LOCAL IHARA'S LEMMA AND APPLICATIONS

by

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Abstract. — Persistence of non-degeneracy is a phenomenon which appears in the theory of $\overline{\mathbb{Q}}_l$-representations of the linear group: every irreducible submodule of the restriction to the mirabolic sub-representation of a non-degenerate irreducible representation is non-degenerate. This is not true anymore in general, if we look at the modulo l reduction of some stable lattice. As in the Clozel-Harris-Taylor generalization of global Ihara's lemma, we show that this property, called non-degeneracy persistence and related to the notion of essentially absolutely irreducible and generic representations in the work of Emerton-Helm, remains true for lattices given by the cohomology of Lubin-Tate spaces. As an global application, we give a new construction of automorphic congruences in the Ribet spirit.

2010 ***Mathematics Subject Classification.*** — 11F70, 11F80, 11F85, 11G18, 20C08.
Key words and phrases. — Rapoport-Zink spaces, mod l representations, mirabolic group, non-degenerate representations, essentially absolutely irreducible and generic representations.

The authors thanks the ANR for his support through the project CoLoss AAPG2019.

Résumé. — La persistence de la non dégénérescence est un phénomène qui apparait dans la théorie des $\overline{\mathbb{Q}}_l$-représentations du groupe linéaire: toute sous-représentation irréductible de la restriction au groupe mirabolique d'une représentation irréductible non dégénérée, est non dégénérée. Ce n'est plus le cas en général pour la réduction modulo l d'un réseau stable. Comme dans la généralisation par Clozel-Harris-Taylor du lemme d'Ihara, nous montrons que cette propriété de non dégénérescence, qui est reliée à la notion de représentation essentiellement absolument générique de Emerton-Helm, reste valide pour les réseaux donnés par la cohomologie des espaces de Lubin-Tate. Nous une application de nature globale en construisant des congruences automorphes dans l'esprit du travail de Ribet.

Contents

Introduction

Before the "Ihara avoidance" argument of Taylor, the proof of Sato-Tate conjecture by Clozel, Harris and Taylor, rested on a conjectural generalization in higher dimension of the classical Ihara's lemma for GL_2. Their formulation can be understood as some persistence of the non-degeneracy property by reduction modulo l of automorphic representations.

Fix prime numbers $l \neq p$ and a finite extension K of $\mathbb{Q}_p$. Recall then [**28**] corollary 6.8, that any irreducible $\overline{\mathbb{Q}}_l$-representation π of $GL_d(K)$ is homogeneous which means, cf. [**28**] definition 5.1, that its restriction to the mirabolic subgroup $M_d(K)$ of matrices such that the last row is $(0, \cdots, 0, 1)$, is homogeneous in the sense that every irreducible sub-$M_d(K)$-representation has the same level of degeneracy, cf. [**28**] 4.3 or [**5**] 3.5. In particular if π is non-degenerate i.e. its level of degeneracy equals d, then any irreducible sub-representation of $\pi_{|M_d(K)}$ is also non-degenerate. Modulo l, for π an irreducible non-degenerate representation of $GL_d(K)$, there might exist stable lattices such that $\pi_{|M_d(K)} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ owns irreducible degenerate subspaces, cf. corollary 1.4.3.

We then propose to prove some persistence of non-degeneracy phenomenons in the cohomology groups of Lubin-Tate spaces. Consider a finite extension $K/\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$. For $d \geq 1$, denote by $\widehat{\mathcal{M}}_{LT,d,n}$ the formal scheme representing the functor of isomorphism classes of deformations by quasi-isogenies of the formal $\mathcal{O}_K$-module over $\overline{\mathbb{F}}_p$ of dimension 1 and height d with n-level structure. We denote by $\mathcal{M}_{LT,d,n}$ its generic fiber over $\widehat{K}^{un}$. For $\Lambda = \overline{\mathbb{Q}}_l, \overline{\mathbb{Z}}_l$ or $\overline{\mathbb{F}}_l$, consider both

$$\mathcal{U}^{d-1}_{LT,d,\Lambda} := \varinjlim_n H^{d-1}(\mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{un}} \widehat{\overline{K}}, \Lambda)$$

and

$$\mathcal{V}^{d-1}_{LT,d,\Lambda} := \varinjlim_n H^{d-1}_c(\mathcal{M}_{LT,d,n} \widehat{\otimes}_{\widehat{K}^{un}} \widehat{\overline{K}}, \Lambda).$$

There is a natural action of $GL_d(K) \times D^\times_{K,d} \times W_K$ on $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ and $\mathcal{V}^{d-1}_{LT,d,\Lambda}$, where $D_{K,d}$ (resp. W_K) is the central division algebra over K with invariant $1/d$ (resp. the Weil group of K). In this paper we focus on the action of $GL_d(K)$ and it appears, cf. [**7**], that every irreducible $GL_d(K)$-subquotient of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ and $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ is either a cuspidal or a generalized Steinberg representation, so it is always non-degenerate. One can then ask if any irreducible $GL_d(K)$-equivariant subspace of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{F}}_l}$ (resp. $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{F}}_l}$) is still non-degenerate or even more if any irreducible $M_d(K)$-equivariant subspace is non-degenerate.

Theorem. — *(cf. corollaries 4.1.9 and 4.2.5)*
The persistence of non-degeneracy property relatively to M_d holds for $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ and $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l,free} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, i.e. any irreducible $M_d(K)$-equivariant subspace is non-degenerate.

Remark: $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l,free}$ is the free quotient of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l}$. In [**10**] we prove that $\mathcal{V}^{i}_{LT,d,\overline{\mathbb{Z}}_l}$ and $\mathcal{U}^{i}_{LT,d,\overline{\mathbb{Z}}_l}$ are free for every i so that

$$\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{F}}_l} \quad \text{and} \quad \mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l,free} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{F}}_l}.$$

Note that we do not use this result to prove the theorem.

The main motivation of this work is to obtain a geometric incarnation of the local Langlands correspondance in families of Emerton-Helm-Moss and we hope to come to this project soon.

The strategy for proving this property of the Lubin-Tate cohomology, is to argue globally on Shimura varieties of Harris-Taylor type, $X_I \to \operatorname{Spec} \mathcal{O}_K$ where $\mathcal{O}_K$ is the ring of integers of K, cf. 2.2. Thanks to Berkovich's comparison theorem in [**4**], we have to understand the stalk of the $\overline{\mathbb{Z}}_l$-vanishing cycle perverse sheaf Ψ_I at some geometric supersingular point of the geometric special fiber $X_{I,\bar{s}}$ of X_I.

Using the Newton stratification of $X_{I,\bar{s}}$ and usual adjunction properties, cf. [**11**], we can construct various filtrations of Ψ_I. The main issue about these general constructions is to understand, with the terminology of §3.2, the phenomenon of saturation which is a blind process consisting of choosing artificially the right sub-perverse sheaves so that all the graded pieces are free. In particular it seems impossible to follow the lattices during this process. One solution is to use the construction of [**14**] based on a coarse filtration of stratification as recalled in §3, which introduce no saturation process during the construction: see lemmas 3.4.1 and 4.1.2. As explained in [**14**] the main reason that this coarse filtration is more interesting, is its link with the small mirabolic induction as defined in (1.3.5) rather than the full parabolic induction appearing in [**7**]. For more details, we advice the reader to look at the introduction of §4.

For z a geometric supersingular point and $i_z : \{z\} \hookrightarrow X_{I,\bar{s}}$, by considering either $i_z^* \mathcal{H}^i \Psi_I$ or $i_z^! \mathcal{H}^i \Psi_I$, where $\mathcal{H}^\bullet$ designates the functor of sheaf cohomology, we then obtain a filtration of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l,free}$ and $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Z}}_l,free}$. The graded pieces of these filtrations are then lattices of the irreducible $\overline{\mathbb{Q}}_l[GL_d(K) \times D^\times_{K,d} \times W_K]$-subquotients of $\mathcal{U}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$ (resp. $\mathcal{V}^{d-1}_{LT,d,\overline{\mathbb{Q}}_l}$), which can be described as a tensorial product of stable lattices $\Lambda_G \otimes \Lambda_D \otimes \Lambda_W$ of respectively $GL_d(K)$, $D^\times_{K,d}$ and W_K. Using the combinatorics of the non supersingular strata and the classical properties of the induced representations, cf. proposition 1.3.7, we are then able to prove that $V := \Lambda_G \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is *an essentially absolutely irreducible and generic representation in the sense of* [**21**] *definition 3.2.1*, i.e.

– the socle $\mathrm{soc}(V)$ of V is absolutely irreducible and generic,
– the quotient $V/\mathrm{soc}(V)$ contains no generic Jordan-Holder factors,
– the representation V is the union of its finite length submodules.

In §4.3, using results of [**10**], we also look at $\mathcal{U}^{d-1-\delta}_{LT,d,\overline{\mathbb{F}}_l}$ (resp. $\mathcal{V}^{d-1+\delta}_{LT,d,\overline{\mathbb{F}}_l}$) for $\delta > 0$. The situation is less pleasant to state but we can find cases where, cf. proposition 4.3.1 and the remarks before and after it, that irreducible subspaces must have minimal derivative order, but among the irreducible quotients of such derivative order, the lattices of Lubin-Tate cohomology groups select the one with non-degenerate highest derivative.

In the last section, we give a global application with new congruences between tempered and non tempered automorphic representations with the same level at l: their level are the same except at one place which can be chosen almost arbitrary.

Finally to give a perspective about this work, we could say, using the terminology cf. §3.2, that in [**10**] we solve the question about positions of the perverse Harris-Taylor sheaves inside the perverse sheaf of nearby cycles, and here we elucidate that of lattices.

1. Review on the representation theory for $GL_n(\mathbb{Q}_p)$

We fix a finite extension $K/\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We denote by $|-|$ its absolute value.

1.1. Induced representations.

— For a representation π of $GL_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \,\mathrm{val} \circ \det}.$$

1.1.1. Notations. — *For π_1 and π_2 representations of respectively $GL_{n_1}(K)$ and $GL_{n_2}(K)$, we will denote by*

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1\{\frac{n_2}{2}\} \otimes \pi_2\{-\frac{n_1}{2}\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$GL_{r_1} \times GL_{r_2-r_1} \times \cdots \times GL_{r_k-r_{k-1}}.$$

The symbol $\times$ being associative, we define inductively $\pi_1 \times \cdots \times \pi_s$ as $(\pi_1 \times \cdots \times \pi_{s-1}) \times \pi_s = \pi_1 \times (\pi_2 \times \cdots \times \pi_s)$.

Recall that a representation ϱ of $GL_d(K)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincide, but this is not true anymore for $\overline{\mathbb{F}}_l$.

1.1.2. Definition. — *(see* [**28**] §*9 and* [**8**] §*1.4) Let g be a divisor of $d = sg$ and π an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(K)$.*

- *The induced representation*

$$\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\}$$

  holds a unique irreducible quotient (resp. subspace) denoted by $\mathrm{St}_s(\pi)$ *(resp.* $\mathrm{Speh}_s(\pi)$*); it's a generalized Steinberg (resp. Speh) representation.*
- *For any integers $t, r \geq 1$, the induced representation* $\mathrm{St}_t(\pi\{\frac{-r}{2}\}) \times \mathrm{Speh}_r(\pi\{\frac{t}{2}\})$ *(resp.* $\mathrm{St}_{t-1}(\pi\{\frac{-r-1}{2}\}) \times \mathrm{Speh}_{r+1}(\pi\{\frac{t-1}{2}\})$*) owns a unique irreducible subspace (resp. quotient), denoted by $LT_\pi(t-1, r)$.*

1.2. Reduction modulo l of a Steinberg representation. — Denote by $e_l(q)$ the order of $q \in \mathbb{F}_l^\times$.

1.2.1. Notation. — *For $\Lambda = \overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, denote by $\mathrm{Scusp}_\Lambda(g)$ the set of equivalence classes of irreducible supercuspidal Λ-representations of $GL_g(K)$.*

1.2.2. Proposition. — *(cf. [**26**] III.5.10) Let π be an irreducible cuspidal representation of $GL_g(K)$ with a stable $\overline{\mathbb{Z}}_l$-lattice*[(1)]*, then its modulo l reduction is irreducible and cuspidal but not necessarily supercuspidal.*

In the following we will denote by r_l the functor of modulo l reduction, i.e. for a $\overline{\mathbb{Q}}_l$-representation π of a group G, with a stable $\overline{\mathbb{Z}}_l$-lattice Λ, then $r_l(\pi)$ is $\Lambda \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ with the induced action of G. Note that such $r_l(\pi)$ should depend on the chosen lattice Λ but its semi-simplification doesn't.

1.2.3. Proposition. — [**20**] *§2.2.3*
Let π be an irreducible entire cuspidal representation, and $s \geq 1$. Then the modulo l reduction of $\mathrm{Speh}_s(\pi)$ is irreducible.

1.2.4. Notation. — *The Zelevinski line associated with some irreducible supercuspidal $\overline{\mathbb{F}}_l$-representation ϱ, is the set $\{\varrho\{i\} \ / \ i \in \mathbb{Z}\}$. It is clearly a finite set and we denote by $\epsilon(\varrho)$ its cardinal which is a divisor of $e_l(q)$. We also introduce, cf.* [**27**] *p.51*

$$m(\varrho) = \begin{cases} \epsilon(\varrho), & \textit{if } \epsilon(\varrho) > 1; \\ l, & \textit{sinon.} \end{cases}$$

1.2.5. Definition. — *Consider a multiset*[(2)] *$\underline{s} = \{\rho_1^{n_1}, \cdots, \rho_r^{n_r}\}$ of irreducible supercuspidal $\overline{\mathbb{F}}_l$-representations. Following* [**27**] *V.7, we then denote by $\mathrm{St}(\underline{s})$ the unique non-degenerate irreducible sub-quotient of the induced representation*

$$\rho(\underline{s}) := \overbrace{\rho_1 \times \cdots \times \rho_1}^{n_1} \times \cdots \times \overbrace{\rho_r \times \cdots \times \rho_r}^{n_r}.$$

Remark: Thanks to [**27**] V.7, every irreducible non-degenerate $\overline{\mathbb{F}}_l$-representation can be written as $\mathrm{St}(\underline{s})$.

(1) We say that π is entire.
(2) meaning we take into account the multiplicities

1.2.6. Notation. — *For $s \geq 1$ and ϱ an irreducible cuspidal $\overline{\mathbb{F}}_l$-representation, we denote by $\underline{s}(\varrho)$ for the multi-segment $\{\varrho, \varrho\{1\}, \cdots, \varrho\{s-1\}\}$ and, cf.* [**27**] *V.4,* $\mathrm{St}_s(\varrho) := \mathrm{St}(\underline{s}(\varrho))$.

1.2.7. Proposition. — *(cf.* [**27**] *V.4)*
With the previous notation, the $\overline{\mathbb{F}}_l$-representation $\mathrm{St}_s(\varrho)$ is cuspidal if and only if $s = 1$ or $m(\varrho)l^k$ for some $k \geq 0$.

Remark: by [**26**] III-3.15 and 5.14, every irreducible cuspidal $\overline{\mathbb{F}}_l$-representation can be written $\mathrm{St}_s(\varrho)$ for some irreducible supercuspidal representation ϱ, and $s = 1$ or $s = m(\varrho)l^k$ with $k \geq 0$.

1.2.8. Notations. — *Let ϱ be an irreducible cuspidal $\overline{\mathbb{F}}_l$-representation of $GL_g(K)$. We then denote*

- *$g_{-1}(\varrho) := g$ and for $i \geq 0$, $g_i(\varrho) := m(\varrho)l^i g$;*
- *$\varrho_{-1} = \varrho$ and for all $i \geq 0$, $\varrho_i = \mathrm{St}_{m(\varrho)l^i}(\varrho)$.*
- $\mathrm{Cusp}(\varrho, i)$ *the set of equivalence classes of irreducible entire $\overline{\mathbb{Q}}_l$-representations such that modulo l it is isomorphic to ϱ_i,*
- *and* $\mathrm{Cusp}(\varrho) = \bigcup_{i \geq -1} \mathrm{Cusp}(\varrho, i)$.

1.2.9. Notation. — *Let $s \geq 1$ and ϱ an irreducible cuspidal $\overline{\mathbb{F}}_l$-representation of $GL_g(K)$. We denote by $\mathcal{I}_\varrho(s)$ the set of sequences $(m_{-1}, m_0, \cdots)$ of non-negative integers such that*

$$s = m_{-1} + m(\varrho) \sum_{k=0}^{+\infty} m_k l^k.$$

We denote by $\lg_\varrho(s)$ the cardinal of $\mathcal{I}_\varrho(s)$. We then define a relation of order on $\mathcal{I}_\varrho(s)$ by

$$(m_{-1}, m_0, \cdots) > (m'_{-1}, m'_0, \cdots) \Leftrightarrow \exists k \geq -1 \text{ s.t. } \forall i > k : m_i = m'_i \text{ and } m_k > m'_k.$$

1.2.10. Definition. — *For $\underline{i} = (i_{-1}, i_0, \cdots) \in \mathcal{I}_\varrho(s)$, we define*

$$\mathrm{St}_{\underline{i}}(\varrho) := \mathrm{St}_{i_{-1}}(\varrho_{-1}) \times \mathrm{St}_{i_0}(\varrho_0) \times \cdots \times \mathrm{St}_{i_u}(\varrho_u)$$

where $i_k = 0$ for all $k > u$.

Remark: we will denote by $\underline{s_{\max}}$ the maximal element of $\mathcal{I}_\varrho(s)$ so that $\mathrm{St}_{\underline{s_{\max}}}(\varrho)$ is non-degenerate.

1.2.11. Theorem. — *(cf.* [**9**] *proposition 3.1.5) Consider π an entire irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(K)$ and let ϱ be its modulo l reduction. In the Grothendieck group of $\overline{\mathbb{F}}_l$-representations of $GL_{sg}(K)$, we have the following equality:*

$$r_l\Big(\mathrm{St}_s(\pi)\Big) = \sum_{\underline{i}\in\mathcal{I}_\varrho(s)} \mathrm{St}_{\underline{i}}(\varrho).$$

Remark: for $s < m(\varrho)$, it is irreducible so, up to isomorphism, it possesses a unique stable lattice, cf. [**2**] proposition 3.3.2 and the following remark.

1.3. Restriction to the mirabolic group. — In this paragraph, we want to state some of the main results of [**5**] §4 about $\overline{\mathbb{Q}}_l$-representations(3): for $\overline{\mathbb{F}}_l$-representations the usual reference is [**26**] §III.

Recall first some notations of [**5**] §3, see also [**26**] §III-1 or [**21**] §3. The mirabolic subgroup $M_d(K)$ of $GL_d(K)$ is the set of matrices with last row $(0,\cdots,0,1)$: we denote

$$V_d(K) = \{(m_{i,j}) \in M_d(K) : \ m_{i,j} = \delta_{i,j} \text{ for } j < d\}.$$

its unipotent radical. We fix a non trivial character ψ of K and let θ the character of $V_d(K)$ defined by $\theta((m_{i,j})) = \psi(m_{d-1,d})$. For $G = GL_r(K)$ or $M_r(K)$, we denote $\mathrm{Alg}(G)$ the abelian category of smooth representations of G and, following [**5**], we introduce

$$\Psi^- : \mathrm{Alg}(M_d(K)) \longrightarrow \mathrm{Alg}(GL_{d-1}(K)),$$

and

$$\Phi^- : \mathrm{Alg}(M_d(K)) \longrightarrow \mathrm{Alg}(M_{d-1}(K)),$$

defined by $\Psi^- = r_{V_d,1}$ (resp. $\Phi^- = r_{V_d,\theta}$) the functor of V_d coinvariants (resp. (V_d,θ)-coinvariants), cf. [**5**] 1.8. We also introduce the unnormalized compact induced functor

$$\Psi^+ := i_{V,1} : \mathrm{Alg}(GL_{d-1}(K)) \longrightarrow \mathrm{Alg}(M_d(K)),$$
$$\Phi^+ := i_{V,\theta} : \mathrm{Alg}(M_{d-1}(K)) \longrightarrow \mathrm{Alg}(M_d(K)).$$

1.3.1. Proposition. — *(*[**5**] *p451,* [**21**] *proposition 3.1.3 or* [**26**] *§III-1)*

– *The functors Ψ^-, Ψ^+, Φ^- and Φ^+ are exact.*

(3)In loc. cit. the author consider complex representations, but for admissible ones, so in particular for irreducible smooth representations, they are defined over a finite extension of $\mathbb{Q}$ so that the facility consisting to fix an isomorphism $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$ is harmless.

– $\Phi^- \circ \Psi^+ = \Psi^- \circ \Phi^+ = 0$.
– Ψ^- *(resp.* Φ^+*) is left adjoint to* Ψ^+ *(resp.* Φ^-*) and the following adjunction maps*

$$\mathrm{Id} \longrightarrow \Phi^-\Phi^+, \qquad \Psi^+\Psi^- \longrightarrow \mathrm{Id},$$

are isomorphisms and the following sequence is exact

$$0 \to \Phi^+\Phi^- \longrightarrow \mathrm{Id} \longrightarrow \Psi^+\Psi^- \to 0.$$

1.3.2. Definition. — *For* $\tau \in \mathrm{Alg}(M_d(K))$*, the representation*

$$\tau^{(k)} := \Psi^- \circ (\Phi^-)^{k-1}(\tau)$$

is called the k*-th derivative of* τ*. If* $\tau^{(k)} \neq 0$ *and* $\tau^{(m)} = 0$ *for all* $m > k$*, then* $\tau^{(k)}$ *is called the highest derivative of* τ*.*

1.3.3. Notation. — *(cf.* [**28**] *4.3) Let* $\pi \in \mathrm{Alg}(GL_d(K))$ *(or* $\pi \in \mathrm{Alg}(M_d(K))$*). The maximal number* k *such that* $(\pi_{|M_d(K)})^{(k)} \neq (0)$ *is called the level of non-degeneracy of* π *and denoted by* $\lambda(\pi)$*.*

Remark: cf [**5**] 3.5, there exists a natural filtration $0 \subset \tau_d \subset \cdots \subset \tau_1 = \tau$ with

$$\tau_k = (\Phi^+)^{k-1} \circ (\Phi^-)^{k-1}(\tau) \text{ and } \tau_k/\tau_{k+1} = (\Phi^+)^{k-1} \circ \Psi^+(\tau^{(k)}).$$

In particular for τ irreducible there is exactly one k such that $\tau^{(k)} \neq (0)$ and then $\tau \simeq (\Phi^+)^{k-1} \circ \Psi^+(\tau^{(k)})$.

1.3.4. Notation. — *In the particular case where* $k = d$*, there is a unique irreducible representation* τ_{nd} *of* $M_d(K)$ *with derivative of order* d*.*

Remark: Note then by [**5**] 4.4, for every irreducible supercuspidal representation π of $GL_d(K)$, we have

$$\pi_{|M_d(K)} \simeq \tau_{nd}.$$

We can moreover understand theorem 1.2.11 as giving a partition of $\mathrm{St}_t(\pi)_{|M_d(K)}$ that associates to each part an irreducible constituent of $r_l(\mathrm{St}_t(\pi))$.

Consider first the following embedding $GL_r(K) \times M_s(K) \hookrightarrow M_{r+s}(K)$ sending

$$(A, M) \mapsto \begin{pmatrix} A & 0 \\ 0 & M \end{pmatrix}.$$

Imposing $\begin{pmatrix} I_r & U \\ 0 & I_s \end{pmatrix}$ acting trivially, and considering the normalized induced functor, we then define

$$\rho \otimes \tau \in \mathrm{Alg}(GL_r(K)) \times \mathrm{Alg}(M_s(K)) \mapsto \rho \times \tau \in \mathrm{Alg}(M_{r+s}(K)). \quad (1.3.5)$$

Secondly we consider $M_r(K) \times GL_s(K) \hookrightarrow M_{r+s}(K)$ sending

$$\Big(\begin{pmatrix} A & V \\ 0 & 1 \end{pmatrix}, B \Big) \mapsto \begin{pmatrix} A & 0 & V \\ 0 & B & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

imposing $\begin{pmatrix} I_{r-1} & U & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acting trivially and considering the normalized compact induction functor, we define

$$\tau \otimes \rho \in \mathrm{Alg}(M_r(K)) \times \mathrm{Alg}(GL_s(K)) \mapsto \tau \times \rho\nu^{-1/2} \in \mathrm{Alg}(M_{r+s}(K)), \quad (1.3.6)$$

where for $g \in GL_s(K)$, we denote by $\nu(g) := q^{\mathrm{val}(\det g)}$.

1.3.7. Proposition. — *(cf.* [**5**] *4.13) Let $\rho \in \mathrm{Alg}(GL_r(K))$, $\sigma \in \mathrm{Alg}(GL_t(K))$ and $\tau \in \mathrm{Alg}(M_s(K))$.*

(a) *In $\mathrm{Alg}(M_{r+t}(K))$, we have*

$$0 \to (\rho_{|M_r(K)}) \times \sigma \longrightarrow (\rho \times \sigma)_{|M_{r+t}(K)} \longrightarrow \rho \times (\sigma_{|M_t(K)}) \to 0.$$

(b) *If Ω is one of the functors $\Psi^{\pm}, \Phi^{\pm}$, then $\rho \times \Omega(\tau) \simeq \Omega(\rho \times \tau)$.*

(c) *$\Psi^-(\tau \times \rho) \simeq \Psi^-(\tau) \times \rho$ and*

$$0 \to \Phi^-(\tau) \times \rho \longrightarrow \Phi^-(\tau \times \rho) \longrightarrow \Psi^-(\tau) \times (\rho_{|M_r(K)}) \to 0.$$

(d) *Suppose $r > 0$. Then for any non-zero $M_{r+s}(K)$-submodule $\omega \subset \tau \times \rho$, we have $\Phi^-(\omega) \neq (0)$.*

We will call the the induced representation $\rho \times \tau$ (resp. $\tau \times \rho$) the small (resp. the big) mirabolic induced representation in the sense that the big one owns the highest derivative as you can see it in the proposition above or in lemma 1.3.11.

1.3.8. Definition. — *(*[**28**] *5.1) A representation $\tau \in \mathrm{Alg}(M_d(K))$ is called homogeneous if for all non-zero submodules $\sigma \subset \tau$, we have $\lambda(\sigma) = \lambda(\tau)$.*

1.3.9. Proposition. — *(cf.* [**28**] *6.8) Let π be an irreducible representation of $GL_d(K)$. Then $\pi_{|M_d(K)}$ is homogeneous.*

In the sequel we will use, in some sense dually, the group $P_d(K)$ with first column equals to ${}^t(1,0,\cdots,0)$. The map $g \mapsto \sigma({}^tg^{-1})\sigma^{-1}$ where σ is the matrix permutation associated with the cycle $(1\ 2\ \cdots\ n)$, induces an isomorphism between $P_d(K)$ and $M_d(K)$. After twisting with this isomorphism, we obtain analogs of the previous results with for example the following short exact sequence

$$0 \to \rho \times (\sigma_{|P_t(K)}) \longrightarrow (\rho \times \sigma)_{|P_{r+t}(K)} \longrightarrow (\rho_{|P_r(K)}) \times \sigma \to 0, \quad (1.3.10)$$

where the first representation is the compact induction relatively to

$$\begin{pmatrix} 1 & 0 & V_{t-1} \\ 0 & GL_r & U \\ 0 & 0 & GL_{t-1} \end{pmatrix},$$

and the second one is the induction from

$$\begin{pmatrix} P_r & U \\ 0 & GL_t \end{pmatrix}.$$

We will particularly use the following case.

1.3.11. Lemma. — *(cf. [**14**] lemme 4.4) Let π be an irreducible cuspidal representation of $GL_g(K)$. Then as a representation of $P_{(t+s)g}(K)$, we have isomorphisms*

$$\mathrm{St}_t(\pi\{-\frac{s}{2}\})_{|P_{tg}(K)} \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\}) \simeq LT_\pi(t-1,s)_{|P_{(t+s)g}(K)},$$

and

$$\mathrm{St}_t(\pi\{-\frac{s}{2}\}) \times \mathrm{Speh}_s(\pi\{\frac{t}{2}\})_{|P_{sg}(K)} \simeq LT_\pi(t,s-1)_{|P_{(t+s)g}(K)}.$$

1.3.12. Notation. — *For $c \in K^d$, we will denote by $M_c(K)$ the mirabolic subgroup stabilizing c.*

Remark: with this notation $P_d(K)$ is $M_c(K)$ for $c=(1,0,\cdots,0)$.

1.4. Some lattices of Steinberg representations. — Let π be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(K)$, supposed to be entire. As its reduction modulo l, denoted by ϱ, is still irreducible, up to isomorphism, it has a unique stable lattice, cf. [**2**] proposition 3.3.2 and its following remark.

1.4.1. Definition. — *(cf.* [**9**]*) Given a stable lattice of* $\mathrm{St}_t(\pi)$*, the surjection (resp. the embedding)*

$$\mathrm{St}_t(\pi) \times \pi\{t\} \twoheadrightarrow \mathrm{St}_{t+1}(\pi), \quad \textit{resp. } \mathrm{St}_{t+1}(\pi) \hookrightarrow \mathrm{St}_t(\pi\{1\}) \times \pi$$

gives a stable lattice of $\mathrm{St}_{t+1}(\pi)$ *so that inductively starting from* $t = 1$*, we construct a lattice denoted by* $RI_{\bar{\mathbb{Z}}_l,-}(\pi, t)$ *(resp.* $RI_{\bar{\mathbb{Z}}_l,+}(\pi, t)$*). We then denote by*

$$RI_{\bar{\mathbb{F}}_l,-}(\pi, t) := RI_{\bar{\mathbb{Z}}_l,-}(\pi, t) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l, \quad \textit{resp. } RI_{\bar{\mathbb{F}}_l,+}(\pi, t) := RI_{\bar{\mathbb{Z}}_l,+}(\pi, t) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l.$$

1.4.2. Proposition. — *(cf.* [**9**] *propositions 3.2.2 and 3.2.7) For every* $0 \leq k \leq \lg_\varrho(s)$*, there exists a unique length* k *sub-representation* $V_{\varrho,\pm}(s; k)$ *of* $RI_{\bar{\mathbb{F}}_l,\pm}(\pi, s)$

$$(0) = V_{\varrho,\pm}(s; 0) \subsetneq V_{\varrho,\pm}(s; 1) \subsetneq \cdots \subsetneq V_{\varrho,\pm}(s; \lg_\varrho(s)) = RI_{\bar{\mathbb{F}}_l,\pm}(\pi, s),$$

such that the image of $V_{\varrho,-}(s; k)$ *(resp.* $V_{\varrho,+}(s; k)$*) in the Grothendieck group verifies the following property: all its irreducible constituents are strictly greater (resp. smaller) than any irreducible constituent of*

$$W_{\varrho,-}(s; k) := V_{\varrho,-}(s; \lg_\varrho(s))/V_{\varrho,-}(s; k)$$

(resp. $W_{\varrho,+}(s; k) := V_{\varrho,+}(s; \lg_\varrho(s))/V_{\varrho,+}(s; k)$*), relatively to the relation of order of 1.2.9.*

1.4.3. Corollary. — *If* $\lg_\varrho(s) \geq 2$*, then we have two irreducible subspaces of* $RI_{\overline{\mathbb{Z}}_l,+}(\pi, s)_{|P_{sg}(K)} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ *which are*

- *first some irreducible* $P_{sg}(K)$*-subspaces of* $\mathrm{St}_{\underline{s}}(\varrho)$ *which is necessarily degenerate,*
- *and the non-degenerate irreducible* $P_{sg}(K)$*-representation,* τ_{nd} *which is a subspace of* $\mathrm{St}_{\underline{s_{\max}}}(\varrho)_{|P_{sg}(K)}$*.*

1.4.4. Proposition. — *The only irreducible subspace of the modulo* l *reduction of* $RI_{\overline{\mathbb{Z}}_l,-}(\pi, s)_{|P_{sg}(K)}$ *is the non-degenerate one* τ_{nd}*.*

Proof. — From the previous section, we have

$$RI_{\overline{\mathbb{Z}}_l,-}(\pi, s)_{|P_{sg}(K)} \simeq RI_{\overline{\mathbb{Z}}_l,-}(\pi\{\frac{-1}{2}\}, s-1) \times (\pi\{\frac{s-2}{2}\})_{|P_{sg}(K)},$$

so that the result follows by induction using proposition 1.3.7. □

2. Review on the geometric objects

2.1. Lubin-Tate spaces. — Let $\mathcal{O}_K$ be the ring of integers of K, $\mathcal{P}_K$ its maximal ideal, ϖ_K a uniformizer and $\kappa = \mathcal{O}_K/\mathcal{P}_K$ the residue field of cardinal $q = p^f$. Let K^{nr} be the maximal unramified extension of K and $\hat{K}^{nr}$ its completion with ring of integers $\mathcal{O}_{\hat{K}^{nr}}$. Let $\Sigma_{K,d}$ be the one-dimensional $\mathcal{O}_K$-formal module of Barsotti-Tate over $\overline{\mathbb{F}}_p$ with height d, cf. [**22**] §II. We consider the category C of artinian local $\mathcal{O}_{\hat{K}^{nr}}$-algebras with residue field $\overline{\kappa}$.

2.1.1. Definition. — *The functor $\mathcal{M}_{LT,d,n}$ which associates to an object R of C, the set of isomorphism classes of deformations by quasi-isogenies over R of $\Sigma_{K,d}$, equipped with a n-level structure, is a disjoint union of sub-functors $\mathcal{M}^{(h)}_{LT,d,n}$ of deformations by a quasi-isogeny of height h which is representable by a formal scheme $\widehat{\mathcal{M}}^{(h)}_{LT,d,n}$ where $\widehat{\mathcal{M}}^{(h)}_{LT,d,n}$.*

Remark: each of the $\widehat{\mathcal{M}}^{(h)}_{LT,d,n}$ is non canonically isomorphic to the formal scheme $\widehat{\mathcal{M}}^{(0)}_{LT,d,n}$ denoted by $\mathrm{Spf}\,\mathrm{Def}_{d,n}$ in [**7**]. We will use the notations without hat for the Berkovich generic fibers which are $\widehat{K^{nr}}$-analytic spaces in the sense of [**3**] and we note $\mathcal{M}^{d/K}_{LT,n} := \mathcal{M}_{LT,d,n}\hat{\otimes}_{\hat{K}^{nr}}\hat{\overline{K}}$.

The group of quasi-isogenies of $\Sigma_{K,d}$ is isomorphic to the unit group $D^\times_{K,d}$ of the central division algebra over K with invariant $1/d$, which then acts on $\mathcal{M}^{d/K}_{LT,n}$. For all $n \geq 1$, we have a natural action of $GL_d(\mathcal{O}_K/\mathcal{P}^n_K)$ on the level structures and then on $\mathcal{M}^{d/K}_{LT,n}$. This action can be extended to $GL_d(K)$ on the projective limit $\lim\limits_{\leftarrow n}\mathcal{M}^{d/K}_{LT,n}$ which is then equipped with the action of $GL_d(K) \times D^\times_{K,d}$ which factorises by $\Big(GL_d(K) \times D^\times_{K,d}\Big)/K^\times$ where $K^\times$ is embedded diagonally.

2.1.2. Definition. — *Let $\Psi^i_{K,\Lambda,d,n} \simeq H^i(\mathcal{M}^{(0)}_{LT,d,n}\hat{\otimes}_{\hat{K}^{nr}}\hat{\overline{K}}, \Lambda)$, be the Λ-module of finite type associated, by the vanishing cycle theory of Berkovich, to the structural morphism $\widehat{\mathcal{M}}^{(0)}_{LT,d,n} \longrightarrow \mathrm{Spf}\,\hat{\mathcal{O}}^{nr}_K$.*

We also introduce $\mathcal{U}^i_{K,\Lambda,d,n} := H^i(\mathcal{M}^{d/K}_{LT,n}, \Lambda)$ and $\mathcal{U}^i_{K,\Lambda,d} = \varinjlim_n \mathcal{U}^i_{K,\Lambda,d,n}$ as well as the cohomology groups with compact supports

$$\mathcal{V}^i_{K,\Lambda,d,n} := H^i_c(\mathcal{M}^{d/K}_{LT,n}, \Lambda), \text{ and } \mathcal{V}^i_{K,\Lambda,d} = \varinjlim_n \mathcal{V}^i_{K,\Lambda,d,n}.$$

As $\mathfrak{K}_n := \mathrm{Ker}(GL_d(\mathcal{O}_K) \longrightarrow GL_d(\mathcal{O}_K/\mathcal{P}_K^n))$ is pro-p for all $n \geq 1$, then we have $\mathcal{U}^i_{K,\Lambda,d,n} = (\mathcal{U}^i_{K,\Lambda,d})^{\mathfrak{K}_n}$ and $\mathcal{V}^i_{K,\Lambda,d,n} = (\mathcal{V}^i_{K,\Lambda,d})^{\mathfrak{K}_n}$.

The description of the $\mathcal{U}^i_{K,\overline{\mathbb{Q}}_l,d}$ is given in [**7**] theorem 2.3.5. We will denote by $\mathcal{U}^i_{K,\overline{\mathbb{Z}}_l,d,free}$ (resp. $\mathcal{V}^i_{K,\overline{\mathbb{Z}}_l,d,free}$) the free quotient which is the whole of $\mathcal{U}^i_{K,\overline{\mathbb{Z}}_l,d}$ (resp. $\mathcal{V}^i_{K,\overline{\mathbb{Z}}_l,d}$) by the main result of [**10**].

2.2. KHT-Shimura varieties. — Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary. For B/F a central division algebra with dimension d^2 equipped with an involution of second kind $*$ and $\beta \in B^{*=-1}$, consider the similitude group $G/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra R by

$$G(R) := \{(\lambda, g) \in R^\times \times (B^{op} \otimes_\mathbb{Q} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$ where $c = *_{|F}$ is the complex conjugation and $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$. For $p = uu^c$ decomposed in E, we have

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$$

where w describes the places of F above u. We suppose as in [**22**] that

- the associated unitary group $G_0(\mathbb{R})$ has signatures $(1, d-1) \times (0, d) \times \cdots \times (0, d)$;
- for any place x of $\mathbb{Q}$ inert or ramified in E, then $G(\mathbb{Q}_x)$ is quasi-split.
- We moreover suppose that u is chosen so that there exists a fixed place $v|u$ with $B_v \simeq M_d(F_v)$.

2.2.1. Notations. — *– Denote by $\mathbb{A}$ the adele ring of $\mathbb{Q}$. For a finite set S of places of $\mathbb{Q}$, we then introduce $\mathbb{A}^S$ the adele ring of $\mathbb{Q}$ outside S.*

- *The set of places p of $\mathbb{Q}$ decomposed in E is denoted* Spl *and we also introduce* Spl^S *the subset of places of* Spl *which does not belong to S.*

For all open compact subgroups U^p of $G(\mathbb{A}^{\infty,p})$ and $m=(m_w)_{w|u}$ a collection of non-negative integers, we consider

$$U^p(m)=U^p\times\mathbb{Z}_p^\times\times\prod_{w|u}\mathrm{Ker}(\mathcal{O}_{B_w}^\times\longrightarrow(\mathcal{O}_{B_w}/\mathcal{P}_w^{m_w})^\times),$$

where $\mathcal{O}_{B_w}$ is the maximal order of B_w.

We then denote by $\mathcal{I}$ the set of these $U^p(m)$ such that it exists a place x for which the projection from U^p to $G(\mathbb{Q}_x)$ doesn't contain any element with finite order except the identity, cf. [**22**] bellow of page 90.

Attached to each $I\in\mathcal{I}$ is a Shimura variety $X_I\to\mathrm{Spec}\,\mathcal{O}_v$ of type Kottwitz-Harris-Taylor. The projective system $X_{\mathcal{I}}=(X_I)_{I\in\mathcal{I}}$ is then equipped with a Hecke action of $G(\mathbb{A}^\infty)$, the transition morphisms $r_{J,I}:X_J\to X_I$ for $J\subset I$ being finite flat and even etale when $m_v(J)=m_v(I)$.

2.2.2. Notations. — *(cf.* [**7**] *§1.3) Let $I\in\mathcal{I}$,*

- *the special (resp. generic) fiber of X_I at v will be denoted by $X_{I,s}$ (resp. $X_{I,\eta}$) and its geometric special (resp. generic) fiber $X_{I,\bar s}:=X_{I,s}\times\mathrm{Spec}\,\overline{\mathbb{F}}_p$ (resp. $X_{I,\bar\eta}$).*
- *For $1\leq h\leq d$, let $X_{I,\bar s}^{\geq h}$ (resp. $X_{I,\bar s}^{=h}$) be the closed (resp. open) Newton stratum of height h, defined as the subscheme where the connected component of the universal Barsotti-Tate group is of height greater or equal to h (resp. equal to h).*

Remark: $X_{\mathcal{I},\bar s}^{\geq h}$ is of pure dimension $d-h$. For $1\leq h<d$, the Newton stratum $X_{\mathcal{I},\bar s}^{=h}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d}(\mathcal{O}_v)$ in the sense where there exists a closed subscheme $X_{I,\bar s,\overline{1_h}}^{=h}$ stabilized by the Hecke action of $P_{h,d}(\mathcal{O}_v)$ and such that

$$X_{\mathcal{I},\bar s}^{=h}\simeq X_{\mathcal{I},\bar s,\overline{1_h}}^{=h}\times_{P_{h,d}(\mathcal{O}_v)}GL_d(\mathcal{O}_v).$$

Let $\mathcal{G}(h)$ denote the universal Barsotti-Tate group over $X_{I,\bar s,\overline{1_h}}^{=h}$:

$$0\to\mathcal{G}(h)^c\longrightarrow\mathcal{G}(h)\longrightarrow\mathcal{G}(h)^{et}\to 0$$

where $\mathcal{G}(h)^c$ (resp. $\mathcal{G}(h)^{et}$) is connected (resp. étale) of height h (resp. $d-h$). Denote by $\iota_{m_v}:(\mathcal{P}_v^{-m_v}/\mathcal{O}_v)^d\longrightarrow\mathcal{G}(h)[\mathcal{P}_v^{m_v}]$ the universal level structure. If we denote by $(e_i)_{1\leq i\leq d}$ the canonical basis of $(\mathcal{P}_v^{-m_v}/\mathcal{O}_v)^d$, then the Newton stratum $X_{I,\bar s,\overline{1_h}}^{=h}$ is defined by asking $\big\{\iota_{m_v}(e_i):\ 1\leq i\leq h\big\}$ to be a Drinfeld basis of $\mathcal{G}(h)^c[\mathcal{P}_v^{m_v}]$.

2.2.3. Notation. — *In the following, we won't make any distinction between an element $a \in GL_d(F_v)/P_{h,d}(F_v)$ and the subspace $\langle a(e_1), \cdots, a(e_h)\rangle$ generated by the image through a of the first h vectors $e_1, \cdots, e_h$ of the canonical basis of F_v^d. Denote by $P_a(F_v) := aP_{h,d}(F_v)a^{-1}$ the parabolic subgroup of elements of $GL_d(F_v)$ stabilizing $a \subset F_v^d$.*

For $I \in \mathcal{I}$, the element $a \in GL_d(F_v)/P_{h,d}(F_v)$ gives a direct factor a_{m_v} of $(\mathcal{P}_v^{-m_v}/\mathcal{O}_v)^d$ and so a stratum $X^{=h}_{I,\bar{s},a}$ which is defined by asking for a basis $(f_1, \cdots, f_h)$ of a_{m_v}, that $\big\{\iota_{m_v}(f_i) :\ 1 \leq i \leq h\big\}$ is a Drinfeld basis of $\mathcal{G}(h)^c[\mathcal{P}_v^{m_v}]$. We also denote by $X^{\geq h}_{I,\bar{s},a}$ its closure in $X^{\geq h}_{I,\bar{s}}$. Such a stratum is said pure compared to the following situation. For a pure stratum $X^{=h}_{\mathcal{I},\bar{s},c}$ and $h' \geq h$, denote by

$$X^{=h'}_{\mathcal{I},\bar{s},c} := \coprod_{\substack{a:\ \dim a = h' \\ c \subset a}} X^{=h'}_{\mathcal{I},\bar{s},a}$$

and $X^{\geq h'}_{\mathcal{I},\bar{s},c}$ its closure.

2.3. Harris-Taylor perverse sheaves. — We recall now some notations about Harris-Taylor local systems of [**22**]. Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$. Fix $t \geq 1$ such that $tg \leq d$. The Jacquet-Langlands correspondence associates to $\mathrm{St}_t(\pi_v)$, an irreducible representation $\pi_v[t]_D$ of $D^\times_{v,tg}$. For $\mathcal{D}_{v,tg}$ the maximal order of $D_{v,tg}$, we decompose $(\pi_v[t]_D)_{|\mathcal{D}^\times_{v,tg}} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with the $\rho_{v,i}$ irreductible. Thanks to Igusa varieties, we then have a local system $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1_{tg}}}$ on $X^{=tg}_{\mathcal{I},\bar{s},\overline{1_{tg}}}$ associated with each $\rho_{v,i}$ and we write

$$\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} := \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_v)_{\overline{1_{tg}}} \qquad (2.3.1)$$

where ρ_v is any of the $\rho_{v,i}$, cf. [**7**] §2.4.4. Note that the Hecke action of $P_{tg,d}(F_v)$ on (2.3.1) is given through its quotient $GL_{d-tg}(F_v) \times \mathbb{Z}$.

2.3.2. Notation. — *Let e_{π_v} denote the order of the set of π'_v inertially equivalent to π_v in the sense there exists a character $\xi : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi' \simeq \pi \otimes (\xi \circ \mathrm{val} \circ \det)$ where* $\mathrm{val} : F_v \longrightarrow \mathbb{Z}$ *is the valuation of F_v; cf. definition 1.1.3 of* [**7**].

2.3.3. Notations. — *For Π_t any representation of $GL_{tg}(F_v)$ and $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce*

$$\widetilde{HT}_{\overline{1_{tg}}}(\pi_v, \Pi_t) := \mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

living over $X^{=tg}_{\mathcal{I},\bar{s},\overline{1_{tg}}}$ and its induced version living over $X^{=tg}_{\mathcal{I},\bar{s}}$

$$\widetilde{HT}(\pi_v, \Pi_t) := \Big(\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}\Big) \times_{P_{tg,d}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d}(F_v) \times W_{F_v}$$

is given

- *by the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, and*
- *the action of $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ on $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Xi^{\frac{tg-d}{2}}$.*

We also introduce

$$HT_{\overline{1_{tg}}}(\pi_v, \Pi_t) := \widetilde{HT}_{\overline{1_{tg}}}(\pi_v, \Pi_t)[d - tg],$$

and the perverse sheaf

$$P(t, \pi_v)_{\overline{1_{tg}}} := j^{=tg}_{\overline{1_{tg}},!*} HT_{\overline{1_{tg}}}(\pi_v, \mathrm{St}_t(\pi_v)) \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT(\pi_v, \Pi_t)$ and $P(t, \pi_v)$, where

- *for any $1 \leq h \leq d$ we denote by*

$$j^{=h} := i^h \circ j^{\geq h} : X^{=h}_{\mathcal{I},\bar{s}} \hookrightarrow X^{\geq h}_{\mathcal{I},\bar{s}} \hookrightarrow X_{\mathcal{I},\bar{s}},$$

  and

$$j^{=h}_{\overline{1_h}} := i^h_{\overline{1_h}} \circ j^{\geq h}_{\overline{1_h}} : X^{=h}_{\mathcal{I},\bar{s},\overline{1_h}} \hookrightarrow X^{\geq h}_{\mathcal{I},\bar{s},\overline{1_h}} \hookrightarrow X_{\mathcal{I},\bar{s}}.$$

- *The contragredient $\mathbb{L}^\vee$ of $\mathbb{L}$, is the local Langlands correspondence.*

More notations: for $a \in GL_d(F_v)/P_{tg,d}(F_v)$ as in notation 2.2.3, we will also denote by $HT_a(\pi_v, \Pi_t)$ (resp. $P_a(\pi_v, t)$) the image of $HT_{\overline{1_{tg}}}(\pi_v, \Pi_t)$ (resp. $P_{\overline{1_{tg}}}(\pi_v, t)$) under the action of a, which is then equivariant for $P_a(F_v)$. We will also consider, cf. the end of the previous paragraph, non necessarily pure strata $X^{\geq tg}_{\mathcal{I},\bar{s},c}$ associated with some pure strata $X^{=h}_{\mathcal{I},\bar{s},c}$ with $h \leq tg$ and more specifically to the case where $h = 1$ to which we then restrict ourselves. Denote by

$$HT_c(\pi_v, \Pi_t) := \mathrm{ind}^{P_c(F_v)}_{P_{c \subset a}(F_c)} HT_a(\pi_v, \Pi_t) \qquad (2.3.4)$$

(resp. $P_c(\pi_v,t) := \mathrm{ind}_{P_{c\subset a}(F_c)}^{P_c(F_v)} P_a(\pi_v,t)$) where, using notation 2.2.3,

- the index $a \in GL_d(F_v)/P_{tg,d}(F_v)$ is any element containing $c \in GL_d(F_v)/P_{1,d}(F_v)$ and $P_c(F_v)$ (resp. $P_{c\subset a}(F_v)$) is the parabolic subgroup of $GL_d(F_v)$ stabilizing c (resp. $c\subset a$);
- we restrict the natural action of $P_a(F_v)$ on $HT_a(\pi_v,\Pi_t)$ (resp. on $P_a(\pi_v,t)$) to $P_{c\subset a}(F_v)$ before inducing.

Consider a geometric supersingular point z and denote by $i_z:\{z\}\hookrightarrow X_{\mathcal{I},\bar{s}}$. We then have an action of $P_c(F_v)$ on $\mathrm{ind}_{(D_{v,d}^\times)^0\varpi_v^{\mathbb{Z}}}^{D_{v,d}^\times} \mathcal{H}^i i_z^{*p} j_{c,!*}^{=tg} HT_c(\pi_v,\Pi_t)$ (resp. $\mathrm{ind}_{(D_{v,d}^\times)^0\varpi_v^{\mathbb{Z}}}^{D_{v,d}^\times} \mathcal{H}^i i_z^{!p} j_{c,!*}^{=tg} HT_c(\pi_v,\Pi_t)$). From (2.3.4) we then obtain the following abstract description.

2.3.5. Lemma. — *For any $tg-d<i\leq 0$, $\mathrm{ind}_{(D_{v,d}^\times)^0\varpi_v^{\mathbb{Z}}}^{D_{v,d}^\times} \mathcal{H}^i i_z^{*p} j_{c,!*}^{=tg} HT_c(\pi_v,\Pi_t)$ as a representation of the mirabolic group $M_c(F_v)$ associated with c, cf. notation 1.3.12, is isomorphic to a small mirabolic induced representation of 1.3.7, cf. also the remark after loc. cit.*

$$(\Pi_t)_{|M_c(F_v)}\times\tau,$$

for τ some representation of $GL_{d-tg}(F_v)$ and where, by abuse of notation in the term $(\Pi_t)_{|M_c(F_v)}$ of the above formula, $M_c(F_v)$ is the mirabolic subgroup of $GL_{tg}(F_v)$.

2.3.6. Notation. — *Let $X_{\mathcal{I},\bar{s},c}^{\geq 1}$ be a pure stratum and denote by*

$$j_{\neq c} := j_{\neq c}^{\geq 1} : X_{\mathcal{I},\bar{s}}^{\geq 1}\setminus X_{\mathcal{I},\bar{s},c}^{\geq 1}\hookrightarrow X_{\mathcal{I},\bar{s}}^{\geq 1}.$$

For $X_{\mathcal{I},\bar{s},c}^{\geq 1}\neq X_{\mathcal{I},\bar{s},c'}^{\geq 1}$ two distinct pure strata, and for $h\geq 2$, we write $\langle c,c'\rangle$ the subspace of F_v^d generated by $\{c,c'\}$ and

$$X_{\mathcal{I},\bar{s},\langle c,c'\rangle}^{=h} = \coprod_{\substack{a:\dim a=h\\ \langle c,c'\rangle\subset a}} X_{\mathcal{I},\bar{s},a}^{=h},$$

with $j_{\langle c,c'\rangle}^{=h}: X_{\mathcal{I},\bar{s},\langle c,c'\rangle}^{=h}\hookrightarrow X_{\mathcal{I},\bar{s},\langle c,c'\rangle}^{\geq h}\hookrightarrow X_{\mathcal{I},\bar{s}}^{\geq 1}$.

Consider a pure stratum $X_{\mathcal{I},\bar{s},a}^{=h}$ with $a\supset\langle c,c'\rangle$. For $HT_a(\pi_v,\Pi_t)$ a Harris-Taylor local system on $X_{\mathcal{I},\bar{s},a}^{=h}$, we will denote by

$$HT_{\langle c,c'\rangle}(\pi_v,\Pi_t) := \mathrm{ind}_{P_{\langle c,c'\rangle\subset a}(F_v)}^{P_{\langle c,c'\rangle}(F_v)} HT_a(\pi_v,\Pi_t), \qquad (2.3.7)$$

where $P_{\langle c,c'\rangle}(F_v)$ (resp. $P_{\langle c,c'\rangle\subset a}(F_v)$) is the parabolic subgroup of elements of $GL_d(F_v)$ stabilizing $\langle c,c'\rangle$ (resp. $\langle c,c'\rangle \subset a$).

Consider now the subgroup $P_{c,c'}(F_v)$ of $P_c(F_v)$ stabilizing c'. Every element of $P_c(F_v)$ induces a endomorphism of F_v^d/c and the image of $P_{c,c'}(F_v)$ is then the parabolic $P_{c'}(F_v)$ of $GL(F_v^d/c)$.

2.3.8. Lemma. — *With the previous notations, and π_v an irreducible cuspidal entire $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$, we have the following short exact sequence of $P_{c,c'}(F_v)$-equivariant perverse sheaves*

$$0 \to {}^p j^{=(t+1)g}_{\langle c,c'\rangle,!*} HT_{\langle c,c'\rangle}\big(\pi_v, \mathrm{St}_{t+1}(\pi_v)\big)(\frac{1}{2}) \longrightarrow j^{=1}_{\neq c',!}\, j^{=1,*}_{\neq c'}\big({}^p j^{=tg}_{c,!*} HT_c(\pi_v,\mathrm{St}_t(\pi_v))\big) \longrightarrow {}^p j^{=1}_{\neq c',!*}\, j^{=1,*}_{\neq c'}\big({}^p j^{=tg}_{c,!*} HT_c(\pi_v,\mathrm{St}_t(\pi_v))\big) \to 0.$$

Remark: using the main results of [**10**], we could easily proved that the lemma is still valid over $\overline{\mathbb{Z}}_l$.

Proof. — Recall that for P a perverse sheaf on $X_{I,\bar{s}}$, the kernel of $j^{=1}_{\neq c',!} j^{=1,*}_{\neq c'} P \twoheadrightarrow j^{=1}_{\neq c',!*} j^{=1,*}_{\neq c'} P$ is given by ${}^p\mathcal{H}^{-1} i^{1,*}_{c'} P$ so that we are reduced to prove that

$${}^p\mathcal{H}^{-1} i^{1,*}_{c'}({}^p j^{=tg}_{c,!*} HT_c(\pi_v,\mathrm{St}_t(\pi_v))) \simeq {}^p j^{=(t+1)g}_{\langle c,c'\rangle,!*} HT_{\overline{\mathbb{Q}}_l,\langle c,c'\rangle}\big(\pi_v,\mathrm{St}_{t+1}(\pi_v)\big)(\frac{1}{2}).$$

In [**7**] 4.3.1, we described $j^{=tg}_{a,!} HT_a(\pi_v,\mathrm{St}_t(\pi_v))$ in the Grothendieck group of equivariant perverse sheaves and the weight filtration gives us a filtration, cf. also [**14**] (5.4), with the successive graded parts

$${}^p j^{=(t+\delta)g}_{a,!*} HT_a(\pi_v, \mathrm{St}_t(\pi_v)\{\frac{\delta(g-1)}{2}\} \otimes \mathrm{St}_\delta(\pi_v)\{\frac{t(1-g)}{2}\})(\delta/2).$$

By inducing from $P_{c\subset a}(F_v)$ to $P_c(F_v)$, we then obtain a filtration $\mathrm{Fil}^\bullet_c(t)$ of ${}^p j^{=tg}_{c,!} HT_c(\pi_v,\mathrm{St}_t(\pi_v))$ with graded parts

$$\mathrm{gr}_c^{-\delta}(t) := {}^p j^{=(t+\delta)g}_{c,!*} HT_c(\pi_v, \mathrm{St}_t(\pi_v)_{|P_c(F_v)}\{\frac{-\delta}{2}\} \times \mathrm{St}_\delta(\pi_v)\{\frac{t}{2}\})(\delta/2),$$

where by lemma 1.3.11, and taking into account the notation $\times$ in 1.1.1,

$$\big(\mathrm{St}_t(\pi_v\{\frac{-\delta}{2}\})\big)_{|P_c(F_v)} \times \mathrm{St}_\delta(\pi_v\{\frac{t}{2}\}) \simeq \mathrm{St}_{t+\delta}(\pi_v)_{|P_c(F_v)}.$$

We then apply then the functor ${}^p\mathcal{H}^{-1} i^{1,*}_{c'}$ to this filtration of $j^{=tg}_{c,!} HT_c(\pi_v, \mathrm{St}_t(\pi_v))$, so that we obtain

$$ {}^p\mathcal{H}^{-1} i^{1,*}_{c'} \big({}^p j^{=tg}_{c,!*} HT_c(\pi_v, \mathrm{St}_t(\pi_v)) \big) \simeq {}^p\mathcal{H}^0 i^{1,*}_{c'} \mathrm{Fil}^{-1}_c(t) \twoheadrightarrow {}^p\mathcal{H}^0 i^{1,*}_{c'} \mathrm{gr}^{-1}_c(t), $$

with ${}^p\mathcal{H}^0 i^{1,*}_{c'} \mathrm{gr}^{-1}_c(t) \simeq {}^p j^{=(t+1)g}_{\langle c,c'\rangle,!*} HT_{\langle c,c'\rangle}(\pi_v, \mathrm{St}_{t+1}(\pi_v))(\frac{1}{2})$.

Now we use the computations of [**14**] corollary 6.6, where we proved the same result for the whole of $X^{\geq 1}_{I,\bar{s}}$ instead of $X^{\geq 1}_{I,\bar{s},c}$, i.e.

$$ {}^p\mathcal{H}^{-1} i^{1,*}_{c'} \big({}^p j^{=tg}_{!*} HT(\pi_v, \mathrm{St}_t(\pi_v)) \big) \simeq {}^p j^{=(t+1)g}_{c',!*} HT_c(\pi_v, \mathrm{St}_{t+1}(\pi_v))(\frac{1}{2}). $$

Note also, cf. [**14**] lemma 6.2, that ${}^p\mathcal{H}^{-k} i^{1,*}_{c'} \big({}^p j^{=tg}_{!*} HT(\pi_v, \mathrm{St}_t(\pi_v)) \big)$ is zero for any $k \neq -1$. We then apply the functor ${}^p\mathcal{H}^{-1} i^{1,*}_{c'}$ to the short exact sequence of perverse sheaves

$$ \begin{aligned} 0 \rightarrow {}^p j^{=tg}_{c,!*} HT_c(\pi_v, \mathrm{St}_t(\pi_v)) \longrightarrow {}^p j^{=tg}_{!*} HT(\pi_v, \mathrm{St}_t(\pi_v)) \\ \longrightarrow {}^p j^{=tg}_{\neq c,!*} HT_{\neq c}(\pi_v, \mathrm{St}_t(\pi_v)) \rightarrow 0, \end{aligned} $$

which gives us in particular

$$ {}^p\mathcal{H}^{-1} i^{tg+1,*}_{c'} \big({}^p j^{=tg}_{c,!*} HT_c(\pi_v, \mathrm{St}_t(\pi_v)) \big) \hookrightarrow {}^p j^{=(t+1)g}_{c',!*} HT_c(\pi_v, \mathrm{St}_{t+1}(\pi_v))(\frac{1}{2}), $$

which factorizes to ${}^p j^{=(t+1)g}_{\langle c,c'\rangle,!*} HT_{\langle c,c'\rangle}(\pi_v, \mathrm{St}_{t+1}(\pi_v))(\frac{1}{2})$, just by considering the supports.

□

3. Some coarse filtrations of $\Psi_{\mathcal{I}}$

3.1. Filtrations of free perverse sheaves. — Let $S = \mathrm{Spec}\, \mathbb{F}_q$ and X/S of finite type, then the usual t-structure on $\mathcal{D}(X, \overline{\mathbb{Z}}_l) := D^b_c(X, \overline{\mathbb{Z}}_l)$ is

$$ \begin{aligned} A \in {}^p\mathcal{D}^{\leq 0}(X, \overline{\mathbb{Z}}_l) \Leftrightarrow \forall x \in X,\ \mathcal{H}^k i_x^* A = 0,\ \forall k > -\dim \overline{\{x\}} \\ A \in {}^p\mathcal{D}^{\geq 0}(X, \overline{\mathbb{Z}}_l) \Leftrightarrow \forall x \in X,\ \mathcal{H}^k i_x^! A = 0,\ \forall k < -\dim \overline{\{x\}} \end{aligned} $$

where $i_x : \mathrm{Spec}\, \kappa(x) \hookrightarrow X$ and $\mathcal{H}^k(K)$ is the k-th sheaf of cohomology of K.

3.1.1. Notation. — *Denote by ${}^p\mathcal{C}(X, \overline{\mathbb{Z}}_l)$ the heart of this t-structure with associated cohomology functors ${}^p\mathcal{H}^i$. For a functor T we denote by ${}^pT := {}^p\mathcal{H}^0 \circ T$.*

The category ${}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ is abelian equipped, cf. [**11**] §1.1, with a torsion theory $(\mathcal{T},\mathcal{F})$ where $\mathcal{T}$ (resp. $\mathcal{F}$) is the full subcategory of objects T (resp. F) such that $l^N 1_T$ is trivial for some large enough N(resp. $l.1_F$ is a monomorphism). Recall that this means in particular that for every object A there exists a short exact sequence

$$0 \to A_{tor} \longrightarrow A \longrightarrow A_{free} \to 0$$

with $A_{tor} \in \mathcal{T}$ and $A_{free} \in \mathcal{F}$. Applying Grothendieck-Verdier duality, we obtain

$$\begin{aligned} {}^{p+}\mathcal{D}^{\leq 0}(X,\overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\leq 1}(X,\overline{\mathbb{Z}}_l) :\ {}^p\mathcal{H}^1(A) \in \mathcal{T}\} \\ {}^{p+}\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l) &:= \{A \in {}^p\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l) :\ {}^p\mathcal{H}^0(A) \in \mathcal{F}\} \end{aligned}$$

with heart${}^{p+}\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ equipped with its torsion theory $(\mathcal{F},\mathcal{T}[-1])$.

3.1.2. Definition. — *(cf.* [**11**] *§1.3) Let*

$$\mathcal{F}(X,\overline{\mathbb{Z}}_l) := {}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{C}(X,\overline{\mathbb{Z}}_l) = {}^p\mathcal{D}^{\leq 0}(X,\overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X,\overline{\mathbb{Z}}_l)$$

the quasi-abelian category of free perverse sheaves over X.

Remark: for an object L of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$, we will consider filtrations

$$L_1 \subset L_2 \subset \cdots \subset L_e = L$$

such that for every $1 \leq i \leq e-1$, $L_i \hookrightarrow L_{i+1}$ is a strict monomorphism, i.e. L_{i+1}/L_i is an object of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$.

Consider an open subscheme $j : U \hookrightarrow X$ and $i : F := X \backslash U \hookrightarrow X$. Then

$${}^{p+}j_!\mathcal{F}(U,\overline{\mathbb{Z}}_l) \subset \mathcal{F}(X,\overline{\mathbb{Z}}_l) \quad \text{and} \quad {}^p j_*\mathcal{F}(U,\overline{\mathbb{Z}}_l) \subset \mathcal{F}(X,\overline{\mathbb{Z}}_l).$$

Moreover, if j is affine then $j_!$ is t-exact and $j_! = {}^p j_! = {}^{p+}j_!$.

3.1.3. Lemma. — *Consider $L \in \mathcal{F}(X,\overline{\mathbb{Z}}_l)$ such that $j_!j^*L \in \mathcal{F}(X,\overline{\mathbb{Z}}_l)$. Then $i_*{}^p\mathcal{H}^{-\delta}i^*L$ is trivial for every $\delta \neq 0,1$; for $\delta = 1$ it belongs to $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$.*

Remark: If j is affine then the condition $j_!j^*L \in \mathcal{F}(X,\overline{\mathbb{Z}}_l)$ is fulfilled.

Proof. — Start from the following distinguished triangle $j_!j^*L \longrightarrow L \longrightarrow i_*i^*L \rightsquigarrow$. From the perversity of L and $j_!j^*L$, the long exact sequence of perverse cohomology is

$$0 \to i_*{}^p\mathcal{H}^{-1}i^*L \longrightarrow {}^p j_!j^*L \longrightarrow L \longrightarrow i_*{}^p\mathcal{H}^0 i^*L \to 0.$$

The freeness of $i_*{}^p\mathcal{H}^{-1}i^*L$ then follows from those of ${}^p j_!j^*L = j_!j^*L$. $\square$

3.1.4. Definition. — *Recall the following notions, cf.* [**11**] *definition 1.3.4.*

- *A* bimorphism *of $\mathcal{F}(X,\overline{\mathbb{Z}}_l)$, written $L \hookrightarrow\!\!\!\!\rightarrow L'$, is both a monomorphism in ${}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ and an epimorphism in ${}^{p+}\mathcal{C}(X,\overline{\mathbb{Z}}_l)$. If moreover the cokernel in ${}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ is of dimension strictly less than those of the support of L, we will write $L \hookrightarrow\!\!\!\!\rightarrow_+ L'$.*
- *A morphism $L \longrightarrow L'$ is a strict monomorphism (resp. a strict epimorphism) if it is a monomorphism (resp. an epimorphism) in ${}^{p+}\mathcal{C}(X,\overline{\mathbb{Z}}_l)$ (resp. in ${}^p\mathcal{C}(X,\overline{\mathbb{Z}}_l)$) in which case we denote it by $L \hookrightarrow\!\!\!\!\!+\!\!\!\rightarrow L'$ (resp. $L \twoheadrightarrow\!\!\!\!\!\!+\,\, L'$).*

For a free $L \in \mathcal{F}(X,\overline{\mathbb{Z}}_l)$, we consider the following diagram

$$\begin{tikzcd}
 & L \arrow[drr, "\mathrm{can}_{*,L}"] & & \\
{}^{p+}j_!j^*L \arrow[ur, "\mathrm{can}_{!,L}"] \arrow[r, two heads, "+"] & {}^pj_{!*}j^*L \arrow[r, hook, two heads, "+"'] & {}^{p+}j_{!*}j^*L \arrow[r, hook, "+"] & {}^pj_*j^*L
\end{tikzcd}$$

where the bottom row is, cf. the remark following 1.3.10 of [**11**], the canonical factorisation of ${}^{p+}j_!j^*L \longrightarrow {}^pj_*j^*L$ and where the maps $\mathrm{can}_{!,L}$ and $\mathrm{can}_{*,L}$ are given by the adjunction property.

3.1.5. Notation. — *(cf. lemma 2.1.2 of* [**11**]*) We introduce the filtration $\mathrm{Fil}^{-1}_{U,!}(L) \subset \mathrm{Fil}^0_{U,!}(L) \subset L$ with*

$$\mathrm{Fil}^0_{U,!}(L) = \mathrm{Im}_{\mathcal{F}}(\mathrm{can}_{!,L}) \quad \textit{and} \quad \mathrm{Fil}^{-1}_{U,!}(L) = \mathrm{Im}_{\mathcal{F}}\Big((\mathrm{can}_{!,L})_{|\mathcal{P}_L}\Big),$$

where $\mathcal{P}_L := i_*{}^p\mathcal{H}^{-1}_{free} i^*j_*j^*L$ is the kernel of $\mathrm{Ker}_{\mathcal{F}}\Big({}^{p+}j_!j^*L \twoheadrightarrow {}^pj_{!*}j^*L\Big)$.

Remark: we have $L/\mathrm{Fil}^0_{U,!}(L) \simeq i_*{}^{p+}i^*L$ and ${}^pj_{!*}j^*L \hookrightarrow\!\!\!\!\rightarrow_+ \mathrm{Fil}^0_{U,!}(L)/\mathrm{Fil}^{-1}_{U,!}(L)$, which gives, cf. lemma 1.3.11 of [**11**], a commutative triangle

$$\begin{tikzcd}
{}^pj_{!*}j^*L \arrow[r, hook, two heads, "+"'] \arrow[dr, hook, two heads, "+"'] & \mathrm{Fil}^0_{U,!}(L)/\mathrm{Fil}^{-1}_{U,!}(L) \arrow[d, hook, two heads, "+"'] \\
 & {}^{p+}j_{!*}j^*L.
\end{tikzcd}$$

3.1.6. Notation. — *(cf.* [**11**] *2.1.4) Dually there is a cofiltration $L \twoheadrightarrow \mathrm{CoFil}_{U,*,0}(L) \twoheadrightarrow \mathrm{CoFil}_{U,*,1}(L)$ where*

$$\mathrm{CoFil}_{U,*,0}(L) = \mathrm{Coim}_{\mathcal{F}}(\mathrm{can}_{*,L}) \quad \textit{and} \quad \mathrm{CoFil}_{U,*,1}(L) = \mathrm{Coim}_{\mathcal{F}}(p_L \circ \mathrm{can}_{*,L}),$$

with $p_L : {}^{p+}j_{!*}j^*L \twoheadrightarrow Q_L := i_*{}^p\mathcal{H}^0_{free}i^*j_*j^*L$.

Remark: the kernel $\mathrm{cogr}_{U,*,1}(L)$ of $\mathrm{CoFil}_{U,*,0}(L) \twoheadrightarrow \mathrm{CoFil}_{U,*,1}(L)$ verifies

$$ {}^pj_{!*}j^*L \hookrightarrow\!\!\!\!\twoheadrightarrow_+ \mathrm{cogr}_{U,*,1}(L) \hookrightarrow\!\!\!\!\twoheadrightarrow_+ {}^{p+}j_{!*}j^*L. $$

The kernel $\mathrm{cogr}_{U,*,0}(L)$ of $L \twoheadrightarrow \mathrm{CoFil}_{U,*,0}(L)$ is isomorphic to $i_*{}^pi^!L$.

Consider now X equipped with a stratification consisting of closed subsets

$$ X = X^{\geq 1} \supset X^{\geq 2} \supset \cdots \supset X^{\geq d,} $$

and let $L \in \mathcal{F}(X, \overline{\mathbb{Z}}_l)$. For $1 \leq h < d$, denote by $X^{1\leq h} := X^{\geq 1} - X^{\geq h+1}$ and $j^{1\leq h} : X^{1\leq h} \hookrightarrow X^{\geq 1}$. We then define

$$ \mathrm{Fil}^r_!(L) := \mathrm{Im}_{\mathcal{F}}\Big({}^{p+}j^{1\leq r}_! j^{1\leq r,*}L \longrightarrow L \Big), $$

which gives a filtration

$$ 0 = \mathrm{Fil}^0_!(L) \subset \mathrm{Fil}^1_!(L) \subset \mathrm{Fil}^2_!(L) \cdots \subset \mathrm{Fil}^{d-1}_!(L) \subset \mathrm{Fil}^d_!(L) = L. $$

Dually, the following

$$ \mathrm{CoFil}_{*,r}(L) = \mathrm{Coim}_{\mathcal{F}}\Big(L \longrightarrow {}^pj^{1\leq r}_* j^{1\leq r,*}L \Big), $$

define a cofiltration

$$ L = \mathrm{CoFil}_{*,d}(L) \twoheadrightarrow \mathrm{CoFil}_{*,d-1}(L) \twoheadrightarrow \cdots $$
$$ \cdots \twoheadrightarrow \mathrm{CoFil}_{*,1}(L) \twoheadrightarrow \mathrm{CoFil}_{*,0}(L) = 0, $$

and a filtration

$$ 0 = \mathrm{Fil}^{-d}_*(L) \subset \mathrm{Fil}^{1-d}_*(L) \subset \cdots \subset \mathrm{Fil}^0_*(L) = L $$

where

$$ \mathrm{Fil}^{-r}_*(L) := \mathrm{Ker}_{\mathcal{F}}\big(L \twoheadrightarrow \mathrm{CoFil}_{*,r}(L) \big). $$

Note these two constructions are exchanged by Grothendieck-Verdier duality,

$$ D\Big(\mathrm{CoFil}_{!,-r}(L)\Big) \simeq \mathrm{Fil}^{-r}_*(D(L)) \text{ and } D\Big(\mathrm{CoFil}_{*,r}(L)\Big) \simeq \mathrm{Fil}^r_!(D(L)). $$

We can also refine the previous filtrations with the help of $\mathrm{Fil}^{-1}_{U,!}(L)$, cf. [**11**] proposition 2.3.2, to obtain exhaustive filtrations

$$ 0 = \mathrm{Fill}^{-2^{d-1}}_!(L) \subset \mathrm{Fill}^{-2^{d-1}+1}_!(L) \subset \cdots $$
$$ \cdots \subset \mathrm{Fill}^0_!(L) \subset \cdots \subset \mathrm{Fill}^{2^{d-1}-1}_!(L) = L, \quad (3.1.7) $$

such that the graded pieces $\mathrm{grr}^k(L)$ are simple over $\overline{\mathbb{Q}}_l$, i.e. verify ${}^p j^{=h}_{!*} j^{=h,*}\mathrm{grr}^k(L) \hookrightarrow\!\!\!\!\rightarrow_+ \mathrm{grr}^k(L)$ for some h. Dually using : $\mathrm{coFil}_{U,*,1}(L)$, we construct a cofiltration

$$L = \mathrm{CoFill}_{*,2^{d-1}}(L) \twoheadrightarrow \mathrm{CoFill}_{*,2^{d-1}-1}(L) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{CoFill}_{*,-2^{d-1}}(L) = 0$$

and a filtration $\mathrm{Fill}_*^{-r}(L) := \mathrm{Ker}_{\mathcal{F}}\big(L \twoheadrightarrow \mathrm{CoFill}_{*,r}(L)\big)$. These two constructions are exchanged by duality

$$D\big(\mathrm{CoFill}_{*,r}(L)\big) \simeq \mathrm{Fill}_!^r\big(D(L)\big) \text{ and } D\big(\mathrm{CoFill}_{!,r}(L)\big) \simeq \mathrm{Fill}_*^r\big(D(L)\big)$$

and can be mixed if we want to.

3.2. Remarks and terminology about perverse sheaves. — Be cause the previous definitions come from the geometry, it is then possible to construct filtrations whatever is the ring of coefficients. Moreover when you want to understand the graded pieces,

- you can first look at these filtrations over $\overline{\mathbb{Q}}_l$ which gives you the simple perverse sheaves described in terms of an intermediate extension $i_* j_{!*}\mathcal{L}[-\delta]$ of some local system $\mathcal{L}$ living in some locally closed stratum $U \overset{j}{\hookrightarrow} \overline{U} \overset{i}{\hookrightarrow} X$ where δ is the dimension of U.
- Then you have to understand the $\overline{\mathbb{Z}}_l$-lattice of $\mathcal{L}$; in the following we will speak about *the lattice of the perverse sheaf*.
- And finally determine *the position* of the graded piece between the two natural intermediate extension $i_*{}^p j_{!*}\mathcal{L}[-\delta] \hookrightarrow\!\!\!\!\rightarrow_+ i_*{}^{p+} j_{!*}\mathcal{L}[-\delta]$.

One also have to take into account that the lattices and the positions depend strongly on the order of the graded pieces, i.e. for two different filtrations $\mathrm{Fil}_1^\bullet$ and $\mathrm{Fil}_2^\bullet$, then two graded pieces $\mathrm{gr}_1^{k_1}$ and $\mathrm{gr}_2^{k_2}$ which are isomorphic over $\overline{\mathbb{Q}}_l$, might be not isomorphic over $\overline{\mathbb{Z}}_l$ either because the lattices, or their positions, are different.

Finally as remarked in the introduction, when taking image in $\mathcal{F}$ or kernel in $\mathcal{F}^+$, you loose control of lattices and positions:[(4)] we then speak of a *saturation* process as it corresponds to the usual saturation of lattices in the case where the geometric support is of dimension 0. In the following we will focus on graded pieces concentrated on the supersingular locus so that there is no issue about the positions. Concerning the lattices of these graded pieces, we advise the reader when reading the arguments of §4 to focus on this issue to understand how we manage to recover the lattices.

(4) The understanding of positions in the previous meaning, is solved in [**10**].

Remark: by arguing inductively on the Lubin-Tate spaces, we will in fact be able, cf. the proof of theorem 4.1.6, to understand among all the lattices of Harris-Taylor perverse sheaf $P(t,\pi_v)(\frac{1-t+2k}{2})$ with $0 \leq k \leq t-1$, given by filtrations of stratification of $\Psi_{\mathcal{I}}$, those for $k=0$ and $k=t-1$. It would then be possible, and quite easy using the coarse filtrations of §3, to describe the others which then depend on how we decide to filtrate $\Psi_{\mathcal{I}}$, as opposed to the cases where $k=0, t-1$ considered here. We postpone this work to the day when we will find an application.

3.3. Supercuspidal decomposition of $\Psi_{\mathcal{I}}$. —

3.3.1. Notation. — *For $I \in \mathcal{I}$, let*

$$\Psi_{I,\Lambda} := R\Psi_{\eta_v,I}(\Lambda[d-1])(\frac{d-1}{2})$$

be the vanishing cycle autodual perverse sheaf on $X_{I,\bar{s}}$. When $\Lambda = \overline{\mathbb{Z}}_l$, we will simply write Ψ_I.

As before, we will use the notation $\Psi_{\mathcal{I}}$ for the system $(\Psi_I)_{I \in \mathcal{I}}$. Recall the following result of [**22**] relating $\Psi_{\mathcal{I}}$ with Harris-Taylor local systems.

3.3.2. Proposition. — *(cf. [**22**] proposition IV.2.2 and §2.4 of [**7**]) There is an isomorphism $G(\mathbb{A}^{\infty,v}) \times P_{h,d}(F_v) \times W_{F_v}$-equivariant*

$$\mathrm{ind}_{(D_{v,h}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,h}^\times} \big(\mathcal{H}^{h-d-i}\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}\big)_{|X^{=h}_{\mathcal{I},\bar{s},\overline{1_h}}} \simeq \bigoplus_{\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)} \mathcal{L}_{\overline{\mathbb{Z}}_l,\overline{1_h}}(\mathcal{U}^{h-1-i}_{\bar{\tau},\mathbb{N}}),$$

where

- *$\mathcal{L}_{\overline{\mathbb{Z}}_l,\overline{1_h}}(\mathcal{U}^{h-1-i}_{\bar{\tau},\mathbb{N}})$ is the local system over $X^{=h}_{\mathcal{I},\bar{s},\overline{1_h}}$ associated with $\mathcal{U}^{h-1-i}_{\bar{\tau},\mathbb{N}}$ viewed as a representation of $\mathcal{D}^\times_{v,h}$, cf. the remark before 2.3.3;*
- *$\mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ is the set of equivalence classes of irreducible $\overline{\mathbb{F}}_l$-representations of $D^\times_{v,h}$;*
- *for $\bar{\tau} \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$ and V a $\overline{\mathbb{Z}}_l$-representation of $D^\times_{v,h}$, then $V_{\bar{\tau}}$ denotes, cf. [**19**] §B.2, the direct factor of V whose irreducible subquotients are isomorphic to a subquotient of $\bar{\tau}_{|\mathcal{D}^\times_{v,h}}$ where $\mathcal{D}_{v,h}$ is the maximal order of $D_{v,h}$.*
- *With the previous notation, $\mathcal{U}^i_{\bar{\tau},\mathbb{N}} := \big(\mathcal{U}^i_{F_v,\overline{\mathbb{Z}}_l,d,n}\big)_{\bar{\tau}}$.*
- *The matching at finite levels between the system indexed by $\mathcal{I}$ and those by $\mathbb{N}$ is given by the map $m_v : \mathcal{I} \longrightarrow \mathbb{N}$.*

Remark: for $\bar\tau \in \mathcal{R}_{\overline{\mathbb{F}}_l}(h)$, and a lifting τ which by Jacquet-Langlands correspondence can be written $\tau \simeq \pi[t]_D$ for π irreductible cuspidal, let $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ be in the supercuspidal support. Then the inertial class of ϱ depends only on $\bar\tau$ and we will use the following notation.

3.3.3. Notation. — *With the previous notation, we write V_ϱ for $V_{\bar\tau}$.*

The description of the various filtration from the previous section applied to $\Psi_{\mathcal{I},\overline{\mathbb{Q}}_l}$ is given in [**11**] §3.4. Over $\overline{\mathbb{Z}}_l$, first note that $\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$ is an object of $\mathcal{F}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l)$. Indeed, by [**1**] proposition 4.4.2, $\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$ is an object of ${}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l)$. By [**23**] variant 4.4 of theorem 4.2, we have $D\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l} \simeq \Psi_{\mathcal{I},\overline{\mathbb{Z}}_l}$, so that

$$\Psi_{\mathcal{I},\overline{\mathbb{Z}}_l} \in {}^p\mathcal{D}^{\leq 0}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l) \cap {}^{p+}\mathcal{D}^{\geq 0}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l) = \mathcal{F}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l).$$

We can then deduce from the description of the filtrations of $\Psi_{\mathcal{I},\overline{\mathbb{Q}}_l}$ the same sort of description except that first we have no control on the bimorphism ${}^p j^{=h}_{!*} j^{=h,*}\mathrm{grr}^k(L) \hookrightarrow\!\!\!\!\twoheadrightarrow_+ \mathrm{grr}^k(L)$ and secondly all the contribution relatively to irreducible cuspidal $\overline{\mathbb{Q}}_l$-representations should be considered altogether. About this last point, we have the following result.

3.3.4. Proposition. — *We have a decomposition*

$$\Psi_{\mathcal{I}} \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_\varrho$$

with $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} \Psi_{\pi_v}$ where the irreducible constituent of Ψ_{π_v} are exactly the perverse Harris-Taylor sheaves attached to π_v, i.e. with the notations of 2.3.3, the $P(\pi_v,t)(\frac{1-t+\delta}{2})$ with $1 \leq t \leq d/g$ and $0 \leq \delta \leq t-1$.

Remark: the graded pieces $\mathrm{gr}^h_!(\Psi_\varrho)$ of the previous filtration of stratification of Ψ_ϱ verify

$$j^{=h,*}\mathrm{gr}^h_!(\Psi_\varrho) \simeq \begin{cases} 0 & \text{if } g \nmid h \\ \mathcal{L}_{\overline{\mathbb{Z}}_l}(\varrho[t]_D) & \text{for } h = tg. \end{cases}$$

Proof. — We argue by induction on r to show that there exists a decomposition

$$\mathrm{Fil}^r_!(\Psi_{\mathcal{I}}) = \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \mathrm{Fil}^r_{!,\varrho}(\Psi_{\mathcal{I}}).$$

The case $r=0$ being trivial, we suppose it is true for $r-1$. From $j^{=r,*}\mathrm{gr}^r_!(\Psi_{\mathcal{I}}) \simeq \bigoplus_{g|r=tg} \bigoplus_{\varrho\in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \mathcal{L}_{\overline{\mathbb{Z}}_l}(\varrho[t]_D)$, we obtain

$$\mathrm{gr}^r_!(\Psi_{\mathcal{I}}) \simeq \bigoplus_{g|r} \bigoplus_{\varrho\in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \mathrm{gr}^r_{!,\varrho}(\Psi_{\mathcal{I}})$$

with $j^{=r}_!\mathcal{L}_{\overline{\mathbb{Z}}_l}(\varrho[t]_D)[d-r] \twoheadrightarrow \mathrm{gr}^r_{!,\varrho}(\Psi_{\mathcal{I}})$ where the irreducible constituents of $\mathrm{gr}^r_{!,\varrho}(\Psi_{\mathcal{I}}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are of type ϱ.

Consider two free perverse sheaves A_1 and A_2 and let A be an extension

$$0 \to A_1 \longrightarrow A \longrightarrow A_2 \to 0,$$

supposed to be split over $\overline{\mathbb{Q}}_l$. Denote then the pull back A'_2

$$\begin{tikzcd}
A'_2 \arrow[r, hook, dashed] \arrow[d, hook, dashed] & A \arrow[d, hook] \\
A_2 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \arrow[r, hook] & A \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l
\end{tikzcd}$$

so that

$$\begin{tikzcd}
 & A_1 \arrow[r, equal] \arrow[d, hook] & A_1 \arrow[d, hook] \\
A'_2 \arrow[r, hook] \arrow[d, equal] & A \arrow[r, two heads] \arrow[d] & A'_1 \arrow[d] \\
A'_2 \arrow[r, hook] & A_2 \arrow[r, two heads] & T
\end{tikzcd} \tag{3.3.5}$$

where T is the common cokernel of $A_1 \hookrightarrow A'_1$ and $A'_2 \hookrightarrow A_2$. Then $T=0$ if and only the extension A is split. Now suppose that A_1 (resp. A_2) is a Harris-Taylor perverse sheaf of type ϱ_1 (resp. ϱ_2) with ϱ_1 and ϱ_2 not belonging to the same Zelevinsky line. Then the action of the Weil group on $T[l]$ seen as a quotient of A'_1 (resp. of A_2) is isotypic relatively to the galois representation associated with ϱ_1 (resp. ϱ_2) by the Langlands-Vigneras correspondence, which imposes that $T=0$.

By applying this general remark to $\mathrm{gr}^r_{!,\varrho_2}(\Psi_{\mathcal{I}})$, we conclude it is in a direct sum with $\mathrm{Fil}^{r-1}_{!,\varrho_1}(\Psi_{\mathcal{I}})$, which, by varying ϱ_1 and ϱ_2, proves the result. $\square$

In order to understand the next computations on $\Psi_{\mathcal{I}}$, it might be useful for the reader to recall the following description in the Grothendieck group of Ψ_{π_v} given in [**7**]

$$\big[\Psi_{\pi_v}\big]=\sum_{t=1}^{\lfloor \frac{d}{g}\rfloor}\sum_{k=0}^{t-1} P(t,\pi_v)(\frac{1-t+2k}{2}), \tag{3.3.6}$$

where π_v is an irreducible cuspidal representation of $GL_g(F_v)$. Then, cf. [**11**] §3.4, the graded pieces $\mathrm{gr}_!^r(\Psi_{\pi_v})$ are zero if $r\not\in\{g,2g,\cdots,\lfloor \frac{d}{g}\rfloor g\}$ and otherwise its image in the Grothendieck group is

$$\big[\mathrm{gr}_!^{kg}\big(\Psi_{\pi_v}\big)\big]=\sum_{t=k}^{\lfloor \frac{d}{g}\rfloor} P(t,\pi_v)(\frac{1+t-2k}{2}). \tag{3.3.7}$$

In particular for $k=1$, then

$$\big[\mathrm{gr}_!^{g}\big(\Psi_{\pi_v}\big)\big]=\sum_{t=1}^{\lfloor \frac{d}{g}\rfloor} P(t,\pi_v)(\frac{t-1}{2}). \tag{3.3.8}$$

3.4. Filtrations with the use of $j_{\neq c}$. — Denote by

$$\bar{j}:X_{\mathcal{I},\bar\eta}\hookrightarrow \overline{X}_{\mathcal{I}}\hookleftarrow X_{\mathcal{I},\bar s}:\bar{i},$$

and consider the following t-structure on $\overline{X}_{\mathcal{I}}:=X_{\mathcal{I}}\times_{\operatorname{Spec}\mathcal{O}_v}\operatorname{Spec}\overline{\mathcal{O}}_v$ obtained by glueing

$$\Big({}^pD^{\leq -1}(X_{\mathcal{I},\bar\eta},\overline{\mathbb{Z}}_l),{}^pD^{\geq -1}(X_{\mathcal{I},\bar\eta},\overline{\mathbb{Z}}_l)\Big)\quad\text{and}\quad\Big({}^pD^{\leq 0}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l),{}^pD^{\geq 0}(X_{\mathcal{I},\bar s},\overline{\mathbb{Z}}_l)\Big).$$

The functors $\bar{j}_!$ and $\bar{j}_*={}^p\bar{j}_{!*}$ are then t-exact with

$$0\to\Psi_{\mathcal{I}}\longrightarrow \bar{j}_!\overline{\mathbb{Z}}_l[d-1](\frac{d-1}{2})\longrightarrow \bar{j}_*\overline{\mathbb{Z}}_l[d-1](\frac{d-1}{2})\to 0.$$

Consider now a pure stratum $X^{\geq 1}_{\mathcal{I},\bar s,c}$. Note then that the morphism $\bar{j}_{\neq c}:\overline{X}_{\mathcal{I}}\setminus X^{\geq 1}_{\mathcal{I},\bar s,c}\hookrightarrow \overline{X}_{\mathcal{I}}$ is affine, cf. [**14**] beginning of §7.

3.4.1. Lemma. — *The perverse sheaf $\Psi_c:=i^1_{c,*}{}^p\mathcal{H}^0 i^{1,*}_c\big(\Psi_{\mathcal{I}}\big)$ is free.*

Proof. — Let $\overline{F}:=\bar{j}_*\overline{\mathbb{Z}}_l[d-1](\frac{d-1}{2})=\bar{j}_{!*}\overline{\mathbb{Z}}_l[d-1](\frac{d-1}{2})$ over $\overline{X}_{\mathcal{I}}$. Denote by $i^1_c:X^{\geq 1}_{\mathcal{I},\bar s,c}\hookrightarrow X^{\geq 1}_{\mathcal{I},\bar s}$, and $\bar{i}_c:=\bar{i}\circ i^{\geq 1}_c$. As $\Psi_{\mathcal{I}}={}^p\mathcal{H}^{-1}\bar{i}^*\bar{j}_*\overline{\mathbb{Z}}_l[d-1](\frac{d-1}{2})$, we have to prove that $i^{1,*}_c{}^p\mathcal{H}^{-1}\bar{i}^*\overline{F}$ is perverse for the t-structures p and $p+$. Consider the spectral sequence

$$E_2^{r,s}={}^p\mathcal{H}^r i^{1*}_c\Big({}^p\mathcal{H}^s\bar{i}^*\overline{F}\Big)\Rightarrow {}^p\mathcal{H}^{r+s}\bar{i}^*_c\overline{F}.$$

As $\bar{j}$ is affine, by lemma 3.1.3, we know that ${}^p\mathcal{H}^s\bar{i}^*\overline{F}$ is trivial for $s < -1$. The epimorphism $\bar{j}_!\bar{j}^*\overline{F} \twoheadrightarrow \overline{F}$, gives also that ${}^p\mathcal{H}^0\bar{i}^*\overline{F} = 0$ so that the previous spectral sequence degenerates at E_2 with

$$ {}^p\mathcal{H}^r\bar{i}_c^*\overline{F} \simeq {}^p\mathcal{H}^{r+1}i_c^{1,*}\big({}^p\mathcal{H}^{-1}\bar{i}^*\overline{F}\big). $$

In the same way as $\bar{j}_{\neq c} : X_{\mathcal{I}} \setminus X^{\geq 1}_{\mathcal{I},\bar{s},c} \hookrightarrow X_{\mathcal{I}}$ is affine, then, by lemma 3.1.3, ${}^p\mathcal{H}^r\bar{i}_c^*\overline{F}$ is trivial for $r < -1$ and free for $r = -1$ which finishes the proof. □

The decomposition of 3.3.4 gives $\Psi_c \simeq \bigoplus_{g=1}^d \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_{\varrho,c}$. For any $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ we then have the following short exact sequence of free perverse sheaves

$$ 0 \to j_{\neq c,!}j_{\neq c}^*\Psi_\varrho \longrightarrow \Psi_\varrho \longrightarrow \Psi_{\varrho,!,c} \to 0, \tag{3.4.2} $$

where $j_{\neq c} : X^{\geq 1}_{\mathcal{I},\bar{s}} \setminus X^{\geq 1}_{\mathcal{I},\bar{s},c} \hookrightarrow X^{\geq 1}_{\mathcal{I},\bar{s}}$.

Remark: applied to Ψ_{π_v}, the equality (3.3.6) becomes, cf. [**11**] §3.4

$$ \big[\Psi_{\pi_v,!,c}\big] = \sum_{t=1}^{\lfloor \frac{d}{g} \rfloor} P(t,\pi_v)_c(\frac{1-t}{2}). \tag{3.4.3} $$

Consider the filtration of stratification

$$ 0 = \mathrm{Fil}_*^{-d}(\Psi_{\varrho,!,c}) \subset \mathrm{Fil}_*^{1-d}(\Psi_{\varrho,!,c}) \subset \cdots \subset \mathrm{Fil}_*^0(\Psi_{\varrho,!,c}) = \Psi_{\varrho,!,c}. $$

3.4.4. Proposition. — *The graded pieces $\mathrm{gr}_*^h(\Psi_{\varrho,!,c})$ verify the following properties*

- *it is trivial if h is not equal to some$-g_i(\varrho) + 1 > -d$ for $i \geq -1$;*
- *for such $i \geq -1$ with $g_i(\varrho) \leq d$, then*

$$ \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho,!,c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho,i)} \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c}) $$

where $\mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c})$ is the push forward

$$ \begin{tikzcd} \Psi_{\pi_v} \arrow[r, two heads] \arrow[d] & \Psi_{\pi_v,!,c} \arrow[d, dashed] \\ \mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v}) \arrow[r, dashed, two heads] & \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c}). \end{tikzcd} $$

Remark: from [**14**] proposition 7.1, the graded pieces $\mathrm{gr}_!^h\big(\mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c})\big)$ of the filtration of stratification are then

– trivial if h is not of the shape $tg_i(\varrho) \leq d$,
– and for $h = tg_i(\varrho) \leq d$, we have, if we consider for simplicity $c = \overline{1_1}$

$$\mathrm{gr}_!^{tg_i(\varrho)}\big(\mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c})\big) \simeq \mathrm{ind}_{P_{1,h,d}(F_v)}^{P_{1,d}(F_v)} P(t,\pi_v)_{\overline{1_h}}(\frac{1-t}{2}).$$

Proposition 7.1 of [**14**] only deals with the Grothendieck group and not with the filtration, which is the main point of our proposition here.

Proof. — As explained in the remark before §3.3, as long as the statement do not speak about the lattices and the positions, then it is only a statement on Ψ_{π_v} and the precise description of Ψ_{π_v}. For simplicity we suppose $c = \overline{1_1}$. From [**11**], the graded piece $\mathrm{gr}_!^h\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big)$ of the filtration $\mathrm{Fil}_!^\bullet\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big)$ are trivial for all $h \neq tg_i(\varrho) \leq d$ and

$$\mathrm{gr}_!^{tg_i(\varrho)}\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big) \simeq P(t,\pi_v)(\frac{1-t}{2}).$$

3.4.5. Lemma. — *For $0 \leq r \leq d$, the*

$$\mathrm{gr}_!^h\Big(i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\,\mathrm{Fil}_!^r\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big)\Big)$$

verify the following properties

– *they are trivial if $r < g_i(\varrho)$;*
– *for $tg_i(\varrho) \leq r < (t+1)g_i(\varrho)$, they are trivial if $h \neq tg_i(\varrho)$;*
– *otherwise, for $a \in GL_d(F_v)/P_{h,d}(F_v)$ such that $c \subset a$, it is isomorphic to*

$$\mathrm{ind}_{P_{c\subset a}(F_v)}^{P_c(F_v)} P(t,\pi_v)_a(\frac{1-t}{2}).$$

Remark: in the last point with $h = tg_i(\varrho)$, for $c = \overline{1_1}$ and $a = \overline{1_h}$ the formula is

$$\mathrm{ind}_{P_{1,h,d}(F_v)}^{P_{1,d}(F_v)} P(t,\pi_v)_{\overline{1_h}}(\frac{1-t}{2}).$$

Proof. — Note first that the statement is trivially true for $r < g_i(\varrho)$. Recall moreover that

$$i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}P(t,\pi_v) \simeq P(t,\pi_v)_c := \mathrm{ind}_{P_{c\subset a}(F_v)}^{P_c(F_v)} P(t,\pi_v)_a(\frac{1-t}{2}),$$

where $a \in GL_d(F_v)/P_{tg_i(\varrho),d}(F_v)$ is such that $c \subset a$. We then argue by induction through the short exact sequence

$$0 \to \mathrm{Fil}_!^{r-1}\Big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\Big)\Big) \longrightarrow \mathrm{Fil}_!^r\Big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\Big)\Big) \longrightarrow \mathrm{gr}_!^r\Big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\Big)\Big) \to 0.$$

If r is not of the shape $tg_i(\varrho)$ there is nothing to prove, otherwise as

- the irreducible constituents of $i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\,\mathrm{Fil}_!^{r-1}\Big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\Big)\Big)$ are, by induction, intermediate extensions of Harris-Taylor local systems on $X^{=i}_{\mathcal{I},\bar{s}}$ for $i \leq r$,
- and $i^1_{c,*}\,{}^p\mathcal{H}^{-1} i_c^{1,*} P(t,\pi_v)$ is supported on $X^{\geq r+1}_{\mathcal{I},\bar{s}}$,

then the cone map $i^1_{c,*}\,{}^p\mathcal{H}^{-1} i_c^{1,*} P(t,\pi_v) \longrightarrow i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\,\mathrm{Fil}_!^{r-1}\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big)$ is trivial. The result follows then from the short exact sequence

$$0 \to i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\,\mathrm{Fil}_!^{r-1}\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big) \longrightarrow i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\,\mathrm{Fil}_!^{r}\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big) \longrightarrow i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\mathrm{gr}_!^{r}\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big) \to 0.$$

□

It suffices now to prove that the epimorphism

$$i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\Psi_{\pi_v} \twoheadrightarrow i^1_{c,*}\,{}^p\mathcal{H}^0 i_c^{1,*}\big(\mathrm{coFil}_{*,g_i(\varrho)}(\Psi_{\pi_v})\big)$$

is an isomorphism. For that it suffices to prove that, for every geometric point z, the germs at z of the sheaves cohomology groups of these two perverse sheaves, are the same.

Let then z be a geometric point of $X^{=h}_{\mathcal{I},\bar{s},\overline{1_h}}$. By [**7**], the germ at z of the i-th sheaf of cohomology $\mathcal{H}^i j^{=kg}_{\overline{1_{kg}},*} HT_{\overline{1_{kg}}}(\pi_v, \mathrm{St}_k(\pi_v)) \otimes \Xi^{\frac{1-k}{2}}$ is zero if (h,i) is not of the shape $(d-tg, tg-d+k-t)$ with $k \leq t \leq \lfloor \frac{d}{g} \rfloor$ and otherwise isomorphic to those of

$$HT_{\overline{1_{tg}}}\big(\pi_v, \mathrm{St}_k(\pi_v\{\frac{k-t}{2}\}) \otimes \mathrm{Speh}_{t-k}(\pi_v\{\frac{k}{2}\})\big) \otimes \Xi^{\frac{1+t-2k}{2}}.$$

Then for $h = d-tg$ and $i = tg-d+k-t$, the fiber at z of $\mathcal{H}^i j^{=kg}_{\overline{1_1},*} HT_{\overline{1_1}}(\pi_v, \mathrm{St}_k(\pi_v)) \otimes \Xi^{\frac{1-k}{2}}$ is isomorphic to those of

$$HT_{\overline{1_{tg}}}\Big(\pi_v, \big(\mathrm{St}_k(\pi_v\{\frac{k-t}{2}\})_{|P_{1,kg}(F_v)} \times \mathrm{Speh}_{t-k}(\pi_v\{\frac{k}{2}\})\big)\Big) \otimes \Xi^{\frac{1+t-2k}{2}},$$

where we induce from $P_{1,kg}(F_v) \otimes GL_{(t-k)g}(F_v)$ to $P_{1,tg}(F_v)$. Moreover considering the weights, we see that the spectral sequence computing the fibers of sheaves of cohomology of $\Psi_{\pi_v,c}$ from those of $\mathrm{gr}_!^k(\Psi_{\pi_v,c})$ degenerate at E_1. From 1.3.11, we have

$$\big(\mathrm{St}_k(\pi_v\{\frac{k-t}{2}\})\big)_{|P_{1,kg}(F_v)}\times\mathrm{Speh}_{t-k}(\pi_v\{\frac{k}{2}\})\simeq\big(LT_\pi(k,t-1-k)_{\pi_v}\big)_{|P_{1,tg}(F_v)}$$

so that, by the main result of [**7**], the fiber at z of $\mathcal{H}^i\Psi_{\pi_v,c}$ is isomorphic to those of $\mathcal{H}^i\big({}^p\mathcal{H}^0 i_c^{1,*}\Psi_{\pi_v}\big)$, so we are done. □

Dually we have

$$0\rightarrow\Psi_{\varrho,*,c}\longrightarrow\Psi_\varrho\longrightarrow j_{\neq c,*}j_{\neq c}^*\Psi_\varrho\rightarrow 0, \tag{3.4.6}$$

such that the graded pieces $\mathrm{gr}_!^h(\Psi_{\varrho,*,c})$ verify the following properties

- it is trivial if h is not equal to some $g_i(\varrho)\leq d$ for $i\geq -1$;
- for such $i\geq -1$ with $g_i(\varrho)\leq d$, then

$$\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\varrho,*,c})\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l\simeq\bigoplus_{\pi_v\in\mathrm{Cusp}(\varrho,i)}\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$$

  where $\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ is the pull back

$$\begin{tikzcd}
\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c}) \arrow[r, hook, dashed] \arrow[d, hook, dashed] & \Psi_{\pi_v,*,c} \arrow[d, hook] \\
\mathrm{Fil}_!^{g_i(\varrho)}(\Psi_{\pi_v}) \arrow[r, hook] & \Psi_{\pi_v}.
\end{tikzcd}$$

Remark: applied to Ψ_{π_v}, the equality (3.3.6) becomes, cf. [**11**] §3.4

$$\big[\Psi_{\pi_v,*,c}\big]=\sum_{t=1}^{\lfloor\frac{d}{g}\rfloor}P(t,\pi_v)_c(\frac{t-1}{2}). \tag{3.4.7}$$

More precisely for $\pi_v\in\mathrm{Cusp}(\varrho,i)$, then $\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ has a filtration $\mathrm{Fil}^k(\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c}))$ for $0\leq k\leq s_i(\varrho):=\lfloor\frac{d}{g_i(\varrho)}\rfloor$ with graded pieces $\mathrm{gr}^k(\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c}))\simeq P(s_i(\varrho)-k+1,\pi_v)_c(\frac{s_i(\varrho)-k}{2})$.

4. Non-degeneracy property for submodules

Recall first that, for a fixed irreducible $\overline{\mathbb{F}}_l$-representation ϱ of $D_{v,d}^\times$, the notation $\mathcal{V}_{\varrho,\mathbb{N}}^{d-1}$ (resp. $\mathcal{U}_{\varrho,\mathbb{N}}^{d-1}$) designates the direct factor of $\mathcal{V}_{F_v,\overline{\mathbb{Z}}_l,d}^{d-1}$ (resp. the free quotient $\mathcal{U}_{F_v,\overline{\mathbb{Z}}_l,d,free}^{d-1}$) associated with ϱ in the sense of [**19**] §B.2.

Let $i_z : z \hookrightarrow X^{=d}_{\mathcal{I},\bar{s}}$, be any supersingular point. Then, from the main theorem of Berkovitch in [**4**], we have an isomorphism

$$\mathrm{ind}_{(D^\times_{v,d})^0\varpi_v^{\mathbb{Z}}}^{D^\times_{v,d}} {}^p\mathcal{H}^0 i_z^! \Psi_\varrho \simeq \mathcal{V}^{d-1}_{\varrho,\mathbb{N}}, \tag{4.0.1}$$

and respectively

$$\big(\mathrm{ind}_{(D^\times_{v,d})^0\varpi_v^{\mathbb{Z}}}^{D^\times_{v,d}} {}^p\mathcal{H}^0 i_z^* \Psi_\varrho\big)_{free} \simeq \mathcal{U}^{d-1}_{\varrho,\mathbb{N}}, \tag{4.0.2}$$

which are equivariant for $D^\times_{v,d} \times GL_d(F_v) \times W_{F_v}$, so that we are led to compute ${}^p\mathcal{H}^0 i_z^! \Psi_\varrho$ (resp. the free quotient of ${}^p\mathcal{H}^0 i_z^* \Psi_\varrho$).
Remark: ${}^{p+}\mathcal{H}^0 i_z^! \Psi_\varrho$ might have torsion(5) but by definition ${}^p\mathcal{H}^0 i_z^! \Psi_\varrho$ is necessarily free.

Recall that our strategy is to construct a filtration of $\mathcal{V}^{d-1}_{\varrho,\mathbb{N}}$ (resp. $\mathcal{U}^{d-1}_{\varrho,\mathbb{N}}$) with irreducible graded pieces as a $\overline{\mathbb{Z}}_l$-representation of $GL_d(F_v) \times D^\times_{F_v,d} \times W_{F_v}$, which means that they are free and irreducible after tensoring with $\overline{\mathbb{Q}}_l$. Of course the idea is to obtain such a filtration from a filtration $\mathrm{Fil}^\bullet(\Psi_\varrho)$ of Ψ_ϱ, constructed using the Newton stratification, so that the associated spectral sequences

$$E^{r,s}_{!,1} := {}^p\mathcal{H}^{r+s} i_z^! \mathrm{gr}^{-r}(\Psi_\varrho) \Rightarrow {}^p\mathcal{H}^{r+s} i_z^! \Psi_\varrho$$

and

$$E^{r,s}_{*,1} := {}^p\mathcal{H}^{r+s} i_z^* \mathrm{gr}^{-r}(\Psi_\varrho) \Rightarrow {}^p\mathcal{H}^{r+s} i_z^* \Psi_\varrho,$$

give us the expected filtrations of $\mathcal{H}^0 i_z^! \Psi_\varrho$ and ${}^p\mathcal{H}^0 i_z^* \Psi_\varrho$, where $\mathrm{gr}^\bullet(\Psi_\varrho)$ are the graded pieces of $\mathrm{Fil}^\bullet(\Psi_\varrho)$.
Remark: as long as we are only interested in $\mathcal{V}^{d-1}_{\varrho,\mathbb{N}}$ (resp. $\mathcal{U}^{d-1}_{\varrho,\mathbb{N}}$) and not with the others $\mathcal{V}^\bullet_{\varrho,\mathbb{N}}$ and $\mathcal{U}^\bullet_{\varrho,\mathbb{N}}$, we have in fact only to bother with the perverse sheaves concentrated on the supersingular locus. More precisely note that for a perverse sheaf $\mathcal{P}$ not concentrated in the supersingular locus, then ${}^p\mathcal{H}^\delta i_z^! \mathcal{P}$ (resp. ${}^p\mathcal{H}^{-\delta} i_z^* \mathcal{P}$) is zero for $\delta \leq 0$ (resp. zero for $\delta < 0$ and torsion for $\delta = 0$).

Meanwhile some of the $E^{r,s}_{!,1}$ (resp. $E^{r,s}_{*,1}$) for $r+s=1$ (resp. $r+s=0$) might be torsion so that to control the lattices, it is better if all the perverse sheaves concentrated on the supersingular locus appears before (resp. after) the others, see the fourth step in the next section.
Remark: of course as long as you are only concerned with perverse sheaves on the supersingular locus, you do not need to bother about the positions

(5)The main theorem of [**10**] tells that this is not the case.

of these perverse sheaves but only about their lattices, cf. §3.2. At the end the non-degeneracy persistence property will be deduced from proposition 1.3.7.

4.1. The case of $\mathcal{V}^{d-1}_{\varrho,\mathbb{N}}$. — The main goal is then to construct a filtration of Ψ_ϱ.

First step: we start with the following three pieces filtration. For every $c \in GL_d(F_v)/P_{1,d}(F_v)$, note that any supersingular point belongs to the pure stratum $X^{\geq 1}_{\mathcal{I},\bar{s},c}$, so that in the short exact sequence (3.4.6)

$$0 \to \Psi_{\varrho,*,c} \longrightarrow \Psi_\varrho \longrightarrow j_{\neq c,*} j^*_{\neq c} \Psi_\varrho \to 0, \qquad (4.1.1)$$

we have, with harmless abuse of notations, $\mathcal{H}^0 i_z^! \Psi_\varrho \simeq \mathcal{H}^0 i_z^! \Psi_{\varrho,*,c}$. Consider then another pure stratum $X^{\geq 1}_{\mathcal{I},\bar{s},c'}$ with $c' \neq c$.

4.1.2. Lemma. — *The perverse sheaf ${}^p\mathcal{H}^i i^{1,*}_{c'} \Psi_{\varrho,*,c}$ is zero for $i \neq 0$ and it is free for $i = 0$.*

Proof. — Note first, cf. lemma 3.4.1, that the result is true for Ψ_ϱ. Moreover for any perverse free sheaf P, we have ${}^p\mathcal{H}^i i^*_{c'} P = 0$ if $i \notin \{0, -1\}$ and it is free for $i = -1$. The result then follows easily from the long exact sequence associated to the previous short exact sequence when we apply $i^*_{c'}$. □

In particular in the following short exact sequence

$$0 \to j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c} \longrightarrow \Psi_\varrho \longrightarrow \Psi_{\varrho,*,c,!,c'} \to 0$$

the perverse sheaf $\Psi_{\varrho,*,c,!,c'}$ is free. Moreover as the cokernel of $j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c} \hookrightarrow \Psi_{\varrho,*,c}$ is $i^1_{c',*} {}^p\mathcal{H}^0 i^{1,*}_{c'} \Psi_{\varrho,*,c}$, we have the following short exact sequence

$$0 \to i^1_{c',*} {}^p\mathcal{H}^0 i^{1,*}_{c'} \Psi_{\varrho,*,c} \longrightarrow \Psi_{\varrho,*,c,!,c'} \longrightarrow j_{\neq c,*} j^*_{\neq c} \Psi_\varrho \to 0.$$

Remark: with the terminology of §3.2, the dual version of lemma 3.4.1 tells us that in the short exact sequence (4.1.1), there is no saturation process: in fact all the questions about saturation keep inside the left and right terms of this short exact sequence, cf. [**10**] for more details. In the same way, the previous lemma tells us that there is no saturation process in considering the adjunction morphism $j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c} \longrightarrow \Psi_{\varrho,*,c}$.

Second step: we want to refine the filtration of the first step in order to compute ${}^p\mathcal{H}^0 i_z^! \Psi_\varrho$ for any supersingular point z. Note first that, as z belongs to any $X^{\geq 1}_{\mathcal{I},\bar{s},c'}$, then ${}^p\mathcal{H}^0 i_z^! j_{\neq c,*} j^*_{\neq c} \Psi_\varrho = (0)$ so in fact we just need to bother about the first two graded pieces of the previous filtration, which corresponds to a filtration of $\Psi_{\varrho,*,c}$. With the terminology of §3.2, at this step we do not want to deal with lattices and positions, but just describe the irreducible sub-quotients over $\overline{\mathbb{Q}}_l$.

Consider the first graded piece $j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}$. As $j_{\neq c'}$ is affine, $j_{\neq c',!}j^*_{\neq c'}$ is an exact functor, so that from the previous section, $j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}$ has a filtration $\mathrm{Fil}^\bullet\big(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}\big)$ with graded pieces $j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}$. (6)

We then have

$$j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\varrho,*,c})\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l\simeq\bigoplus_{\pi_v\in\mathrm{Cusp}(\varrho,i)}j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c}),$$

where $j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ has a filtration whose graded pieces, by combining (3.4.7) and lemma 2.3.8, are, in the order of appearance from the socle to the top,

– for $t=s_i(\varrho),\cdots,2$, the $P(t,\pi_v)_c(\frac{t-1}{2})$ obtained through the following short exact sequence twisted by $(\frac{t-1}{2})$

$$0\rightarrow P(t,\pi_v)_{c,\neq c'}\longrightarrow P(t,\pi_v)_c\longrightarrow P(t,\pi_v)_{\langle c,c'\rangle}\rightarrow 0,$$

where

$$P(t,\pi_v)_{c,\neq c'}:={}^pj^{=1}_{\neq c',!*}j^{=1,*}_{\neq c'}P_{\overline{\mathbb{Q}}_l,c}(t,\pi_v),$$

– and with last quotient $P(1,\pi_v)_{c,\neq c'}$.

The second graded piece $i^1_{c',*}{}^p\mathcal{H}^0i^{1,*}_{c'}\Psi_{\varrho,*,c}$ of the filtration of the first step, is a free perverse sheaf which, over $\overline{\mathbb{Q}}_l$, have then irreducible constituents $P(1,\pi_v)_{\langle c,c'\rangle}$ for $\pi_v\in\mathrm{Cusp}(\varrho)$ an irreducible representation of $GL_g(F_v)$ with $g<d$.

Third step: We now want, using the terminology of §3.2, to understand lattices and positions of the perverse sheaves of the second step. Our aim is to give a filtration over $\overline{\mathbb{Z}}_l$ from which we will be able to compute ${}^p\mathcal{H}^0i^!_z\Psi_\varrho$ thanks to a spectral sequence as usual.

(a) Start first with $i^1_{c',*}{}^p\mathcal{H}^0i^{1,*}_{c'}\Psi_{\varrho,*,c}$. For π_v an irreducible cuspidal representation of $GL_g(F_v)$ and $1\leq t\leq s:=\lfloor d/g\rfloor$, note that, if $tg<d$, then ${}^p\mathcal{H}^0i^!_zP(t,\pi_v)=(0)$. In particular if ${}^p\mathcal{H}^0i^!_zP(1,\pi_v)_{\langle c,c'\rangle}\neq(0)$ then $g=d$ so that

– trivially $P(1,\pi_v)_{\langle c,c'\rangle}=P(1,\pi_v)$,

(6) Again here there is no need of saturation.

– concerning the action of $GL_d(F_v)$, as the modulo l reduction of π_v is still irreducible, all the stable lattices are homothetic and
– there is only one position possible for the intermediate extension as the perverse sheaf is concentrated on a zero dimensional sub-scheme.

In short, we do not have to bother with $i^1_{c',*}{}^p\mathcal{H}^0 i^{1,*}_{c'}\Psi_{\varrho,*,c}$.

(b) We focus then on ${}^p\mathcal{H}^0 i^!_z\big(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}\big)$ by refining the previous filtration $\mathrm{Fil}^\bullet\big(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}\big)$ with graded pieces $j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}$. Recall that

$$j^{=g_i(\varrho),*}j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v\in\mathrm{Cusp}(\varrho,i)} HT_{c,\neq c'}(\pi_v,\pi_v),$$

and by fixing any numbering of $\mathrm{Cusp}(\varrho,i)=\{\pi_{v,1},\cdots\}$ the pull-back

$$\begin{tikzcd}
\mathrm{Fil}^k\Big(j^{=g_i(\varrho),*}j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}\Big) \arrow[r, hook, dashed] \arrow[d, hook, dashed] & j^{=g_i(\varrho),*}j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c} \arrow[d, hook] \\
\bigoplus_{i=1}^k HT_{c,\neq c'}(\pi_{v,i},\pi_{v,i}) \arrow[r, hook] & j^{=g_i(\varrho),*}j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l,
\end{tikzcd}$$

define then a naive filtration of $j^{=g_i(\varrho),*}j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}$ such that the graded pieces are some lattices of $HT_{c,\neq c'}(\pi_v,\pi_v)$ for π_v describing $\mathrm{Cusp}(\varrho,i)$. By taking the image by $j_!^{=g_i(\varrho)}$ of this filtration, we obtain a filtration of $j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}\Psi_{\varrho,*,c}$ whose graded pieces are entire versions of $j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}_!^{g_i(\varrho)}(\Psi_{\pi_v,*,c})$ for π_v describing $\mathrm{Cusp}(\varrho,i)$. Then we can filtrate each of these graded pieces to obtain a filtration denoted $\mathrm{Fill}^\bullet\big(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}\big)$ whose graded pieces $\mathrm{grr}^k\big(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}\big)$ are some entire version of the $P(t,\pi_v)_c$ if $2\leq t\leq s_i(\varrho)$ (resp. $P(1,\pi_v)_{c,\neq c'}$), for $\pi_v\in\mathrm{Cusp}(\varrho,i)$ with $i\geq -1$: these entire perverse sheaves may depend on all the choices.

Remark: as pointed out above, we just have to deal with the graded pieces concentrated on the supersingular locus for which with do not have to bother about the position of the intermediate extension, but now as t might be strictly greater than one, we have to describe the lattices. For $\pi_v\in\mathrm{Cusp}(\varrho,i)$ and t such that $(t+1)g_i(\varrho)=d$, let then denote by

$$\mathcal{P}_{\mathrm{Fill},!,c}(t+1,\pi_v),$$

the lattice defined above starting from a filtration of $j_{\neq c',!}j^*_{\neq c'}\Psi_{\pi_v,*,c}$, cf. (4.1.8). Note at this point that finally $\mathcal{P}_{\mathrm{Fill},!,c}(t+1,\pi_v)$ is given by the short exact sequence of lemma 2.3.8 so that we can easily identify its lattice as explained in theorem 4.1.6.

Fourth step: now we want to modify the previous filtration of $\Psi_{\varrho,*,c}$ by reorganizing the order of the graded pieces so that those concentrated on the supersingular locus appear first. Let first explain why it is possible to do so. Denote by

$$j^{1\leq d-1} : X^{\geq 1}_{\mathcal{I},\bar{s}} \setminus X^{=d}_{\mathcal{I},\bar{s}} \hookrightarrow X^{\geq 1}_{\mathcal{I},\bar{s}}$$

and consider the adjunction morphism

$$\Psi_{\varrho,*,c} \longrightarrow {}^{p+}j^{1\leq d-1}_{*} j^{1\leq d-1,*}\Psi_{\varrho,*,c},$$

where (3.4.7) described $\Psi_{\varrho,*,c}$, at least over $\overline{\mathbb{Q}}_l$. Then the kernel $K_{\varrho,*,c,d}$ of this morphism, is by construction free and in the Grothendieck group we have

$$\Big[K_{\varrho,*,c,d} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l\Big] = \sum_{i\geq -1} \sum_{\substack{\pi_v\in \mathrm{Cusp}(\varrho,i)\\ d=t_ig_i(\varrho)}} \big[P(t_i,\pi_v)(\frac{t_i-1}{2})\big],$$

i.e. $K_{\varrho,*,c,d}$ gathers all the irreducible constituents of $\Psi_{\varrho,*,c}$ concentrated on the supersingular locus. To see this, it suffices to argue on Ψ_{π_v} for $\pi_v \in \mathrm{Cusp}(\varrho,i)$ with $K_{\pi_v,*,c,d}$ the kernel of the adjunction morphism $\Psi_{\pi_v,*,c} \longrightarrow {}^{p+}j^{1\leq d-1}_{*} j^{1\leq d-1,*}\Psi_{\pi_v,*,c}$. We then notice that

- for a pure perverse sheaf P of weight 0 then the irreducible constituents of ${}^{p}j^{1\leq d-1}_{*} j^{1\leq d-1,*}P$ are of non-negative weight;
- and the irreducible constituents of $j^{1\leq d-1,*}\Psi_{\pi_v,*,c}$ are the $P(t,\pi_v)(\frac{t-1}{2})$ with $t\leq s_i := \lfloor \frac{d}{g_i(\varrho)}\rfloor$ (resp. $t<s_i$) if $g_i(\varrho)$ do not divide d (resp. if $s_ig_i(\varrho)=d$).

Then when $g_i(\varrho)$ divide d, all the irreducible constituents are of weight strictly greater than those of $P(s_i,\pi_v)(\frac{s_i-1}{2})$ so that $K_{\pi_v,*,c,d} = P(s_i,\pi_v)(\frac{s_i-1}{2})$. Otherwise $K_{\pi_v,*,c,d}$ is trivial.

We start then from the previous filtration which is $P_{c,c'}(F_v)$-equivariant and where the order of its graded pieces verifies the following property:

- $P(1,\pi_v)$ for $\pi_v\in \mathrm{Cusp}(\varrho,i)$ with $g_i(\varrho)=d$ appears after, i.e. in higher graded pieces than those associated with either $P(t,\pi'_v)_{c,\neq c'}$ or $P(t,\pi'_v)_{\langle c,c'\rangle}$ for $\pi'_v\in\mathrm{Cusp}(\varrho,i')$ where $tg_i(\varrho)<d$ or $t>1$;
- let k be the index of graded piece associated with $P(t,\pi_v)$ with $t>1$, $\pi_v\in\mathrm{Cusp}(\varrho,i)$ with $tg_i(\varrho)=d$. Then for any graduate piece

of index $k' < k$ associated with some $P(t', \pi'_v)_{c, \neq c'}$ or $P(t', \pi'_v)_{\langle c, c' \rangle}$ with $\pi'_v \in \mathrm{Cusp}(\varrho, i')$ and $t' g_{i'}(\varrho) < d$, then $i' < i$.

Note also that, for any $\pi_v \in \mathrm{Cusp}(\varrho, i)$ and t such that $t g_i(\varrho) < d$, then if $k_{\neq c'} > k_{c'}$ are the indexes of the graded pieces associated with respectively $P(t', \pi'_v)_{c, \neq c'}$ and $P(t', \pi'_v)_{\langle c, c' \rangle}$, then for any $k_{c'} < k < k_{\neq c'}$, the associated graded piece is never concentrated in the supersingular locus. This implies that we can modify the filtration such that

- the graded pieces are of the shape $P(t, \pi_v)_c$ without modifying the lattices of the graded pieces concentrated in the supersingular locus.
- Now the filtration is equivariant for the action of $P_c(F_v)$ as all the graduate and the whole of the perverse sheaf, are $P_c(F_v)$-equivariant.

4.1.3. Lemma. — *Consider a $P_c(F_v)$-equivariant perverse sheaf X which can be written*

$$0 \to A_1 \longrightarrow X \longrightarrow A_2 \to 0$$

where

- *A_2 is a free perverse sheaf with $A_2 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = P(t, \pi_v)$ for $\pi_v \in \mathrm{Cusp}(\varrho, i)$ and $t g_i(\varrho) = d$,*
- *and A_1 is some free perverse sheaf with $A_1 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ isomorphic to $P(t', \pi'_v)_c$ with $\pi'_v \in \mathrm{Cusp}(\varrho, i')$, $h = t' g_{i'}(\varrho)$ and $i' < i$.*

Suppose moreover that

$$X \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq (A_1 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l) \oplus (A_2 \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l),$$

then $X \simeq A_1 \oplus A_2$.

Proof. — We have a diagram like (3.3.5) where T is supported on $X^{=d}_{\mathcal{I}, \bar{s}}$ so that $A_1 \hookrightarrow A'_1 \twoheadrightarrow T$ is obtained through

$${}^p j^{=h}_{c,!*} j^{=h,*}_c A_1 \hookrightarrow\!\!\!\!\rightarrow_+ A_1 \hookrightarrow\!\!\!\!\rightarrow_+ A'_1 \hookrightarrow\!\!\!\!\rightarrow_+ {}^{p+} j^{=h}_{c,!*} j^{=h,*}_c A_1.$$

Suppose by absurdity, that $T \neq (0)$.

- Then as a quotient of A'_1 and as a representation of $P_c(F_v)$ $T \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is isomorphic to a small mirabolic induced representation $(r_l(\mathrm{St}_{t'}(\pi'_v))_{|P_c(F_v)} \times \tau$ for some $\overline{\mathbb{F}}_l$-representation τ of $GL_{d - t' g_{i'}(\varrho)}(F_v)$, and where r_l designates the modulo l reduction functor. In particular, proposition 1.3.7 imposes that $T \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ must have an irreducible sub-quotient with derivative of order $g_{i'}(\varrho)$.

– On the other side, note that as a quotient of A_2 all of its derivative have order $\geq g_i(\varrho) > g_{i'}(\varrho)$.

So T must be trivial, i.e. $X = A_1 \oplus A_2$. □

We now sum up what we have done until now.

4.1.4. ***Proposition***. — *There exists a filtration*

$$(0) = \mathrm{Fil}^0(\Psi_{\varrho,*,c}) \subset \mathrm{Fil}^1(\Psi_{\varrho,*,c}) \subset \mathrm{Fil}^2(\Psi_{\varrho,*,c}) = \Psi_{\varrho,*,c}$$

such that

– *the irreducible constituents of* $\mathrm{gr}^i(\Psi_{\varrho,*,c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ *for* $i = 1$ *(resp.* $i = 2$*) are all with support in* $X^{=d}_{\mathcal{I},\bar{s}}$ *(resp. are of the form* $P(t,\pi_v)_c$ *with* $\pi_v \in \mathrm{Cusp}(\varrho, i)$ *and* $tg_i(\varrho) < d$*).*
– *Moreover there is a filtration of*

$$(0) = \mathrm{Fil}^{-2}(\mathrm{gr}^1(\Psi_{\varrho,*,c})) \subset \mathrm{Fil}^{-1}(\mathrm{gr}^1(\Psi_{\varrho,*,c})) \subset \cdots \\ \cdots \subset \mathrm{Fil}^s(\mathrm{gr}^1(\Psi_{\varrho,*,c})) = \mathrm{gr}^1(\Psi_{\varrho,*,c})$$

whose graded pieces $\mathrm{gr}^i(\mathrm{gr}^1(\Psi_{\varrho,*,c}))$ *are zero except if there exists* t *such that* $tg_i(\varrho) = d$ *in which case with the previous notations,* $\mathrm{gr}^i(\mathrm{gr}^1(\Psi_{\varrho,*,c}))$ *admits a naive filtration, cf. point (b) in the third step, indexed by* $\pi_v \in \mathrm{Cusp}(\varrho, i)$ *such that its graded pieces are, cf. (4.1.8), the* $\mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v)$*.*

Remark: we will next identify the $\mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v)$ and see that, at least for the action of $P_c(F_v)$, they are independent of the choice of the naive filtration, so that we will simply write

$$\mathrm{gr}^i(\mathrm{gr}^1(\Psi_{\varrho,*,c})) \approx \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho,i)} \mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v), \tag{4.1.5}$$

in place of the long statement in the second point of the proposition above.

Last step: as explained in the introduction of §4, for a perverse sheaf $P(t,\pi_v)$ not concentrated in the supersingular locus, we have ${}^p\mathcal{H}^0 i_z^! P(t,\pi_v) = (0)$. In the long exact sequence associated with the ${}^p\mathcal{H}^\bullet i_z^!$ applied to the short exact sequence

$$0 \to \mathrm{gr}^1(\Psi_{\varrho,*,c}) \longrightarrow \mathrm{Fil}^2(\Psi_{\varrho,*,c}) \longrightarrow \mathrm{gr}^2(\Psi_{\varrho,*,c}) \to 0,$$

we then have ${}^p\mathcal{H}^0 i_z^! \Psi_{\varrho,*,c} \simeq {}^p\mathcal{H}^0 i_z^! \mathrm{gr}^1(\Psi_{\varrho,*,c})$. From the previous proposition we obtain a filtration

$$(0) = \mathrm{Fil}^{-2}({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) \subset \mathrm{Fil}^{-1}({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) \subset \cdots \\ \cdots \subset \mathrm{Fil}^s({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) = {}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)$$

whose non zero graded pieces coincide with the indexes $i \geq -1$ such that there exists t with $tg_i(\varrho) = d$ and then with the notation of (4.1.5)

$$\mathrm{gr}^i({}^p\mathcal{H}^0 i_z^!(\Psi_\varrho)) \approx \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho,i)} \mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v)_z.$$

We will now simply denote by $\big(\mathrm{Fill}^k(\Psi_{\varrho,*,c})\big)_{0 \leq k \leq r}$ the filtration of $\mathrm{gr}^1(\Psi_{\varrho,*,c})$ obtained above such that its graded pieces $\mathrm{grr}^k(\Psi_{\varrho,*,c})$ are irreducible after tensoring with $\overline{\mathbb{Q}}_l$. We then also denote by

$$\mathrm{Fill}^k(\mathcal{V}^{d-1}_{\varrho,\mathbb{N}}) := \mathrm{ind}_{(D_{v,d}^\times)^0 \varpi_v^{\mathbb{Z}}}^{D_{v,d}^\times} {}^p\mathcal{H}^0 i_z^! \, \mathrm{Fill}^k(\Psi_{\varrho,*,c}).$$

4.1.6. Theorem. — *As a $\overline{\mathbb{Z}}_l[P_d(F_v) \times D_{v,d}^\times \times W_{F_v}]$-module, the successive graded pieces $\mathrm{gr}^k(\mathcal{V}^{d-1}_{\varrho,\mathbb{N}})$ are such that there exists i, $\pi_v \in \mathrm{Cusp}(\varrho,i)$ and t such that $tg_i(\varrho) = d$ with*

$$\mathrm{gr}^k(\mathcal{V}^{d-1}_{\varrho,\mathbb{N}}) \simeq \Gamma_{GDW}(\pi_v),$$

with $\Gamma_{GDW}(\pi_v) \simeq \Gamma_G(\pi_v) \otimes \Gamma_D(\pi_v) \otimes \Gamma_W(\pi_v)$ where

- *$\Gamma_D(\pi_v)$ (resp. $\Gamma_W(\pi_v)$) is a stable lattice of $\pi_v[t]_D$ (resp. $\mathbb{L}_{g_i(\varrho)}(\pi_v)$);*
- *$\Gamma_G(\pi_v)$ is isomorphic to the stable lattice $\big(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v,t)\big)_{|P_d(F_v)}$ of definition 1.4.1.*

Remark: in other words, the lattice $\mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v)$, which is a sheaf on the supersingular locus, is with the previous notations, fiber by fiber isomorphic to $\Gamma_G(\pi_v) \otimes \Gamma_D(\pi_v) \otimes \Gamma_W(\pi_v)$.

Proof. — We argue by induction on d. As the result is trivial for $g_{-1}(\varrho)$ because, as the modulo l reduction is irreducible, there is, up to isomorphism, only one stable lattice, we suppose the result true for all $h < d$. We then use the statement of the theorem through the isomorphism (4.0.1), to obtain informations on the lattices of our perverse sheaves not concentrated on the supersingular locus. Thus arguing like before on $j^{1 \leq h,*}\Psi_\varrho$, we can conclude that, for any $i \geq -1$, $\pi_v \in \mathrm{Cusp}(\varrho,i)$ and $tg_i(\varrho) < d$, the lattices of $HT_c(\pi_v, \mathrm{St}_t(\pi_v))(\frac{t-1}{2})$ of the graded pieces $\mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c})$ are of the shape

$$\mathcal{L}_D \otimes \big(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v, t)\big)_{|P_{tg_i(\varrho)}(F_v)} \otimes \Gamma_W, \tag{4.1.7}$$

where $\mathcal{L}_D$ is some stable $\overline{\mathbb{Z}}_l$-lattice sheaf of $\mathcal{L}(\pi_v[t]_D)_c$.

To prove the theorem, by the isomorphism (4.0.1), we now have to show that the lattice $\mathcal{P}_{\mathrm{Fill},!,c}(t, \pi_v)$ is a tensorial product of lattices equipped with action by $P_d(F_v)$, $D^\times_{v,d}$ and W_{F_v}, and the lattice for the action of $P_d(F_v)$ is isomorphic to $\big(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v, t)\big)_{|P_d(F_v)}$.

By hypothesis we have $\pi_v \in \mathrm{Cusp}(\varrho, i)$ and $tg_i(\varrho) = d$. Recall also, using the exactness of $j_{c,\neq c',!}$, that $\mathcal{P}_{\mathrm{Fill},!,c}(t, \pi_v)$ fits in the following short exact sequence of lemma 2.3.8

$$\begin{aligned} 0 \to \mathcal{P}_{\mathrm{Fill},!,c}(t, \pi_v) \longrightarrow\ & j_{\neq c',!} j^*_{\neq c'} \mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c}) \\ & \longrightarrow {}^p j_{\neq c',!*} j^*_{\neq c'} \mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c}) \to 0 \end{aligned} \tag{4.1.8}$$

where

$$\mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq P(t-1, \pi_v)_c(\frac{t-2}{2}) \hookrightarrow P(t-1, \pi_v)(\frac{t-2}{2}).$$

Over $\overline{\mathbb{Z}}_l$, we have seen that $j^{=(t-1)g_i(\varrho),*} \mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c})$ is a tensorial product of lattices where those relatively to the action of $P_{(t-1)g_i(\varrho)}(F_v)$ is $\big(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v, t-1)\big)_{|P_{(t-1)g_i(\varrho)}(F_v)}$. Moreover, as the supersingular locus belongs to $X^{\geq 1}_{\mathcal{I},\bar{s},c'}$, relatively to the supersingular locus, $\mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c})$ behaves like a p-intermediate extension, i.e.

$$\mathcal{H}^0 i^*_z \mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c}) = (0),$$

for all geometric supersingular point z. For such a p-intermediate extension we materialize this property by writing $p(ss)$ as a left exponent. By inducing we can then write

$$\mathrm{gr}^k(j_{\neq c',!} j^*_{\neq c'} \Psi_{\varrho,*,c}) \hookrightarrow {}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v)(\frac{t-2}{2})$$

where ${}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v)$ is a lattice of $P_{\overline{\mathbb{Q}}_l}(t-1, \pi_v)$ verifying the following two properties:

- for all geometric supersingular point z, the perverse sheaf behaves like a p-intermediate extension, i.e. $\mathcal{H}^0 i^*_z \big({}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1, \pi_v)\big) = (0)$;

– $j^{=(t-1)g_i(\varrho),*}\big({}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1,\pi_v)\big)$ is a tensorial stable lattice where the lattice associated with the action of $P_{(t-1)g_i(\varrho)}(F_v)$ is isomorphic to $\big(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v,t-1)\big)_{|P_{(t-1)g_i(\varrho)}(F_v)}$.

Remark: we prefer to introduce the full of ${}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1,\pi_v)$ instead of $j_c^{\geq 1,*\,p(ss)}\mathcal{P}_{RI,\otimes}(t-1,\pi_v)$ which is then isomorphic to $\mathrm{gr}^k(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c})$, see the next diagram.

We then have

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v) \arrow[r, hook] \arrow[d, dashed, "\simeq"] & j_{\neq c',!}j^*_{\neq c'}\mathrm{gr}^k(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c}) \arrow[d, hook] \\
{}^p\mathcal{H}^{-1}i^*_{c'}\,{}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1,\pi_v)(\frac{t-2}{2}) \arrow[r, hook] & j_{\neq c',!}j^*_{\neq c'}\,{}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1,\pi_v)(\frac{t-2}{2}).
\end{tikzcd}$$

By (4.1.8) $\mathcal{P}_{\mathrm{Fill},!,c}(t,\pi_v) \simeq {}^p\mathcal{H}^{-1}i^*_{c'}\mathrm{gr}^k(j_{\neq c',!}j^*_{\neq c'}\Psi_{\varrho,*,c})$ and as for a Harris-Taylor perverse sheaf P supported on the supersingular locus, we have $P = P_c = P_{\langle c,c'\rangle}$, then the left map of the diagram is an isomorphism over $\overline{\mathbb{Q}}_l$. Moreover as each of the other maps of this diagram are strict so is the dotted one which is then an isomorphism.

By lemma 2.3.5 and (4.1.7), the lattice relatively to the action of $P_d(F_v)$ on ${}^p\mathcal{H}^{-1}i^*_{c'}\,{}^{p(ss)}\mathcal{P}_{RI,\otimes}(t-1,\pi_v)(\frac{t-2}{2})$ is given by the induced representation

$$\begin{tikzcd}
RI_{\overline{\mathbb{Z}}_l,-}(\pi_v\{\frac{-1}{2}\},t-1)\times(\pi_v\{\frac{t-1}{2}\})_{|P_{g_i(\varrho)}(F_v)} \arrow[r, hook] \arrow[dr, "\sim"] & \big(RI_{\overline{\mathbb{Z}}_l,-}(\pi_v\{\frac{-1}{2}\},t-1)\times\pi_v\{\frac{t-1}{2}\}\big)_{|P_d(F_v)} \arrow[d, two heads] \\
 & RI_{\overline{\mathbb{Z}}_l,-}(\pi_v,t)_{|P_d(F_v)},
\end{tikzcd}$$

which finishes the proof. □

From proposition 1.4.2, we obtain the expected non-degeneracy property.

4.1.9. Corollary. — *Any irreducible $P_d(F_v)$-equivariant subspace of $\mathcal{V}^{d-1}_{\varrho,\mathbb{N}}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$, is non-degenerate and so isomorphic to τ_{nd}.*

4.2. The case of $\mathcal{U}^{d-1}_{\varrho,\mathbb{N}}$. — In [**10**], we prove that for any supercuspidal $\overline{\mathbb{F}}_l$-representation ϱ, then $\mathcal{U}^{d-1}_{\varrho,\mathbb{N}}$ is free. As at this stage we do not want to use [**10**], we introduce its free quotient $\mathcal{U}^{d-1}_{\varrho,\mathbb{N},free}$. We then follow exactly the same steps than in the previous section, but dually. Precisely fix a supersingular point z and denote as before $i_z : z \hookrightarrow X^{=d}_{\mathcal{I},\bar{s}}$. From the $D^\times_{v,d} \times GL_d(F_v) \times W_{F_v}$-equivariant isomorphism

$$\mathrm{ind}_{(D^\times_{v,d})^0 \varpi_v^{\mathbb{Z}}}^{D^\times_{v,d}} {}^p\mathcal{H}^0 i_z^* \Psi_\varrho \simeq \mathcal{U}^{d-1}_{\varrho,\mathbb{N}}, \tag{4.2.1}$$

we are led to compute the free quotient of ${}^p\mathcal{H}^0 i_z^* \Psi_\varrho$. As explained in the introduction of §4, we first construct a filtration of Ψ_ϱ. Dually to the previous section consider first the short exact sequences

$$0 \to j_{\neq c,!} j^*_{\neq c} \Psi_\varrho \longrightarrow \Psi_\varrho \longrightarrow \Psi_{\varrho,!,c} \to 0,$$

and

$$0 \to i^1_{c',*} {}^{p+}\mathcal{H}^0 i^{1,!}_{c'} \Psi_{\varrho,!,c} \longrightarrow \Psi_{\varrho,!,c} \longrightarrow j_{c,\neq c',*} j^*_{c,\neq c'} \Psi_{\varrho,!,c} \to 0.$$

One can also introduce $\Psi_{\varrho,!,c,*,c'}$ as the pull-back

$$\begin{tikzcd} j_{\neq c,!} j^*_{\neq c} \Psi_\varrho \arrow[r, hook] \arrow[d, equal] & \Psi_{\varrho,!,c,*,c'} \arrow[r, dashed, two heads] \arrow[d, hook, dashed] & i^1_{c',*} {}^{p+}\mathcal{H}^0 i^{1,!}_{c'} \Psi_{\varrho,!,c} \arrow[d, hook] \\ j_{\neq c,!} j^*_{\neq c} \Psi_\varrho \arrow[r, hook] & \Psi_\varrho \arrow[r, two heads] & \Psi_{\varrho,!,c}. \end{tikzcd}$$

(a) Using the exactness of $j_{\neq c',*} j^*_{\neq c'}$, as in second step of the previous section, the filtration $\mathrm{Fil}^\bullet_*(\Psi_{\varrho,!,c})$ of proposition 3.4.4, gives a filtration $\mathrm{Fil}^\bullet(j_{\neq c',*} j^*_{\neq c'} \Psi_{\varrho,!,c})$ with graded pieces $j_{\neq c',*} j^*_{\neq c'} \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho,!,c})$ such that

$$j_{\neq c',*} j^*_{\neq c'} \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho,!,c}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho,i)} j_{\neq c',*} j^*_{\neq c'} \mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c}),$$

where $j_{\neq c',*} j^*_{\neq c'} \mathrm{gr}_!^{-g_i(\varrho)+1}(\Psi_{\pi_v,!,c})$ has a filtration whose graded pieces, by lemma 2.3.8, are, from quotient to subspaces,

- for $s_i(\varrho) \geq t \geq 2$, the $P(t,\pi_v)_{\langle c,c' \rangle}(\frac{1-t}{2})$ and $P(t,\pi_v)_{c,\neq c'}(\frac{1-t}{2})$ allowing to reconstruct $P(t,\pi_v)_c(\frac{1-t}{2})$ by the short exact sequence
$$0 \to P(t,\pi_v)_{\langle c,c' \rangle} \longrightarrow P(t,\pi_v)_c \longrightarrow P(t,\pi_v)_{c,\neq c'} \to 0,$$
- and $P(1,\pi_v)_{c,\neq c'}$.

(b) Concerning $i^1_{c',*} {}^{p+}\mathcal{H}^0 i^{1,!}_{c'} \Psi_{\varrho,!,c}$, after tensoring with $\overline{\mathbb{Q}}_l$, its irreducible sub-quotients are the $P(1,\pi_v)_{\langle c,c' \rangle}$ for $\pi_v \in \mathrm{Cusp}(\varrho,i)$ with $g_i(\varrho) < d$.

By arguing like in the third and fourth steps of the previous section, we can manage to modify the previous filtration to another one such that

- it is equivariant for the action of $P_c(F_v)$;

- its graded pieces are the $P(t,\pi_v)_c$ without modifying the lattices, given by the starting filtration, of the perverse sheaves concentrated on the supersingular locus;
- the perverse sheaves concentrated on the supersingular locus appears in the last graded pieces;
- if $P(t,\pi_v)(\frac{1-t}{2})$ (resp. $P(t',\pi'_v)(\frac{1-t'}{2})$) for $\pi_v \in \mathrm{Cusp}(\varrho,i)$ (resp. $\pi'_v \in \mathrm{Cusp}(\varrho,i')$) are concentrated in the supersingular locus. If $i<i'$ then the indexes k and k' respectively associated to them verify $k>k'$.

Remark: As before we do not pay attention to the position of these perverse sheaves between the p and $p+$ intermediate extensions, but we merely concentrate on the lattice of the associated local systems.

4.2.2. Notation. — *For $\pi_v \in \mathrm{Cusp}(\varrho,i)$ and t such that $tg_i(\varrho)=d$, denote by $\mathcal{P}_{\mathrm{Fill},*,c}(t,\pi_v)$ the lattice obtained by the previous construction starting from the filtration of $j_{\neq c',*}j^*_{\neq c'}\Psi_{\pi_v,!,c}$.*

To sum up we state the analogous of proposition 4.1.4.

4.2.3. Proposition. — *There exists a filtration of*

$$(0)=\mathrm{Fil}^{-2}(\Psi_{\varrho,!,c})\subset \mathrm{Fil}^{-1}(\Psi_{\varrho,!,c})\subset \mathrm{Fil}^{0}(\Psi_{\varrho,!,c})=\Psi_{\varrho,!,c}$$

such that

- *the irreducible constituents of $\mathrm{gr}^i(\Psi_{\varrho,!,c})\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ for $i=0$ (resp. $i=-1$) are all with support in $X^{=d}_{\mathcal{I},\bar s}$ (resp. are of the form $P(t,\pi_v)_c$ with $\pi_v\in\mathrm{Cusp}(\varrho,i)$ and $tg_i(\varrho)<d$).*
- *Moreover there is a filtration of*

$$(0)=\mathrm{Fil}^{-s-1}(\mathrm{gr}^0(\Psi_{\varrho,!,c}))\subset \mathrm{Fil}^{-s}(\mathrm{gr}^0(\Psi_{\varrho,!,c}))\subset\cdots$$
$$\cdots\subset \mathrm{Fil}^{-1}(\mathrm{gr}^0(\Psi_{\varrho,!,c}))=\mathrm{gr}^0(\Psi_{\varrho,!,c})$$

whose graded pieces $\mathrm{gr}^{-i}(\mathrm{gr}^0(\Psi_{\varrho,!,c}))$ are zero except if there exists t such that $tg_i(\varrho)=d$ in which case with the notation of 4.2.2 and (4.1.5),

$$\mathrm{gr}^{-i}(\mathrm{gr}^0(\Psi_{\varrho,!,c}))\approx\bigoplus_{\pi_v\in\mathrm{Cusp}(\varrho,i)}\mathcal{P}_{\mathrm{Fill},*,c}(t,\pi_v).$$

We denote by $\big(\mathrm{Fill}^k(\Psi_{\varrho,!,c})\big)_{0\leq k\leq r}$ the filtration of $\mathrm{gr}^0(\Psi_{\varrho,!,c})$ obtained above such that its graded pieces $\mathrm{grr}^k(\Psi_{\varrho,!,c})$ are irreducible after tensoring with $\overline{\mathbb{Q}}_l$. We also denote by

$$\mathrm{Fill}^k(\mathcal{U}^{d-1}_{\varrho,\mathbb{N}}) := {}^p\mathcal{H}^0 i_z^* \,\mathrm{Fill}^k(\Psi_{\varrho,!,c}).$$

Using (4.0.2) and arguing by induction we obtain the $\mathcal{U}^{d-1}_{\varrho,\mathbb{N}}$-version of theorem 4.1.6.

4.2.4. Theorem. — *As a $\overline{\mathbb{Z}}_l[P_d(F_v)\times D^\times_{v,d}\times W_{F_v}]$-module, the successive graded pieces $\mathrm{grr}^k(\mathcal{U}^{d-1}_{\varrho,\mathbb{N}})$ are such that there exists i, $\pi_v\in\mathrm{Cusp}(\varrho,i)$ and t such that $tg_i(\varrho)=d$ with*

$$\mathrm{grr}^k(\mathcal{U}^{d-1}_{\varrho,\mathbb{N},free})\simeq\Gamma_{GDW}(\pi_v),$$

with $\Gamma_{GDW}(\pi_v)\simeq\Gamma_G(\pi_v)\otimes\Gamma_D\otimes\Gamma_W(\pi_v)$ where

- *Γ_D (resp. Γ_W) is a stable lattice of $\pi_v[s_i(\varrho)]_D$ (resp. $\mathbb{L}_{g_i(\varrho)}(\pi_v)$);*
- *Γ_G is isomorphic to a stable $P_d(F_v)$-equivariant lattice of $\mathrm{St}_{s_i(\varrho)}(\pi_v)$ such that every irreducible subspace of its modulo l reduction, is isomorphic to $r_l(\tau_{nd})$.*

The only difference from the previous section concerns the lattice Γ_G which is obtained through

$$\begin{tikzcd}
j_{\neq c',*}j^*_{\neq c'}{}^{p+(ss)}\mathcal{P}_{RI,\otimes}(s_i(\varrho)-1,\pi_v)(\frac{2-s_i(\varrho)}{2}) \arrow[r, two heads] & {}^{p+}\mathcal{H}^1 i^!_{c'}\mathcal{P}_{RI,\otimes}(s_i(\varrho)-1,\pi_v)(\frac{s_i(\varrho)-2}{2}) \\
j_{\neq c',*}j^*_{\neq c'}\mathrm{gr}^k(j_{\neq c',*}j^*_{\neq c'}\Psi_{\varrho,!,c}) \arrow[u, hook] \arrow[r, two heads] & \mathcal{P}_{\mathrm{Fill},*,c}(s_i(\varrho),\pi_v) \arrow[u, dashed, two heads, "\simeq"']
\end{tikzcd}$$

and where, by induction, ${}^{p+}\mathcal{H}^1 i^!_{c'}\mathcal{P}_{RI,\otimes}(s_i(\varrho)-1,\pi_v)(\frac{s_i(\varrho)-2}{2})$ is given by $\Gamma'_G\times(\pi_v\{\frac{1-s_i(\varrho)}{2}\})_{|P_{g_i(\varrho)}(F_v)}$ where by the induction hypothesis Γ'_G is a $P_{(s_i(\varrho)-1)g_i(\varrho)}(F_v)$-equivariant lattice of $\mathrm{St}_{s_i(\varrho)-1}(\pi_v\{\frac{1}{2}\})$ such that every subspace of its modulo l reduction is isomorphic to $r_l(\tau_{nd})$. The persistence of non-degeneracy property then follows from the exactness of Φ^- and Ψ^- and from proposition 1.3.7.

Remark: It is not so easy than in the previous situation, to identify the lattice as now we only have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{St}_{s_i(\varrho)}(\pi_v)_{|P_d(F_v)} \arrow[d, hook] \\
\Gamma'_G \times (\pi_v\{\frac{1-s_i(\varrho)}{2}\})_{|P_{g_i(\varrho)}(F_v)} \arrow[r, hook] \arrow[dr, two heads] & \big(\Gamma'_G \times \pi_v\{\frac{1-s_i(\varrho)}{2}\}\big)_{|P_d(F_v)} \arrow[d, two heads] \\
 & LT_{\pi_v}(s_i(\varrho)-2,1)_{|P_d(F_v)}.
\end{tikzcd}$$

4.2.5. Corollary. — *Any irreducible $P_d(F_v)$-equivariant subspace of $\mathcal{U}^{d-1}_{\varrho,\mathbb{N},free} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l$, is non-degenerate and so isomorphic to τ_{nd}.*

4.3. Other orders of cohomology groups. — As the situations of $\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N}}$ and $\mathcal{V}^{d-1+\delta}_{\varrho,\mathbb{N}}$ are dual, consider for example the case of $\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N}}$ for $\delta > 0$. Remember the strategy explained in the introduction of §4 which consists in computing ${}^p\mathcal{H}^{-\delta} i_z^* \Psi_\varrho$ through the spectral sequence

$$E^{r,s}_{*,1} := {}^p\mathcal{H}^{r+s} i_z^* \mathrm{gr}^{-r}(\Psi_\varrho) \Rightarrow {}^p\mathcal{H}^{r+s} i_z^* \Psi_\varrho,$$

associated with some filtration $\mathrm{Fil}^\bullet(\Psi_\varrho)$ of Ψ_ϱ. For $\delta > 0$ we now need to consider all the perverse sheaves and not only those supported on the supersingular locus. In the previous sections, arguing inductively on the Lubin-Tate spaces, we essentially understood the lattices but now the question is about the positions of the Harris-Taylor perverses sheaves which is solved in [**10**].

Start again from

$$0 \to j_{\neq c,!} j^*_{\neq c} \Psi_\varrho \longrightarrow \Psi_\varrho \longrightarrow \Psi_{\varrho,!,c} \to 0,$$

and with the filtration of $\mathrm{Fil}^\bullet_*(\Psi_{\varrho,!,c})$ with graded pieces $\mathrm{gr}_*^{-g_i(\varrho)+1}(\Psi_{\varrho,!,c})$ which can be refined as before, such that to obtain graded pieces $\mathrm{grr}^k(\Psi_{\varrho,!,c})$ verifying

$$\begin{aligned}
&{}^p j^{=tg_i(\varrho)}_{c,!*} j_c^{=tg_i(\varrho),*} \mathrm{grr}^k(\Psi_{\varrho,!,c}) \\
&\qquad\qquad \hookrightarrow\!\!\!\!\rightarrow_+ \mathrm{grr}_k(\Psi_{\varrho,*,c}) \hookrightarrow\!\!\!\!\rightarrow_+ \\
&\qquad\qquad\qquad\qquad {}^{p+} j^{=tg_i(\varrho)}_{c,!*} j_c^{=tg_i(\varrho),*} \mathrm{grr}^k(\Psi_{\varrho,!,c}),
\end{aligned}$$

with $\mathrm{grr}^k(\Psi_{\varrho,!,c}) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l \simeq P(t,\pi_v)(\frac{1-t}{2})$. In [**10**], we prove

– $\mathrm{grr}^k(\Psi_{\varrho,*,c}) \simeq {}^p j^{=tg_i(\varrho)}_{c,!*} j^{=tg_i(\varrho),*}_c \mathrm{grr}^k(\Psi_{\varrho,!,c})$,
– and the sheaf cohomology group of $\mathrm{grr}_k(\Psi_{\varrho,*,c})$ are free.

In particular for a supersingular point z, the spectral sequence computing $\mathcal{H}^{-\delta} i_z^* \Psi_\varrho \simeq \mathcal{H}^{-\delta} i_z^* \Psi_{\varrho,!,c}$ through the $\mathcal{H}^\bullet i_z^* \mathrm{grr}^k(\Psi_{\varrho,*,c})$, degenerates at E_1. Note then that the $P_d(F_v)$-lattice is given by the induced representation

$$\Gamma_G \times \mathrm{Speh}_\delta(\pi_v\{\frac{s_i(\varrho)-\delta-1}{2}\})$$

where

– Γ_G is a stable $P_{(s_i(\varrho)-\delta)g_i(\varrho)}(F_v)$-lattice of $\mathrm{St}_{(s_i(\varrho)-\delta)g_i(\varrho)(F_v)}(\pi_v)$ such that any irreducible subspace is isomorphic to τ_{nd};
– $\mathrm{Speh}_\delta(\pi_v)$ has, up to isomorphism, only one stable $GL_{\delta g_i(\varrho)}(F_v)$-stable lattice.

Like in the previous sections, we then obtain the following description of $\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N}}$, which is free by the main result of [**10**].

4.3.1. Proposition. — *As a $\overline{\mathbb{Z}}_l[P_d(F_v) \times D^\times_{v,d} \times W_{F_v}]$-module, $\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N}}$ has a filtration with successive graded pieces $\mathrm{grr}^k(\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N},free})$ where there are an associated i, $\pi_v \in \mathrm{Cusp}(\varrho,i)$ and t such that $tg_i(\varrho)=d$ and*

$$\mathrm{gr}^{-i}(\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N},free}) \simeq \Gamma_{GDW}(\pi_v),$$

with $\Gamma_{GDW}(\pi_v) \simeq \Gamma_G(\pi_v) \otimes \Gamma_D \otimes \Gamma_W(\pi_v)$ where

– *Γ_D (resp. Γ_W) is a stable lattice of $\pi_v[s_i(\varrho)]_D$ (resp. $\mathbb{L}_{g_i(\varrho)}(\pi_v)$);*
– *Γ_G is isomorphic to a stable $P_d(F_v)$-equivariant lattice of $LT_{\pi_v}(s_i(\varrho)-\delta-1_i,\delta)$ such that any irreducible $P_d(F_v)$-equivariant subspace of $\Gamma_G \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ has order of derivative equal to $g_i(\varrho)$.*

Remark: consider the case where $s=-1$, that is $g_0(\varrho)$ does not divide d. Then we see that the non-degeneracy property which would advocate that irreducible subspaces of $\mathcal{U}^{d-1-\delta}_{\varrho,\mathbb{N}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ should be the less possible degenerate among all the others, is no longer true for $\delta>0$, even more this is the exact opposite as $g_i(\varrho)$ is the smallest derivative order of all irreducible subquotients of $\big(\Gamma_G \times \mathrm{Speh}_\delta(\pi_v\{\frac{s_i(\varrho)-\delta-1}{2}\})\big) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. One way to keep trace of the non-degeneracy property might be the following statement which follows trivially from the isomorphism $(\tau\times\pi)^{(k)} \simeq \tau^{(k)} \times \pi$ for τ (resp. π) a representation of $P_d(F_v)$ (resp. $GL_s(F_v)$), and the short exact sequence (1.3.10).

4.3.2. Proposition. — *Let τ be an irreducible subspace of the modulo l reduction of* $\Gamma_G \times \mathrm{Speh}_\delta(\pi_v\{\frac{s_i(\varrho)-\delta-1}{2}\})$. *Then $\tau^{(g_i(\varrho))}$ is non-degenerate.*

To sum up, we have seen that an irreducible subspace of $\big(\Gamma_G \times \mathrm{Speh}_\delta(\pi_v\{\frac{s_i(\varrho)-\delta-1}{2}\})\big) \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l$ is necessarily with derivative order $g_i(\varrho)$, but among all of them it is the less degenerated one.

5. Automorphic congruences

The class number formula for number fields (resp. the Birch-Swinnerton-Dyer conjecture) asserts that the order of vanishing of the Dedekind zeta function at $s = 0$ of a number field K (resp. the order of vanishing at $s = 1$ of the L-function of some elliptic curve E over a number field K) is given by the rank of its group of units (resp. by the rank of the Mordell-Weil group $E(K)$). Both of these statements can be restated in terms of the rank of Selmer groups and is generalized for p-adic motivic Galois representations in the Bloch-Kato conjecture.

Since the work of Ribet, one strategy to realize a part of this conjecture is to consider some automorphic tempered representation Π of a reductive group $G/\mathbb{Q}$ and take a prime divisor l of some special values of its L-function. We try then to construct an automorphic non tempered representation Π' of G congruent to Π modulo l in some sense so that such an automorphic congruence produces a non trivial element in some Selmer group.

For G a similitude group as in §2.2, in [**13**] we show how to produce automorphic congruences from torsion classes in the cohomology of Sh_K with coefficients in the local system V_ξ. For example,

- see corollary 2.9 of [**13**], to each non trivial torsion cohomology class of level I, we can associate an infinite collection of non isomorphic weakly congruent irreducible automorphic representations of the same weight and level but each of them being tempered.
- In section 3 of [**13**], we obtained automorphic congruences between tempered and non tempered automorphic representations but with distinct weights.
- In [**15**], using completed cohomology, we construct automorphic congruences between tempered and non tempered automorphic representations of the same weight but without any control of their

respective level at l which might be an issue to construct then non trivial elements in some Selmer groups, cf. loc. cit.

Another way to interpret the computations of [**15**], is to say that, whatever is the weight ξ, if you take the level at l large enough, then the cohomology groups of your KHT Shimura variety with coefficients in V_ξ can not be all free, there must exist some non trivial cohomology classes. The main aim of this section is then to find explicit conditions for the existence of non trivial cohomology classes with coefficients in V_ξ, with the control of the level at l.

5.1. Cohomology groups over $\overline{\mathbb{Q}}_l$. —

5.1.1. Definition. — *(cf.* [**24**]*) For* Π *an automorphic irreducible representation* ξ*-cohomological of* $G(\mathbb{A})$*, then, see for example lemma 3.2 of* [**12**]*, there exists an integer* s *called the degeneracy depth of* Π*, such that through the Jacquet-Langlands correspondence and base change, its associated representation of* $GL_d(\mathbb{A}_\mathbb{Q})$ *is isobaric of the following form*

$$\mu|\det|^{\frac{1-s}{2}} \boxplus \mu|\det|^{\frac{3-s}{2}} \boxplus \cdots \boxplus \mu|\det|^{\frac{s-1}{2}}$$

where μ *is an irreducible cuspidal representation of* $GL_{d/s}(\mathbb{A}_\mathbb{Q})$*.*

Remark: For a place v such that $G(F_v) \simeq GL_d(F_v)$ in the sense of our previous convention, the local component Π_v of Π at v is isomorphic to some $\mathrm{Speh}_s(\pi_v)$ where π_v is an irreducible non degenerate representation, $s \geq 1$ is an integer and $\mathrm{Speh}_s(\pi_v)$ is the Langlands quotient of the parabolic induced representation $\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\}$. In terms of the Langlands correspondence, $\mathrm{Speh}_s(\pi_v)$ corresponds to $\sigma \oplus \sigma(1) \oplus \cdots \oplus \sigma(s-1)$ where σ is the representation of $\mathrm{Gal}(\bar{F}/F)$ associated with π_v by the local Langlands correspondence.

5.1.2. Notation. — *For* π_v *an irreducible cuspidal* $\overline{\mathbb{Q}}_l$*-representation of* $GL_g(F_v)$ *and* $t \geq 1$ *such that* $tg \leq d$*, write*

$$H^i_{I^v(\infty),!,1}(\pi_v,t,\xi) := \varinjlim_n H^i_c(\mathrm{Sh}^{=tg}_{I^v(n),\bar{s}_v,\overline{1_{tg}}}, V_\xi \otimes j^{=tg,*}_{\overline{1_{tg}}} P(\pi_v,t)_{\overline{1_{tg}}})$$

and its induced version

$$\begin{aligned} H^i_{I^v(\infty),!}(\pi_v,t,\xi) &:= \varinjlim_n H^i_c(\mathrm{Sh}^{=tg}_{I^v(n),\bar{s}_v}, V_\xi \otimes j^{=tg,*} P(\pi_v,t)) \\ &\simeq H^i_{I^v(\infty),!,1}(\pi_v,t) \times_{P_{tg,d}(F_v)} GL_d(F_v). \end{aligned}$$

We also consider

$$H^i_{I^v(\infty),!*,1}(\pi_v,t,\xi) := \varinjlim_n H^i(\mathrm{Sh}^{\geq tg}_{I^v(n),\bar{s}_v,\overline{1_{tg}}}, V_\xi \otimes P(\pi_v,t)_{\overline{1_{tg}}})$$

and

$$H^i(\pi_v,t,\xi)_{I^v(\infty),!*} := \varinjlim_n H^i(\mathrm{Sh}^{\geq tg}_{I^v(n),\bar{s}_v}, V_\xi \otimes P(\pi_v,t)) \simeq H^i_{I^v(\infty),!*,1}(\pi_v,t) \times_{P_{tg,d}(F_v)} GL_d(F_v).$$

In this section we only consider the $\overline{\mathbb{Q}}_l$-cohomology groups and we recall the computations of [**8**].

5.1.3. Notation. — *Let $\mathbb{T}^S_\xi$ be the image of $\mathbb{T}^S_{abs}$ inside*

$$\bigoplus_{i=0}^{2d-2} \varinjlim_I H^i(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\overline{\mathbb{Q}}_l})$$

where the limit concerned the ideals I which are maximal at each places outside S.

For $\Pi^{\infty,v}$ an irreducible representation of $G(\mathbb{A}^{\infty,v})$, consider the set S of finite places w of $\mathbb{Q}$ such that G, I and $\Pi^{\infty,v}$ are unramified at w, We then consider $\Pi^{\infty,v}$ as a $\mathbb{T}^S_{abs}$-module and we denote by $[H^i_{I^v(\infty),!}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\}$ the associated $\mathbb{T}^S_{abs}$-isotypic component of $H^i_{I^v(\infty),!}(\pi_v,t,\xi)$. We will use similar notations with the cohomology groups introduced above. Consider now a fixed irreducible cuspidal representation π_v of $GL_g(F_v)$.

5.1.4. Proposition. — *(cf. [**12**] §3.2 and 3.3) Let Π be an irreducible automorphic representation of $G(\mathbb{A})$ which is ξ-cohomological and with degeneracy depth $s \geq 1$.*

- *If $s=1$ then $[H^i_{I^v(\infty),!}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\}$ and $[H^i_{I^v(\infty),!,*}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\}$ are all zero for $i \neq 0$. For $i=0$, if $[H^i_{I^v(\infty),!}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\} \neq (0)$ (resp. $[H^i_{I^v(\infty),!*}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\} \neq (0)$) then*

$$\Pi_v \simeq \mathrm{St}_k(\widetilde{\pi}_v) \times \Pi'_v,$$

where Π'_v is any irreducible representation, $\widetilde{\pi}_v$ is inertially equivalent to π_v and $k \leq t$ (resp. $k=t$).

– *For* $s \geq 1$, *and* $\Pi_v \simeq \mathrm{Speh}_s(\pi_v \times \pi'_v)$ *for* π'_v *any irreducible representation of* $GL_{\frac{d-sg}{s}}(F_v)$, *then* $[H^i_{I^v(\infty),!}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\}$ *(resp.* $[H^i_{I^v(\infty),!*}(\pi_v,t,\xi)]\{\Pi^{\infty,v}\}$*) is non zero if and only if* $i = s-1$ *and* $t \geq s$ *(resp.* $t = s$ *and* $i \equiv s-1 \mod 2$ *with* $|i| \leq s-1$*).*

Remark: In [**12**], we give the complete description of these cohomology groups.

5.2. Torsion for Harris-Taylor perverse sheaves. — From now on, we fix an irreducible supercuspidal $\overline{\mathbb{F}}_l$-representation ϱ and all the irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation π_v considered will be of type ϱ. In [**11**], using the adjunction maps $\mathrm{Id} \longrightarrow j^{=h}_* j^{=h,*}$, we construct a filtration of stratification

$$0 = \mathrm{Fil}^{-d}_*(\pi_v,\Pi_t) \subset \mathrm{Fil}^{1-d}_*(\pi_v,\Pi_t) \subset \cdots \subset \mathrm{Fil}^0_*(\pi_v,\Pi_t) = j^{=tg}_! HT(\pi_v,\Pi_t),$$

with free gradutates $\mathrm{gr}^{-r}_*(\pi_v,\Pi_t) := \mathrm{Fil}^{-r}_*(\pi_v,\Pi_t)/\mathrm{Fil}^{-r-1}_*(\pi_v,\Pi_t)$ which are trival except for $r = kg-1$ with $t \leq k \leq s$ and then verifying

$$\begin{aligned}
{}^p j^{=kg}_{!*} HT(\pi_v, \Pi_t \overrightarrow{\times} \mathrm{St}_{k-t}(\pi_v)) \otimes \Xi^{(t-k)/2} \\
\hookrightarrow\!\!\!\!\rightarrow \mathrm{gr}^{1-kg}_*(\pi_v,\Pi_t) \hookrightarrow\!\!\!\!\rightarrow \\
{}^{p+} j^{=kg}_{!*} HT(\pi_v, \Pi_t \overrightarrow{\times} \mathrm{St}_{k-t}(\pi_v)) \otimes \Xi^{(t-k)/2},
\end{aligned}$$

where we recall that $\hookrightarrow\!\!\!\!\rightarrow_+$ means a bimorphism, i.e. both a mono and a epi-morphism, whose cokernel has support in $\mathrm{Sh}^{\geq kg+1}_{\mathcal{I},\bar{s}}$.
Remark: In [**10**], we in fact proved that each of these graded parts are isomorphic to the p-intermediate extensions.

5.2.1. Lemma. — *When* $g = 1$, *i.e.* $\pi_v = \chi_v$ *is a character, then for all* $1 \leq t \leq d$, *whatever is the representation* Π_t *of* $GL_t(F_v)$, *we have*

$${}^p j^{=t}_{!*} HT(\chi_v,\Pi_t) \simeq {}^{p+} j^{=t}_{!*} HT(\chi_v,\Pi_t).$$

Proof. — For π_v a character, the associated Harris-Taylor local system on $\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}}$ is just the trivial one $\overline{\mathbb{Z}}_l$ where the fundamental group $\Pi_1(\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}})$ acts by its quotient $\Pi_1(\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}}) \twoheadrightarrow \mathcal{D}^\times_{v,h}$ with $\mathcal{D}^\times_{v,h}$ acting by the character χ_v. Then as $\mathrm{Sh}^{\geq h}_{\mathcal{I},\bar{s}_v,\overline{1_h}}$ is smooth over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, then this Harris-Taylor local system shifted by the dimension $d-h$, is perverse for both t-structures p and $p+$, in particular the two intermediate extensions are equal. □

Remark: One of the main result of [**10**] is that this equality of perverse extensions remains true for every Harris-Taylor local systems associated with any irreducible cuspidal representation π_v such that its modulo l reduction is still supercuspidal, i.e. is of ϱ-type -1.

5.2.2. Proposition. — *For any representation Π_t of $GL_t(F_v)$, we have the following resolution of ${}^p j^{=t}_{!*} HT(\chi_v,\Pi_t)$*

$$0 \to j^{=d}_! HT(\chi_v, \Pi_t\{\frac{t-s}{2}\}) \times \mathrm{Speh}_{d-t}(\chi_v\{t/2\})) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \cdots \longrightarrow j^{=t+1}_! HT(\chi_v, \Pi_t\{-1/2\} \times \chi_v\{t/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow j^{=t}_! HT(\chi_v,\Pi_t) \longrightarrow {}^p j^{=t}_{!*} HT(\chi_v,\Pi_t) \to 0. \quad (5.2.3)$$

Proof. — As explained in [**10**], the statement is equivalent to the freeness of the sheaf cohomology groups of ${}^p j^{=t}_{!*} HT(\chi_v,\Pi_t)$ which is trivial when χ_v is a character. Indeed, as the strata $\mathrm{Sh}^{\geq h}_{I^v,\bar{s}_v,1}$ are smooth, then the constant sheaf, up to shift, is perverse and so equals to the intermediate extension of the constant sheaf, shifted by $d-h$, on $\mathrm{Sh}^{=h}_{I^v,\bar{s}_v,\overline{1_h}}$. In particular we have trivially the following resolution

$$0 \to j^{=d}_! HT(1_v, \Pi_t\{\frac{t-s}{2}\}) \otimes \mathrm{Speh}_{d-t}(1_v\{t/2\})) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \cdots \longrightarrow j^{=t+1}_! HT(1_v, \Pi_t\{-1/2\} \otimes 1_v\{t/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow j^{=t}_! HT(1_v,\Pi_t) \longrightarrow {}^p j^{=t}_{!*} HT(1_v,\Pi_t) \to 0,$$

where we recall that $\mathrm{Speh}_\delta(1_v)$ is the trivial representation of $GL_\delta(F_v)$. The resolution (5.2.3) is then just the induced version of the previous one twisted by χ_v, as $HT(\chi_v,\Pi_t)$ is the $HT(1_v,\Pi_t)$ where the action of the fundamental group factors through

$$\pi_1(\mathrm{Sh}^{=t}_{I,\bar{s}_v}) \twoheadrightarrow \mathcal{D}^\times_{v,t} \xrightarrow{\chi_v} F_v^\times.$$

□

Remark: In [**10**], we prove the previous resolution more generally for every irreducible cuspidal representation π_v of $GL_g(F_v)$,

$$0 \to j^{=sg}_! HT(\pi_v, \Pi_t \overrightarrow{\times} \mathrm{Speh}_{s-t}(\pi_v)) \otimes \Xi^{\frac{s-t}{2}} \longrightarrow \cdots \longrightarrow j^{=(t+2)g}_! HT(\pi_v, \Pi_t \overrightarrow{\times} \mathrm{Speh}_2(\pi_v)) \otimes \Xi^1 \longrightarrow j^{=(t+1)g}_! HT(\pi_v, \Pi_t \overrightarrow{\times} \pi_v) \otimes \Xi^{\frac{1}{2}} \longrightarrow j^{=tg}_! HT(\pi_v,\Pi_t) \longrightarrow {}^p j^{=tg}_{!*} HT(\pi_v,\Pi_t) \to 0, \quad (5.2.4)$$

which is again equivalent to the property that the sheaf cohomology groups of ${}^p j^{=tg}_{!*} HT(\pi_v, \Pi_t)$ are torsion free.

In [**13**] we prove that torsion classes arising in some cohomology group of the whole Shimura variety, can be raised in characteristic zero to some automorphic tempered representation of $G(\mathbb{A})$ in the following sense.

5.2.5. Definition. — *A torsion class either in $H^i_{I^v(\infty),!*}(\pi_v, t, \xi)_{\mathfrak{m}}$ or in $H^i_{I^v(\infty),!}(\pi_v, t, \xi)_{\mathfrak{m}}$, is said tempered ξ-cohomological if there exists an irreducible automorphic and ξ-cohomological tempered representation Π unramified outside I and p with Π^∞ a sub-quotient of* $\lim_{\rightarrow n} H^{d-1}(\mathrm{Sh}_{I^v(n),\bar\eta}, V_{\xi,\overline{\mathbb{Q}}_l})_{\mathfrak{m}}$.

From now on we denote by ϱ a $\overline{\mathbb{F}}_l$-character of $F_v^\times$ which could be, if we admit the results of [**10**], any irreducible $\overline{\mathbb{F}}_l$-supercuspidal representation of $GL_{g_{-1}(\varrho)}(F_v)$. We will write the statements and the proofs in the general case. We moreover suppose that

$$d = g_{-1}(\varrho) m(\varrho) l^u$$

and we will pay attention to irreducible $GL_d(F_v)$-sub-quotients of either $H^i_{I^v(\infty),!*}(\pi_v, t, \xi)_{\mathfrak{m}}[l]$ or $H^i_{I^v(\infty),!}(\pi_v, t, \xi)_{\mathfrak{m}}[l]$, isomorphic to ρ_u.

5.2.6. Lemma. — *Consider $\pi_{v,i} \in \mathrm{Cusp}_i(\varrho)$ for $i \geq -1$. Suppose there exists a $GL_d(F_v)$-irreducible sub-quotient of $H^j_{I^v(\infty),!*}(\pi_{v,i}, t, \xi)_{\mathfrak{m}}[l]$ (resp. $H^j_{I^v(\infty),!}(\pi_{v,i}, t, \xi)_{\mathfrak{m}}[l]$), isomorphic to ρ_u, then $j \in \{0, 1\}$ (resp. $j = 1$).*

Proof. — (a) Consider first the case of $i = -1$. We argue by induction from $t = s = m(\varrho) l^u$ to $t = 1$ with both $H^j_{I^v(\infty),!,1}(\pi_{v,-1}, t, \xi)_{\mathfrak{m}}$ and $H^j_{I^v(\infty),!*}(\pi_{v,-1}, t, \xi)_{\mathfrak{m}}$. Concerning $H^j_{I^v(\infty),!*}(\pi_{v,-1}, t, \xi)_{\mathfrak{m}}$, recall that, as $\pi_{v,-1} \in \mathrm{Cusp}_{-1}(\varrho)$ so that[(7)] whatever is the representation Π_t of $GL_{tg_{-1}(\varrho)}(F_v)$,

$${}^p j^{=tg}_{!*} HT(\pi_{v,-1}, \Pi_t) \simeq {}^{p+} j^{=tg}_{!*} HT(\pi_{v,-1}, \Pi_t),$$

then we only have to consider the case $j \leq 0$. By Artin's theorem, see for example theorem 4.1.1 of [**6**], using the affiness of $\mathrm{Sh}^{=h}_{I,\bar s_v}$, we know that $H^j_{I^v(\infty),!}(\pi_{v,-1}, t, \xi)_{\mathfrak{m}}$ is zero for every $j < 0$ and is torsion free for $j = 0$.

(7) cf. the lemma 5.2.1 for a character and [**10**] for the general case.

- Note first that for $t=s$, then $HT(\pi_{v,-1},\Pi_s)$ has support in dimension zero, so that $H^j_{I^v(\infty),!*}(\pi_{v,-1},s,\xi)_{\mathfrak{m}}=H^j_{I^v(\infty),!}(\pi_{v,-1},s,\xi)_{\mathfrak{m}}$ is zero for $j\neq 0$ and free for $j=0$, so the result is trivially true.
- Suppose by induction, the result is true for all $t'>t$ and consider the case of $H^j_{I^v(\infty),!*}(\pi_{v,-1},t,\xi)_{\mathfrak{m}}$ through the spectral sequence associated with the resolution (5.2.3). Note first that concerning irreducible sub-quotients of the l-torsion of the cohomology groups which are $GL_d(F_v)$-isomorphic to ρ_u, then we can truncate (5.2.3) to the short exact sequence of its last three terms.

$$0 \dashrightarrow j_!^{=(t+1)g} HT(\pi_{v,-1},\Pi_t \overrightarrow{\times} \pi_{v,-1})\otimes \Xi^{\frac{1}{2}} \longrightarrow$$
$$j_!^{=tg}HT(\pi_{v,-1},\Pi_t)\longrightarrow {}^p j_{!*}^{=tg}HT(\pi_{v,-1},\Pi_t)\to 0. \quad (5.2.7)$$

Then considering our problem for $H^j_{I^v(\infty),!*}(\pi_{v,-1},t,\xi)_{\mathfrak{m}}$, for $j\leq -1$, there is no torsion with an irreducible sub-quotient isomorphic to ρ_u. We are then done with $H^j_{I^v(\infty),!*}(\pi_{v,-1},t,\xi)_{\mathfrak{m}}$. The result about $H^j_{I^v(\infty),!}(\pi_{v,-1},t,\xi)_{\mathfrak{m}}$, then follows from the long exact sequence associated with (5.2.7) using the fact that for $j=0$, it is torsion free.

(b) Consider now the case $i\geq 0$. Recall, cf. [**20**] proposition 2.3.3, that the semi-simplification of the modulo l reduction of $\pi_{v,i}[t]_D$, does not depend of the choice of a stable lattice, and is equal to

$$\sum_{k=0}^{m(\varrho)l^i-1}\tau\{-\frac{m(\varrho)l^i-1}{2}+k\}$$

where τ is the modulo l reduction of $\pi_{v,-1}[tm(\varrho)l^i]_D$ which is irreducible, and $\tau\{n\}:=\tau\otimes q^{-n\,\mathrm{val}\circ\mathrm{nrd}}$ where nrd is the reduced norm. In particular for any representation Π_t of $GL_{tg_{-1}(\varrho)}(F_v)$, we have

$$m(\varrho)l^i\mathbb{F}\Big[j_!^{=tm(\varrho)l^ig_{-1}(\varrho)}HT(\pi_{v,-1},\Pi)\Big]=m(\varrho)l^ij_!^{=tm(\varrho)l^ig_{-1}(\varrho)}\Big[\mathbb{F}HT(\pi_{v,-1},\Pi)\Big]$$
$$=j_!^{=tg_i(\varrho)}\Big[\mathbb{F}HT(\pi_{v,i},\Pi)\Big]=\mathbb{F}\Big[j_!^{=tg_i(\varrho)}HT(\pi_{v,i},\Pi)\Big], \quad (5.2.8)$$

where $\mathbb{F}(\bullet)=\bullet\otimes^{\mathbb{L}}_{\mathbb{Z}_l}\overline{\mathbb{F}}_l$. By the computation of [**8**] §5, we note that for $j>0$, the irreducible sub-quotients of $H^j(\mathrm{Sh}_{I,\bar{s}_v},j_!^{=tg}HT(\pi_{v,-1},\Pi_t)\otimes V_\xi)\otimes_{\mathbb{Z}_l}\overline{\mathbb{Q}}_l$ are not tempered except if $t=s-1$ and $j=1$. Then concerning sub-quotients isomorphic to ρ_u, the only case where it can appeared in the modulo l reduction of some irreducible sub-quotient of the free quotient of $H^j(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=tg} HT(\pi_{v,-1}, \Pi_t) \otimes V_\xi)$ is when either $(t,j) = (s-1,1)$ or $j=0$. The result about $H^j_{I^v(\infty),!}(\pi_{v,i}, t, \xi)_{\mathfrak{m}}[l]$ then follows from the previous case where $i=-1$ using (5.2.8) and the following wellknown short exact sequence

$$0 \to H^n(X,\mathcal{P}) \otimes_{\mathbb{Z}_l} \mathbb{F}_l \longrightarrow H^n(X, \mathbb{F}\mathcal{P}) \longrightarrow H^{n+1}(X,\mathcal{P})[l] \to 0,$$

for any $\mathbb{F}_q$-scheme X and any $\mathbb{Z}_l$-perverse free sheaf $\mathcal{P}$.

Then the result about the cohomology of ${}^p j_{!*}^{=tg} HT(\pi_{v,i}, \Pi_t)$ follows from the resolution analog of (5.2.7), and the case of ${}^{p+} j_{!*}^{=tg} HT(\pi_{v,i}, \Pi_t)$ is obtained by Grothendieck-Verdier duality. $\square$

5.3. Tempered and non tempered congruences. —

5.3.1. Proposition. — *Let Π be an irreducible automorphic cuspidal representation of $G(\mathbb{A})$ verifying the following properties:*

- *it is ξ-cohomological with non trivial invariant under some fixed $I \in \mathcal{I}$;*
- *its degeneracy depth is equal to $s > 1$;*
- *its local component at v is isomorphic to $\mathrm{Speh}_s(\pi_v)$ with $\pi_v \in \mathrm{Cusp}(\varrho, -1)$ and where[(8)] $d = g_u(\varrho)$ for some $u \geq 0$.*

Denote by $\mathfrak{m}$ the maximal ideal of $\mathbb{T}_I$ associated with Π. Then for any $w \in \mathrm{Spl}$ such that I_w is maximal, and distinct from l, there exists an irreducible tempered representation $\Pi(w)$ of $G(\mathbb{A})$ such that:

- *it is ξ-cohomological,*
- *of level $I(w) = I^w I_w$ where I_w is the subgroup of elements of $GL_d(\mathcal{O}_w)$ which, modulo the maximal ideal of $\mathcal{O}_w$, belong to the parabolic subgroup $P_{1,d}(\kappa(w))$;*
- *$\Pi(w)$ is weakly $\mathfrak{m}$-congruent to Π in the sense it shares the same multiset of Satake's parameters than Π outside $I(w)$.*

Remark: In particular for $s=2$, as in Ribet's proof of Herbrand theorem, we should obtain a non trivial element in the Selmer group of the adjoint representation of the Galois $\overline{\mathbb{F}}_l$-representation associated with $\mathfrak{m}$.

Thanks to the main result of [**13**], it suffices to prove that under the previous hypothesis, the torsion of $H^1(\mathrm{Sh}_{I,\bar{\eta}_v}, V_\xi[d-1])_{\mathfrak{m}}$ is non trivial. Note moreover that $\Pi(w)_w$ looks like $\mathrm{St}_2(\chi_w) \times \chi_{w,1} \times \cdots \times \chi_{w,d-2}$ for unramified characters $\chi_w, \chi_{w,1}, \cdots, \chi_{w,d-2}$.

[(8)] For π_v the trivial character, the hypothesis $d = g_u(\varrho)$ for $u=0$ is equivalent to ask that the order of $q \in \mathbb{F}_l$, which is the cardinal of the residue field of F_v, is equal to d.

Proof. — Thanks to the main result of [**13**], it suffices to prove that under the previous hypothesis, the torsion of $H^1(\mathrm{Sh}_{I,\bar\eta_v}, V_\xi[d-1])_{\mathfrak{m}}$ is non trivial. To do so, consider the spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{I,\bar s_v}, \mathrm{grr}^{-p}_{!,\varrho})_{\mathfrak{m}} \Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar\eta_v}, V_\xi[d-1])_{\mathfrak{m}}$$

associated with the filtration $\mathrm{Fill}^\bullet_{!,\varrho}$ of Ψ_ϱ. Up to translation we may suppose that $E_1^{p,q}=0$ for all $p<0$.

- The first idea to construct torsion classes, could be to find some non trivial torsion classes in the E_1-page, i.e. in the cohomology of the Harris-Taylor perverse sheaves. For example in [**12**] proposition 4.5.1, we prove that if the modulo l reduction of such π_v is cuspidal but not supercuspidal, then, for a well chosen level, the cohomology groups of the associated Harris-Taylor perverse sheaves, can not be all free, so there is torsion on the E_1 page. Unfortunately it seems not so clear that such torsion cohomology class remains in the E_∞-page.
- The idea is then to produce torsion in the E_2 page by finding a map $d_1^{p,q}$ with

$$\begin{tikzcd} E_1^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \arrow[r, "d_1^{p,q}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l"] \arrow[d, two heads] & E_1^{p+1,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \\ Q \arrow[r, "\sim"] & Q' \arrow[u, hook] \end{tikzcd}$$

  such that the $\overline{\mathbb{Z}}_l$-lattices of Q and Q' respectively induced by $E_1^{p,q}$ and $E_1^{p+1,q}$, are not isomorphic.

First note that over $\mathbb{Q}_l$:

- $E_1^{-r,r} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ has a direct factor isomorphic to $(\Pi^{\infty,v})^I \otimes \mathrm{St}_s(\pi_v) \otimes \mathbb{L}(\pi_v)(\frac{1-s}{2})$ where we recall that the contragredient of $\mathbb{L}(\pi_v)$ is the Galois representation attached to π_v by the local Langlands correspondance;
- $d_1^{-r,r} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ induces a injection from the previous direct factor into a direct factor of $E_1^{-r+1,r}$ which, as a representation of $GL_d(F_v)$, is parabolically induced from $P_{(s-1)g_{-1}(\varrho),d}(F_v)$ to $GL_d(F_v)$.

From the last remark of the previous section, $\mathrm{ind}_{(D_{v,d}^\times)^0\varpi_v^{\mathbb{Z}}}^{D_{v,d}^\times} {}^p\mathcal{H}^0 i_z^* E_1^{-r,r}$ as a $GL_d(F_v)$-representation, has a sub-space isomorphic to $\Gamma_G(\pi_v)$ where $\Gamma_G(\pi_v)$ is a stable lattice of $\mathrm{St}_s(\pi_v)$ such that ρ_u is the only irreducible sub-representation of $\Gamma_G(\pi_v) \otimes_{\mathbb{Z}_l} \overline{\mathbb{Q}}_l$. Moreover we know that ρ_u can not be a sub-space of a parabolically induced representation. From these facts we conclude that the torsion of $E_2^{-r+1,r}$ is non trivial and more precisely that ρ_u is a sub-quotient of $E_2^{-r+1,r}[l]$.

If ρ_u as a sub-quotient of $E_2^{-r+1,r}[l]$ remains a subquotient of $E_\infty^1[l]$ then we are done. Suppose by absurdity it is not the case. First about the free quotient $E_{k,free}^{p,q}$ of the $E_k^{p,q}$, we know from[9] [**8**] that:

- if ρ_u is a sub-quotient of $E_{1,free}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ with $p+q \neq 0$, then $\mathrm{grr}_{!,\varrho}^{-p}$ is isomorphic to some $P(\pi_v, s_i(\varrho)-1)$ with $\pi_v \in \mathrm{Cusp}(\varrho, i)$ and then $p+q = \pm 1$;
- for $k \geq 2$ and $p+q \neq 0$, as the $\overline{\mathbb{Q}}_l$-spectral sequence degenerates in E_2 and that for $n \neq 0$, $E_\infty^n \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ does not have a tempered sub-quotient, then ρ_u is never a sub-quotient of $E_{k,free}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

Then there must exist (p,q) and a torsion class in $(E_{1,tor}^{p,q})_{\mathfrak{m}}$ with $p+q = 2$ such that ρ_u is a sub-quotient of its l-torsion which contradicts lemma 5.2.6. □

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[9]see also §3.2 of [**12**] and more specially proposition 3.2.5

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