Regular de Rham representations in the cohomology of modular curves — Lue Pan

Main Results

p prime

Completed cohomology

$K \subseteq GL_2(\mathbb{A}_f)$ $\quad Y_K(\mathbb{C}) = GL_2(\mathbb{Q}) \backslash (\mathbb{C} \setminus \mathbb{R}) \times GL_2(\mathbb{A}_f) / K$

$Y_K / \mathbb{Q}$, $X_K = \overline{Y_K}$

Fix $K^p \subseteq GL_2(\mathbb{A}_f^p)$. $\rightsquigarrow K_p \leq GL_2(\mathbb{Q}_p)$

$$\varprojlim_n \varinjlim_{K_p \leq GL_2(\mathbb{Q}_p)} H^i_{et}(Y_{K^pK_p, \overline{\mathbb{Q}}}, \mathbb{Z}/p^n) =: \tilde{H}^i(K^p, \mathbb{Z}_p) \circlearrowleft G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$$

$E/\mathbb{Q}_p < \infty \rightsquigarrow \tilde{H}^i(K^p, E) = \tilde{H}^i(K^p, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} E$

E-Banach space rep

Thm A: $\rho : G_{\mathbb{Q}} \to GL_2(E)$ $\mathcal{C}^0$, abs. irr. Suppose (i) $\pi_\rho := \mathrm{Hom}_{E[G_{\mathbb{Q}}]}(\rho, \tilde{H}^1(K^p, E)) \neq 0$

(ii) $\rho|_{G_{\mathbb{Q}_p}}$ is de Rham of HT weights $0, k \geq 1$

Then ρ comes from an eigenform of weight $k+1$.

Rmk. •) $\Pi_p \neq 0 \Rightarrow \rho|_{G_{\mathbb{Q}_\ell}}$ unramified $\forall \ell$. Proved by Emerton via p-adic local Langlands.

ρ is odd, i.e. $\det \rho(C) = -1$

$\downarrow$ $\mathbb{C}$-lex conjugation

•) Our approach is purely geometric (avoid p-adic LLC)

as in Thm A

$\rho \rightsquigarrow \pi = \bigotimes' \pi_\ell$

For simplicity: $E = \mathbb{Q}_p$; $k = 1$

$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p} \xrightarrow{\sim} \mathbb{C}$

D: quaternion algebra $/\mathbb{Q}$ ramified at p, ∞

$D_p = D \otimes \mathbb{Q}_p$

I) $\boxed{\pi_p \text{ supercuspidal}}$

$$\begin{array}{c} GL_2(\mathbb{Q}_p)^0 \circlearrowright \mathcal{M}_\infty \longleftrightarrow \det(\ ) \in \mathbb{Z}_p^\times \\ \downarrow \pi_\infty \\ \Omega = \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{Q}_p) \quad \text{adic space} / \mathrm{Spa}(\mathbb{C}_p, \mathcal{O}_{\mathbb{C}_p}) \\ \circlearrowleft \\ GL_2(\mathbb{Q}_p) \end{array}$$

③

Thm B: $\exists$ natural map, $GL_2(\mathbb{Q}_p)$ equivariant

$$\ell_\pi : \pi_p \longrightarrow \left(\pi_p' \otimes_{\mathbb{C}_p} \varinjlim_n H^1_c(\mathcal{M}_n, \mathcal{O}_{\mathcal{M}_n})\right)^{\mathcal{O}_{D_p}^\times}$$

smooth

similar statement to enhance to $GL_2(\mathbb{Q}_p)$ action

$JL : \pi \mapsto \pi' = \otimes' \pi_\ell' \circlearrowleft (D\otimes A)^\times$

$\pi_p' \overset{JL}{\longleftrightarrow} \pi_p$

s.t. $$\Pi_p^{la} \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p \simeq \left(\quad \Big/ \ell_\pi(\pi_p)\right) \otimes_{\mathbb{C}_p} (\pi^{p,\infty})^{K^p}$$

find a canonical ident. $H^1_{dR,c}(\mathcal{M}_n) \simeq \pi_p \otimes D_{dR}$ π_p'

Rmk: 1) $j\ \Omega \hookrightarrow \mathbb{P}^1$

$$H^1_c(\mathcal{M}_n, \mathcal{O}_{\mathcal{M}_n}) = H^1(\mathbb{P}^1, j_!(\pi_{n,*}\mathcal{O}_{\mathcal{M}_n}))$$

2) ℓ_π comes from $\mathcal{L}$-invariant

$$\varinjlim_{K_p} H^0(X_{K^pK_p,\mathbb{C}_p}, \Omega^1)[\pi] \overset{Fil^1}{\hookrightarrow} \varinjlim_{K_p} H^1_{dR}(X_{K^pK_p,\mathbb{C}_p})[\pi]$$

$$\text{II}\wr \qquad (\pi^\infty)^{K_p}$$

restrict to sving locus

$$\left(\pi_p' \otimes \varinjlim_n \underbrace{H^1_{dR,c}(\mathcal{M}_n)}\right)^{\mathcal{O}_{D_p}^\times} \otimes (\pi^{\infty,p})^{K^p}$$

$$\downarrow$$

$$H^1_c(\mathcal{M}_n, \mathcal{O}_{\mathcal{M}_n})$$

(3) Conjecture of Breuil - Strauch (proved by Dospinescu - Le Bras)

II) π_p Principal Series

Igusa curve K^p, n $\rightsquigarrow$ $Ig(K^p, n)/\overline{\mathbb{F}}_p$

$$\circlearrowright\ H^1_{rig}(I_g) = \varinjlim_n \left(H^1_{rig}\left(Ig(K^p, n)/W(\overline{\mathbb{F}}_p)[1/p]\right)\right) \otimes \mathbb{C}_p$$

rigid cohomology

$$B = \left\{\begin{pmatrix} * & * \\ & * \end{pmatrix}\right\} \subseteq GL_2(\mathbb{Q}_p)$$

Thm C: $\exists\ \iota_\pi$: $(\pi^\infty)^{K^p} \hookrightarrow \text{sm-Ind}_B^{GL_2(\mathbb{Q}_p)}\left(\underbrace{H^1_{rig}(I_g)}_{(\pi^\infty)^{K^p} \otimes D_{pst}(\rho|G_{\mathbb{Q}_p})}\right)[\pi]$ $\quad T_\ell,\ \ell \nmid p$ spherical Hecke op

$$\& \quad \pi_p^{la} \hat{\otimes} \mathbb{C}_p = \text{Ind}_B^{GL_2(\mathbb{Q}_p)} H^1_{rig}(I_g)^{[\pi]} / \iota_\pi\left((\pi^\infty)^{K^p}\right)$$

loc. analytic induction

Rmk:

1. ℓ_{π} :

$$
\begin{array}{c}
H^0(X_{K^pK_p}, \mathbb{C}_p, \Omega^1)[\pi] \\
\Big\downarrow \scriptstyle \mathrm{Fil}^1 \\
\varinjlim_{K_p} H^1_{dR}(X_{K^pK_p}, \mathbb{C}_p)[\pi] \\
\Big\downarrow \text{restrict to ordinary locus} \\
\mathrm{sm\,Ind}_B^G H^1_{rig}(I_g)[\pi]
\end{array}
$$

2. $H^1_{rig}(I_g)[\pi] \cong (\pi^{\infty})^{K^p} \otimes D_{\substack{cris \\ pst}}(\rho|G_{\mathbb{Q}_p})$

$\circlearrowleft_B$ congruence relation (cf. Coraggi's paper

$\rightsquigarrow \Pi_p^{la} \otimes \mathbb{C}_p \cong (\pi^{\infty})^{K^p} \otimes \left(\mathrm{Ind}_B^G D_{\substack{cris \\ pst}}(\rho|G_{\mathbb{Q}_p})\right) / \pi_p$

conjecture of Berger-Breuil, Emerton

proved by Berger, Liu-Xie-Zhang

$\overline{OO}$ more complicated in the "Steinberg" case

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<u>Upshot</u> $\Pi_p^{la} \hat{\otimes} \mathbb{C}_p \cong H^1(\mathbb{P}^1, \mathrm{DR})[\pi]$ / $\boxed{\mathrm{Fil}^1 \cong \pi_p}$

$\downarrow$ de Rham complex (the " $k>1$ " will be with nontrivial coefficients)

<u>Today</u>: Hodge-Tate

<u>Thm</u> (Dospinescu-Paškūnas-Schraen, Pan) $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_p)$ $\mathcal{C}^0$, abs irr

$0 \neq \Pi_p^{la}$ $\circlearrowleft$ $\mathcal{Z} := Z(U(\mathfrak{gl}_2(\mathbb{Q}_p)))$ has ∞-al character $\tilde{\chi}_k$ $\Leftrightarrow$ $\rho|_{G_{\mathbb{Q}_p}}$ Hodge-Tate of weights $(0,k)$

$\tilde{\chi}_k : Z \to \mathbb{Q}_p$ ∞-al character of $\left(\mathrm{Sym}^{k-1} \mathbb{Q}_p^{\oplus 2}\right)^*$

$(0, 1-k)$

Note: $\tilde{H}^{la, \tilde{\chi}_k}$: Hodge-Tate of weights $0, k$: $\tilde{H}^{la,\tilde{\chi}_k} \hat{\otimes} \mathbb{C}_p \cong W_0 \oplus W_k$

where $W_0 = W_0^{G_{\mathbb{Q}_p}} \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p$, $W_k(k) = W_k(k)^{G_{\mathbb{Q}_p}} \otimes \mathbb{C}_p$

$\mathcal{X}_K$: adic space associated to $X_K \times_{\mathbb{Q}} \mathbb{C}_p$

Schulze : $\mathcal{X}_{K^p} \sim \varprojlim_{K_p} \mathcal{X}_{K^pK_p}$ $\quad$ $\pi_{HT} : \mathcal{X}_{K^p} \to \mathbb{P}^1 = \mathcal{F}\ell$ flag variety of GL_2

$\circlearrowleft$ $GL_2(\mathbb{Q}_p)$ $\quad$ perfectoid space $\quad$ $GL_2(\mathbb{Q}_p)$-equivariant

Define $\mathcal{O} = \pi_{HT\,*} \mathcal{O}_{\mathcal{X}_{K^p}}$

Scholze : $H^1(\mathcal{F}\ell, \mathcal{O}) \cong \widetilde{H}^i(K^p, \mathbb{Z}_p) \hat{\otimes}_{\mathbb{Z}_p} \mathbb{C}_p$

Def : $\mathcal{O}^{la} \subseteq \mathcal{O}$ subsheaf of $GL_2(\mathbb{Q}_p)$ loc an. sections

$$\boxed{\text{Fact} \quad H^1(\mathcal{F}\ell, \mathcal{O}^{la}) \cong \widetilde{H}^{la} \hat{\otimes} \mathbb{C}_p}$$

Key $\mathcal{O}^{la}$ satisfies a first order differential equation

$\boxed{\text{line bundle}}$

$$\mathfrak{g}^\circ = \mathfrak{gl}_2(\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{F}\ell} \qquad \mathfrak{b}^\circ = \{f \in \mathfrak{g}^\circ,\ f_x \in \mathfrak{b}_x\ \forall x\}, \quad \mathfrak{n}^\circ = \{f \in \mathfrak{g}^\circ,\ f_x \in \mathfrak{n}_x\ \forall x\} \cong \Omega^1_{\mathcal{F}\ell}$$

$x \in \mathcal{F}\ell(\mathbb{C}_p)$

Theorem: $\mathfrak{n}^\circ$ annihilates $\mathcal{O}^{la}$ (p-adic Cauchy Riemann eq^n)

(should be generalizable for $\mathbb{Q}_p$-analytic)

Consequence of Kodaira-Spencer isom

e.g. $SL_2 \supseteq N$

$N\backslash SL_2 = \mathbb{A}^2 \setminus \{(0,0)\}$

$\circlearrowleft \mathbb{G}_m$ via $\lambda \cdot (a,b) = (\lambda a, \lambda b)$

Consequences: $\mathfrak{b}^\circ / \mathfrak{n}^\circ \curvearrowright \mathcal{O}^{la} \Rightarrow \mathfrak{h} \curvearrowright \mathcal{O}^{la}$ commuting with $GL_2(\mathbb{Q}_p)$

$\longleftrightarrow$ Fix a Cartan subalgebra $\mathfrak{h} = \begin{pmatrix} * & \\ & * \end{pmatrix} \subseteq \begin{pmatrix} * & * \\ & * \end{pmatrix}$

$\begin{matrix} \mathfrak{b}^\circ/\mathfrak{n}^\circ \\ \parallel \\ \mathfrak{h} \otimes \mathcal{O}_{\mathcal{FL}} \\ \uparrow \\ \mathfrak{h} \end{matrix}$

action: $\lambda \longmapsto -w(\rho - \lambda) + \rho$ centered at $\rho := \left(\frac{1}{2}, -\frac{1}{2}\right)$

Harish-Chandra $\mathcal{Z} \cong S(\mathfrak{h})^W$

$\Omega \in \mathcal{Z}$

$\mathcal{O}^{la} \curvearrowleft S(\mathfrak{h})$

$\theta_{\mathfrak{h}}$

Thm: $\theta_{\mathfrak{h}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \theta_{Sen}$ Sen Operator $\in$ "$(\mathrm{Lie}\, G_{\mathbb{Q}_p} \otimes \mathbb{C}_p)^{G_{\mathbb{Q}_p}}$"

(should be true for F-analytic)

$1 \longleftrightarrow \theta_{Sen_G}$ $\quad \mathcal{O}^{la}(\mathcal{U})^{H_K, T_K\text{-}la}$

$\mathbb{Q}_p = \mathrm{Lie}\, T_K$ $\quad \mathcal{U}$: G_K-stable, $K/\mathbb{Q}_p$ $\quad H_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p / K(\mu_{p^\infty}))$ $\quad T_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$

$$\mathcal{O}^{la,\tilde{\chi}_k} = \mathcal{O}^{la,\chi_1} \oplus \mathcal{O}^{la,\chi_2} \qquad \chi_1 \neq \chi_2$$

$$\tilde{\chi}_k : \mathfrak{z} \to \mathbb{Q}_p, \quad \mathfrak{z} = \mathfrak{S}(\mathfrak{h})^{W} = \mathfrak{S}(\mathfrak{h}) \xrightarrow{\chi_1,\ \chi_2} \mathbb{Q}_p$$

$$\chi_1 = (1-k, 0) : \mathfrak{h} \to \mathbb{Q}_p$$

$$\chi_2 = (1, -k) \quad \mathfrak{h} \to \mathbb{Q}_p$$

$$\theta_{Sen} = 0 \qquad \theta_{Sen} = -k$$

$$H^1(\mathcal{FL}, \mathcal{O}^{la,\tilde{\chi}_k}) \cong H^1(\mathcal{FL}, \mathcal{O}^{la})^{\tilde{\chi}_k} \quad \overset{\sim}{\not=} \quad k \geq 2$$

$$\| $$

$$H^1(\mathcal{FL}, \mathcal{O}^{la,(1-k,0)}) \oplus H^1(\mathcal{FL}, \mathcal{O}^{la,(1,-k)}) = \tilde{H}^{la,\tilde{\chi}_k} \hat{\otimes} \mathbb{C}_p$$

Hodge–Tate decomposition