pj_{I,!*}\Lambda_I\Big) \longrightarrow \bigoplus_{\substack{\sharp I=h\\ s\notin I}} {}^pj_{I,!*}\Lambda_I \to 0,$$

in which X appears. □

In other words the adjunction morphism giving rise to

$$j_!^{(k)}\Lambda^{(k)}(\frac{1-k}{2}) \twoheadrightarrow \mathrm{gr}_!^k(\Psi(\Lambda))$$

is surjective when $\Lambda = \overline{\mathbb{Z}}_l$, i.e. without using the saturation process of definition 4.9. As a consequence, using (6), the adjunction morphisms give epimorphism

$$j_!^{(k)}\Lambda^{(k)}(\frac{k-1-2(h-1)}{2}) \twoheadrightarrow \mathrm{Fill}^{-k}(\mathrm{gr}_!^h(\Psi(\Lambda)), \tag{8}$$

without using saturation when $\Lambda = \overline{\mathbb{Z}}_l$.

Dually the cofiltration of stratification

$$\Psi(\Lambda) = \mathrm{coFil}_{*,r}(\Psi(\Lambda)) \twoheadrightarrow \mathrm{coFil}_{*,r-1}(\Psi(\Lambda)) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{coFil}_{*,0}(\Psi(\Lambda)) = 0,$$

with graded parts $\mathrm{cogr}_{*,k}(\Psi(\Lambda))$ having a filtration

$$0 = \mathrm{Fill}^{r-k}(\mathrm{cogr}_{*,k}(\Psi(\Lambda))) \hookrightarrow \mathrm{Fill}^{r-k+1}(\mathrm{cogr}_{*,k}(\Psi(\Lambda))) \hookrightarrow \cdots \\ \hookrightarrow \mathrm{Fill}^{r}(\mathrm{cogr}_{*,k}(\Psi(\Lambda))) = \mathrm{cogr}_{*,k}(\Psi(\Lambda))$$

with graded parts $\mathrm{grr}^h(\mathrm{cogr}_{*,k}(\Psi(\Lambda))) \cong {}^p j_{!*}^{(h)} \Lambda^{(h)}(\frac{1-h+2(r-k))}{2})$. Moreover the adjunction morphisms give monomorphism with torsion free cokernels

$$\mathrm{cogr}_{k,*}(\Psi(\Lambda)) \hookrightarrow j_!^{(r-k+1)} \Lambda^{(r-k+1)}(\frac{r-k}{2}).$$

As above, using (7), the extensions inside $\mathrm{Fill}^\bullet(\mathrm{cogr}_{*,k}(\Psi(\Lambda)))$ do not split so that we have monomorphisms with torsion free cokernels

$$\mathrm{cogr}_{*,h}(\Psi(\Lambda))/\,\mathrm{Fill}^k(\mathrm{cogr}_{*,h}(\Psi(\Lambda)) \hookrightarrow i_*^{(k+1)} j_*^{(k+1)} \Lambda^{(k+1)}(\frac{1-k+2(h-1)}{2}). \tag{9}$$

Note finally that we never need to use the saturation process of definition 4.9, for $\Lambda = \overline{\mathbb{Z}}_l$, when constructing the filtrations of stratification.

Combining lemma 6.3 with proposition 5.4 we then deduce the following result which was proved in [**Dat12**] theorem 2.2 or [**Zhe08**] lemma 5.6 using different tools.

Proposition 6.4. — *(cf.* [**Dat12**] *theorem 2.2 or* [**Zhe08**] *lemma 5.6)*
For every $I \subseteq \{1, \cdots, r\}$, the adjunction morphism

$$i_{I,*} i_I^* \Psi(\Lambda) \longrightarrow j_{I,*} j_I^* \Psi(\Lambda)$$

is an isomorphism.

Remark. j_I being affine so that $j_{I,*} j_I^* \Psi(\Lambda)$ is perverse, the map of the proposition factorizes through ${}^p h^0 i_I^* \Psi(\Lambda)$.

Remark. as

$$i_*^{(t)}\, {}^p h^0 i^{(t),*} \Psi(\Lambda) \cong \Psi(\Lambda)/\,\mathrm{Fil}^t(\Psi(\Lambda))$$

we can obtain an explicit filtration of it with graded parts the perverse sheaves ${}^p j_{!*}^{(k)} \Lambda^{(k)}$. Similarly

$$i_*^{(t)}\, {}^p h^{-1} i^{(t),*} \Psi(\Lambda) \cong \ker_{\mathcal{F}}\big(j_!^{(t)} j^{(t),*} \Psi(\Lambda) \twoheadrightarrow \mathrm{Fil}^t(\Psi(\Lambda))\big),$$

can also be described explicitly. Dually we can also give a filtration of

$$i_*^{(t)}\, {}^p h^1 i^{(t),!} \Psi(\Lambda) = j_*^{(t)} j^{(t),*} \Psi(\Lambda)/\,\mathrm{coFil}_{*,t}(\Psi(\Lambda))$$

and

$$i_*^{(t)}\, {}^p h^0 i^{(t),!} \Psi(\Lambda) = \ker_{\mathcal{F}}(\Psi(\Lambda) \longrightarrow \mathrm{coFil}_{*,t}(\Psi(\Lambda))).$$

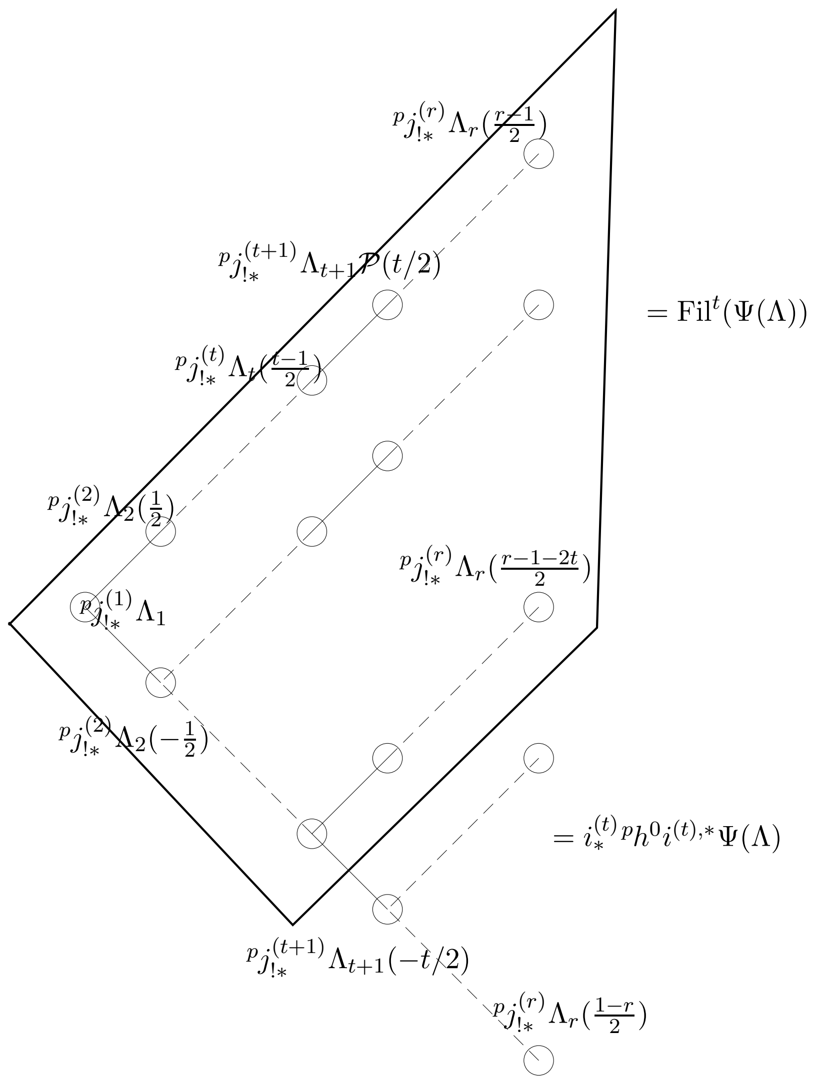

FIGURE 2. Filtration of stratification of $\Psi(\Lambda)$.

7. The nilpotent monodromy operator

First start with the knowledge of the existence of a non zero nilpotent monodromy operator over $\Lambda = \overline{\mathbb{Q}}_l$

$$N : \Psi(\Lambda) \longrightarrow \Psi(\Lambda)(1).$$

Note that the kernel of N contains necessary $\mathrm{Fil}_!^1(\Psi(\Lambda))$: indeed for any of its graded parts $\mathrm{grr}^k(\mathrm{gr}_!^1(\Psi(\Lambda))$ then $\mathrm{grr}^k(\mathrm{gr}_!^1(\Psi(\Lambda))(1)$ is not a constituant of $\Psi(\Lambda)$.

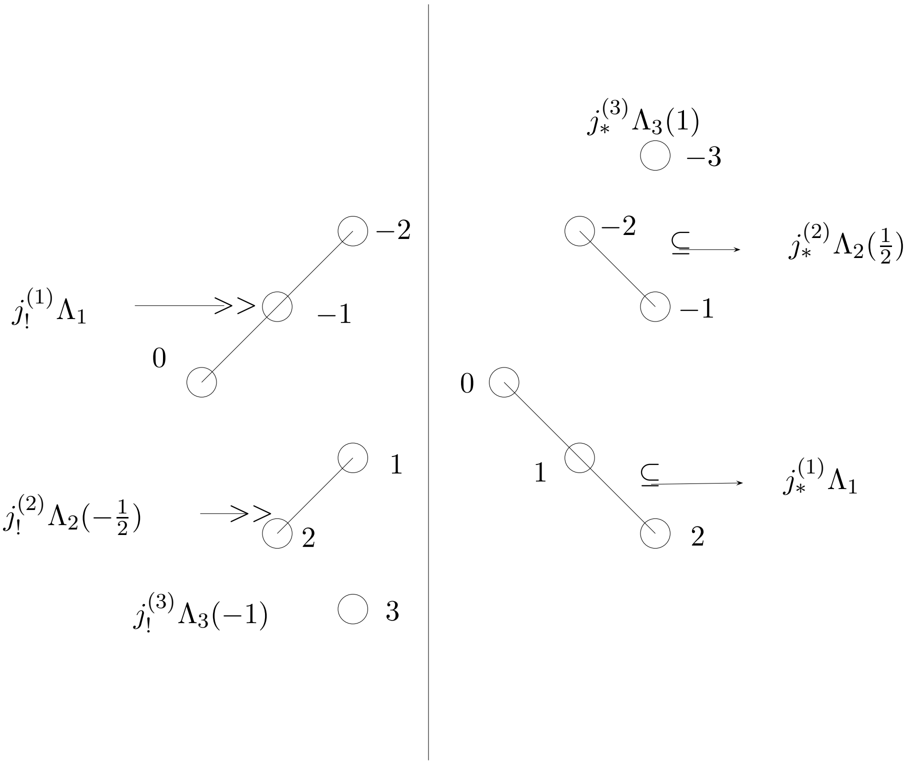

Figure 3. Filtration (fig. on the left) and cofiltration (fig. on the right) of stratification of $\Psi(\Lambda)$ with $r=3$: the number correspond to the weights.

As the socle of $\Psi(\Lambda)$ is $i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-1}{2})$, then $i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-3}{2})$ does not belong to the kernel of N: note that it appears with multiplicity one in $\Psi(\Lambda)$ as all the other constituants.

Lemma 7.1. — *The socle of $i_*^{(1)}{}^p h^0 i^{(1),*}\Psi(\Lambda)$ is equal to $i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-3}{2})$.*

Proof. — We have already seen that the socle of the $\mathrm{grr}^h(\mathrm{gr}_!^k(\Psi(\Lambda))$ are equal to $i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-1+2(k-1)}{2})$. Dually we already remarked that the socle of the

$$\mathrm{grr}^h(\mathrm{cogr}_{*,k}(\Psi(\Lambda))/i_*^{(k)}{}^p j_{!*}^{(k)}\Lambda^{(k)}(\frac{k-1}{2}),$$

is $i_*^{(k+1)}{}^p j_{!*}^{(k+1)}\Lambda^{(k+1)}(\frac{k-3}{2})$. Then the only possibility for being a subspace of

$$i_*^{(1)}{}^p h^0 i^{(1),*}\Psi(\Lambda) \cong \Psi(\Lambda)/\,\mathrm{Fil}_!^1(\Psi(\Lambda))$$

is $i_*^{(r)}{}^p j_{!*}^{(r)}\Lambda^{(r)}(\frac{r-3}{2})$. □

Finally we then obtain that

$$\ker N = \mathrm{Fil}_!^1(\Psi(\Lambda)),$$

and N induces an isomorphism

$$\begin{array}{ccc} N : i^{(1)}_* {}^p h^0 i^{(1),*}\Psi(\Lambda) & \xrightarrow{\sim} & i^{(1)}_* {}^p h^0 i^{(1),!}\Psi(\Lambda)(1) \\ \downarrow \wr & & \downarrow \wr \\ \Psi(\Lambda) \cong \Psi(\Lambda)/\operatorname{Fil}^1_!(\Psi(\Lambda)) & & \ker(\Psi(\Lambda) \to \operatorname{coFil}_{r-1,*}(\Psi(\Lambda))(1). \end{array}$$

As the socle of $i^{(1)}_* {}^p h^0 i^{(1),*}\Psi(\overline{\mathbb{Q}}_l)$ and $i^{(1)}_* {}^p h^0 i^{(1),!}\Psi(\overline{\mathbb{Q}}_l)(1)$ are irreducible isomorphic to $i^{(r)}_* {}^p j^{(r)}_{!*}\overline{\mathbb{Q}}_l^{(r)}(\frac{r-1}{2})$, this isomorphism is unique up to a scalar so that we can find an $\overline{\mathbb{Z}}_l$ version of N inducing a $\overline{\mathbb{Z}}_l$-isomorphism on $i^{(r)}_* {}^p j^{(r)}_{!*}\overline{\mathbb{Z}}_l^{(r)}(\frac{r-1}{2})$. By reducing modulo l, we obtain a $\overline{\mathbb{F}}_l$-version

$$\overline{N} : \Psi(\overline{\mathbb{F}}_l) \longrightarrow \Psi(\overline{\mathbb{F}}_l)(1)$$

factorizing through

$$\overline{N} : i^{(1)}_* {}^p h^0 i^{(1),*}\Psi(\overline{\mathbb{F}}_l) \longrightarrow i^{(1)}_* {}^p h^0 i^{(1),!}\Psi(\overline{\mathbb{F}}_l)(1). \tag{10}$$

As the socle of $i^{(1)}_* {}^p h^0 i^{(1),*}\Psi(\overline{\mathbb{F}}_l)$ and $i^{(1)}_* {}^p h^0 i^{(1),!}\Psi(\overline{\mathbb{F}}_l)(1)$ are irreducible both isomorphic to $i^{(r)}_* {}^p j^{(r)}_{!*}\overline{\mathbb{F}}_l^{(r)}(\frac{r-1}{2})$ and as we choose $\overline{N}$ so that it induces an isomorphism on this socle, then (10) is an isomorphism.

Finally we obtain version of N over $\Lambda \in \{\overline{\mathbb{Q}}_l, \overline{\mathbb{Z}}_l, \overline{\mathbb{F}}_l\}$ which are nilpotent with order of nilpotency equal to r.

Remark. It is also possible to prove directly that $i^{(1)}_* {}^p h^0 i^{(1),*}\Psi(\Lambda)$ and $i^{(1)}_* {}^p h^0 i^{(1),!}\Psi(\Lambda)(1)$ are isomorphic using the filtration of stratification. One can argue by induction starting from the fact that $\operatorname{Fil}^2_!$ of these two perverse sheaves are isomorphic obtained as a quotient of $j^{(2)}_! \Lambda^{(2)}(\frac{1}{2})$. Then one can define N directly

$$\begin{tikzcd} & N : \Psi(\Lambda) \arrow[r, two heads] & \Psi(\Lambda)/\operatorname{Fil}^1_!(\Psi(\Lambda) \cong i^{(1)}_* {}^p h^0 i^{(1),*}\Psi(\Lambda) \arrow[dll, "\sim"] \\ i^{(1)}_* {}^p h^0 i^{(1),!}\Psi(\Lambda)(1) \arrow[rr, hook] & & \ker\Big(\Psi(\Lambda) \to \operatorname{cogr}_{*,r-1}(\Psi(\Lambda))\Big)(1) \arrow[d, hook] \\ & & \Psi(\Lambda)(1) \end{tikzcd}$$

References

[Boy14] P. Boyer, *Filtrations de stratification de quelques variétés de Shimura simples*, Bulletin de la SMF **142, fascicule 4** (2014), 777–814.

[Boy23] ———, *La cohomologie des espaces de Lubin-Tate est libre*, Duke Math. Journal (june 2023), 1531–1622.

[Dat12] J.-F. Dat, *A lemma on nearby cycles and its application to the tame Lubin-Tate space*, Math. Res. Lett. (2012), 1–9.

[Ill94] L. Illusie, *Autour du théorème de monodromie locale*, Périodes p-adiques, Astérisque, no. 223, 1994.

[Jut09] D. Juteau, *Decomposition numbers for perverse sheaves*, Annales de l'Institut Fourier, 59 (3) (2009), 1177–1229.

[Zhe08] W. Zheng, *Sur la cohomologie des faisceaux l-adiques entiers sur les corps locaux*, Bull. Soc. Math. France (2008), 465–503.

Pascal Boyer • *E-mail :* `boyer@math.univ-paris13.fr`, Université Paris 13, Sorbonne Paris Nord, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse (France), PPAL: ANR-25-PRC