IHARA LEMMA AND LEVEL RAISING IN HIGHER DIMENSION

by

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Abstract. — A key ingredient in the Taylor-Wiles proof of Fermat last theorem is the classical Ihara lemma which is used to raise the modularity property between some congruent galoisian representations. In their work on Sato-Tate, Clozel-Harris-Taylor proposed a generalization of the Ihara lemma in higher dimension for some similitude groups. The main aim of this paper is then to prove some new instances of this generalized Ihara lemma by considering some particular non pseudo Eisenstein maximal ideals of unramified Hecke algebras. As a consequence, we prove a level raising statement.

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Introduction

Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary. For $\overline{B}/F$ a central division algebra with dimension d^2 equipped with a involution of second kind $*$ and $\beta \in \overline{B}^{*=-1}$, consider the similitude group $\overline{G}/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra R by

$$\overline{G}(R) := \big\{(\lambda, g) \in R^\times \times (\overline{B}^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\big\}$$

with $\overline{B}^{op} = \overline{B} \otimes_{F,c} F$ where $c = *_{|F}$ is the complex conjugation and $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$. For $p = uu^c$ decomposed in E, we have

$$\overline{G}(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (\overline{B}_v^{op})^\times$$

where w describes the places of F above u. We suppose:

- the associated unitary group $\overline{G}_0(\mathbb{R})$ being compact,
- for any place x of $\mathbb{Q}$ inert or ramified in E, then $G(\mathbb{Q}_x)$ is quasi-split.
- There exists a place v_0 of F above u such that $\overline{B}_{v_0} \simeq D_{v_0,d}$ is the central division algebra over the completion F_{v_0} of F at v_0, with invariant $\frac{1}{d}$.

Fix a prime number $l \neq p$ and consider a finite set S of places of F containing the ramification places Bad of $\overline{B}$. Let denote by $\mathbb{T}_S/\overline{\mathbb{Z}}_l$ the unramified Hecke algebra of G outside S. For a cohomological minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_S$, we can associate both a near equivalence class of $\overline{\mathbb{Q}}_l$-automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ and a Galois representation

$$\rho_{\widetilde{\mathfrak{m}}} : G_F := \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

such that the eigenvalues of the Frobenius morphism at an unramified place w are given by the Satake parameter of the local component $\Pi_{\widetilde{\mathfrak{m}},w}$ of $\Pi_{\widetilde{\mathfrak{m}}}$. The semi-simple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$ containing $\widetilde{\mathfrak{m}}$. For all prime x of $\mathbb{Z}$ split in E and a place $w \notin S$ of F above x, we then denote by $P_{\mathfrak{m},w}(X)$ the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w)$.

Conjecture. — ***(Generalized Ihara lemma by Clozel-Harris-Taylor)*** *Consider*

- *an open compact subgroup $\overline{U}$ of $\overline{G}(\mathbb{A})$ such that outside S, its local component is the maximal compact subgroup;*
- *a place $w_0 \notin S$ decomposed in E;*
- *a maximal $\mathfrak{m}$ of $\mathbb{T}_S$ such that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.*

Let $\bar{\pi}$ be an irreducible sub-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/\overline{U}^{w_0},\overline{\mathbb{F}}_l)_{\mathfrak{m}}$, where $\overline{U}=\overline{U}_{w_0}\overline{U}^{w_0}$, then its local component $\bar{\pi}_{w_0}$ at w_0 is generic.

Remark. In its classical version for GL_2, Ihara's lemma is used to raise the modularity property between some congruent galoisian representations and so was the role of this higher dimensional version in Clozel-Harris-Taylor paper on the Sato-Tate conjecture. Shortly after Taylor found an argument to avoid Ihara's lemma. However this conjecture remains highly interesting, see for example works of Clozel-Thorne [**14**], or Emerton-Helm [**18**].

The main result of this paper is the following instances of the previous conjecture.

Theorem. — *The previous generalized Ihara conjecture is true if the maximal ideal $\mathfrak{m}$ verifies the following extra properties.*

(H1) *$\mathfrak{m}$ is KHT-free, cf. the next remark;*

(H2) *The image of $\overline{\rho}_{\mathfrak{m},w_0}$ in its Grothendieck group is multiplicity free*(1) *and*(2) *does not contain any full Zelevinsky line.*

(H3) *$\overline{\rho}_{\mathfrak{m},v_0}$ is multiplicity free in the following meaning. It corresponds, cf. §1.2, by Jacquet-Langlands correspondence to some superspeh representation, $\mathrm{Speh}_s(\varrho_{v_0})$, where ϱ_{v_0} is a supercuspidal $\overline{\mathbb{F}}_l$-representation of $GL_g(F_{v_0})$ with $d=sg$, cf. theorem 3.1.4 of* [**16**]*. We then ask, cf. the notation in §1.2, that*

$$\varrho_{v_0}, \varrho_{v_0}\{1\}, \cdots, \varrho_{v_0}\{s-1\},$$

are pairwise distinct.

(1) In particular q_{w_1} can't be congruent to 1 modulo l.

(2) Using the main result of [**9**] we could take off the condition about not containing a full Zelevinsky line, cf. the last remark of §3.2.

Remark. Concerning (H1), we say that $\mathfrak{m}$ is KHT-free if the cohomology groups of the Kottwitz-Harris-Taylor Shimura variety of §1.1, localized at $\mathfrak{m}$, are free. From [**8**], any of the following properties insure KHT-freeness of $\mathfrak{m}$, cf. §2.2

(1) There exists $w_1 \in \mathrm{Spl}(I)$ such that, cf. §2.1, the multi-set $S_{\mathfrak{m}}(w_1)$ of roots of $P_{\mathfrak{m},w_1}(X)$ does not contain any sub-multi-set of the shape $\{\alpha, q_{w_1}\alpha\}$ where q_{w_1} is the cardinality of the residue field. This hypothesis is called *generic* in [**12**].

(2) When $[F(\exp(2i\pi/l) : F] > d$, if we suppose the following property to be true, cf. [**8**] 4.17. If $\theta : G_F \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ is an irreducible continuous representation such that for all place $w \notin S$ above a prime $x \in \mathbb{Z}$ split in E, then $P_{\mathfrak{m},w}(\theta(\mathrm{Frob}_w)) = 0$ (resp. $P_{\mathfrak{m}^\vee,w}(\theta(\mathrm{Frob}_w)) = 0$) then θ is equivalent to $\overline{\rho}_{\mathfrak{m}}$ (resp. $\overline{\rho}_{\mathfrak{m}^\vee}$), where $\mathfrak{m}^\vee$ is the maximal ideal of $\mathbb{T}_S$ associated to the dual mutliset of Satake parameters, cf. [**8**] notation 4.4. In [**17**], the authors proved that the previous property is verified in each of the following cases:
   - either $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K where K/F is a cyclic galoisian extension;
   - or $l \geqslant d$ and $SL_d(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^\times GL_d(k)$ for some subfield $k \subset \overline{\mathbb{F}}_l$.

(3) Finally in [**11**], we announce $\mathfrak{m}$ to be KHT-free when $\overline{\rho}_{\mathfrak{m}}$ is irreducible and $[F(\exp(2i\pi/l) : F] > d$.

By Cebotarev's theorem, the hypothesis $[F(\exp(2i\pi/l) : F] > d$ allows to pick places v of F such that the order q_v of the residue field of F at v, is of order strictly greater than d in $\mathbb{Z}/l\mathbb{Z}$.

Remark. About (H2) note that the first condition also appears in section 4.5 of [**13**] in the statements of level raising. Concerning the second condition of (H2), using the main result of [**9**], we can remove it, cf. the remark after the proof of lemma 3.2.2.

To prove this result, we first translate such a property to the cohomology group of middle degree of the Kottwitz-Harris-Taylor Shimura variety $X_{\mathcal{I}}$ associated to the similitude group $G/\mathbb{Q}$ such that

– $G(\mathbb{A}^\infty) = \overline{G}(\mathbb{A}^{\infty,p}) \times GL_d(F_{v_0}) \times \prod_{\substack{w|u \\ w \neq v_0}} (\overline{B}_w^{op})^\times$,

– the signatures of $G(\mathbb{R})$ are $(1, d-1) \times (0, d) \times \cdots \times (0, d)$.

In particular to each prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_S$, is associated a $\overline{\mathbb{Q}}_l$-irreducible automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ of $G(\mathbb{A}_{\mathbb{Q}})$ whose Satake parameters at finite places outside S are prescribed by $\widetilde{\mathfrak{m}}$. We then compute the cohomology groups of the geometric generic fiber of $X_{\mathcal{I}}$ through the spectral sequence of vanishing cycles at the place v_0. Thanks to (H1), the $H^i(X_U, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ are free and so, $H^i(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} = (0)$ for $i \neq d-1$.

Remark. Moreover (H2) (resp. (H3)) insures that the graded parts of the filtration of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, given by the integral version of the weight-monodromy filtration, at the place w_0 (resp. v_0) are also free.

The contribution of the supersingular points of the special fiber at v_0, using (H3), allows us to associate to an irreducible sub-representation $\overline{\pi}$ of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/\overline{U}^{w_0}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, an irreducible sub-representation π of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, such that $\pi^{\infty,v_0} \simeq \overline{\pi}^{\infty,v_0}$. We then try to prove the genericness of π_{w_0} by proving, using (H2), such genericness property of irreducible submodules of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. On ingredient in §3.1, comes from [**21**] §5, where the hypothesis that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, insures that the lattices of isotypic components of $H^{d-1}(X_U, \overline{\mathbb{Q}}_l)_{\mathfrak{m}}$ given by the $\overline{\mathbb{Z}}_l$-cohomology, can be written as a tensorial product of stable lattices for the $G(\mathbb{A}^\infty)$ and the Galois actions.

Finally (H2) is needed to control the combinatorics.

Remark. As pointed out to us by M. Harris, the case where the cardinality q_{w_0} of the residue field at w_0, is congruent to 1 modulo l should be of crucial importance for the applications. Meanwhile our strategy relies on the construction of a filtration of $H^{d-1}(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ which each graded parts verify the genericness property of irreducible submodule and where these graded parts are parabolically induced. When $q_{w_0} \equiv 1 \mod l$, parabolically induced $\overline{\mathbb{F}}_l$-representations are often semi-simple and so they can't verify such genericness property of irreducible submodule. It seems that our approach is not well adapted to treat this fundamental case.

To state our application to level raising, denote by $\mathcal{S}_{w_0}(\mathfrak{m})$ the supercuspidal support of the modulo l reduction of $\Pi_{\widetilde{\mathfrak{m}},w_0}$ for any prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$: it depends only on $\mathfrak{m}$. By (H2) this support is multiplicity free and we first break it $\mathcal{S}_{w_0}(\mathfrak{m}) = \coprod_{\varrho \in \mathcal{Z}} \mathcal{S}_\varrho(\mathfrak{m})$ according to the set of Zelevinsky lines $ZL(\varrho) = \{\varrho\{k\} : k \in \mathbb{Z}\}$, where $\mathcal{Z}$ is the set of equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations ϱ of some $GL_{g(\varrho)}(F_{w_0})$ with $1 \leqslant g(\varrho) \leqslant d$, under the equivalence relation $\varrho \sim \varrho\{k\}$ for any $k \in \mathbb{Z}$.

For any such ϱ we then denote by $l_1(\varrho) \geqslant \cdots \geqslant l_{r(\varrho)}(\varrho) \geqslant 1$, such that $S_\varrho(\mathfrak{m})$ can be written as the union of $r(\varrho)$ Zelevinsky unlinked segments of length $l_i(\varrho)$

$$\big[\varrho\nu^k, \bar{\rho}\nu^{k+l_i(\varrho)-1}\big] = \big\{\varrho\nu^k, \varrho\nu^{k+1}, \cdots, \varrho\nu^{k+l_i(\varrho)-1}\big\}.$$

Then for any minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, and $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$, we write its local component $\Pi_{w_0} \simeq \times_\varrho \Pi_{w_0}(\varrho)$ and $\Pi_{w_0}(\varrho) \simeq \times_{i=1}^{r(\varrho)} \Pi_{w_0}(\varrho, i)$ where for each $1 \leqslant i \leqslant r(\varrho)$, the modulo l reduction of the supercuspidal support of $\Pi_{w_0}(\varrho, i)$ is, with the notations of §2.2, those of the Zelevinsky segment $\big[\varrho\nu^{\delta_i}, \varrho\nu^{\delta_i+l_i(\varrho)-1}\big]$.

Proposition. — *Take a maximal ideal $\mathfrak{m}$ verifying the hypothesis (H1) and (H2). Let ϱ_0 such that $\mathcal{S}_{\varrho_0}(\mathfrak{m})$ is non empty and consider $1 \leqslant i \leqslant r(\varrho_0)$. Then there exists a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and an automorphic representation $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ such that, with the previous notations, $\Pi_{w_0}(\varrho_0, i)$ is non degenerate, i.e. isomorphic to $\mathrm{St}_{l_i(\varrho)}(\pi_{w_0})$ for some irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation π_{w_0}.*

In particular when there is only one segment, which is always the case for GL_2, then the result is optimal.

Remark. In the previous proposition, we could also prove that for any such $\widetilde{\mathfrak{m}}$ and any $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$, then $\Pi_{w_0}(\varrho_0, i)$ is non degenerate, which looks similar to theorem 2.1 of [**1**] where $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible and decomposed generic which also imposes the cohomology groups to be free.

1. Shimura variety of Kottwitz-Harris-Taylor type

1.1. Geometry. — Recall from the introduction that a prime number l is fixed, distinct from all others prime numbers which will be considered in the following. Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ imaginary quadratic such that l is unramified, and F^+ totally real with a fixed embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we denote by F_v the completion of F at v, with ring of integers $\mathcal{O}_v$, uniformizer ϖ_v and residual field $\kappa(v)$ with cardinality q_v.

Let B be a central division algebra over F of dimension d^2 such that at any place x of F, either B_x is split or it is a division algebra. We moreover suppose the existence of an involution of second kind $*$ on B

such that $*_{|F}$ is the complex conjugation c. For $\beta \in B^{*=-1}$, we denote by $\sharp_\beta : x \mapsto \beta x^* \beta^{-1}$ and let $G/\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra R,

$$G(R) = \big\{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\big\},$$

with $B^{op} = B \otimes_c F$. If $x = yy^c$ is split in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times Q_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times$$

where, identifying the places of F^+ above x with those of F above y, we write $x = \prod_i z_i$. Moreover we can impose that

- if x is inert in E then $G(\mathbb{Q}_x)$ is quasi-split,
- the signature of $G(\mathbb{R})$ are $(1, d-1) \times (0, d) \times \cdots \times (0, d)$.

With the notations of the introduction, we have

$$G(\mathbb{A}^\infty) = \overline{G}(\mathbb{A}^{\infty,p}) \times \Big(\mathbb{Q}_{p_{v_0}}^\times GL_d(F_{v_0}) \times \prod_{\substack{w|u \\ w \neq v_0}} (\overline{B}_w^{op})^\times\Big).$$

1.1.1. Definition. — *We denote by* Bad *the set of places w of F such that B_w is non split. Let* Spl *the set of finite places w of F not in* Bad *such that $w_{|\mathbb{Q}}$ is split in E. For such a place w with $p = w_{|\mathbb{Q}}$, we write abusively*

$$G(\mathbb{A}^w) = G(\mathbb{A}^p) \times \mathbb{Q}_p^\times \times \prod_{\substack{u|p \\ u \neq w}} (B_u^{op})^\times,$$

and $G(F_w) = GL_d(F_w)$.

Remark. With the notations of the introduction, the role of w in the previous definition will be taken by either by v_0, v_1 or w_0.

1.1.2. Notation. — *For all open compact subgroups U^p of $G(\mathbb{A}^{\infty,p})$ and $m = (m_1, \cdots, m_r) \in \mathbb{Z}_{\geqslant 0}^r$, we consider*

$$U^p(m) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r \mathrm{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times).$$

For w_0 one of the v_i and $n \in \mathbb{N}$, we also introduce $U^{w_0}(n) := U^p(0, \cdots, 0, n, 0, \cdots, 0)$.

We then denote by $\mathcal{I}$ for the set of these $U^p(m)$ such that there exists a place x for which the projection from U^p to $G(\mathbb{Q}_x)$ does not contain any element with finite order except the identity, cf. [**19**] bellow of page 90.

Attached to each $I \in \mathcal{I}$ is a Shimura variety $X_I \to \operatorname{Spec} \mathcal{O}_v$ of type Kottwitz-Harris-Taylor and we denote by $X_{\mathcal{I}} = (X_I)_{I \in \mathcal{I}}$ the projective system: recall that the transition morphisms $r_{J,I} : X_J \to X_I$ are finite flat and even etale when $m_1(J) = m_1(I)$. This projective system is then equipped with a Hecke action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$, where the action of z in the Weil group W_v of F_v is given by $-\deg(z) \in \mathbb{Z}$ where $\deg = \mathrm{val} \circ \mathrm{Art}_v^{-1}$, where $\mathrm{Art}_v^{-1} : W_v^{ab} \simeq F_v^\times$ is the Artin isomorphism which sends geometric Frobenius to uniformizers.

1.1.3. Notations. — *(cf. [**3**] §1.3) Let $I \in \mathcal{I}$,*

- *the special fiber of X_I will be denoted by $X_{I,s}$ and its geometric special fiber $X_{I,\bar{s}} := X_{I,s} \times \operatorname{Spec} \overline{\mathbb{F}}_p$.*
- *For $1 \leqslant h \leqslant d$, let $X^{\geqslant h}_{I,\bar{s}}$ (resp. $X^{=h}_{I,\bar{s}}$) be the closed (resp. open) Newton stratum of height h, defined as the subscheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to h (resp. equal to h).*

Remark: $X^{\geqslant h}_{\mathcal{I},\bar{s}}$ is of pure dimension $d-h$. For $1 \leqslant h < d$, the Newton stratum $X^{=h}_{\mathcal{I},\bar{s}}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d-h}(F_v)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretly this means there exists a closed subscheme $X^{=h}_{I,\bar{s},\overline{1_h}}$ stabilized by the Hecke action of $P_{h,d-h}(F_v)$ and such that

$$X^{=h}_{\mathcal{I},\bar{s}} \simeq X^{=h}_{\mathcal{I},\bar{s},\overline{1_h}} \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

1.1.4. Notations. — *Denote by*

$$i^h : X^{\geqslant h}_{\mathcal{I},\bar{s}} \hookrightarrow X^{\geqslant 1}_{\mathcal{I},\bar{s}}, \quad j^{\geqslant h} : X^{=h}_{\mathcal{I},\bar{s}} \hookrightarrow X^{\geqslant h}_{\mathcal{I},\bar{s}}$$

and $j^{=h} = i^h j^{\geqslant h}$.

1.2. Jacquet-Langlands correspondence and ϱ-type. — For a representation π_v of $GL_d(F_v)$ and $n \in \frac{1}{2}\mathbb{Z}$, set $\pi_v\{n\} := \pi_v \otimes q_v^{-n \,\mathrm{val} \circ \det}$.

Recall that the normalized induction of two representations $\pi_{v,1}$ and $\pi_{v,2}$ of respectively $GL_{n_1}(F_v)$ and $GL_{n_2}(F_v)$ is

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\{\frac{n_2}{2}\} \otimes \pi_{v,2}\{-\frac{n_1}{2}\}.$$

A representation π_v of $GL_d(F_v)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true for $\overline{\mathbb{F}}_l$.
Remark. The modulo l reduction of an irreducible $\overline{\mathbb{Q}}_l$-representation is still irreducible and cuspidal but not necessarily supercuspidal. In this last case, its supercuspidal support is a Zelevinsky segment associated to some unique inertial equivalent class ϱ, where ϱ is an irreducible supercuspidal $\overline{\mathbb{F}}_l$-representation. Thanks to (H2) we will not be concerned by this subtlety.

1.2.1. Definition. — *We say that π_v is of type ϱ when the supercuspidal support of its modulo l reduction is contained in the Zelevinsky line of ϱ.*

1.2.2. Definition. — *(see* [**24**] *§9 and* [**4**] *§1.4) Let g be a divisor of $d = sg$ and π_v an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$. Then the normalized induced representation*

$$\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\}$$

holds a unique irreducible quotient (resp. subspace) denoted by $\mathrm{St}_s(\pi_v)$ (resp. $\mathrm{Speh}_s(\pi_v)$); it is a generalized Steinberg (resp. Speh) representation.

Remark. If χ_v is a character of $F_v^\times$ then $\mathrm{Speh}_s(\chi_v) = \chi_v \circ \det$.

The local Jacquet-Langlands correspondance is a bijection between irreducible essentially square integrable representations of $GL_d(F_v)$, i.e. representations of the type $\mathrm{St}_s(\pi_v)$ for π_v cuspidal, and irreducible representations of $D_{v,d}^\times$ where $D_{v,d}$ is the central division algebra over F_v with invariant $\frac{1}{d}$.

1.2.3. Notation. — *We will denote by* $\pi_v[s]_D$ *the irreducible representation of* $D_{v,d}^\times$ *associated to* $\mathrm{St}_s(\pi_v^\vee)$ *by the local Jacquet-Langlands correspondance.*

We denote by $\mathcal{R}_{\overline{\mathbb{F}}_l}(d)$ the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D_{v,d}^\times$. For $\bar\tau \in \mathcal{R}_{\overline{\mathbb{F}}_l}(d)$, let $\mathcal{C}_{\bar\tau}$ be the sub-category of smooth $\mathbb{Z}_l^{nr}$-representations of $D_{v,d}^\times$ with objects whose irreducible sub-quotients are isomorphic to a sub-quotient of $\bar\tau_{|\mathcal{D}_{v,d}^\times}$. Note that $\mathcal{C}_{\bar\tau}$ is a direct factor inside $\mathrm{Rep}^\infty_{\mathbb{Z}_l^{nr}}(D_{v,d}^\times)$ so that every $\mathbb{Z}_l^{nr}$-representation $V_{\mathbb{Z}_l^{nr}}$ of $D_{v,d}^\times$ can be decomposed as a direct sum

$$V_{\mathbb{Z}_l^{nr}} \simeq \bigoplus_{\bar\tau \in \mathcal{R}_{\overline{\mathbb{F}}_l}(d)} V_{\mathbb{Z}_{l,\bar\tau}^{nr}}$$

where $V_{\mathbb{Z}_{l,\bar\tau}^{nr}}$ is an object of $\mathcal{C}_{\bar\tau}$.

Let π_v be an irreducible cuspidal representation of $GL_g(F_v)$ and fix an integer $s \geqslant 1$. Then the modulo l reduction of $\mathrm{Speh}_s(\pi)$ is irreducible, cf. [**16**] §2.2.3.

1.2.4. Notation. — *When the modulo* l *reduction of* π, *denoted by* ϱ, *is supercuspidal, then we will denote by* $\mathrm{Speh}_s(\varrho)$ *the modulo* l *reduction of* $\mathrm{Speh}_s(\pi)$*: we call it a* $\overline{\mathbb{F}}_l$*- superspeh representation.*

By [**16**] 3.1.4, we have a bijection

$$\Big\{\bar{\mathbb{F}}_l - \text{superspeh irreducible representations of } GL_d(F_v)\Big\}$$
$$\simeq$$
$$\Big\{\bar{\mathbb{F}}_l - \text{representations irreducible of } D_{v,d}^\times\Big\} \quad (1.2.5)$$

compatible with the modulo l reduction in the sense that if π_v is a lifting of ϱ, then the modulo l reduction of $\pi^\vee[s]_D$ matches through the previous bijection, with the superspeh $\mathrm{Speh}_s(\varrho)$.

1.2.6. Definition. — *A* $\overline{\mathbb{F}}_l$*-representation of* $D_{v,d}^\times$ *(resp. an irreducible cuspidal representation of* $GL_d(F_v)$*) is said to be of type* ϱ *if all its irreducible sub-quotients are, through the previous bijection, associated to some superspeh* $\mathrm{Speh}_s(\varrho)$ *(resp. its supercuspidal support belongs to the Zelevinsky line of* ϱ*).*

Recall that if $\epsilon(\varrho)$ is the cardinality of the Zelevinsky line associated to ϱ and, cf. [**23**] p.51, then let

$$m(\varrho)=\begin{cases}\epsilon(\varrho), & \text{if } \epsilon(\varrho)>1;\\ l, & \text{otherwise.}\end{cases}$$

1.2.7. Notation. — *Let $r(\varrho)$ be the biggest integer i such that l^i divides $\frac{d}{m(\varrho)g}$. We then define*

$$g_{-1}(\varrho)=g \quad \text{ and } \quad \forall 0\leqslant i\leqslant r(\varrho),\ g_i(\varrho)=m(\varrho)l^ig.$$

We also denote by $s_i(\varrho):=\lfloor \frac{d}{g_i(\varrho)}\rfloor$.

Then if π_v is an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_k(F_v)$ with type ϱ, then there exists i such that $k=g_i$. We say that π_v is of ϱ-type i and we denote by $\mathrm{Scusp}_i(\varrho)$ the set of inertial equivalence classes of irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations of ϱ-type i.

1.2.8. Notation. — *For ϱ an irreducible supercuspidal $\overline{\mathbb{F}}_l$- representation of $GL_g(F_v)$, we denote by $\mathcal{R}_\varrho=\coprod_{s\geqslant 1}\mathcal{R}_\varrho(sg)$ where $\mathcal{R}_\varrho(sg)$ is the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D_{v,sg}^\times$ of type ϱ.*

1.3. Harris-Taylor local systems. — Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ and fix $t\geqslant 1$ such that $tg\leqslant d$. Thanks to Igusa varieties, Harris and Taylor constructed a local system on $X^{=tg}_{\mathcal{I},\bar{s},\overline{1_h}}$

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_h}}=\bigoplus_{i=1}^{e_{\pi_v}}\mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1_h}}$$

where $(\pi_v[t]_D)_{|\mathcal{D}_{v,h}^\times}=\bigoplus_{i=1}^{e_{\pi_v}}\rho_{v,i}$ with $\rho_{v,i}$ irreducible. The Hecke action of $P_{tg,d-tg}(F_v)$ is then given through its quotient $GL_{d-tg}\times\mathbb{Z}$. These local systems have stable $\overline{\mathbb{Z}}_l$-lattices and we will write simply $\mathcal{L}(\pi_v[t]_D)_{\overline{1_h}}$ for any $\overline{\mathbb{Z}}_l$-stable lattice that we do not want to specify.

1.3.1. Notations. — *For Π_t any representation of GL_{tg} and $\Xi:\frac{1}{2}\mathbb{Z}\longrightarrow\overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2})=q^{1/2}$, we introduce*

$$\widetilde{HT}_1(\pi_v,\Pi_t):=\mathcal{L}(\pi_v[t]_D)_{\overline{1_h}}\otimes\Pi_t\otimes\Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}(\pi_v, \Pi_t) := \Big(\mathcal{L}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \Big) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

is given

- *by the action of g_v^c on Π_t and* $\deg(\sigma_v) \in \mathbb{Z}$ *on $\Xi^{\frac{tg-d}{2}}$, and*
- *the action of* $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ *on* $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_h}} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT(\pi_v, \Pi_t)_{\overline{1_h}} := \widetilde{HT}(\pi_v, \Pi_t)_{\overline{1_h}}[d-tg],$$

and the perverse sheaf

$$P(t, \pi_v)_{\overline{1_h}} := j^{=tg}_{\overline{1_h},!*} HT(\pi_v, \mathrm{St}_t(\pi_v))_{\overline{1_h}} \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT(\pi_v, \Pi_t)$ and $P(t, \pi_v)$, where

$$j = h = i^h \circ j^{\geqslant h} : X^{=h}_{\mathcal{I},\bar{s}} \hookrightarrow X^{\geqslant h}_{\mathcal{I},\bar{s}} \hookrightarrow X_{\mathcal{I},\bar{s}}$$

and $\mathbb{L}^\vee$ is the local Langlands correspondence.

Remark. Recall that π'_v is said to be inertially equivalent to π_v if there exists a character $\zeta : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \mathrm{val} \circ \det)$. Note, cf. [**3**] 2.1.4, that $P(t, \pi_v)$ depends only on the inertial class of π_v and

$$P(t, \pi_v) = e_{\pi_v} \mathcal{P}(t, \pi_v)$$

where $\mathcal{P}(t, \pi_v)$ is an irreducible perverse sheaf. When we want to speak of the $\overline{\mathbb{Q}}_l$-versions we will add it on the notations.

1.3.2. Definition. — *We will say that $HT(\pi_v, \Pi_t)$ or $\mathcal{P}(t, \pi_v)$ is of type ϱ if π_v is.*

1.3.1. Lemma. — *If $\rho \otimes \sigma$ is a $GL_d(F_v) \times W_v$ equivariant irreducible sub-quotient of $H^i(X_{\mathcal{I},\bar{s}_v}, \mathcal{P}(\pi_v, t) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$ then*

- *σ is an irreducible sub-quotient of the modulo l reduction of $\mathbb{L}(\pi_v \otimes \chi_v)$, where χ_v is an uramified character of F_v, and*

– *ρ is an irreducible sub-quotient of the modulo l reduction of a induced representation of the following shape* $\mathrm{St}_t(\pi_v \otimes \chi_v) \times \psi_v$ *where* ψ_v *is an integral irreducible representation of* $GL_{d-tg}(F_v)$.

Proof. — The result follows directly from the description of the actions given previously. □

As usually for σ a representation of W_v and $n \in \frac{1}{2}\mathbb{Z}$, we will denote by $\sigma(n)$ the twisted representation $g \mapsto \sigma(g)|\operatorname{Art}_v^{-1}(g)|^n$ where $| - |$ is the absolute value of F_v.

1.4. Free perverse sheaf. — Let $S = \operatorname{Spec} \mathbb{F}_q$ and X/S of finite type, then the usual t-structure on $\mathcal{D}(X, \overline{\mathbb{Z}}_l) := D^b_c(X, \overline{\mathbb{Z}}_l)$ is

$$\begin{aligned} A \in {}^p\mathcal{D}^{\leqslant 0}(X, \overline{\mathbb{Z}}_l) &\Leftrightarrow \forall x \in X,\ \mathcal{H}^k i_x^* A = 0,\ \forall k > -\dim \overline{\{x\}} \\ A \in {}^p\mathcal{D}^{\geqslant 0}(X, \overline{\mathbb{Z}}_l) &\Leftrightarrow \forall x \in X,\ \ \mathcal{H}^k i_x^! A = 0,\ \forall k < -\dim \overline{\{x\}} \end{aligned}$$

where $i_x : \operatorname{Spec} \kappa(x) \hookrightarrow X$ and $\mathcal{H}^k(K)$ is the k-th sheaf of cohomology of K.

1.4.1. Notation. — *Let ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ denote the heart of this t-structure with associated cohomology functors ${}^p\mathcal{H}^i$. For a functor T we denote ${}^pT := {}^p\mathcal{H}^0 \circ T$.*

The category ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ is abelian equipped with a torsion theory $(\mathcal{T}, \mathcal{F})$ where $\mathcal{T}$ (resp. $\mathcal{F}$) is the full subcategory of objects T (resp. F) such that $l^N 1_T$ is trivial for some large enough N(resp. $l.1_F$ is a monomorphism). Applying Grothendieck-Verdier duality, we obtain

$$\begin{aligned} {}^{p+}\mathcal{D}^{\leqslant 0}(X, \overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\leqslant 1}(X, \overline{\mathbb{Z}}_l) :\ {}^p\mathcal{H}^1(A) \in \mathcal{T}\} \\ {}^{p+}\mathcal{D}^{\geqslant 0}(X, \overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\geqslant 0}(X, \overline{\mathbb{Z}}_l) :\ {}^p\mathcal{H}^0(A) \in \mathcal{F}\} \end{aligned}$$

with heart${}^{p+}\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ equipped with its torsion theory $(\mathcal{F}, \mathcal{T}[-1])$.

1.4.2. Definition. — *(cf.* [**6**] *§1.3) Let*

$$\mathcal{F}(X, \overline{\mathbb{Z}}_l) := {}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{C}(X, \overline{\mathbb{Z}}_l) = {}^p\mathcal{D}^{\leqslant 0}(X, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geqslant 0}(X, \overline{\mathbb{Z}}_l)$$

the quasi-abelian category of free perverse sheaves over X.

Remark. For an object L of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$, we will consider filtrations
$$L_1 \subset L_2 \subset \cdots \subset L_e = L$$
such that for every $1 \leqslant i \leqslant e-1$, $L_i \hookrightarrow L_{i+1}$ is a strict monomorphism, i.e. L_{i+1}/L_i is an object of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$.

For a free $L \in \mathcal{F}(X,\Lambda)$, we consider the following diagram
$$\begin{tikzcd}
 & L \arrow[drr, "\mathrm{can}_{*,L}"] & & \\
{}^{p+}j_!j^*L \arrow[ur, "\mathrm{can}_{!,L}"] \arrow[r, "+", two heads] & {}^{p}j_{!*}j^*L \arrow[r, hook, two heads, "+"'] & {}^{p+}j_{!*}j^*L \arrow[r, hook, "+"] & {}^{p}j_*j^*L
\end{tikzcd}$$
where below is, cf. the remark following 1.3.12 of [**6**], the canonical factorisation of ${}^{p+}j_!j^*L \longrightarrow {}^{p}j_*j^*L$ and where the maps $\mathrm{can}_{!,L}$ and $\mathrm{can}_{*,L}$ are given by the adjonction property. Consider now X equipped with a stratification
$$X = X^{\geqslant 1} \supset X^{\geqslant 2} \supset \cdots \supset X^{\geqslant d,}$$
and let $L \in \mathcal{F}(X,\overline{\mathbb{Z}}_l)$. For $1 \leqslant h < d$, denote by $X^{1\leqslant h} := X^{\geqslant 1} - X^{\geqslant h+1}$ and $j^{1\leqslant h} : X^{1\leqslant h} \hookrightarrow X^{\geqslant 1}$. We then define
$$\mathrm{Fil}^r_!(L) := \mathrm{Im}_{\mathcal{F}}\Big({}^{p+}j^{1\leqslant r}_! j^{1\leqslant r,*}L \longrightarrow L\Big),$$
which gives a filtration
$$0 = \mathrm{Fil}^0_!(L) \subset \mathrm{Fil}^1_!(L) \subset \mathrm{Fil}^1_!(L) \cdots \subset \mathrm{Fil}^{d-1}_!(L) \subset \mathrm{Fil}^d_!(L) = L.$$
Dually $\mathrm{CoFil}_{*,r}(L) = \mathrm{Coim}_{\mathcal{F}}\Big(L \longrightarrow {}^{p}j^{1\leqslant r}_* j^{1\leqslant r,*}L\Big)$, define a cofiltration
$$\begin{aligned} L = \mathrm{CoFil}_{\mathfrak{G},*,d}(L) \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{G},*,d-1}(L) \twoheadrightarrow \cdots \\ \cdots \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{G},*,1}(L) \twoheadrightarrow \mathrm{CoFil}_{\mathfrak{G},*,0}(L) = 0,\end{aligned}$$
and a filtration
$$0 = \mathrm{Fil}^{-d}_*(L) \subset \mathrm{Fil}^{1-d}_*(L) \subset \cdots \subset \mathrm{Fil}^0_*(L) = L$$
where $\mathrm{Fil}^{-r}_*(L) := \mathrm{Ker}_{\mathcal{F}}\big(L \twoheadrightarrow \mathrm{CoFil}_{*,r}(L)\big)$.
Remark. These two constructions are exchanged by Grothendieck-Verdier duality, $D\Big(\mathrm{CoFil}_{\mathfrak{G},!,-r}(L)\Big) \simeq \mathrm{Fil}^{-r}_{\mathfrak{G},*}(D(L))$ and $D\Big(\mathrm{CoFil}_{\mathfrak{G},*,r}(L)\Big) \simeq \mathrm{Fil}^r_{\mathfrak{G},!}(D(L))$.

We can also refine the previous filtrations, cf. [**6**] proposition 2.3.3, to obtain exhaustive filtrations

$$0 = \mathrm{Fill}_!^{-2^{d-1}}(L) \subset \mathrm{Fill}_!^{-2^{d-1}+1}(L) \subset \cdots$$
$$\cdots \subset \mathrm{Fill}_!^{0}(L) \subset \cdots \subset \mathrm{Fill}_!^{2^{d-1}-1}(L) = L, \quad (1.4.3)$$

such that the graded parts $\mathrm{grr}^k(L)$ are simple over $\overline{\mathbb{Q}}_l$, i.e. verify ${}^p j^{=h}_{!*} j^{=h,*}\mathrm{grr}^k(L) \hookrightarrow\!\!\!\!\rightarrow_+ \mathrm{grr}^k(L)$ for some h. Dually we can construct a cofiltration

$$L = \mathrm{CoFill}_{*,2^{d-1}}(L) \twoheadrightarrow \mathrm{CoFill}_{*,2^{d-1}-1}(L) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{CoFill}_{*,-2^{d-1}}(L) = 0$$

and a filtration $\mathrm{Fill}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}\big(L \twoheadrightarrow \mathrm{CoFill}_{*,r}(L)\big)$.

1.5. Vanishing cycles perverse sheaf. —

1.5.1. Notation. — *For $I \in \mathcal{I}$, let*

$$\Psi_{I,\Lambda} := R\Psi_{\eta_v,I}(\Lambda[d-1])(\frac{d-1}{2})$$

be the vanishing cycle autodual perverse sheaf on $X_{I,\bar{s}_v}$. When $\Lambda = \overline{\mathbb{Z}}_l$, we will simply write $\Psi_{\mathcal{I}}$.

Recall the following result of [**19**] relating $\Psi_{\mathcal{I}}$ with Harris-Taylor local systems.

1.5.2. Proposition. — *(cf. [**19**] proposition IV.2.2 and §2.4 of [**3**]) There is a $G(\mathbb{A}^{\infty,v}) \times P_{h,d-h}(F_v) \times W_v$-equivariant isomorphism*

$$\mathrm{ind}_{(D_{v,h}^\times)^0\varpi_v^{\mathbb{Z}}}^{D_{v,h}^\times}\big(\mathcal{H}^{h-d-i}\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}\big)_{|X^{=h}_{\mathcal{I},\bar{s},\overline{1_h}}} \simeq \bigoplus_{\bar{\tau}\in\mathcal{R}_{\overline{\mathbb{F}}_l}(h)} \mathcal{L}_{\overline{\mathbb{Z}}_l,\overline{1_h}}(\mathcal{U}^{h-1-i}_{\bar{\tau},\mathbb{N}}),$$

where

- *$\mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ is the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D_{v,h}^\times$;*
- *for $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ and V a $\overline{\mathbb{Z}}_l$-representation of $D_{v,h}^\times$, $V_{\bar{\tau}}$ denotes, cf. [**15**] §B.2, the direct factor of V whose irreducible subquotients are isomorphic to a subquotient of $\bar{\tau}_{|\mathcal{D}_{v,h}^\times}$ where $\mathcal{D}_{v,h}$ is the maximal order of $D_{v,h}$.*
- *With the previous notation, $\mathcal{U}^i_{\bar{\tau},\mathbb{N}} := \big(\mathcal{U}^i_{F_v,\overline{\mathbb{Z}}_l,d}\big)_{\bar{\tau}}$.*

– *The matching between the system indexed by $\mathcal{I}$ and those by $\mathbb{N}$ is given by the map $m_1 : \mathcal{I} \longrightarrow \mathbb{N}$.*

Remark. For $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$, and a lifting τ which by Jacquet-Langlands correspondence can be written $\tau \simeq \pi[t]_D$ for π irreducible cuspidal, let $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ be in the supercuspidal support. Then the inertial class of ϱ depends only on $\bar{\tau}$ and we will use the following notation.

1.5.3. Notation. — *With the previous notation, we denote by V_ϱ for $V_{\bar{\tau}}$.*

Remark. $\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$ is an object of $\mathcal{F}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l)$. Indeed, by [**2**] proposition 4.4.2, $\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$ is an object of ${}^p\mathcal{D}^{\leqslant 0}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l)$. By [**20**] variant 4.4 of theorem 4.2, we have $D\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l} \simeq \Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$, so that

$$\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l} \in {}^p\mathcal{D}^{\leqslant 0}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geqslant 0}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l) = \mathcal{F}(X_{\mathcal{I},\bar{s}}, \overline{\mathbb{Z}}_l).$$

1.5.4. Proposition. — *(cf. [**9**] §3.2) We have a decomposition*

$$\Psi_{\mathcal{I}} \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_\varrho$$

where all the Harris-Taylor perverse sheaves of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are of type ϱ.

Remark. In [**3**], we decomposed $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ as a direct sum $\bigoplus_{\pi_v} \Psi_{\pi_v}$ where π_v described the set of equivalent inertial classes of irreducible cuspidal representations. The $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} \Psi_{\pi_v}$ where $\mathrm{Cusp}(\varrho)$ is the set of equivalent inertial classes of irreducible cuspidal representation of type ϱ in the sense of definition 1.2.6.

In [**10**], we give the precise description of the $\mathrm{gr}^r_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho})$ which is defined over $\overline{\mathbb{Z}}_l$. By construction they are supported on $X^{\geqslant r}_{\mathcal{I},\bar{s}_v}$ and trivial if g does not divide r. Otherwise for $r = tg$, we have

$$\mathrm{ind}_{(D^\times_{v,tg})^0 \varpi_v^{\mathbb{Z}}}^{D^\times_{v,tg}} \Big(j^{=tg,*} \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \Big) \simeq \bigoplus_{\substack{i=-1 \\ t_i g_i(\varrho) = tg}}^{r(\varrho)} \bigoplus_{\pi_v \in \mathrm{Scusp}_i(\varrho)} HT(\pi_v, \mathrm{St}_{t_i}(\pi_v)) \otimes \mathbb{L}(\pi_v)(-\frac{t_i - 1}{2}).$$

We can then consider the following naive ϱ-filtration

$$\mathrm{Fil}^*_{\varrho,r(\varrho),\overline{\mathbb{Q}}_l}(\Psi,tg) \subset \cdots \subset \mathrm{Fil}^*_{\varrho,-1,\overline{\mathbb{Q}}_l}(\Psi,tg) = j^{=tg,*}\mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$$

where $\mathrm{ind}_{(D^\times_{v,tg})^0\varpi_v^{\mathbb{Z}}}^{D^\times_{v,tg}}\Big(\mathrm{Fil}^*_{\varrho,k,\overline{\mathbb{Q}}_l}(\Psi,tg)\Big)$ is isomorphic to

$$\bigoplus_{\substack{i=k \\ t_ig_i(\varrho)=tg}}^{r(\varrho)} \bigoplus_{\pi_v\in\mathrm{Scusp}_i(\varrho)} HT(\pi_v,\mathrm{St}_{t_i}(\pi_v))\otimes\mathbb{L}(\pi_v)(-\frac{t_i-1}{2}),$$

and the associated integral filtration of $j^{=tg,*}\mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho})$ defined by pull-back

$$\begin{tikzcd}
\mathrm{Fil}^*_{\varrho,k}(\Psi,tg) \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \mathrm{Fil}^*_{\varrho,k,\overline{\mathbb{Q}}_l}(\Psi,tg) \arrow[d, hook] \\
j^{=tg,*}\mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho}) \arrow[r, hook] & j^{=tg,*}\mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l.
\end{tikzcd}$$

For $k=-1,\cdots,r(\varrho)$, the graded parts $\mathrm{gr}_{\varrho,k}(\Psi,tg)$ are then of ϱ-type k. We can then refine these filtrations by separating the $\pi_v \in \mathrm{Scusp}_k(\varrho)$ to obtain

$$(0) = \mathrm{Fil}^{*,0}_{\varrho}(\Psi,tg) \subset \mathrm{Fil}^{*,1}_{\varrho}(\Psi,tg) \subset \cdots \subset \mathrm{Fil}^{*,r}_{\varrho}(\Psi,tg) = j^{=tg,*}\mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho}).$$

By taking the iterated images of $j^{=tg}_! \,\mathrm{Fil}^k_\varrho(\Psi,tg) \longrightarrow \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho})$, we then construct a filtration

$$(0) = \mathrm{Fil}^0_\varrho(\Psi,tg) \subset \mathrm{Fil}^1_\varrho(\Psi,tg) \subset \cdots \subset \mathrm{Fil}^r_\varrho(\Psi,tg) = \mathrm{gr}^{tg}_{\mathfrak{S},!}(\Psi_{\mathcal{I},\varrho}).$$

Finally we can filtrate each of these graded part using an exhaustive filtration of stratification to obtain a filtration of $\Psi_{\mathcal{I},\varrho}$ whose graded parts are the $\mathfrak{P}(\pi_v,t)(\frac{1-s_i(\varrho)+2k}{2})$ for $\pi_v \in \mathrm{Scusp}_i(\varrho)$ with $i \geqslant -1$, and $k = 0,\cdots,s_i(\varrho)-1$.

2. Cohomology of KHT Shimura varieties

2.1. Localization at a non pseudo-Eisenstein ideal. —

2.1.1. Definition. — *Let* Spl *be the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write* $\mathrm{Spl}(I)$ *the subset of* Spl *of places which do not divide the level I.*

Let $\mathrm{Unr}(I)$ be the union of

- places $q \neq l$ of $\mathbb{Q}$ inert in E not below a place of Bad and where I_q is maximal,
- and places $w \in \mathrm{Spl}(I)$.

2.1.2. Notation. — *For $I \in \mathcal{I}$ a finite level, write*

$$\mathbb{T}_I := \prod_{x \in \mathrm{Unr}(I)} \mathbb{T}_x$$

where for x a place of $\mathbb{Q}$ (resp. $x \in \mathrm{Spl}(I)$), $\mathbb{T}_x$ is the unramified Hecke algebra of $G(\mathbb{Q}_x)$ (resp. of $GL_d(F_x)$) over $\overline{\mathbb{Z}}_l$.

Example: for $w \in \mathrm{Spl}(I)$, we have

$$\mathbb{T}_w = \overline{\mathbb{Z}}_l\big[T_{w,i} : \ i = 1, \cdots, d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w, \cdots, \varpi_w}^{i}, \overbrace{1, \cdots, 1}^{d-i})GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

More generally, the Satake isomorphism identifies $\mathbb{T}_x$ with $\overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ where

- T_x is a split torus,
- W_x is the spherical Weyl group
- and $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.

Consider a fixed maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ and for every $x \in \mathrm{Unr}(I)$ denote by $S_{\mathfrak{m}}(x)$ the multi-set(3) of modulo l Satake parameters at x associated to $\mathfrak{m}$.

Example: for every $w \in \mathrm{Spl}(I)$, the multi-set of Satake parameters at w corresponds to the roots of the Hecke polynomial

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^{d} (-1)^i q_w^{\frac{i(i-1)}{2}} \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_l[X]$$

i.e. $S_{\mathfrak{m}}(w) := \big\{\lambda \in \mathbb{T}_I/\mathfrak{m} \simeq \overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda) = 0\big\}$. For a maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, we also have the multi-set of Satake parameters

$$S_{\widetilde{\mathfrak{m}}}(w) := \big\{\lambda \in \mathbb{T}_I \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l/\widetilde{\mathfrak{m}} \simeq \overline{\mathbb{Q}}_l \text{ such that } P_{\widetilde{\mathfrak{m},w}}(\lambda) = 0\big\}.$$

(3) A multi-set is a set with multiplicities.

2.1.3. Notation. — *Let Π be an irreducible automorphic representation of $G(\mathbb{A})$ of level I which means here, that Π has non trivial invariants under I and for every $x \in \mathrm{Unr}(I)$, then Π_x is unramified. Then Π defines*

- *a maximal ideal $\widetilde{\mathfrak{m}}(\Pi)$ of $\mathbb{T}_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, or*
- *a minimal prime ideal $\widetilde{\mathfrak{m}}(\Pi)$ of $\mathbb{T}_{I^l}$ contained in a maximal ideal $\mathfrak{m}(\Pi)$ of $\mathbb{T}_{I^l}$ which corresponds to its the modulo l Satake parameters.*

A minimal prime ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_{I^l}$ is said to be cohomological if there exists a cohomological automorphic $\overline{\mathbb{Q}}_l$-representation Π of $G(\mathbb{A})$ of level I with $\widetilde{\mathfrak{m}} = \widetilde{\mathfrak{m}}(\Pi)$. Such a Π is not unique but $\widetilde{\mathfrak{m}}$ defines a unique near equivalence class in the sense of [**22**], we denote it by $\Pi_{\widetilde{\mathfrak{m}}}$. Let then

$$\rho_{\widetilde{\mathfrak{m}},\overline{\mathbb{Q}}_l} : \mathrm{Gal}(\bar{F}/F) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

be the galoisian representation associated to such a Π thanks to [**19**] and [**22**], which by the Cebotarev theorem, can be defined over some finite extension $K_{\widetilde{\mathfrak{m}}}$, i.e. $\rho_{\widetilde{\mathfrak{m}},\overline{\mathbb{Q}}_l} \simeq \rho_{\widetilde{\mathfrak{m}}} \otimes_{K_{\widetilde{\mathfrak{m}}}} \overline{\mathbb{Q}}_l$.

It's well known that $\rho_{\widetilde{\mathfrak{m}}}$ has stable lattices and the semi-simplication of its modulo l reduction is independent of the chosen stable lattice. Moreover it depends only of the maximal ideal $\mathfrak{m}$; we denote by

$$\overline{\rho}_{\mathfrak{m}} : G_F \longrightarrow GL_d(\overline{\mathbb{F}}_l),$$

its extension to $\overline{\mathbb{F}}_l$. For every $w \in \mathrm{Spl}(I)$, recall that the multiset of eigenvalues of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_w)$ is $S_{\mathfrak{m}}(w)$ obtained from $S_{\widetilde{\mathfrak{m}}}(w)$ by taking modulo l reduction.

Assume moreover that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible. Then the $\overline{\mathbb{Q}}_l$-cohomology group $H^{d-1}(X_{U,\bar{\eta}}, \overline{\mathbb{Q}}_l)_{\mathfrak{m}}$ gives a continuous d-dimensional Galois representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}_{S,\mathfrak{m}}[1/l]),$$

where $\mathrm{Gal}_{F,S}$ is the Galois group of the maximal extension of F which is unramified outside S. As all characteristic polynomials of Frobenius take values in $\mathbb{T}_{S,\mathfrak{m}}$ and as $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible, using the classical theory of pseudo-representations, we know that $\rho_{\mathfrak{m}}$ takes values in $GL_d(\mathbb{T}_{S,\mathfrak{m}})$.

2.2. Freeness of the cohomology. — From now on we fix such a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ verifying one of the following three conditions, cf. the introduction also.

(1) There exists $w_1 \in \mathrm{Spl}(I)$ such that $S_{\mathfrak{m}}(w_1)$ does not contain any sub-multi-set of the shape $\{\alpha, q_{w_1}\alpha\}$ where q_{w_1} is the cardinality of the residue field. This hypothesis is called *generic* in [**12**].

(2) With $[F(\exp(2i\pi/l) : F] > d$, we ask $\mathfrak{m}$ to verify one of the two following hypothesis
- either $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K for a cyclic galoisian extension K/F;
- or $SL_n(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^{\times} GL_n(k)$ for a sub-field $k \subset \overline{\mathbb{F}}_l$.

Remark. If the main result of [**11**] is true, one may only suppose, beside the irreducibility of $\overline{\rho}_{\mathfrak{m}}$, that $[F(\exp(2i\pi/l) : F] > d$.

2.2.1. Theorem. — *(cf.* [**8**]*) For* $\mathfrak{m}$ *as above, the localized cohomology groups* $H^i(X_{I,\bar\eta}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$ *are free.*

As $X_I \longrightarrow \operatorname{Spec} \mathcal{O}_v$ is proper, we have a $G(\mathbb{A}^\infty) \times W_v$-equivariant isomorphism $H^{d-1+i}(X_{\mathcal{I},\bar\eta_v}, \overline{\mathbb{Z}}_l) \simeq H^i(X_{I,\bar s_v}, \Psi_{\mathcal{I}})$. Using the previous filtration of $\Psi_{\mathcal{I}}$, we can compute $H^{p+q}(X_{I,\bar s_v}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}}$ through a spectral sequence whose entries[(4)] $E_1^{p,q}$ are the $H^{p+q}(X_{\mathcal{I},\bar s_v}, \mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2}))_{\mathfrak{m}}$ for $\pi_v \in \mathrm{Scusp}_i(\varrho)$ with $i \geqslant -1$, and $k = 0, \cdots, s_i(\varrho) - 1$. Over $\overline{\mathbb{Q}}_l$, it follows from [**4**], thanks to the hypothesis (1) above on $\mathfrak{m}$, that all these cohomology groups are concentrated in degree 0 so that this $\overline{\mathbb{Q}}_l$-spectral sequence degenerates in E_1. In this section, under (H2), we want to prove the same result on $\overline{\mathbb{F}}_l$ which is equivalent to the freeness of the $H^j(X_{\mathcal{I},\bar s_v}, \mathfrak{P}(\pi_v, t)(\frac{1-s_i(\varrho)+2k}{2}))_{\mathfrak{m}}$.

We need first some notations from [**4**] §1.2. For all $t \geqslant 0$, we denote by

$$\Gamma^t := \Big\{(a_1, \cdots, a_r, \epsilon_1, \cdots, \epsilon_r) \in \mathbb{N}^r \times \{\pm\}^r : \ r \geqslant 1, \ \sum_{i=1}^r a_i = t\Big\}.$$

A element of Γ^t will be denoted by $(\overleftarrow{a_1}, \cdots, \overrightarrow{a_r})$ where for the arrow above each integer a_i is $\overleftarrow{a_i}$ (resp. $\overrightarrow{a_i}$) if $\epsilon_i = -$ (resp. $\epsilon_i = +$). We then consider

(4) We do not need here to give the precise relations between (p, q) and i, t, k in the formula.

on Γ^t the equivalence relation induced by

$$(\cdots, \overleftarrow{a}, \overleftarrow{b}, \cdots) = (\cdots, \overleftarrow{a+b}, \cdots), \quad (\cdots, \overrightarrow{a}, \overrightarrow{b}, \cdots) = (\cdots, \overrightarrow{a+b}, \cdots),$$

and $(\cdots, \overleftarrow{0}, \cdots) = (\cdots, \overrightarrow{0}, \cdots)$. We denote by $\overrightarrow{\Gamma}^t$ the set of these equivalence classes whose elements are denoted by $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]$.

Remark. In each class $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_k}] \in \overrightarrow{\Gamma}^t$, there exists a unique reduced element $(b_1, \cdots, b_r, \epsilon_1, \cdots, \epsilon_r) \in \Gamma^t$ such that for all $1 \leqslant i \leqslant r$ $b_i > 0$ and for $1 \leqslant i < r$, $\epsilon_i \epsilon_{i+1} = -$.

2.2.1. Definition. — *Let $(b_1, \cdots, b_r, \epsilon_1, \cdots, \epsilon_r)$ be the reduced element in $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_k}] \in \overrightarrow{\Gamma}^t$. We then define*

$$\mathcal{S}\Big([\overleftarrow{a_1}, \cdots, \overrightarrow{a_k}]\Big)$$

as the subset of permutations σ of $\{0, \cdots, t-1\}$ such that for all $1 \leqslant i \leqslant r$ with $\epsilon_i = +$ (resp. $\epsilon_i = -$) and for all $b_1 + \cdots + b_{i-1} \leqslant k < k' \leqslant b_1 + \cdots + b_i$ then $\sigma^{-1}(k) < \sigma^{-1}(k')$ (resp. $\sigma^{-1}(k) > \sigma^{-1}(k')$).

We also introduce $\mathcal{S}^{op}\Big([\overleftarrow{a_1}, \cdots, \overrightarrow{a_k}]\Big)$, by imposing under the same conditions, $\sigma^{-1}(k) > \sigma^{-1}(k')$ (resp. $\sigma^{-1}(k) < \sigma^{-1}(k')$).

2.2.2. Proposition. — *(cf. [**24**] §2) Let g be a divisor of $d = sg$ and π an irreducible cuspidal representation of $GL_g(F_v)$. There exists a bijection*

$$[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}] \in \overrightarrow{\Gamma}^{s-1} \mapsto [\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\pi$$

into the set of irreducible sub-quotients of the induced representation

$$\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\}$$

characterized by the following property

$$J_{P_{g,2g,\cdots,sg}}([\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\pi) = \sum_{\sigma \in \mathcal{S}\Big([\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]\Big)} \pi\{\frac{1-s}{2} + \sigma(0)\} \otimes \cdots \otimes \pi\{\frac{1-s}{2} + \sigma(s-1)\},$$

or equivalently by

$$J_{P^{op}_{g,2g,\cdots,sg}}([\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\pi) = \sum_{\sigma \in \mathcal{S}^{op}\Big([\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]\Big)} \pi\{\frac{1-s}{2} + \sigma(0)\} \otimes \cdots \otimes \pi\{\frac{1-s}{2} + \sigma(s-1)\}.$$

Remark. With this notation $\mathrm{St}_s(\pi)$ (resp. $\mathrm{Speh}_s(\pi)$) is $[\overleftarrow{s-1}]_\pi$ (resp. $[\overrightarrow{s-1}]_\pi$).

2.2.3. Lemma. — *Let π be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ such that its modulo l reduction ϱ is supercuspidal. Suppose the cardinality of the Zelevinsky line of ϱ is greater or equal to s. Then the irreducible sub-quotients of the modulo l reduction of $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\pi$ for $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]$ describing $\overrightarrow{\Gamma}^{s-1}$, are pairwise distinct.*

Proof. — By hypothesis on the cardinality of the Zelevinsky line, all these irreducible $\overline{\mathbb{F}}_l$-sub-quotient have a non trivial image under $J_{P_{g,2g,\cdots,sg}}$. The result then follows directly

- from the commutation of Jacquet functors with the modulo l reduction and
- from the fact that the $r_l(\pi)\{\frac{1-s}{2}+k\}$ are pairwise distinct for $0 \leqslant k < s$ so that the image under $J_{P_{g,2g,\cdots,sg}}$ of $\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\}$ is multiplicity free.

□

2.2.2. Notation. — *We will denote by $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\varrho$ any irreducible sub-quotient of the modulo l reduction of $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_r}]_\pi$.*

Remark. If moreover the cardinality of the Zelevinsky line of ϱ is strictly greater than s, then, cf. [**5**], the modulo l reduction of $[\overleftarrow{s-1}]_\pi$ is irreducible and non degenerate, i.e. $[\overleftarrow{s-1}]_\varrho$ is well determinate and non degenerate.

2.2.4. Theorem. — *Consider a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_S$ such that for all i, the $\overline{\mathbb{Z}}_l$-module $H^i(X_{U,\bar\eta}, \overline{\mathbb{Z}}_l)_\mathfrak{m}$ is free. We suppose moreover (H2) that the image of $\bar\rho_{\mathfrak{m},w_0}$ in the Grothendieck group is multiplicity free. Then for all $\overline{\mathbb{Z}}_l$-Harris-Taylor local system $HT(\pi_{w_0}, t)$, the $H^i(X_{U,\bar s_{w_0}}, {}^p j_{!*}^{=tg} HT(\pi_{w_0}, t))_\mathfrak{m}$ are free.*

Remark. In the previous statement, we just need the multiplicity free part of (H2) as it is used in the previous lemma. Note moreover that, by [**7**] §4.5, the multiplicity free hypothesis is necessarily.

Proof. — First denote by $\mathrm{Scusp}_{w_0}(\mathfrak{m})$ the set of inertial equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations belonging to the supercuspidal support of the modulo l reduction of the local component at w_0 of a representation Π in the near equivalence class $\Pi_{\mathfrak{m}}$ associated to $\mathfrak{m}$.

We then consider the vanishing cycle spectral sequence at w_0, localized at $\mathfrak{m}$

$$H^i(X_{U,\bar{\eta}_{w_0}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}} \simeq \bigoplus_{\varrho \in \mathrm{Scusp}_{w_0}(\mathfrak{m})} H^i(X_{U,\bar{s}_{w_0}}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}}.$$

Then for every $\varrho \in \mathrm{Scusp}_{w_0}(\mathfrak{m})$, the $H^i(X_{U,\bar{s}_{w_0}}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}}$ are free. For π_v of type ϱ, the strategy to prove the freeness of the $H^i(X_{U,\bar{s}_{w_0}}, {}^p j^{=tg}_{!*} HT(\pi_{w_0}, t))_{\mathfrak{m}}$ is then to argue by absurdity and to produce some torsion cohomology class in one of the $H^i(X_{U,\bar{s}_{w_0}}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}}$. Let then t be minimal such that there exists $i \neq 0$ with

$$H^i(X_{\mathcal{I},\bar{s}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \neq (0)$$

for $0 \leqslant k < t$, and where $\mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2})$ is a graded part of some filtration of Ψ_ϱ.

Consider for example the filtration constructed before using the adjunction $j^{1\leqslant h}_! j^{1\leqslant h,*} \longrightarrow \mathrm{Id}$. As remarked before and considering also $\Psi_{\varrho^\vee}$ and its filtration constructed using $\mathrm{Id} \longrightarrow j^{1\leqslant h}_* j^{1\leqslant h,*}$, we can suppose that such i is strictly negative and we denote by i_0 such a minimal i.

By lemma 1.3.1, an irreducible $GL_d(F_{w_0}) \times W_{w_0}$-equivariant subquotient of $H^i(X_{\mathcal{I},\bar{s}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is one of the modulo l reduction of a representation we can write in the following shape

$$\Big([\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overleftrightarrow{1}, \overleftarrow{t-1}, \overleftrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overleftarrow{a_r}]_\pi \times \Upsilon_{w_0} \Big) \otimes \mathbb{L}(\pi\{\frac{\delta}{2}\})$$

where the a_j are some integers, Υ_{w_0} is an irreducible $\overline{\mathbb{Q}}_l$-representation whose modulo l reduction has a supercuspidal support away from those of the previous segment and where

– the symbol $\overleftrightarrow{1}$ before (resp. after) the $\overleftarrow{t-1}$ can be $\overleftarrow{1}$ or $\overrightarrow{1}$ if $\sum_{j=1}^{i-1} a_i > 0$ (resp. $\sum_{j=i+1}^{r} a_i > 0$). We will write moreover $a_i = t+1$.

– Let $\Big\{\pi\{\frac{\alpha}{2}\}, \pi\{\frac{\alpha}{2}+1\}, \cdots, \pi\{\frac{\alpha}{2}+t-1\}\Big\}$ denote the supercuspidal support of $\overleftarrow{t-1}$ inside $[\overleftarrow{a_1}, \cdots, \overleftarrow{t-1}, \cdots \overrightarrow{a_r}]_\pi$. The $\frac{\delta}{2} = \frac{\alpha}{2} + k$ where k is the integer in $\mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2})$.

Remark. In particular we can suppose that the previous k is equal to 0.

Consider then such an irreducible sub-quotient $\tau \times \psi_{w_0} \otimes \sigma$ where

– ψ_{w_0} (resp. σ) is any irreducible sub-quotient of the modulo l reduction of Υ_{w_0} (resp. $\mathbb{L}(\pi\{\frac{\delta}{2}\})$), and
– τ is an irreducible sub-quotient of the modulo l reduction of some

$$[\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overrightarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overrightarrow{a_r}]_\pi.$$

By the previous lemma, we can recover the a_i from τ.

Let show now that this $\tau \times \psi_{w_0} \otimes \sigma$ is also a sub-quotient of $H^{i_0}(X_{\mathcal{I},\bar{s}_{w_0}}, \Psi_{\mathcal{I},\varrho})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, which contradicts our hypothesis on $\mathfrak{m}$. Let denore $\mathrm{Fil}^{k-1} \subset \mathrm{Fil}^k \subset \Psi_{\mathcal{I},\varrho}$ such that $\mathrm{gr}^k = \mathrm{Fil}^k / \mathrm{Fil}^{k-1} \simeq \mathcal{P}(\pi_{w_0}, t)(\frac{1-t}{2})$. By the hypothesis (H2), all the irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations $\pi'_{w_0} \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(\varrho)$ such that one of the $H^i(X_{\mathcal{I},\bar{s}_{w_0}}, \mathcal{P}(\pi'_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \neq (0)$ are necessarily of ϱ-type -1. Then in particular all the Harris-Taylor perverse sheaves $\mathcal{P}(\pi'_{w_0}, t')$ which are sub-quotient of Fil^{k-1}, must verify $t' > t$. The spectral sequence which computes $H^{i_0+1}(X_{\mathcal{I},\bar{s}_{w_0}}, \mathrm{Fil}^{k-1})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ thanks to a filtration of Fil^{k-1}, allows to describe it as extensions between irreducible sub-quotients of the modulo l reduction of some

$$\Big([\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overleftrightarrow{1}, \overleftarrow{t'-1}, \overleftrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overrightarrow{a_r}]_{\pi'} \times \psi_{w_0}\Big) \otimes \mathbb{L}(\pi'\{\frac{\delta'}{2}\})$$

with $t' > t$ and where $\pi'\{\frac{\delta'}{2}\}$ belongs to the supercuspidal support of $\overleftarrow{t'-1}$ in the previous writing. But using the inequality $t' > t$, we see that τ can't be a sub-quotient of the modulo l reduction of any $[\overleftarrow{a_1}, \cdots, \overrightarrow{a_{i-1}}, \overrightarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overrightarrow{a_{i+1}}, \cdots \overrightarrow{a_r}]_\pi$.

Now using the filtration $\mathrm{Fil}^{k-1} \subset \mathrm{Fil}^k \subset \Psi_\varrho$, to conclude it suffices to look at $H^{i_0-1}(X_{\mathcal{I},\bar{s}_{w_0}}, \Psi_{\mathcal{I},\varrho}/\mathrm{Fil}^k)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. For the Harris-Taylor perverse sheaves $\mathcal{P}(\pi'_{w_0}, t)(\frac{1-t+2k}{2})$ with $t' > t$ we argue as before, and for the others we invoke the minimality of t and i_0. $\square$

2.3. From Ihara lemma to the cohomology. — Recall first that

$$X^{=d}_{\mathcal{I},\bar{s}_{v_0}} = \coprod_{i \in \mathrm{Ker}^1(\mathbb{Q},G)} X^{=d}_{\mathcal{I},\bar{s}_{v_0},i},$$

and that for a $G(\mathbb{A}^\infty)$-equivariant sheaf $\mathcal{F}_{\mathcal{I},i}$ on $X^{=d}_{\mathcal{I},\bar{s}_{v_0},i}$, its fiber at some compatible system $z_{i,\mathcal{I}}$ of supersingular points, has an action of $\overline{G}(\mathbb{Q}) \times GL_d(F_{v_0})^0$ where $GL_d(F_{v_0})^0$ is the kernel of the valuation of the determinant so that, cf. [**3**] proposition 5.1.1, as a $G(\mathbb{A}^\infty) \simeq \overline{G}(\mathbb{A}^{\infty,v_0}) \times GL_d(F_{v_0})$-module, we have

$$H^0(\bar{X}^{=d}_{\mathcal{I},\bar{s}_{v_0},i}, \mathcal{F}_{\mathcal{I},i}) \simeq \mathrm{ind}_{\overline{G}(\mathbb{Q})}^{\overline{G}(\mathbb{A}^{\infty,v_0}) \times \mathbb{Z}} z_i^* \mathcal{F}_{\mathcal{I},i}$$

with $\delta \in \overline{G}(\mathbb{Q}) \mapsto (\delta^{\infty,v_0}, \mathrm{val} \circ \mathrm{rn}(\delta_{v_0})) \in \overline{G}(\mathbb{A}^{\infty,v_0}) \times \mathbb{Z}$ and where the action of $g_{v_0} \in GL_d(F_{v_0})$ is given by those of $(g_0^{-\mathrm{val}\det g_{v_0}} g_{v_0}, \mathrm{val}\det g_{v_0}) \in GL_d(F_{v_0})^0 \times \mathbb{Z}$ where $g_0 \in GL_d(F_{v_0})$ is any fixed element with $\mathrm{val}\det g_0 = 1$. Moreover, cf. [**3**] corollaire 5.1.2, if $z_i^* \mathcal{F}_{\mathcal{I}}$ is provided with an action of the kernel $(D^\times_{v_0,d})^0$ of the valuation of the reduced norm, action compatible with those of $\overline{G}(\mathbb{Q}) \hookrightarrow D^\times_{v_0,d}$, then as a $G(\mathbb{A}^\infty)$-module, we have

$$H^0(X^{=d}_{\mathcal{I},\bar{s}_{v_0},i}, \mathcal{F}_{\mathcal{I},i}) \simeq \mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}^\infty), \Lambda) \otimes_{D^\times_{v_0,d}} \mathrm{ind}_{(D^\times_{v_0,d})^0}^{D^\times_{v_0,d}} z_i^* \mathcal{F}_{\mathcal{I},i} \quad (2.3.1)$$

2.3.1. Lemma. — *Le $\overline{\pi}$ be an irreducible sub-$\overline{\mathbb{F}}_l$-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/U^{v_0}, \overline{\mathbb{F}}_l)$. Denote its local component $\bar{\pi}_{v_0}$ at v_0 as $\pi_{v_0}[s]_D$ with π_{v_0} an irreducible cuspidal representation of $GL_g(F_{v_0})$ with $d = sg$. Then $\overline{\pi}^{v_0}$ is a sub-representation of $H^0(X^{=d}_{U^{v_0},\bar{s}_{v_0}}, HT(\pi^\vee_{v_0}, s)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.*

Proof. — Clearly we have $\overline{\pi}^{v_0} \subset \mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/U^{v_0}, \overline{\mathbb{F}}_l) \otimes \bar{\pi}^\vee_{v_0}$. The result then follows from (2.3.1) and the definition of the Harris-Taylor local system $HT(\pi^\vee_{v_0}, s)$ with support on the supersingular stratum. □

2.3.2. Proposition. — *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_S$ verifying (H1) and (H3), and let $\bar{\pi}$ be an irreducible sub-$\overline{\mathbb{F}}_l$-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/U^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. Then $\bar{\pi}^{\infty,v}$ is a sub-$\overline{\mathbb{F}}_l$-representation of $H^{d-1}(X_{U,\bar{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$.*

Proof. — By [**16**] theorem 3.1.4, $\bar{\pi}_{v_0}$ is associated, through the modulo l Jacquet-Langlands correspondence, to some superspeh, $\mathrm{Speh}_s(\varrho)$ with ϱ an irreducible supercuspidal representation of $GL_g(F_{v_0})$ with $d = sg$.

Recall that $H^i(X_{U,\bar{s}_{v_0}},\Psi_\varrho)_\mathfrak{m}$ is a direct factor of $H^{d-1}(X_{U,\bar{\eta}_{v_0}},\overline{\mathbb{F}}_l)_\mathfrak{m}$, so that it suffices to prove that $\bar{\pi}^{\infty,v}$ is a sub-$\overline{\mathbb{F}}_l$-representation of $H^i(X_{U,\bar{s}_{v_0}},\Psi_\varrho)_\mathfrak{m}$.

As in the proof of theorem 2.2.4 using now (H3), consider then the filtration of Ψ_ϱ introduced before so that its graded parts are some Harris-Taylor perverse sheaves of type ϱ and whose $\mathfrak{m}$-localized cohomology groups are free concentrated in degree 0. Note in particular that $\mathcal{P}(\pi_{v_0}^\vee,s)(\frac{s-1}{2})$ is its first graded part so that, using the spectral sequence computing $H^i(X_{U,\bar{s}_{v_0}},\Psi_\rho)_\mathfrak{m}$ with $E_1^{p,q}$ given by the $H^i(X_{U,\bar{s}_{v_0}},\mathcal{P}(\pi'_{v_0},t)(\frac{1-t+2k}{2}))_\mathfrak{m}$, we see that

$$H^i(X_{U,\bar{s}_{v_0}},\mathcal{P}(\pi_{v_0}^\vee,s)(\frac{s-1}{2}))_\mathfrak{m} \hookrightarrow H^i(X_{U,\bar{s}_{v_0}},\Psi_\rho)_\mathfrak{m}$$

with free cokernel, so that $H^0(X^{=d}_{U^{v_0},\bar{s}_{v_0}},\mathcal{P}(\pi_{v_0}^\vee,s)(\frac{s-1}{2}))_\mathfrak{m}$ is a subspace of $H^{d-1}(X_{U,\bar{\eta}_{v_0}},\overline{\mathbb{F}}_l)_\mathfrak{m}$. The result then follows from the previous lemma. □

The strategy now to prove Ihara lemma, under our restrictive hypothesis on $\mathfrak{m}$, is then to prove the same statement on $H^{d-1}(X_{U,\bar{\eta}_{v_0}},\overline{\mathbb{F}}_l)_\mathfrak{m}$, i.e. if π^{∞,v_0} is a subspace of it, then its local component $\pi_{w_0}^{\infty,v_0}$ at the place w_0, is generic. Finally our Ihara lemma statement will follows from proposition 3.2.1.

3. Non degeneracy property for global cohomology

3.1. Global lattices are tensorial product. — From now on we suppose that $\bar{\rho}_\mathfrak{m}$ is absolutely irreducible.

3.1.1. Proposition. — *Let* $\Pi^{\infty,U}\otimes L_g(\Pi_{v_1}^\vee)$ *be a direct factor of* $H^{d-1}(X_{U,\bar{\eta}_{v_1}},\overline{\mathbb{Q}}_l)_\mathfrak{m}$*, and consider its lattice given by the* $\overline{\mathbb{Z}}_l$*-cohomology. Then this lattice is a tensorial product* $\Gamma_G\otimes\Gamma_W$ *of a stable lattice* Γ_G *(resp.* Γ_W*) of* $\Pi^{\infty,U}$ *(resp. of* $L_d(\Pi_{v_1}^\vee)$*).*

Proof. — The result is classical and we resume the arguments of [**21**] §5. With the definition 5.2 of loc. cit., as $\bar{\rho}_\mathfrak{m}$ is supposed to be absolutely irreducible, $\Pi^{\infty,U}\otimes L_g(\Pi_{v_1}^\vee)$ is $\sigma_{\overline{\mathbb{Z}}_l}$-typic where $\sigma_{\overline{\mathbb{Z}}_l}$ is the only, up to isomorphism, stable $\overline{\mathbb{Z}}_l$-lattice of $L_g(\Pi_{v_1}^\vee)$. The statement then follows from the proposition 5.4 of [**21**]. □

Reasonably it should be possible to prove the higher dimensional version of [**21**] theorem 5.6, i.e. to prove that as a $\mathbb{T}_{S,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,

$$H^{d-1}(X_{U,\bar{\eta}},\mathbb{Z}_l)_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}} \otimes_{\mathbb{T}_{S,\mathfrak{m}}} \rho_{\mathfrak{m}},$$

for some $\mathbb{T}_{S,\mathfrak{m}}$-module $\sigma_{\mathfrak{m}}$ on which Gal_F acts trivially.

3.2. Proof of the main result. — Let $\mathcal{S}(\mathfrak{m})$ be the supercuspidal support of the modulo l reduction of any $\Pi_{\widetilde{m},w_0}$ in the near equivalence class associated to a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$. Recall that $\mathcal{S}(\mathfrak{m})$ depends only on $\mathfrak{m}$ and by (H2) it is multiplicity free, we decompose it according to the set $\mathcal{Z}$ of Zelevinsky lines defined as the set of equivalence classes of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations ϱ of some $GL_{g(\varrho)}(F_{w_0})$ with $1 \leqslant g(\varrho) \leqslant d$, under the equivalence relation $\varrho \sim \varrho\{k\}$ for any $k \in \mathbb{Z}$:

$$\mathcal{S}(\mathfrak{m}) = \coprod_{\varrho \in \mathcal{Z}} \mathcal{S}_{\varrho}(\mathfrak{m}).$$

Recall that for such ϱ, its associated Zelevinsky line $ZL(\varrho) = \{\varrho\{k\} : k \in \mathbb{Z}\}$ is of cardinality $\epsilon(\varrho)$. We then denote by $l_1(\varrho) \geqslant \cdots \geqslant l_{r(\varrho)}(\varrho) > 0$, the integers so that $S_{\varrho}(\mathfrak{m})$ can be written as a disjoint union of $r(\varrho)$ unlinked Zelevinsky segments

$$[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho) - 1\}] = \big\{\varrho\{\delta_i\}, \varrho\{\delta_i+1\}, \cdots, \varrho\{\delta_i + l_i(\varrho) - 1\}\big\}.$$

An irreducible $\overline{\mathbb{F}}_l$-representation τ_{w_0} of $GL_d(F_{w_0})$ whose supercuspidal support is equal to $\mathcal{S}(\mathfrak{m})$, can be written as a full induced $\tau_{w_0} \simeq \bigtimes_{\varrho} \tau_{\varrho}$ where each τ_{ϱ} is also a full induced representation

$$\tau_{\varrho} \simeq \bigtimes_{i=1}^{r(\varrho)} \tau_{\varrho,i}$$

with $\tau_{\varrho,i}$ of supercuspidal support equals to those of $[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho) - 1\}]$. With the notations of 2.2.2, each of these $\tau_{\varrho,i}$ can be written as

$$\tau_{\varrho,i} \simeq [\overleftarrow{a_1(\varrho)}, \overrightarrow{a_2(\varrho)}, \cdots, \overrightarrow{a_{t_i(\varrho)}(\varrho)}]_{\varrho\{\delta_i\}},$$

with $\sum_{j=1}^{t_i(\varrho)} a_j = l_i(\varrho) - 1$.

3.2.1. Definition. — *We say, cf. the remark following notation 2.2.2, that τ_{w_0} is non degenerate if for all ϱ and for all $1 \leqslant i \leqslant r(\varrho)$, then $\tau_{\varrho,i} \simeq [\overleftarrow{l_i(\varrho) - 1}]_{\varrho\{\delta_i\}}$.*

3.2.1. Proposition. — *Let τ_{w_0} be an irreducible representation of $GL_d(F_{w_0})$ which is a subspace of*

$$H^{d-1}(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}} := \varinjlim_n H^{d-1}(X_{U^{w_0}(n),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}.$$

Then π_{w_0} is non degenerate.

Proof. — Note first that the supercuspidal support of τ_{w_0} must be $\mathcal{S}(\mathfrak{m})$. The exhaustive filtration of Ψ_{ϱ_0}, cf. §1.5, whose graded parts are Harris-Taylor perverse sheaves, gives a filtration of $H^{d-1}(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\overline{\mathbb{F}}_l)_{\mathfrak{m}}$, whose graded part are, thanks to theorem 2.2.4, the

$$H^0(X_{U^{w_0}(\infty),\bar\eta_{w_0}},\mathcal{P}(\pi_{w_0},t)(\frac{1-t+2k}{2}))_{\mathfrak{m}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$$

for $\pi_{w_0}\in \mathrm{Scusp}_{-1}(\varrho)$ with ϱ such that $S_\varrho(\mathfrak{m})$ is non empty. Then τ_{w_0} must be a subspace of one of these graded parts. We argue by absurdity using the following lemma.

3.2.2. Lemma. — *If ρ is a subspace of $[\overleftarrow{t-1}]_{\varrho\{\frac{-\delta}{2}\}}\times\rho'$ then with the notation 2.2.2,*

- *if $\delta=s-t$ then $\rho=[\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t-1}]_\varrho$;*
- *if $\delta=t-s$ then $\rho=[\overbrace{\cdots}^{s-t-1},\overleftarrow{1},\overleftarrow{t-1}]_\varrho$;*
- *otherwise, i.e. $t-s<\delta<s-t$, then*

$$\rho=[\overbrace{\cdots}^{s-t-\delta-1},\overleftarrow{1},\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t+\delta-1}]_\varrho.$$

Proof. — The result is well known over $\overline{\mathbb{Q}}_l$ and we can easily argue in the same way using

- the fact that all the $\varrho\{\frac{1-s}{2}+k\}$ for $0\leqslant k\leqslant s-1$ are pairwise distinct
- and the property of commutation between the modulo l reduction and the Jacquet functors.

Consider for example the case $t-s<\delta<s-t$. By Frobenius reciprocity we see that the subspace we look for, is some undetermined irreducible subspace of the modulo l reduction of $[\overbrace{\cdots}^{s-t-\delta-1},\overleftarrow{1},\overleftarrow{t-1},\overrightarrow{1},\overbrace{\cdots}^{s-t+\delta-1}]_\pi$.

By convention, cf. notation 2.2.2, we denote by such a subquotient $[\overbrace{\cdots}^{s-t-\delta-1}, \overleftarrow{1}, \overleftarrow{t-1}, \overrightarrow{1}, \overbrace{\cdots}^{s-t+\delta-1}]_{\varrho}$. □

Suppose now, by absurdity, there exists an irreducible supercuspidal $\overline{\mathbb{F}}_l$-representation ϱ_0 such that τ_{ϱ_0} is degenerate and take i with

$$\tau_{\varrho_0,i} \simeq [\cdots, \overrightarrow{a}, \cdots]_{\varrho_0},$$

and let $\beta \in \frac{1}{2}\mathbb{Z}$ such that $\varrho_0\{\beta\}$ is the supercuspidal corresponding to the end of the arrow $\overrightarrow{a}$ in the previous notation. From proposition 3.1.1, we see that $\tau_{w_0} \otimes \overline{\rho}_{\mathfrak{m}}$ is a $\overline{\mathbb{F}}_l[GL_d(F_{w_0}) \times \mathrm{Gal}(\overline{F}/F)]$-submodule of $H^{d-1}(X_{U^{w_0}(\infty),\bar{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. After restricting the Galois action to the Weil group at w_0, we see that $\tau_{w_0} \otimes \mathbb{L}(\varrho_0\{\beta\})$ has to be a $\overline{\mathbb{F}}_l[GL_d(F_{w_0}) \times W_{w_0}]$-submodule of $H^{d-1}(X_{U^{w_0}(\infty),\bar{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ and, as before, of one of the

$$H^0(X_{U^{w_0}(\infty),\bar{\eta}_{w_0}}, \mathcal{P}(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$$

for $\pi_{w_0} \in \mathrm{Scusp}_{-1}(\varrho_0)$. Recall that this last cohomology group is parabolically induced from

$$H^0(X^{\geqslant tg}_{U^{w_0}(\infty),\bar{\eta}_{w_0},\overline{1_{tg}}}, \mathcal{P}_1(\pi_{w_0}, t)(\frac{1-t+2k}{2}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$$

where by lemma 1.3.1, every irreducible $\overline{\mathbb{F}}_l[P_{tg,d}(F_{w_0}) \times W_{w_0}]$-subquotient of it can be written as $[\overleftarrow{t-1}]_{\varrho_0\{-\frac{\delta}{2}\}} \otimes \tau \otimes \mathbb{L}(\varrho_0\{\alpha\})$ where τ is any irreducible representation of $GL_{d-tg}(F_{w_0})$ and $\alpha \in \frac{1}{2}\mathbb{Z}$ is such that $\varrho_0\{\alpha\}$ belongs to the supercuspidal support of $[\overleftarrow{t-1}]_{\varrho_0\{-\frac{\delta}{2}\}}$.

The contradiction then follows from the previous lemma. □

Finally our restricted version of Ihara lemma given in the introduction, follows from propositions 2.3.2 and 3.2.1.

Remark. Note that in the previous proof we used the second part of (H2) to say that the modulo l reduction of $[\overleftarrow{s-1}]_{\pi}$ is irreducible and so any of its subspace is non degenerate, cf. the remark just before theorem 2.2.4. Using the main result of [**9**], we have this last property without any hypothesis so, as this is the only place where we use the second part of (H2), we can remove it.

3.3. Level raising. — Before dealing with the general case, consider first the case $d = 2$ and take $l \geqslant 3$ such that the order of q_{w_0} modulo l is 2. Suppose then, by absurdity, there exists a maximal ideal $\mathfrak{m}$ such that

(a) for every prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, the local component at w_0 of $\Pi_{\widetilde{\mathfrak{m}}}$ is unramified;
(b) for such prime ideal, write $\Pi_{\widetilde{\mathfrak{m}},w_0} \simeq \chi_{w_0,1} \times \chi_{w_0,2}$, and suppose $\chi_{w_0,1}\chi_{w_0,2}^{-1} \equiv \nu \mod l$.

Using (a) and the spectral sequence of vanishing cycles at w_0, we obtain

$$H^1(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}} \simeq H^1(X^{=1}_{U,\bar{s}_{w_0}}, \Psi(\overline{\mathbb{F}}_l))_{\mathfrak{m}}$$

where $X^{=1}_{U,\bar{s}_{w_0}}$ is the ordinary locus of the geometric special fiber of X_U at w_0. It is well known that this cohomology group is parabolic induced. Moreover the only non degenerate irreducible representation of $GL_d(F_{w_0})$ which is a subquotient of the modulo l reduction of $\chi_{w_0,1} \times \chi_{w_0,1}\nu$ is cuspidal, because of the fact that q_{w_0} is of order 2 modulo l, this non degenerate representation can not be a subspace of the induced representation $H^1(X_U, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. The contradiction is then given by the Ihara lemma.

In higher dimension, recall first the notations of the beginning of the previous section. For a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and an automorphic representation $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ in the near equivalence class associated to $\widetilde{\mathfrak{m}}$, we write its local component at w_0

$$\Pi_{w_0} \simeq \bigtimes_{\varrho} \Pi_{w_0}(\varrho)$$

and $\Pi_{w_0}(\varrho) \simeq \bigtimes_{i=1}^{r(\varrho)} \Pi_{w_0}(\varrho, i)$ where for each $1 \leqslant i \leqslant r(\varrho)$, the modulo l reduction of the supercuspidal support of $\Pi_{w_0}(\varrho, i)$ is, with the notations of the previous section, those of the Zelevinsky segment $[\varrho\{\delta_i\}, \varrho\{\delta_i + l_i(\varrho) - 1\}]$.

3.3.1. Proposition. — *Take a maximal ideal $\mathfrak{m}$ verifying the hypothesis (H1) and (H2). Let ϱ_0 such that $\mathcal{S}_{\varrho_0}(\mathfrak{m})$ is non empty and consider $1 \leqslant i \leqslant r(\varrho_0)$. Then there exists a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and an automorphic representation $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ such that with the previous notation $\Pi_{w_0}(\varrho_0, i)$ is non degenerate, i.e. isomorphic to $\mathrm{St}_{l_i(\varrho)}(\pi_{w_0})$ for some irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation π_{w_0}.*

Remark. In particular if $\mathcal{S}(\mathfrak{m}) = \mathcal{S}_{\varrho_0}(\mathfrak{m})$ and $r(\varrho_0) = 1$, i.e. the supercuspidal support of the modulo l reduction of the local component at w_0 of any $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$ for any $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, is a Zelevinsky segment, then Π_{w_0} is non degenerate. This is the case considered in section 4.5 of [**13**]. In an incoming work, we intend to explain how to raise the level simultaneously for all $1 \leqslant i \leqslant r(\varrho_0)$ and all ϱ_0 together.

Proof. — For a minimal prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and $\Pi \in \Pi_{\widetilde{\mathfrak{m}}}$, we write

$$\Pi_{w_0}(\varrho_0, i) \simeq \mathrm{St}_{s_1}(\pi_{w_0,1}) \times \cdots \mathrm{St}_{s_a}(\pi_{w_0,a})$$

where $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_a \geqslant 1$ and $\pi_{w_0,1}, \cdots, \pi_{w_0,a}$ irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations of type ϱ_0 of $GL_{g_i}(F_{w_0})$. We then argue by absurdity, i.e. we suppose $a \geqslant 2$ for all $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and we choose such $\widetilde{\mathfrak{m}}$ so that s_1 is maximal. The strategy is then, using lemma 3.2.2, to construct a degenerate $\overline{\mathbb{F}}_l[GL_d(F_{w_0})]$-subspace of $H^{d-1}(X_{U^{w_0}(\infty),\bar{\eta}_{w_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ which contradicts the genericness property of irreducible sub-modules of this cohomology group as proved before. In [**4**] §3.6, we prove that for all minimal prime $\widetilde{\mathfrak{m}}' \subset \mathfrak{m}$

$$H^i(X_{U,\bar{s}_{w_0}}, HT_{\overline{\mathbb{Q}}_l}(\pi_{w_0,1}, t))_{\widetilde{\mathfrak{m}}'} = (0)$$

either if $t > s_1$ or for $t = s_1$, if $i \neq 0$. Consider now the filtration

$$\mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}) \hookrightarrow \mathrm{Fil}_*^{1-s_1 g_1}(\Psi_{\varrho_0}) \hookrightarrow \Psi_{\varrho_0}$$

and recall that, by construction,

– $\mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0})$ is supported in $X^{>s_1 g_1}_{\mathcal{I},\bar{s}_{w_0}}$

– $\mathrm{gr}_*^{1-s_1 g_1}(\Psi_{\varrho_0}) \simeq \bigoplus_{\pi_{w_0} \in \mathrm{Scusp}_{-1}(\varrho_0)} \mathcal{P}(\pi_{w_0}, s_1)(\frac{s_1-1}{2})$.

By the theorem 2.2.4 we know the cohomology groups of Harris-Taylor perverse sheaves to be free, so

– $H^i(X_{U,\bar{s}_{w_0}}, \mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}))_{\mathfrak{m}} = (0)$;

– $H^i(X_{U,\bar{s}_{w_0}}, \Psi_{\varrho_0} / \mathrm{Fil}_*^{-s_1 g_1}(\Psi_{\varrho_0}))_{\mathfrak{m}}$ is free.

Recall moreover, cf. [**4**] §3.6, that $\Pi_{w_0} \otimes \mathbb{L}(\pi_{w_0,1})(\frac{s_1-1}{2})$ is a direct factor of

$$H^i(X_{U^{w_0}(\infty),\bar{s}_{w_0}}, HT_{\overline{\mathbb{Q}}_l}(\pi_{w_0,1}, s_1)(\frac{s_1-1}{2}))_{\widetilde{\mathfrak{m}}},.$$

Moreover the stable lattice given by the $\overline{\mathbb{Z}}_l$-cohomology looks like $\big(\Gamma(\varrho_0, 1) \times \Gamma^{\varrho_0,1}\big) \times \Gamma_W$ where

– $\Gamma(\varrho_0, 1)$ is a stable lattice of $\mathrm{St}_{s_1}(\pi_{w_0,1})$,

– $\Gamma^{\varrho_0,1}$ is a stable lattice of $\big(\times_{\varrho\neq\varrho_0}\Pi_{w_0}(\varrho)\big)\times\big(\times_{i=2}^{r(\varrho_0)}\Pi_{w_0}(\varrho_0,i)\big)$,
– and Γ_W is a stable lattice of $\mathbb{L}(\pi_{w_0,1})(\frac{s_1-1}{2})$.

The result then follows from lemma 3.2.2.

□

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