Abstract. — Using the freeness of the localized cohomology groups of KHT Shimura varieties, and studying the local monodromy operator acting on it, we exhibit cases of a level fixing phenomenon at some fixed place $v$, for some modulo $l$ automorphic representations $\tilde{\pi}$ in the following sense. We give conditions which do not involve the place $v$, such that whatever is a characteristic zero irreducible automorphic representation whose modulo $l$ reduction is $\tilde{\pi}$, its level at $v$ is necessarily equals to those of $\tilde{\pi}$.

Contents

1. Introduction ................................................. 1
2. The monodromy operator on the cohomology ........ 5
3. Local behavior of monodromy over $\overline{\mathbb{F}}_l$ .......... 16
4. Level fixing property ............................................. 19
References .......................................................... 20

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1. Introduction

Let $F = EF^+$ be a finite CM extension of $\mathbb{Q}$ with $E/\mathbb{Q}$ imaginary quadratic and $F^+$ totally real. Let $\rho: \text{Gal}(\overline{F}/F) \to GL_{d}(V)$ be an irreducible $l$-adic representation of dimension $d$. Let $\Gamma$ be a stable $\mathbb{Z}_l$-lattice of $V$ and suppose that the modulo $l$ reduction $\overline{\rho}$ of $\rho$ is still irreducible so that, up to homothety, $\Gamma$ is uniquely defined.

Consider now $p \neq l$ and a place $v$ of $F$ above $p$: we denote by $F_v$ the completion of $F$ at $v$ with $\mathcal{O}_v$ its ring of integers and $\kappa_v$ its residue field. Thanks to Grothendieck’s theorem, we know that the action of the inertia subgroup $I_v$ at $v$ is unipotent and let $N_v$ be the nilpotent monodromy operator. Such a nilpotent operator defines a standard partition

$$d_{\rho,v} = (n_1 \geq n_2 \geq \cdots \geq n_r \geq 1)$$

of $d = n_1 + \cdots + n_r$ corresponding to the decomposition of the restriction $\rho_v$ of $\rho$ to the decomposition group at $v$:

$$\rho_v \cong \text{Sp}_{n_1}(\rho_{v,1}) \oplus \cdots \oplus \text{Sp}_{n_r}(\rho_{v,r}),$$

where the $\rho_{v,i}$ are irreducible and

$$\text{Sp}_{n_i}(\rho_{v,i}) = \rho_{v,i}(1 - n_i) \oplus \rho_{v,i}(\frac{3 - n_i}{2}) \oplus \cdots \oplus \rho_{v,i}(\frac{n_i - 1}{2}),$$

where $N_v$ induces isomorphisms $\rho_{v,i}(\frac{1 - n_i + 2\delta}{2}) \to \rho_{v,i}(\frac{1 - n_i + 2(\delta + 1)}{2})$ for $0 \leq \delta < n_i - 1$ and is trivial on $\rho_{v,i}(\frac{n_i - 1}{2})$.

If $l \leq n_1$, then we can also consider the nilpotent operator $\overline{N}_v$ of the action of $I_v$ modulo $l$ which also defines a partition $d_{\overline{\rho},v}$ of $d$ which is smaller than $d_{\rho,v}$ for the Bruhat order. As in [8], one may ask for a condition so that $d_{\overline{N},v} = d_{\overline{\rho},v}$, especially when there is as much irreducible constituents in the Frobenius semi-simplification of $\overline{\rho}_v$ as in $\rho_v$. To be relevant such a condition should not be about the place $v$.

To state our result, consider a similitude group $G/\mathbb{Q}$ as in §2, and for any open compact subgroup $I$ of $G(\mathbb{A}^\infty)$, let denote by $\text{Sh}_I \to \text{Spec} \mathcal{O}_v$ the KHT-Shimura variety with level $I$, cf. definition 2.4. For any finite set $S$ of places of $F$, let $\mathbb{T}_S^\xi$ be the Hecke algebra defined, cf. 2.5, as the image of the abstract unramified $\mathbb{Z}_l$-Hecke algebra of $G(\mathbb{A}^S)$ outside $S$ acting on the cohomology of $\text{Sh}_I$ where $I$ is any open compact subgroup unramified at any place of $S$.

Consider then a maximal cohomological ideal $m$ of $\mathbb{T}_S$, and for a prime ideal $\mathfrak{m} \subset m$, we denote by $\rho_{\mathfrak{m}}$ for the Galois $\overline{\mathbb{Q}}_l$-representation associated
to \( \hat{m} \), cf. [14]. Recall that the semi-simple class \( \overline{\rho}_m \) of the reduction modulo \( l \) of \( \rho_m \) depends only of the maximal ideal \( m \) of \( \mathbb{T} \) containing \( \hat{m} \). For all prime \( x \) of \( \mathbb{Z} \) split in \( E \) and a place \( w \notin S \) of \( F \) above \( x \), we moreover denote by \( P_{m,w}(X) \) the characteristic polynomial of \( \overline{\rho}_m(\text{Frob}_w) \) and let \( S_m(w) \) be its multi-set of roots.

1.1. Definition. — (cf. the introduction of [5])
We say that \( m \) is KHT-free if the cohomology groups of the Kottwitz-Harris-Taylor Shimura variety of notation 2.4, localized at \( m \), are free.

From [4], any of the following properties insure KHT-freeness of \( m \).

1. There exists a place \( w_1 \notin S \) of \( F \) above a prime \( p_1 \) splits in \( E \), such that the multi-set \( S_m(w_1) \) of roots of \( P_{m,w_1}(X) \) does not contain any sub-multi-set of the shape \( \{\alpha, q_{w_1} \alpha\} \) where \( q_{w_1} \) is the order of the residue field of \( F \) at \( w_1 \). This hypothesis is called generic in [10].

2. When \( [F(\exp(2i\pi/l)) : F] > d \), if we suppose the following property to be true, cf. [4] 4.17. If \( \theta : G_F \rightarrow GL_d(\mathbb{F}_l) \) is an irreducible continuous representation such that for all place \( w \notin S \) above a prime \( x \in \mathbb{Z} \) split in \( E \), then \( P_{m,w}(\theta(\text{Frob}_w)) = 0 \) (resp. \( P_{m,w}^\vee(\theta(\text{Frob}_w)) = 0 \)) implies that \( \theta \) is equivalent to \( \overline{\rho}_m \) (resp. \( \overline{\rho}_m^\vee \)), where \( m^\vee \) is the maximal ideal of \( \mathbb{T}_S \) associated to the dual multi-set of Satake parameters, cf. [4] notation 4.4. In [11], the authors proved that the previous property is verified in each of the following cases:

- either \( \overline{\rho}_m \) is induced from a character of \( G_K \) where \( K/F \) is a cyclic galoisian extension;
- or \( l \geq d \) and \( SL_d(k) \subset \overline{\rho}_m(G_F) \subset \mathbb{F}_l^\times GL_d(k) \) for some subfield \( k \subset \mathbb{F}_l \).

3. Finally in [9], we announce \( m \) to be KHT-free when \( \overline{\rho}_m \) is irreducible and \( [F(\exp(2i\pi/l)) : F] > d \).

Remark. By Cebotarev’s theorem, the hypothesis \( [F(\exp(2i\pi/l)) : F] > d \) allows to pick places \( v \) of \( F \) such that the order \( q_v \) of the residue field of \( F \) at \( v \), is of order strictly greater than \( d \) in \( \mathbb{Z}/l\mathbb{Z} \).

1.2. Theorem. — Suppose that

- \( \overline{\rho}_m \) is irreducible,
- the set \( S_v(m) \) of irreducible sub-quotients of the restriction of \( \overline{\rho}_m \) to the decomposition group at \( v \) is made of characters. Moreover, whatever is the \( \mathbb{F}_l \)-character \( \chi_v \), we assume that \( S_v(m) \) does not contain
\(\{\chi_v, \chi_v(1), \ldots, \chi_v(m(\chi_v) - 1)\}\), where, cf. notation 3.1, \(m(\chi_v)\) is equal the order of \(q_v\) modulo \(l\) if \(q_v \neq 1 \mod l\) otherwise \(m(\chi_v) = l\).

- \(m\) is KHT-free in the sense of definition 1.1.

Then the two partitions \(d_{m,v}\) and \(d_{\bar{m},v}\), associated respectively to the unipotent operator at \(v\) for \(\rho_{m,v}\) and \(\rho_{\bar{m},v}\), are the same.

**Remark.** Using the main results of [6], we can replace in the second assumption, the character \(\chi_v\) by any irreducible \(\overline{F}_l\)-representation \(\varrho\) of the Galois group of \(F_v\). In the following we will write the proof in this more general setting and give self contained arguments for the case of a character.

**Remark.** What is striking is that we find conditions away from \(v\) such that the global lattice of the Galois representation\(^{(1)}\) behaves uniformly at \(v\) whatever are \(\bar{m} \subset m\).

**Remark.** Such a statement is orthogonal to those of [8] to have some level lowering property and it can be translated to some level fixing statement, cf. proposition 4.1 which roughly says that for any \(\bar{m} \subset m\), if \(\Pi_{\bar{m}}\), the irreducible automorphic cohomological \(\overline{U}_l\)-representation with Satake’s parameters prescribed by \(\bar{m}\), has non trivial \(I\)-invariants then the local component \(I_v\) of the open compact subgroup \(I\) at the place \(v\) depends only on \(m\).

The main idea to prove the theorem, is to analyze the monodromy action at the place \(v\), inside the cohomology groups of some compact Kottwitz-Harris-Taylor Shimura variety \(\text{Sh}_{I,v} \to \text{Spec} \mathcal{O}_v\) with level \(I\), an open compact subgroup of \(G(\mathbb{A}^\infty)\) where \(G/\mathbb{Q}\) is some similitude group with signatures \((1, n - 1), (0, n), \ldots, (0, n)\) at infinity, cf. [14].

This cohomology localized at \(m\) is computed through the spectral sequence of vanishing cycles at the place \(v\)

\[
E^{p,q}_{1,m} \implies E^{p+q}_{\infty,m},
\]

which, as \(\overline{\rho}_m\) is supposed to be irreducible, degenerates at \(E_1\) after tensoring with \(\overline{U}_l\). Moreover, by definition of KHT-freeness of \(m\), we know that the \(E^{p,q}_{\infty,m}\) are all free. The main step is then to prove that all the \(E^{p,q}_{1,m}\) are also free so that the monodromy action on the cohomology groups is purely of local nature and can be read on the sheaf of nearby cycles, cf. proposition 3.2 and the proof of proposition 4.1. The results follow\(^{(1)}\)

\(^{(1)}\)It is unique up to isomorphism as the modulo \(l\) reduction is supposed to be irreducible.
then from the analysis on the link between the monodromy action on the Galois representations $\rho_m$ and those on the cohomology of our Shimura variety localized at $m$.

2. The monodromy operator on the cohomology

Let $F = F^+ E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary and $F^+$ totally real. Let $B/F$ be a central division algebra with dimension $d^2$ with an involution of second kind $\ast$. For $\beta \in B^{\ast -1}$, consider the similitude group $G/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra $R$ by

$$G(R) := \{(\lambda, g) \in R^\times \times (B^\text{op} \otimes_\mathbb{Q} R)^\times \text{ such that } gg^\ast\beta = \lambda\},$$

with $B^\text{op} = B \otimes_{F, c} F$ where $c = \ast|_F$ is the complex conjugation and $g^\ast\beta$ is the involution $x \mapsto x^{g\ast\beta} = \beta x^\ast \beta^{-1}$. Following [14], we can manage so that $G(\mathbb{R})$ has signatures $(1, d-1), (0, d), \ldots, (0, d)$.

2.1. Definition. — Let $\text{Spl}$ be the set of places $v$ of $F$ such that $p_v := v|_\mathbb{Q} \neq l$ is split in $E$ and $B_v^\times \simeq GL_d(F_v)$.

We now suppose that $p = uu^c$ splits in $E$ so that

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (B_w^\text{op})^\times$$

where $w$ describes the places of $F$ above $u$. We ask then $B_w$ to be split, isomorphic to $GL_d(F_w)$.

2.2. Definition. — For a finite set $S$ of places of $\mathbb{Q}$ containing the places where $G$ is ramified, denote by $\mathbb{T}_S^\text{abs} := \prod_{x \notin S} \mathbb{T}_{x, \text{abs}}$ the abstract unramified Hecke algebra where $\mathbb{T}_{x, \text{abs}} \simeq \mathbb{Z}[X^\text{un}(T_x)]^W_x$ for $T_x$ a split torus, $W_x$ the spherical Weyl group and $X^\text{un}(T_x)$ is the set of $\mathbb{Z}_l$-unramified characters of $T_x$.

Example. For $w \in \text{Spl}$, we have

$$\mathbb{T}_{w, \text{abs}} = \mathbb{Z}[T_{w, i} : i = 1, \ldots, d],$$

where $T_{w, i}$ is the characteristic function of

$$GL_d(O_w) \text{ diag}(\overline{w}_{w, 1}, \overline{w}_{w, i}, 1, \ldots, 1) GL_d(O_w) \subset GL_d(F_w).$$
We then denote by $\mathcal{I}$ the set of open compact subgroups

$$U^p(m_1, \cdots, m_r) = U^p \times \mathbb{Z}_p^r \times \prod_{i=1}^r \text{Ker}(\mathcal{O}_{B_{v_i}}^\times \to (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i})^\times)$$

where $U^p$ is any small enough open compact subgroup of $G(\mathbb{A}^{p,\infty})$ and $\mathcal{O}_{B_{v_i}}$ is the maximal order of $B_{v_i}$ with maximal ideal $\mathcal{P}_{v_i}$ and where $v_1, \cdots, v_r$ are the places of $F$ above $u$ with $p = uu^c$.

2.3. Notation. — For $I = U^p(m_1, \cdots, m_r) \in \mathcal{I}$, we will denote by $I^p(n) := U^p(n, m_2, \cdots, m_r)$. We also denote by $\text{Spl}(I)$ the subset of $\text{Spl}$ of places which does not divide the level $I$.

2.4. Definition. — Attached to each $I \in \mathcal{I}$ is a Shimura variety

$$\text{Sh}_I \to \text{Spec} \, \mathcal{O}_v$$

where $\mathcal{O}_v$ denote the ring of integers of the completion $F_v$ of $F$ at $v$.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}$ be a fixed embedding and write $\Phi$ for the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ whose restriction to $E$ equals $\sigma_0$. There exists then, cf. [14] p.97, an explicit bijection between irreducible algebraic representations $\xi$ of $G$ over $\overline{\mathbb{Q}}$ and $(d+1)$-uple $(a_0, (\overline{a}_\sigma)_{\sigma \in \Phi})$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overline{a}_\sigma = (a_{\sigma,1} \leq \cdots \leq a_{\sigma,d})$. We then denote by

$$V_{\xi,\overline{a}_\sigma}$$

the associated $\mathbb{Z}_l$-local system on $\text{Sh}_I$.

2.5. Notation. — Let $\mathbb{T}^\xi_S$ be the image of $\mathbb{T}^S_{\text{abs}}$ inside

$$\bigoplus_{i=0}^{2d-2} \lim_{\to} H^i(\text{Sh}_{I,\overline{a}_\sigma}, V_{\xi,\overline{a}_\sigma})$$

where the limit concerned the ideals $I$ which are maximal at each places outside $S$.

To each maximal ideal $\hat{m}$ as before, is associated an irreducible automorphic representation $\Pi_{\hat{m}}$ which is $\xi$-cohomological, i.e. there exists an integer $i$ such that

$$H^i((\text{Lie } G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_{\hat{m}} \otimes \xi^\vee) \neq (0),$$

where $U$ is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$. 
2.6. Notation. — Let denote by $\text{Scusp}_v(\hat{m})$, the supercuspidal support of its local component at $v$, denoted $\Pi_{\hat{m}, v}$. Note that the modulo $l$ reduction of $\text{Scusp}_v(\hat{m})$ is independent of the choice of $\hat{m} \subseteq m$: we denote it $\text{Scusp}_v(m)$.

Recall that $Sh_{I, \overline{s}_v}$ is equipped with the Newton stratification

$$Sh_{I, \overline{s}_v}^d \subset Sh_{I, \overline{s}_v}^{d-1} \subset \cdots \subset Sh_{I, \overline{s}_v}^1 = Sh_{I, \overline{s}_v},$$

where for $1 \leq h \leq d$, $Sh_{I, \overline{s}_v}^h$ (resp. $Sh_{I, \overline{s}_v}^{\leq h}$) is the closed (resp. the open) Newton stratum of height $h$ and of pure dimension $d - h$, defined as the sub-scheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to $h$ (resp. equal to $h$).

Moreover for $1 \leq h < d$, the Newton stratum $Sh_{I, \overline{s}_v}^h$ is geometrically induced under the action of the parabolic subgroup $P_{h, d-h}(F_v)$, defined as the stabilizer of the first $h$ vectors of the canonical basis of $F_v^d$. Concretely this means that there exists a closed sub-scheme $Sh_{I, \overline{s}_v}^h$ stabilized by the Hecke action of $P_{h, d-h}(F_v)$ and such that

$$\lim_{n} Sh_{I, \overline{s}_v}^h \simeq (\lim_{n} Sh_{I, \overline{s}_v}^h) \times_{P_{h, d-h}(F_v)} GL_d(F_v).$$

2.7. Notation. — For a representation $\pi_v$ of $GL_d(F_v)$ and $n \in \frac{1}{2} \mathbb{Z}$, we set $\pi_v\{n\} := \pi_v \otimes \nu^n$ where $\nu(g) := q_v^{-\text{valdet}(g)}$. Recall that the normalized induction of two representations $\pi_{v, 1}$ and $\pi_{v, 2}$ of respectively $GL_{n_1}(F_v)$ and $GL_{n_2}(F_v)$ is

$$\pi_1 \times \pi_2 := \text{ind}_{P_{n_1, n_1+n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v, 1}\{\frac{n_2}{2}\} \otimes \pi_{v, 2}\{-\frac{n_1}{2}\}.$$

Recall that a representation $\pi_v$ of $GL_d(F_v)$ is called cuspidal (resp. supercuspidal) if it is not a subspace (resp. a subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true over $\mathbb{F}_l$. More precisely let $\pi_v$ be an irreducible $\mathbb{Q}_l$-representation then its modulo $l$ reduction is still cuspidal but not necessary supercuspidal: in this case its supercuspidal support is a segment $[\varrho, \varrho \nu^{s-1}]$ where $\varrho$ is irreducible supercuspidal and $s$ is either equal to 1 or of the following shape $s = m(\varrho)k$ for $k \in \mathbb{N}$.

2.8. Definition. — In the former case we say that $\pi_v$ is of $\varrho$-type $-1$ and otherwise of $\varrho$-type $k$. 

Let \( \pi_v \) be an irreducible cuspidal \( \mathfrak{O}_l \)-representation of \( GL_q(F_v) \) and fix \( t \geq 1 \) such that \( tg \leq d \). Thanks to Igusa varieties, Harris and Taylor constructed a local system on \( Sh_{L,s,L_h} \)

\[
\mathcal{L}_{\mathfrak{D};}(\pi_v[t]D)_{L_h} = \bigoplus_{i=1}^{e_v} \mathcal{L}_{\mathfrak{D};} (\rho_{v,i})_{L_h}
\]

where \( (\pi_v[t]D)_{\mathfrak{D};} = \bigoplus_{i=1}^{e_v} \rho_{v,i} \) with \( \rho_{v,i} \) irreducible. The Hecke action of \( P_{t,q,d-t,q}(F_v) \) is then given through its quotient

\[
P_{t,q,d-t,q}(F_v) \rightarrow GL_{t,q}(F_v) \times GL_{d-t,q}(F_v) \rightarrow GL_{d-t,q}(F_v) \times \mathbb{Z},
\]

where \( GL_{t,q}(F_v) \times GL_{d-t,q}(F_v) \) is the Levi quotient of the parabolic \( P_{t,q,d-t,q}(F_v) \) and the second map is given by the valuation of the determinant map \( GL_{t,q}(F_v) \rightarrow \mathbb{Z} \). These local systems have stable \( \mathbb{Z}_d \)-lattices and we will write simply \( \mathcal{L}(\pi_v[t]D)_{L_h} \) for any \( \mathbb{Z}_d \)-stable lattice that we do not want to specify.

2.9. Notations. — For \( \Pi_t \) any representation of \( GL_{q}(F_v) \) and \( \Xi : \frac{1}{2} \mathbb{Z} \rightarrow \mathbb{Z}_q^\times \) defined by \( \Xi(\frac{1}{2}) = q^{1/2} \), we introduce

\[
\overline{HT}_L(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]D)_{L_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}
\]

and its induced version

\[
\overline{HT}(\pi_v, \Pi_t) := \left( \mathcal{L}(\pi_v[t]D)_{L_h} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}} \right) \times_{P_{t,q,d-t,q}(F_v)} GL_{d}(F_v),
\]

where the unipotent radical of \( P_{t,q,d-t,q}(F_v) \) acts trivially and the action of

\[
(g^{\infty,v}, \begin{pmatrix} g_v & * \\ 0 & g_v^t \end{pmatrix}, \sigma_v) \in G(A^{\infty,v}) \times P_{t,q,d-t,q}(F_v) \times W_v
\]

is given

- by the action of \( g^c_v \) on \( \Pi_t \) and \( \deg(\sigma_v) \in \mathbb{Z} \) on \( \Xi^{\frac{tg-d}{2}} \), and
- the action of \( (g^{\infty,v}, g_v^t, \text{val}(\det g_v^t) - \deg \sigma_v) \in G(A^{\infty,v}) \times GL_{d-t,q}(F_v) \times \mathbb{Z} \) on \( \mathcal{L}_{\mathfrak{D};}(\pi_v[t]D)_{L_h} \otimes \Xi^{\frac{tg-d}{2}} \).

We also introduce

\[
HT(\pi_v, \Pi_t)_{L_h} := \overline{HT}(\pi_v, \Pi_t)_{L_h}[d - tg],
\]

and the perverse sheaf

\[
P(t, \pi_v)_{L_h} := J_{L_h,l_\ast}^t HT(\pi_v, St_t(\pi_v))_{L_h} \otimes \mathbb{L}(\pi_v),
\]
and their induced version, $HT(\pi_v, \Pi_t)$ and $P(t, \pi_v)$, where
\[ j^h = i^h \circ j^{\geq h} : Sh_{\bar{I}, \bar{s}}^{\geq h} \hookrightarrow \bar{\text{Sh}}_{\bar{I}, \bar{s}}^{\geq h} \hookrightarrow \text{Sh}_{\bar{I}, \bar{s}} \]
and $\mathbb{L}$ is dual to the local Langlands correspondence.

Remarks:
- Recall that $\pi_v'$ is said inertially equivalent to $\pi_v$ if there exists a character $\zeta : \mathbb{Z} \longrightarrow \bar{\mathbb{Q}}^\times$ such that $\pi_v' \simeq \pi_v \otimes (\zeta \circ \text{val} \circ \text{det})$. We denote by $e_{\pi_v}$ the order of the inertial class of $\pi_v$.
- Note, cf. [1] 2.1.4, that $P(t, \pi_v)$ depends only on the inertial class of $\pi_v$ and
\[ P_{\mathbb{Q}}(t, \pi_v) = e_{\pi_v}P_{\mathbb{Q}}(t, \pi_v) \]
where $P_{\mathbb{Q}}(t, \pi_v)$ is an irreducible perverse sheaf.
- When the modulo $l$ reduction $g$ of $\pi_v$ is still supercuspidal and if $t < m(g)$, then up to homothety there is only one stable $\mathbb{Z}_l$-stable lattice of $L(\pi_v[t]_D)$. From the description of the modulo $l$ reduction of $\text{St}_s(\pi_v)$ in [2], the same is then true for $P(t, \pi_v)$.
- Over $\mathbb{Z}_l$, we also have the $p^+$-perverse structure which is dual to the usual $p$-structure.

2.10. Notation. — Let denote by
\[ \Psi_v := \text{R} \Psi_{\eta_v}(\mathbb{Z}_l[d - 1])\left(\frac{d - 1}{2}\right) \]
the nearby cycles autodual free perverse sheaf on the geometric special fiber $\text{Sh}_{\bar{I}, \bar{s}_v}$ of $\text{Sh}_I$.

Following the constructions of [7], we can then define a $\mathbb{Z}_l$-filtration $\text{Fil}^\bullet(\Psi_v)$ whose graded parts $\text{gr}^r(\Psi_v)$ are free $\mathbb{Z}_l$-perverse sheaves of the following shape
\[ p^\circ j_{= h}^\circ HT_{\mathbb{Z}_l}(\pi_v, \text{St}_t(\pi_v))(\frac{1 - t + 2\delta}{2}) \hookrightarrow \text{gr}^r(\Psi_v) \]
\[ 
\hookrightarrow p^\circ j_{= h}^\circ HT_{\mathbb{Z}_l}(\pi_v, \text{St}_t(\pi_v))(\frac{1 - t + 2\delta}{2}) \]
for some $0 \leq k \leq t - 1$, where $\hookrightarrow$ means a bimorphism, that is both an epimorphism and a monomorphism, and where $HT_{\mathbb{Z}_l}(\pi_v, \text{St}_t(\pi_v))$ means a certain lattice of the Harris-Taylor local system.

Remarks:
In [6], we proved that if you always use the adjunction maps $j^! = h^! j^* \to \text{Id}$ then all the previous graded parts are isomorphic to $p$-intermediate extensions.

One of the main results of [6] is that these two intermediate extensions coincide if the modulo $l$ reduction of $\pi_v$ is supercuspidal.

Finally we can easily arrange the filtration so that it is compatible with the nilpotent monodromy operator $N_v$, i.e. so that for any $r$ the image of $\text{Fil}^r(\Psi_v) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}_l}$ is some $\text{Fil}^\phi(r)(\Psi_v) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}_l}$ for some decreasing function $\phi$.

When dealing with sheaves, there is no need to introduce the local system $V_{\xi}$, because it suffices to add $\otimes_{\mathbb{Z}_l} V_{\xi}$ to the formulas as in the next notation.

\section*{2.11. Notation.} We write $\Psi_{\xi,v} := \Psi_v \otimes_{\mathbb{Z}_l} V_{\xi,v}$.

We then have a spectral sequence

$$E_1^{p,q} = H^{p+q}(\text{Sh}_{I,\bar{\delta}_v, \Psi_v}) \Rightarrow H^{p+q}(\text{Sh}_{I,\bar{\delta}_v, \Psi_v} \otimes_{\mathbb{Z}_l} V_{\xi,v}) .$$

As pointed out in [8], if for some $m$ the spectral sequence is concentrated in middle degree, i.e. $E_{1,m} = 0$ for $p + q \neq d - 1$, and all the $E_{1,m}^{d-1-p}$ are free, then the action of the monodromy operator $N_{v,m}^{\text{coho}}$ on $H^{d-1}(\text{Sh}_{I,\bar{\delta}_v, \Psi_v} \otimes_{\mathbb{Z}_l} V_{\xi,v})_m$ comes from the action of $N_v$ on $\Psi_v$, cf. the proof of the proposition 4.1.

We now want to understand the relation between $N_{v,m}^{\text{coho}}$ and $N_{v,m}$. We first write

$$H^{d-1}(\text{Sh}_{I,\bar{\delta}_v, \Psi_v} \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}_l})_m \simeq \bigoplus_{\mathfrak{m} \subset m} \Pi_{\mathfrak{m}}^I \otimes \rho_{\mathfrak{m}},$$

where we view $\Pi_{\mathfrak{m}}$ as a $\mathbb{T}^S_{\xi}$-module. We then fix any ordering

$$\{\mathfrak{m} \subset m\} = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\},$$

and we consider the associated filtration $\text{Fil}^k(m)$ of $\Gamma := H^{d-1}(\text{Sh}_{I,\bar{\delta}_v, \Psi_v} \otimes_{\mathbb{Z}_l} V_{\xi,v})_m$ constructed as follows:

- $\text{Fil}^1_m := \Gamma \cap (\Pi_{\mathfrak{m}_1}^I \otimes \rho_{\mathfrak{m}_1})$;
– once \( \text{Fil}^{k-1}_m \) being constructed, let \( \text{gr}^k_m := (\Gamma / \text{Fil}^{k-1}_m) \cap (\Pi^L_{\overline{\rho}} \otimes \rho_{\overline{\rho}}) \), and let \( \text{Fil}^k_m \) be defined by the following pull-back

\[
\begin{array}{cccc}
\text{Fil}^{k-1}_m & \xrightarrow{\text{c}} & \Gamma & \xrightarrow{\phi} & \Gamma / \text{Fil}^{k-1}_m \\
\downarrow & & \downarrow & & \downarrow \\
\text{Fil}^{k-1}_m & \xrightarrow{\text{c}} & \text{Fil}^k_m & \xrightarrow{\phi} & \text{gr}^k_m
\end{array}
\]

Each of the \( \text{gr}^k_m \) defines a stable lattice of \( \Pi^L_{\overline{\rho}} \otimes \rho_{\overline{\rho}} \). By taking a basis of each of these \( \text{gr}^k_m \), we then obtain a basis of \( \Lambda_{\Pi^L_{\overline{\rho}}} \), in which the associated matrix of \( N_{\text{coho}}^{v,m} \) is integral. Note that as \( N_{\text{coho}}^{v,m} \) is Hecke-equivariant, then this matrix is diagonal by block.

With the arguments of \([17]\) §5, as \( \overline{\rho} \) is supposed to be absolutely irreducible, \( \Pi^L_{\overline{\rho}} \otimes \rho_{\overline{\rho}} \) is \( \rho_{\overline{\rho}} \)-typic in the sense of definition 5.2 of \( \text{loc.cit.} \), where \( \rho_{\overline{\rho}} \) is the only, up to isomorphism, stable \( \mathbb{T}_{\overline{\rho}} \)-lattice of \( \rho_{\overline{\rho}} \).

From the proposition 5.4 of \([17]\), we then deduce that \( \text{gr}^k_m \simeq \Gamma_{\overline{\rho}} \otimes \rho_{\overline{\rho}} \), where \( \Gamma_{\overline{\rho}} \) is a stable \( \mathbb{T}^S_{\overline{\rho}} \)-lattice of \( \Pi^L_{\overline{\rho}} \) with a trivial action of the Galois group. Thus the matrix of \( N_{\text{coho}}^{v,m} \) is diagonal by block with the same block, in particular to understand \( N_{\text{coho}}^{v,m} \) we can equivalently work with \( N_{\text{coho}}^{v,m} \).

### 2.12. Proposition

**Proposition.** — Under the hypothesis of theorem 1.2, the \( E_{1,m}^{p,q} \) are torsion free and trivial for \( p + q \neq d - 1 \).

The proof uses Grothendieck-Verdier duality and we need to understand the difference between the \( p \) and \( p+ \) intermediate extensions of Harris-Taylor local systems \( HT(\pi_v, St(\pi_v)) \).

### 2.13. Lemma

**Lemma.** — With the previous notations, we have an isomorphism

\[
P_{j_{h,\overline{d}}^{\geq h}}^j HT(\chi_v, \Pi_h) \simeq P_{j_{h,\overline{d}}^{\geq h}}^j HT(\chi_v, \Pi_h).
\]

**Proof.** — Recall that \( \text{Sh}_{I_v, T_{\overline{d}}}^{h,\overline{d}} \) is smooth over \( \text{Spec} \mathbb{F}_p \). As, up to a modification of the action of the fundamental group through the character \( \chi_v \), we have

\[
HT(\chi_v, \Pi_h)_{T_{\overline{d}}}^{\overline{d}}[h - d] = (\mathbb{Z}_d)_{\text{Sh}_{I_v, T_{\overline{d}}}^{h,\overline{d}}} \otimes \Pi_h.
\]

Then \( HT(\chi_v, \Pi_h)_{T_{\overline{d}}}^{\overline{d}} \) is perverse for the two \( t \)-structures with

\[
\text{P}^{h \leq +1, 1}_{T_{\overline{d}}} HT(\chi_v, \Pi_h)_{T_{\overline{d}}} \in P^D<0 \quad \text{and} \quad \text{P}^{h \leq +1, 1}_{T_{\overline{d}}} HT(\chi_v, \Pi_h)_{T_{\overline{d}}} \in P^D>0.
\]
Remark. When \( \pi_v \) is no more a character, one of the main result of [6] insures that the two intermediate extensions still coincide as soon as the modulo \( l \) reduction of \( \pi_v \) remains supercuspidal. As explained in the introduction, as by now [6] is not yet published, one may have to restrict to the case where \( \text{Scusp}_v(m) \) have only characters, but we will write the proof so that it is valid in the general setting as soon as the results of [6] will be available.

Proof. — (of proposition 2.12)
As \( m \) is supposed to be KHT-free, then all the \( E^n_{x,m} \) are free. Moreover, as \( \overline{\pi}_m \) is irreducible, then the \( E^{p,q}_{1,m} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \) and \( E^{p,q}_{x,m} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \) are all zero if \( p+q \neq d-1 \). Recall, cf. [6] proposition 3.1.3, that we have the following splitting

\[
\Psi_v \cong \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \text{Scusp}_f(g)} \Psi_g
\]

where \( \text{Scusp}_f(g) \) is the set of inertial equivalence classes of irreducible \( \mathbb{F}_l \)-supercuspidal representations of \( GL_g(F_v) \), with the property that the irreducible constituents of \( \Psi_g \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \) are exactly the perverse Harris-Taylor sheaf associated to an irreducible cuspidal \( \mathbb{Q}_l \)-representation of \( \text{Scusp}_f(g) \), cf. definition 2.8. In particular for every \( g \in \text{Scusp}_f(g) \), the cohomology groups of \( \Psi_g \) are torsion free.

Remark. When computing the \( m \)-localized cohomology groups, we are only concerned with \( g \in \text{Scusp}_f(m) \) and, from our hypothesis on \( m \), with irreducible \( \overline{\mathbb{Q}}_l \)-cuspidal representations \( \pi_v \) of \( g \)-type \(-1\), or more precisely, if we do not want to use the results of [6], with characters \( \chi_v \). In particular, by the previous lemma, the \( p \) and \( p+1 \) intermediate extensions coincide.

Start first from the following resolution of \( \pi^{=tg}_{j_{1*}} HT(\pi_{v,-1}, St(t(\pi_{v,-1})) \)

\[
0 \rightarrow j_{1}^{=sg} HT(\pi_{v,-1}, St(t(\pi_{v,-1}\{t-s\})) \times \text{Speh}_{x-t}(\pi_{v,-1}\{t/2\})) \otimes \mathbb{Q}_l \rightarrow \cdots
\]

\[
\rightarrow j_{1}^{=(t+1)g} HT(\pi_{v,-1}, St(t(\pi_{v,-1}\{-1/2\}) \times \pi_{v,-1}\{t/2\}) \otimes \mathbb{Q}_l \rightarrow \cdots
\]

\[
\rightarrow j_{1}^{=tg} HT(\pi_{v,-1}, St(t(\pi_{v,-1})) \rightarrow p_{j_{1*}}^{=tg} HT(\pi_{v,-1}, St(t(\pi_{v,-1})) \rightarrow 0. \quad (2.14)
\]

\]
This result is proved in full generality over \( \mathbb{Q}_l \) in [1]. Over \( \mathbb{Z}_l \), it is
proved in [6] but for the case where \( \pi_{v,-1} \) is a character, the argument
is trivial as we just have to notice that the strata \( \text{Sh}_{I,\tilde{s}_v,1}^{\geq h} \) are smooth
so that the constant sheaf, up to shift, is perverse and so equals to the
intermediate extension of the constant sheaf, shifted by \( d-h \), on \( \text{Sh}_{I,\tilde{s}_v,1}^{\geq h} \).

The previous resolution is then just the induced version of this. By
adjunction property, the map

\[
 j_l^{(t+\delta)g} \ HT(\pi_{v,-1}, St_l(\pi_{v,-1}\{-\delta\}/2)) \times \text{Speh}_d(\pi_{v,-1}\{t/2\}) \otimes \Xi^{\delta/2} \\
\longrightarrow \ j_l^{(t+\delta-1)g} \ HT(\pi_{v,-1}, St_l(\pi_{v,-1}\{1-\delta\}/2)) \times \text{Speh}_{d-1}(\pi_{v,-1}\{t/2\}) \otimes \Xi^{\delta-1/2}
\]

is given by

\[
 HT(\pi_{v,-1}, St_l(\pi_{v,-1}\{-\delta\}/2)) \times \text{Speh}_d(\pi_{v,-1}\{t/2\}) \otimes \Xi^{\delta/2} \\
\longrightarrow \ p_l^{(t+\delta)g} j_l^{(t+\delta-1)g} \ HT(\pi_{v,-1}, St_l(\pi_{v,-1}\{1-\delta\}/2)) \times \text{Speh}_{d-1}(\pi_{v,-1}\{t/2\}) \otimes \Xi^{\delta-1/2}
\]

From [6] for the general case, we have

\[
p_l^{(t+\delta)g} j_l^{(t+\delta-1)g} \ HT(\pi_{v,-1}, St_l(\pi_{v,-1}\{1-\delta\}/2)) \times \text{Speh}_{d-1}(\pi_{v,-1}\{t/2\}) \otimes \Xi^{\delta-1/2}
\]

\[
 \simeq \ HT(\pi_{v,-1}, St_l(\pi_{v,-1}\{-\delta\}/2)) \times \left( \text{Speh}_{d-1}(\pi_{v,-1}\{-1/2\}) \times \pi_{v,-1}\{\delta-1/2\}\{t/2\} \right) \otimes \Xi^{\delta/2}
\] (2.16)

**Remark.** When \( \pi_{v,-1} \) is a character the previous isomorphism is trivial.

Indeed from (2.14) and before inducing from \( \text{Sh}_{I,\tilde{s}_v,1}^{\geq h} \) to \( \text{Sh}_{I,\tilde{s}_v,1}^{\geq h} \), we just need to understand \( p_l^{(t+1)g} j_l^{(t+1)g} \ HT(\chi_v,\Pi_h) \) knowing, cf. the proof of
the lemma 2.13, that, as \( \text{Sh}_{I,\tilde{s}_v,1}^{\geq h} \) is smooth, \( p_l^{(t+1)g} \ HT(\chi_v,\Pi_h) \) is trivially
locally constant

\[
 HT(\chi_v,\Pi_h)_{|\text{Sh}_{I,\tilde{s}_v,1}^{\geq h}}[h-d] = (\mathbb{Z}_l)_{|\text{Sh}_{I,\tilde{s}_v,1}^{\geq h}} \otimes \Pi_h.
\]

**Fact.** In particular, up to homothety, the map (2.17), and so those of
(2.16), is unique. Finally as the maps of (2.14) are strict, the given maps
(2.15) are uniquely determined, that is, if we forget the infinitesimal parts, these maps are independent of the chosen \( t \) in (2.14).

For every \( 1 \leq t \leq d \), let denote by \( i(t) \) the smallest index \( i \) such that \( H^i(\text{Sh}_{I,v}, \rho^t) \) has non trivial torsion: if it does not exist then we set \( i(t) = +\infty \). By duality, as \( p_j = p^t \) for cuspidal representation \( \pi_{v,-1} \) of \( \rho \)-type \(-1\), note that when \( i(t) \) is finite then \( i(t) \leq 0 \). Suppose by absurdity there exists \( t \) with \( i(t) \) finite and denote \( t_0 \) the biggest such \( t \).

2.18. Lemma. — For \( 1 \leq t \leq t_0 \) then \( i(t) = t - t_0 \).

Remark. A similar result is proved in \([4]\) when \( \pi_{v,-1} \) is a character and when the level is maximal at \( v \).

Proof. — Note first that for every \( t_0 \leq t \leq s \), then the cohomology groups of \( j^t \) are torsion free. Indeed there exists a filtration

\[
(0) = \text{Fil}^{-d}(\pi_{v,-1}, t) \hookrightarrow \text{Fil}^{-d}(\pi_{v,-1}, t) \hookrightarrow \cdots
\]

with graded parts

\[
\text{gr}^{-k}(\pi_{v,-1}, t) \simeq \rho^t \text{Fil}^{-t}(\pi_{v,-1}, t) = \rho^t \text{HT}(\pi_{v,-1}, \Pi_t)
\]

Remark. For \( \pi_{v,-1} \) a character as in the proof of lemma 2.13, the statement is easy and for the general case, the result is proved in \([6]\).

The associated spectral sequence localized at \( m \) is then concentrated in middle degree and torsion free. Then the spectral sequence associated to (2.14) has all its \( E_1 \) terms torsion free and degenerates at its \( E_2 \) terms. As by hypothesis the aims of this spectral sequence is free and equals to only one \( E_2 \) terms, we deduce that all the maps

\[
H^0(\text{Sh}_{I,v}, j^t_{I,v} \rho^t) \text{HT}(\pi_{v,-1}, \Pi_t(\pi_{v,-1} \{ \frac{t - \delta}{2} \}) \times \text{Speh}_\delta(\pi_{v,-1} \{ t/2 \})) \otimes \Xi^{\delta/2})_m
\]

\[
H^0(\text{Sh}_{I,v}, j^{t+\delta-1}_I \rho^{t+\delta-1}) \text{HT}(\pi_{v,-1} \{ \frac{1 - \delta}{2} \}) \times \text{Speh}_{\delta-1}(\pi_{v,-1} \{ t/2 \}) \otimes \Xi^{\delta-1/2})_m
\]
are strict. Then from the previous fact stressed after (2.17), this property remains true when we consider the associated spectral sequence for $1 \leq t' \leq t_0$.

Consider now $t = t_0$ where we know the torsion to be non trivial. From what was observed above, we then deduce that the map

$$H^0(\text{Sh}_{I,s,v}, j_{t_0}^!(t_0 + 1) g) H^T(\pi_{v,-1}, \text{St}_{t_0}(\pi_{v,-1}(-1/2)) \times \pi_{v,-1}(t_0/2)) \otimes \Xi^{1/2}_m$$

has a non trivial torsion cokernel so that $i(t_0) = 0$.

Finally for any $1 \leq t \leq t_0$, the map like (2.20) for $t + \delta - 1 < t_0$ are strict so that the $H^t(\text{Sh}_{I,s,v}, p_{j_1}^{=tg} H^T(\pi_{v,-1}, \Pi_t)_m$ are zero for $i < t - t_0$ while when $t + \delta - 1 = t_0$ its cokernel has non trivial torsion which gives then a non trivial torsion class in $H^{t-t_0}(\text{Sh}_{I,s,v}, p_{j_1}^{=tg} H^T(\pi_{v,-1}, \Pi_t))_m$. □

Consider now the filtration of stratification of $\Psi_\varrho$ constructed using the adjunction morphisms $j_{t_0}^{=tg} j_{t_0}^{=t\varrho}$ as in [3]

$$\text{Fil}^0(\Psi_\varrho) \hookrightarrow \text{Fil}^1(\Psi_\varrho) \hookrightarrow \text{Fil}^2(\Psi_\varrho) \hookrightarrow \cdots \hookrightarrow \text{Fil}^m(\Psi_\varrho)$$

where $\text{Fil}^m(\Psi_\varrho)$ is the saturated image of $j_{t_0}^{=t\varrho} j_{t_0}^{=t\varrho} \Psi_\varrho \longrightarrow \Psi_\varrho$. For our fixed $\pi_{v,-1}$, let denote $\text{Fil}^m_0(\pi_{v,-1})(\Psi) \hookrightarrow \text{Fil}^m(\Psi_\varrho)$ such that $\text{Fil}^m_0(\pi_{v,-1})(\Psi) \otimes \mathbb{Z}_l \cong \text{Fil}^m(\Psi_{v,-1})$ where $\Psi_{v,-1}$ is the direct factor of $\Psi \otimes \mathbb{Z}_l \otimes \mathbb{Q}_l$ associated to $\pi_{v,-1}$, cf. [3]. From the main result of [6], we have the following resolution of $\text{Fil}^m_0(\pi_{v,-1})(\Psi)$, which amounts to describe the germs of the $\mathbb{Z}_l$-sheaf cohomology of $\text{Fil}^m_0(\pi_{v,-1})(\Psi)$:

$$0 \longrightarrow j_{t_0}^{=t\varrho} H^T(\pi_{v,-1}, \text{Speh}_s(\pi_{v,-1})) \otimes L_\varphi(\pi_{v,-1}(-1/2)) \longrightarrow$$

$$j_{t_0}^{=(s-1)\varrho} H^T(\pi_{v,-1}, \text{Speh}_{s-1}(\pi_{v,-1})) \otimes L_\varphi(\pi_{v,-1}(-2/2)) \longrightarrow$$

$$\cdots \longrightarrow j_{t_0}^{=\varrho} H^T(\pi_{v,-1}, \pi_{v,-1}) \otimes L_\varphi(\pi_{v,-1}) \longrightarrow \text{Fil}^m_0(\pi_{v,-1})(\Psi) \longrightarrow 0, \quad (2.21)$$

where $L_\varphi$ is the dual of the local Langlands correspondence.

Remark. In the case where $\pi_{v,-1}$ is a character $\chi_v$, (2.21) is proved in [1] over $\mathbb{Q}_l$ and we just need to verify that every map is strict. Consider then the torsion part of the cokernel of one of these map. Note that it must have non trivial invariants under the action the Iwahori sub-group
at $v$: indeed, using lemma 2.13, any irreducible sub-quotient of 
\[ j^{s g}_t \ HT(\chi_v, \ Speh_{g}(\chi_v)) \otimes \mathbb{Z}_q \mathbb{F}, \]
has non trivial invariant at Iwahori level at $v$. We can then work at Iwahori level at $v$. As we say it above, it amounts to understand the germs of the $\mathbb{Z}_q$-sheaf cohomology of $\text{Fil}^g_{\chi_v}(\Psi)$ which are described, cf. [12], by the cohomology of the Lubin-Tate tower, or by the comparison theorem of Faltings-Fargues, cf. [13], those of the Drinfeld tower. The cohomology of the Drinfeld tower in Iwahori level is then described in [16].

We can then apply the previous arguments so that $H^i(\text{Sh}_{\tilde{s}_v}, \text{Fil}^g_{\pi_{v-1}}(\Psi))$ has non trivial torsion for $i = 1 - t_0$ and its free quotient is zero for $i \neq 0$.

Consider now the other graded parts which are either associated to some $\pi_v$ of type $\varrho$ or to $\text{gr}^{\text{tg}}(\Psi_{\varrho})$ for $t \geq 2$. In the last case, we also have a similar resolution like before
\[
0 \to j^{s g}_t \ HT(\pi_{v-1}, LT_{t,s}(\pi_{v-1})) \otimes L_g(\pi_{v-1}(\frac{s - t}{2})) \to \\
j^{(s-1) g}_t \ HT(\pi_{v-1}, LT_{t,s-1}(\pi_{v-1})) \otimes L_g(\pi_{v-1}(\frac{s - t - 1}{2})) \to \\
\cdots \to j^{s g}_t \ HT(\pi_{v-1}, St_t(\pi_{v-1})) \otimes L_g(\pi_{v-1} ) \to \text{Fil}^g_{\pi_{v-1}}(\Psi) \to 0, \tag{2.22}
\]
where
\[ LT_{t,s}(\pi_{v-1}) \hookrightarrow St_t(\pi_{v-1}(\{-\delta/2\}) \times \text{Speh}_{\delta}(\pi_{v-1}(t/2)), \]
is the only irreducible sub-space of this induced representation. By the same arguments, the torsion of $H^i(\text{Sh}_{\tilde{s}_v}, \text{Fil}^g_{\pi_{v-1}}(\Psi))$ is trivial for any $i \leq 1 - t_0$ and the free parts are concentrated for $i = 0$. Using then the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(\text{Sh}_{\tilde{s}_v}, \Psi_{\varrho})$ would have non trivial torsion which is false as $m$ is supposed to be KHT-free.

\[ \square \]

\textbf{3. Local behavior of monodromy over } $\mathbb{F}_l$

Let $\varrho$ be an irreducible supercuspidal $\mathbb{F}_l$-representation of $GL_g(F_v)$ with $1 \leq g \leq d$. Recall that we have a filtration $\text{Fil}^*(\Psi_{\varrho})$ whose graded
parts $\text{gr}^r(\Psi_\varrho)$ are free $\mathbb{Z}_l$-perverse sheaves of the following shape

$$\text{gr}^r(\Psi_\varrho) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \simeq p_{J_{st}}^{-h} \text{HT}_{\mathbb{Q}_l}(\pi_v, \text{St}_l(\pi_v))(1 - t + 2\delta)$$

for some $0 \leq k \leq t - 1$ and $\pi_v$ a cuspidal irreducible $\mathbb{Q}_l$-representation of type $\varrho$. In order to compute the cohomology groups localized at $\mathfrak{m}$ of $\text{Sh}_l$, by hypothesis on $\mathfrak{m}$, we are only concerned with the $\mathfrak{m}$-localized cohomology groups of $\mathcal{P}(t, \chi_v)$ for $\chi_v$ a character and $1 \leq t < m(\chi_v)$ where $m(\chi_v)$ is defined below.

**Remark.** If we allow ourself to use the result of [6], we have to consider $\pi_{v, -1}$ an irreducible cuspidal $\mathbb{Q}_l$-representation of $\varrho$-type $-1$ and $t < m(\varrho)$. In the following we will write the proofs in this general setting.

Denote by $s := \lfloor \frac{t}{v} \rfloor$, and consider for $2 \leq t \leq s$, the integers $r_{\pm}(t)$ such that $\text{gr}^{r_{\pm}(t)}(\Psi_\varrho)$ are isomorphic to $\mathcal{P}(t, \pi_{v, -1})(\pm \frac{t - 1}{2})$. Then $N_{v, t}^{-1}$ induces a map

$$N_{v, t}^{-1}: \text{Fil}^{r_{+}(t)}(\Psi_\varrho) \rightarrow \text{Fil}^{-r_{-}(t)}(\Psi_\varrho) \rightarrow \text{gr}^{r_{-}(t)}(\Psi_\varrho),$$

which after tensoring with $\mathbb{Q}_l$, induces an isomorphism $\text{gr}^{r_{+}(t)}(\Psi_\varrho) \simeq \text{gr}^{r_{-}(t)}(\Psi_\varrho)$. In this section we want to prove that this property remains true modulo $l$.

**Remark.** In order to define $N_v$ on $\text{Fil}^{r_{+}(t)}(\Psi_\varrho)/\text{Fil}^{-r_{-}(t)}(\Psi_\varrho)$ over $\mathbb{Z}_l$, we suppose that $l \geq t$ where $t$ is then the order of nilpotency.

**3.1. Notation.** — We denote by $m(\varrho)$ the order of the Zelevinsky line of $\varrho$ if it is not equal to 1, otherwise $m(\varrho) = l$.

**Remark.** When $\varrho$ is a character and $q_v \not\equiv 1 \mod l$, then $m(\varrho)$ is the order of $q_v$ modulo $l$.

**3.2. Proposition.** — Suppose $m(\varrho) > t$, then the morphism on $\text{Fil}^{r_{+}(t)}(\Psi_\varrho)/\text{Fil}^{-r_{-}(t)}(\Psi_\varrho)$ induced by the monodromy operator $N_v$, is strict.

**Proof.** — Recall first that the filtration $\text{Fil}^*(\Psi_\varrho)$ is constructed so that it is compatible with the action of $N_v$ in the sense that, over $\mathbb{Q}_l$, the image of $\text{Fil}^r(\Psi_\varrho)$ is some $\text{Fil}^{\delta(r)}(\Psi_\varrho)$. We then have to prove that for every $r_{-}(t) \leq r \leq r_{+}(t)$, we have a $p$-epimorphism $N_v : \text{Fil}^r(\Psi_\varrho) \rightarrow \text{Fil}^{\delta(r)}(\Psi_\varrho)$ which is clearly equivalent to prove that for every irreducible cuspidal representation $\pi_v$ of $GL_g(F_v)$, which by hypothesis on $\mathfrak{m}$ we can
suppose to be of \( \varrho \)-type \(-1\), and for every \( 1 \leq t' \leq t \), then \( N_v \) induces a isomorphism

\[
p_j = t' \ast g HT_{\mathbb{Z}}(\pi_{v,-1}, St_{t'}(\pi_{v,-1}))(\frac{1-t' + 2\delta}{2}) \rightarrow \\
p_j = t' \ast g HT_{\mathbb{Z}}(\pi_{v,-1}, St_{t'}(\pi_{v,-1}))(\frac{1-t' + 2(\delta - 1)}{2}), \quad (3.3)
\]

for every \( 1 \leq \delta < t' \), where each of these two perverse sheaf is given by the graded parts \( \text{gr}^r(\Psi_\varrho) \) and \( \text{gr}^{\Phi(\varrho)}(\Psi_\varrho) \).

Remark. Note that as \( \pi_{v,-1} \) is of \( \varrho \)-type \(-1\), then by lemma 2.13 these graded parts are the \( p \)-intermediate extensions. In [6] we proved that it is also the case more generally as soon as you construct your filtration using exclusively the adjunction map \( j_!j^* \rightarrow \text{Id} \).

Recall that under the hypothesis that \( m(\varrho) > t \geq t' \), then the reduction modulo \( l \) of \( \pi_{v,-1}[t]_\varrho \otimes L_\varrho(\pi_{v,-1}) \otimes St_{t'}(\pi_{v,-1}) \) is irreducible so that there exists, up to homothety, an unique stable lattice of \( HT(\pi_{v,-1}, St_{t'}(\pi_{v,-1})) \) which means that to prove (3.3) is a isomorphism, it suffices to prove its reduction modulo \( l \) is non zero. To do so,

– in the case where \( \varrho \) is a character it suffices to work in the Iwahori level and use the arguments of [8] §3.1 where the monodromy action is, thanks to Rapoport-Zink cf. [15], described explicitly and is of maximal nilpotency.

– We now want to explain another proof which works whatever is the irreducible supercuspidal representation \( \varrho \). Consider a maximal ideal \( m \) verifying the previous hypothesis and such that for any \( \bar{m} \subset m \) with \( \Pi_{\bar{m}} \) is cohomological, then its local component at \( v \) is isomorphic to \( St_{t}(\pi_{v}) \times \pi_{v}' \)

\bullet where \( \pi_{v} \) is of \( \varrho \)-type \(-1\),

\bullet and where \( \pi_{v}' \) is a representation such that its modulo \( l \) reduction has a supercuspidal support a set of \( \mathcal{F}_{l} \)-characters in disjoint Zelevinsky lines, pairwise distinct and different from those of \( \varrho \).

Such \( \pi_{v}' \) is then isomorphic to \( \chi_{1,v} \times \cdots \times \chi_{r,v} \) for characters \( \chi_{v,i} \). As before, the monodromy operator \( N_{v,m}^{\coho} \) is given by the local monodromy \( N_v \) on \( \Psi_\varrho \) so that in particular, if it happens that the modulo \( l \) reduction of \( N_{v,m}^{\coho} \) on the block \( \rho_{\bar{m},v} \), indexed by \( \bar{m} \) is of maximal nilpotency \( s \), then (3.3) is necessary non trivial for such \( (\pi_v,t) \). We are then reduced to study the nilpotency of \( \rho_{\bar{m},v} \).
Consider now a finite extension $L_v/F_v$ such that the restriction of $\rho_v$ to the Galois group of $L_v$ is unramified and let $L/F$ be a CM field globalizing $L_v/F_v$. By base change the local component of the automorphic representation $\Pi' := BC_{L/F}(\Pi_{\tilde{m}})$ associated to $\rho' := \rho_{|G_L}$ is of the following form

$$\Pi'_v \simeq \text{St}_s(\xi_1) \times \cdots \times \text{St}_s(\xi_g) \times \chi_1 \times \cdots \times \chi_r$$

where $\xi_1, \ldots, \xi_g$ and $\chi_1, \ldots, \chi_r$ are characters. We are then reduced to the case of $(\rho', \Pi')$; the idea is then to realize $\rho'$ in the cohomology of a KHT Shimura variety over $L$ through a spectral sequence like in the previous section, i.e. which degenerates in $E_1$ with all $E^{p,q}_1$ free.

As remarked before, the monodromy is then obtained from those on the vanishing cycle sheaves. We then recover the previous case of characters where the result follows from the Iwahori case described by Rapoport and Zink.

\[ \square \]

4. Level fixing property

Consider as before a maximal ideal $\mathfrak{m}$ associated to the modulo $l$ reduction of an irreducible automorphic representation $\Pi_{\tilde{m}}$ with $\tilde{m} \subset \mathfrak{m}$: recall that $\mathfrak{m}$ corresponds to the multiset of modulo $l$ Satake parameters at almost all places where $\Pi_{\tilde{m}}$ is unramified.

A classical question is then to know if it is possible to find another prime ideal $\tilde{m}'$ such that at a fixed place $v$, the level of $\Pi_{\tilde{m}',v}$ is strictly lower (resp. higher) of whose of $\Pi_{\tilde{m},v}$: we then speak of level lowering (resp. raising) phenomenon.

In this section we are interested in the opposite situation where whatever is $\tilde{m} \subset \mathfrak{m}$, then the level at $v$ is given by $\mathfrak{m}$: we call it a level fixing phenomenon.

4.1. Proposition. — Let $\mathfrak{m}$ be a maximal ideal of $T^S_{\xi}$ verifying the following properties:

- $\tilde{\mathfrak{m}}$ is irreducible,
- $\mathfrak{m}$ is KHT-free in the sense of definition 1.1,
- $\text{Scusp}_v(\mathfrak{m})$ is made of characters, and does not contain any full Zelevinsky line.
Then for any prime ideals $\mathfrak{m}_1$, $\mathfrak{m}_2$ contained in $\mathfrak{m}$ and for any open compact subgroup $K_v$ of $GL_2(F_v)$, then $\Pi_{\mathfrak{m}_1,v}$ has non trivial $K_v$-invariants vectors if and only if the same is true for $\Pi_{\mathfrak{m}_2,v}$.

Proof. — By KHT-freeness of $\mathfrak{m}$, we are in the situation of proposition 2.12 so that the nilpotency of the monodromy operator on $\bar{\mathfrak{m}}_{\mathfrak{m},v}$ is given by those of $N_{v,m}^{\text{coho}}$ on the cohomology. As said before $N_{v,m}^{\text{coho}}$ is controlled by the action of the local monodromy operator $N_v$ on the vanishing cycle sheaves. More precisely, consider a prime ideal $\tilde{\mathfrak{m}} \subset \mathfrak{m}$ such that

$$\Pi_{\tilde{\mathfrak{m}},v} \simeq \text{St}_{t_1}(\chi_{v,1}) \times \cdots \times \text{St}_{t_r}(\chi_{v,r}),$$

with $t_1 \geq \cdots \geq t_r$ and where $\chi_{v,1}, \cdots , \chi_{v,r}$ are characters, and with the notations of the introduction

$$\rho_{\tilde{\mathfrak{m}},v} \simeq \text{Sp}_{t_1}(\chi_{v,1}) \oplus \cdots \oplus \text{Sp}_{t_r}(\chi_{v,r}).$$

Let denote by $\mathfrak{g}$ the modulo $l$ reduction of $\chi_{v,1}$ and let $r_{\pm}(t_1)$ be the indexes such that $\text{gr}_{r_{\pm}(t_1)}(\Psi_{\mathfrak{g}}) \simeq \mathcal{P}(t_1, \chi_{v,1})(\pm \frac{1-t_1}{2})$. Proposition 3.2 applies

$$N_{v,t_1}^{\dagger} : \text{gr}_{r_{+}(t_1)}(\Psi_{\mathfrak{g}}) \simeq \text{gr}_{r_{-}(t_1)}(\Psi_{\mathfrak{g}}).$$

By proposition 2.12, the spectral sequence associated to the filtration of $\Psi_{\mathfrak{g}}$ induces a strict filtration of $H^0(\text{Sh}_{I,\tilde{\mathfrak{m}},v}, \Psi_{\xi})_m$. Note that $H^0(\text{Sh}_{I,\tilde{\mathfrak{m}},v}, \text{gr}_{r_{\pm}(t_1)}(\Psi_{\xi}))_m \otimes \mathbb{Z}/l$ has a sub-quotient isomorphic to $(\widetilde{\Pi}_{\mathfrak{m}}^{\dagger})_l \otimes \chi_{v,1}(\pm \frac{1-t_1}{2})$ and $(N_{v,m}^{\text{coho}})^{t_1-1}$ induces an isomorphism

$$(N_{v,m}^{\text{coho}})^{t_1-1} : H^0(\text{Sh}_{I,\tilde{\mathfrak{m}},v}, \text{gr}_{r_{\pm}(t_1)}(\Psi_{\xi}))_m \longrightarrow H^0(\text{Sh}_{I,\tilde{\mathfrak{m}},v}, \text{gr}_{r_{-}(t_1)}(\Psi_{\xi}))_m.$$

In particular the action of $N_v$ on the direct factor $\text{Sp}_{t_1}(\chi_{v,1})$ of $\rho_{\tilde{\mathfrak{m}},v}$ is such that the modulo $l$ reduction of $N_v^{t_1-1}$ is non zero. The argument works for all the factors $\text{Sp}_{t_1}(\chi_{v,i})$ of $\rho_{\tilde{\mathfrak{m}},v}$ so that the partition $d_{\mathfrak{m},v}$ is $t_1 \geq \cdots \geq t_r$. Moreover for any $\mathfrak{m}' \subset \mathfrak{m}$ with

$$\Pi_{\mathfrak{m}',v} \simeq \text{St}_{t_1'}(\chi_{v,1}') \times \cdots \times \text{St}_{t_r'}(\chi_{v,r}'),$$

then $d_{\mathfrak{m},v} = (t_1' \geq \cdots \geq t_r')$ is as above equals to $d_{\mathfrak{m},v}$ i.e. $(t_1', \cdots , t_r') = (t_1, \cdots , t_r)$, which trivially implies the statement of the proposition.

Remark. Using the main result of [6], one can state and prove in the same way, the above statement without the hypothesis that $\text{Scusp}_v(\mathfrak{m})$ contains only characters.


References


