

# Regular de Rham representations in the cohomology of modular curves

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## Main Results

$p$  prime

Completed cohomology  $K \subseteq GL_2(\mathbb{A}_f)$   $Y_K(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash (\mathbb{C} \backslash \mathbb{R}) \times GL_2(\mathbb{A}_f) / K$

$Y_K / \mathbb{Q}$ ,  $X_K = \overline{Y_K}$

Fix  $K^p \subseteq GL_2(\mathbb{A}_f^p)$ .  $\rightsquigarrow K_p \subseteq GL_2(\mathbb{Q}_p)$

$$\lim_{\leftarrow} \lim_{\rightarrow} H_{\text{ét}}^i(Y_{K^p K_p}, \overline{\mathbb{Q}}_p, \mathbb{Z}/p^w) =: \tilde{H}^i(K^p, \mathbb{Z}_p) \oplus G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$$

$\lim_{\leftarrow} \lim_{\rightarrow} \rightsquigarrow K_p \subseteq GL_2(\mathbb{Q}_p)$

$E/\mathbb{Q}_p < \infty \rightsquigarrow \tilde{H}^i(K^p, E) = \tilde{H}^i(K^p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E$

$E$ -Borel spec rep

Thm A:  $\rho: G_{\mathbb{Q}} \rightarrow GL_2(E)$   $\rho^{\circ}$ , abs. irr. Suppose (i)  $\Pi_{\rho} = \text{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \tilde{H}^i(K^p, E)) \neq 0$

(ii)  $\rho|_{G_{\mathbb{Q}_p}}$  is de Rham of HT weights 0,  $k \geq 1$

Then  $\rho$  comes from an eigenform of weight  $k+1$ .

Recall  $\pi_p \neq 0 \Rightarrow p \mid G_{\mathbb{Q}}$  unramified VHL

Proved by Evertou via p-adic local Langlands

$p$  is odd, i.e.  $\det \rho(C) = -1$

$\downarrow$   
O-lex conjugation

• Our approach is purely geometric (avoid p-adic LLC)

since then A

$$\rho \mapsto \pi = \otimes \pi_i$$

For simplicity:  $E = \mathbb{Q}_p$ ;  $k=1$

$$\mathbb{Q}_p = \widehat{\mathbb{Z}}_p \simeq \mathbb{B}$$

I)  $\mathbb{Z}_p$  supercuspidal

$$\begin{array}{c} \text{GL}_2(\mathbb{Q}_p) \\ \downarrow \mu \\ \text{GL}_2(\mathbb{Z}_p) \end{array} \xrightarrow{\det} \mathbb{Z}_p^\times$$

$$\begin{array}{c} \text{GL}_2(\mathbb{Q}_p) \\ \downarrow \pi_w \\ \mathbb{Q} = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{O}_p) \end{array}$$

$$\begin{array}{c} \text{GL}_2(\mathbb{Q}_p) \\ \downarrow \end{array}$$

adic space /  $\text{Spa}(\mathbb{Q}_p, \mathbb{O}_p)$

D: quaternion algebra  $\mathbb{Q}$  ramified at  $p, \infty$

$$D_p = D \otimes \mathbb{Q}_p$$



(3) Conjecture of Beaulieu - Steudt (proved by Dospinescu - Le Bras)

II)  $\pi_p$  Principal Series

Yousa curve  $K^p, n \rightsquigarrow \mathcal{I}_g(K^p, n) / \mathbb{F}_p$

$$H_{\text{rig}}^1(\mathcal{I}_g) = \varinjlim_n \left( H_{\text{rig}}^1(\mathcal{I}_g(K^p, n) / W(\mathbb{F}_p)[\frac{1}{p}]) \right) \otimes \mathbb{C}_p$$

$\circlearrowleft$  rigid cohomology

$$B = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\} \in GL_2(\mathbb{Q}_p)$$

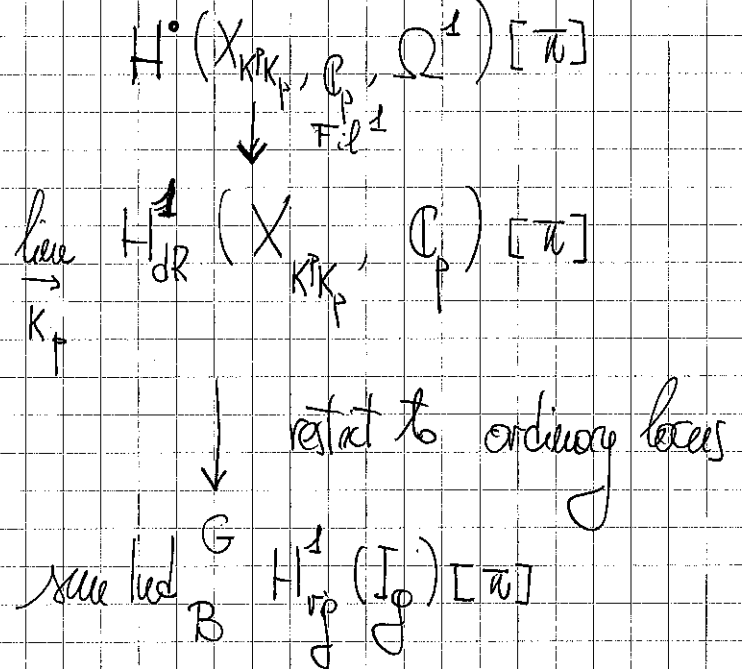
Thm C:  $\exists \varprojlim_n (\pi^\infty)^{K^p} \xrightarrow{\text{sur-luc}_B^{GL_2(\mathbb{Q}_p)}(H_{\text{rig}}^1(\mathcal{I}_g))[\pi]} \mathbb{T}_p$ ,  $\text{LTP}$  spherical Hecke  $\mathfrak{a}$

$$\Delta \pi_p \hat{\otimes} \mathbb{C}_p = \text{luc}_B^{GL_2(\mathbb{Q}_p)}(H_{\text{rig}}^1(\mathcal{I}_g))[\pi] \otimes_{\mathbb{D}_{\text{pst}}(p|G_{\mathbb{Q}_p})} \varprojlim_n (\pi^\infty)^{K^p}$$

loc analytic induction

Recall:

1.  $\mathbb{Z}_p$ :



2.  $H^1_{\text{gp}}(I_g) [\pi] \cong (\pi^\infty)^{K^p} \otimes D_{\text{pst}}^{\text{eff}}(p|G_{\mathbb{Q}_p})$

$\hookrightarrow_B$  conjecture relation (cf. Coray's paper)

$\rightsquigarrow \prod_p \mathbb{Z}_p \otimes \mathbb{C}_p \cong (\pi^\infty)^{K^p} \otimes \left( \text{hd}_B^G D_{\text{pst}}^{\text{eff}}(p|G_{\mathbb{Q}_p}) \right)_{\mathbb{Z}_p}$

conjecture of Berger-Breuil, Fontaine  
proved by Berger, Liu-Xue-Whang

OO more complicated  
in the "Steinberg" case

Update

$$\pi_p^{la} \hat{\otimes} \mathbb{C}_p \cong H^1(\mathbb{P}^1, \mathbb{R}) [\pi]$$

$$\mathbb{R}^1 \cong \pi_p$$

de Rham complex (the "d" will be with <sup>unusual</sup> coefficients)

Today: Hodge-Tate

(Dedekind-Weiss-Schreier) Thm

$$\rho: G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{Q}_p)$$

$\rho^0$  abs irr

$$\neq \pi_p^{la}$$

has  $\infty$ -d character  $\tilde{\chi}_k$

$\Rightarrow \rho|_{G_{\mathbb{Q}_p}}$  Hodge-Tate of weights  $(0, k)$

$$Z := \mathbb{Z}(U(\mathbb{F}_2(\mathbb{Q}_p)))$$

$\tilde{\chi}_k: Z \rightarrow \mathbb{Q}_p$   $\infty$ -d character of  $(\text{Sym}^k \mathbb{Q}_p^{\oplus 2})^*$   
 $(0, 1-k)$

Note:  $\tilde{H}^{la, \tilde{\chi}_k}$ : Hodge-Tate of weights  $0, k$ :  $\tilde{H}^{la, \tilde{\chi}_k} \hat{\otimes} \mathbb{C}_p \cong W_0 \oplus W_k$

where  $W_0 = W_0^{G_{\mathbb{Q}_p}} \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$ ,  $W_k(k) = W_k(k)^{G_{\mathbb{Q}_p}} \hat{\otimes} \mathbb{C}_p$

$X_K$ : ade space associated to  $X_K \times_{\mathbb{Q}} \mathbb{C}_p$

Schulze:  $X_{K^p} \sim \varprojlim_{K_p} X_{K^p/K_p}$   
 $\downarrow$   
 $G_{GL_2(\mathbb{C}_p)}$   $\downarrow$  perfectoid space

$\pi_{HT}: X_{K^p} \rightarrow P^1 = \mathbb{A}^1/G$  loop variety of  $GL_2$   
 $GL_2(\mathbb{C}_p)$ -equivariant

Define:  $\mathcal{O} = \pi_{HT}^* \mathcal{O}_{X_{K^p}}$

Schulze:  $H^1(\mathbb{A}^1, \mathcal{O}) \cong \tilde{H}^1(K^p, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} \mathbb{C}_p$

Def:  $\mathcal{O}^{la} \subseteq \mathcal{O}$  subsheaf of  $GL_2(\mathbb{C}_p)$  loc an sections

Fact:  $H^1(\mathbb{A}^1, \mathcal{O}^{la}) \cong \tilde{H}^{la} \hat{\otimes} \mathbb{C}_p$

Key  $\mathcal{O}^{la}$  satisfies a first order differential equation

$\mathfrak{g}^o = \mathfrak{gl}_2(\mathbb{C}_p) \otimes_{\mathbb{C}_p} \mathcal{O}_{\mathbb{A}^1}$   $\mathfrak{g}^o = \{f \in \mathfrak{g}^o, f_x \in \mathfrak{p}_x \forall x\}$ ,  $\mathfrak{n}^o = \{f \in \mathfrak{g}^o, f_x \in \mathfrak{N}_x \forall x\} \cong \Omega_{\mathbb{A}^1}^1$   
 $x \in \mathbb{A}^1(\mathbb{C}_p)$

line bundle

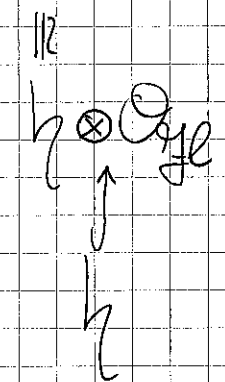
Theorem:  $n^\circ$  accumulates  $\mathcal{O}^{\text{la}}$  (p-adic Cauchy-Picard eq<sup>n</sup>)

eg.  $SL_2 \cong N$   
 $N \backslash SL_2 = A^2 \setminus \{(0,0)\}$   
 $\sigma_{G_{\text{un}}}$  via  $\lambda(a,b) = (a,b)$

(should be generalizable for  $\mathcal{O}_p$ -analytic)  
 Consequence of Rodas-Spencer isom

Consequences:  $\mathcal{O}^\circ / n^\circ \subset G \mathcal{O}^{\text{la}} \Rightarrow h G \mathcal{O}^{\text{la}}$  commutes with  $GL_2(\mathcal{O}_p)$

Fix a Cartan subalgebra  $h = \begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq \begin{pmatrix} * & * \\ & * \end{pmatrix}$



action:  $\lambda \mapsto -w(p-1) + p$  centered at  $p = (\frac{1}{2}, -\frac{1}{2})$

Horish-Chaudra  $\mathcal{Z} \cong S(h)^W$

$\mathcal{O} \cong S(h)$

$\mathcal{O}^{\text{la}} \hookrightarrow S(h)$

$\uparrow$

$h$

Thm:  $\mathcal{O}_h \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{O}_{\text{Sen}} \xrightarrow{\mathcal{O}_{\text{Sen}G}} \mathcal{O}_{\text{Sen}G}$

Sen operator  $\in \text{Lie } G_{\mathcal{O}_p} \otimes \mathbb{C}_p$

(should be true for  $F$ -analytic)

$\mathcal{O}_p = \text{Lie } T_K$

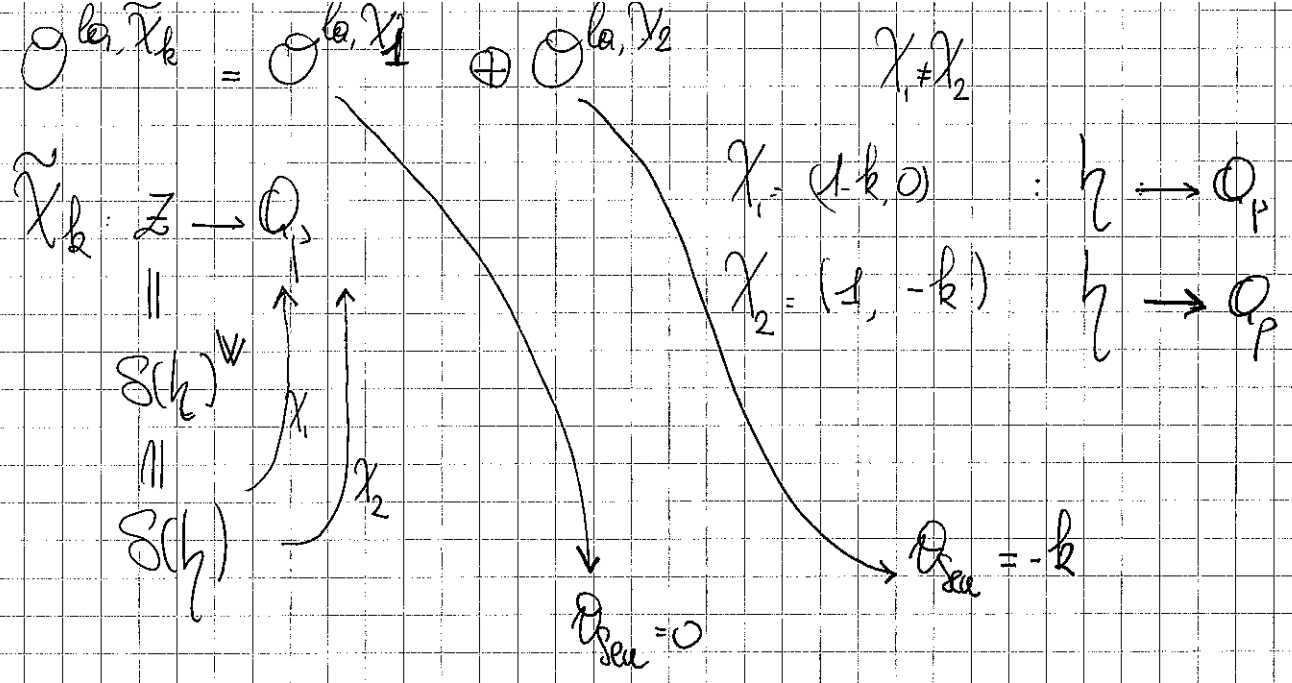
$\mathcal{O}^{\text{la}}(U)$   $H_K, T_K$ -la

$U: G_K$ -stable,  $K/\mathbb{Q}_p$

$H_K = \text{Gal}(\overline{\mathbb{Q}_p}/K(\mu_{p^\infty}))$

$T_K = \text{Gal}(K(\mu_{p^\infty})/K)$





$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\otimes k, \tilde{\chi}_k}) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\otimes k})^{\tilde{\chi}_k} \xrightarrow{\sim} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\otimes k, (1, -k)}) \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\otimes k, (1, k)}) \xrightarrow{\sim} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\otimes k}) \hat{\otimes} \mathbb{C}^2$$

$k \geq 2$

Hodge-Tate decomposition