LEVEL LOWERING: A MAZUR PRINCIPLE IN HIGHER DIMENSION

by

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Abstract. — For a maximal ideal $\mathfrak{m}$ of some anemic Hecke algebra $\mathbb{T}_\xi^S$ of a similitude group of signature $(1, d-1)$, one can associate a Galois $\overline{\mathbb{F}}_l$-representation $\overline{\rho}_\mathfrak{m}$ as well as a Galois $\mathbb{T}_{\xi,\mathfrak{m}}^S$-representation $\rho_\mathfrak{m}$. For $l \geqslant d$, on can also define a monodromy operator $\overline{N}_\mathfrak{m}$ as well as $N_{\widetilde{\mathfrak{m}}}$ for every prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, giving rise to partitions $\underline{\overline{d}_\mathfrak{m}}$ and $\underline{d}_{\widetilde{\mathfrak{m}}}$ of d. As with Mazur's principle for GL_2, analysing the difference between these partitions, we infer informations about the liftings of $\overline{\rho}_\mathfrak{m}$ in characteristic zero known as level lowering problem.

Contents

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1. Introduction

Let $F = EF^+$ be a finite CM extension of $\mathbb{Q}$ with $E/\mathbb{Q}$ an imaginary quadratic field and F^+ totally real. Consider then a similitude group $G/\mathbb{Q}$ as in §2 and a place v of F above a prime number $p = uu^c$ split in E and such that G is split at p with $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times GL_d(F_v) \times \prod_{v \neq w | u} (B_w^{op})^\times$, cf. §2. For any finite set $S \ni v$ of places of F, let $\mathbb{T}^S$ be the anemic Hecke algebra and, for ξ an algebraic representation of $G(\mathbb{Q})$, we denote by $\mathbb{T}^S_\xi$ the quotient of $\mathbb{T}^S$ of ξ-cohomological Satake's parameters. For any maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S_\xi$, and for a prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, we denote by

$$\rho_{\widetilde{\mathfrak{m}}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

the Galois $\overline{\mathbb{Q}}_l$-representation associated to $\widetilde{\mathfrak{m}}$, cf. [**21**], where $\mathrm{Gal}_{F,S}$ is the Galois group of the maximal extension of F which is unramified outside S. By Cebotarev's density theorem and the fact that a semi-simple representation is determined, up to isomorphism, by characteristic polynomials, then the semi-simple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of the maximal ideal $\mathfrak{m}$ of $\mathbb{T}^S_\xi$ containing $\widetilde{\mathfrak{m}}$.

Main assumptions: *we now suppose*

- $[F(\exp(\frac{2i\pi}{l}) : F] > d$: *there exists then v as above such that the order q_v of the residue field at v is of order $> d$ modulo l;*
- $\overline{\rho}_{\mathfrak{m}}$ *is absolutely irreducible,*
- *and as a representation of the Weil group at v, up to the action of the monodromy operator, the semi-simplification of $\overline{\rho}_{\mathfrak{m},v}$ is a multiplicity free direct sum of characters.*

Remark. Note first that for every $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, as $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible, then $\rho_{\widetilde{\mathfrak{m}}}$ has, up to homothety, only one stable $\overline{\mathbb{Z}}_l$-lattice.

We are interested in the set $\{\widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$ and the various partitions $\underline{d_{\widetilde{\mathfrak{m}},v}}$ of d associated to the unipotent operators $N_{\widetilde{\mathfrak{m}},v}$

$$\underline{d_{\widetilde{\mathfrak{m}},v}} = \Big(n_1(\widetilde{\mathfrak{m}}) \geqslant n_2(\widetilde{\mathfrak{m}}) \geqslant \cdots \geqslant n_r(\widetilde{\mathfrak{m}}) \Big)$$

with $d = n_1(\widetilde{\mathfrak{m}}) + \cdots n_r(\widetilde{\mathfrak{m}})$ where the $n_i(\widetilde{\mathfrak{m}})$ are the sizes of Jordan's blocks of the nilpotent operator $N_{\widetilde{\mathfrak{m}}}$. More precisely, the restriction $\rho_{\widetilde{\mathfrak{m}},v}$ of $\rho_{\widetilde{\mathfrak{m}}}$ to the decomposition group at v can be written as a direct sum

$$\rho_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{Sp}_{n_1(\widetilde{\mathfrak{m}})}(\rho_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{n_r(\widetilde{\mathfrak{m}})}(\rho_{v,r}),$$

where $(n_1(\widetilde{\mathfrak{m}}) \geqslant \cdots \geqslant n_r(\widetilde{\mathfrak{m}}))$ is a partition of d and where we suppose the $\rho_{v,i}$ to be characters and

$$\mathrm{Sp}_{n_i(\widetilde{\mathfrak{m}})}(\rho_{v,i}) = \rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})}{2}) \oplus \rho_{v,i}(\frac{3-n_i(\widetilde{\mathfrak{m}})}{2}) \oplus \cdots \oplus \rho_{v,i}(\frac{n_i(\widetilde{\mathfrak{m}})-1}{2}),$$

where $N_{\widetilde{\mathfrak{m}},v}$ induces isomorphisms $\rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})+2\delta}{2}) \longrightarrow \rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})+2(\delta+1)}{2})$ for $0 \leqslant \delta < n_i(\widetilde{\mathfrak{m}}) - 1$ and is trivial on $\rho_{v,i}(\frac{n_i(\widetilde{\mathfrak{m}})-1}{2})$.

1.1. Notation. — *We denote by $T_{\widetilde{\mathfrak{m}},v}$ the Young diagram of*

$$\underline{d_{\widetilde{\mathfrak{m}},v}} := (n_1(\widetilde{\mathfrak{m}}) \geqslant \cdots \geqslant n_1(\widetilde{\mathfrak{m}}))$$

labelled by the $\rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})+2k}{2})$ so that

$$\Big\{\rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})+2k}{2}) : k = 0, \cdots, n_i(\widetilde{\mathfrak{m}}) - 1\Big\}$$

are the labels of i-th line.

As the order of unipotency of the monodromy operator is trivially less than d, for $l \geqslant d$ we can define its logarithm in $\overline{\mathbb{F}}_l$ and so define the modulo l nilpotent monodromy operator $\overline{N}_{\mathfrak{m},v}$ associated to $\overline{\rho}_{\mathfrak{m}}$ at the place v: recall that as $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, each of the $\rho_{\widetilde{\mathfrak{m}}}$ has, up to homothety, a unique stable $\overline{\mathbb{Z}}_l$-lattice so that $\overline{N}_{\mathfrak{m},v}$ is well defined and does not depend on the choice of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$. We then denote by $\underline{\bar{d}_{\mathfrak{m},v}}$ the partition of d given by the sizes of Jordan's blocks of $\overline{N}_{\mathfrak{m},v}$, and $\overline{T}_{\mathfrak{m},v}$ its labelled Young diagram.

1.2. Notation. — *For any $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, we obtain $T_{\mathfrak{m},v}$ from $T_{\widetilde{\mathfrak{m}},v}$ by breaking into $\delta(\widetilde{\mathfrak{m}}, i)$ pieces its i-th line. We then denote by $\delta(\widetilde{\mathfrak{m}})$ the maximum of the $\delta(\widetilde{\mathfrak{m}}, i)$ and speak about degeneration of monodromy when there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ with $\delta(\widetilde{\mathfrak{m}}) > 0$.*

By classical arguments due to Carayol, we can also define a representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}^S_{\xi,\mathfrak{m}}),$$

interpolating the $\rho_{\widetilde{\mathfrak{m}}}$ for all $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$. By Cebotarev such a $\rho_{\mathfrak{m}}$ is, up to isomorphism, uniquely determined and by construction

$$\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\widetilde{\mathfrak{m}} \subset \mathfrak{m}} \rho_{\widetilde{\mathfrak{m}}}$$

is semi-simple.

Main results: *Under the previous assumptions,*

- **Mazur's principle:** *there exists* $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ *such that* $\underline{d_{\widetilde{\mathfrak{m}},v}} = \underline{\bar{d}_{\mathfrak{m},v}}$.
- *The lenght of* $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ *which is equal to* $\sharp\{\widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$, *is greater than* $1 + \max_{\widetilde{\mathfrak{m}} \subset \mathfrak{m}} \delta(\widetilde{\mathfrak{m}})$.

As usual this statement should be translated in its automorphic version.

- In the case $d = 2$ when there exists a lifting $\widetilde{\mathfrak{m}}$ with $\pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_2(\chi_v)$, our multiplicity free hypothesis means that $q_v \not\equiv 1 \mod l$. When the modulo l monodromy operator $\overline{N}_{\mathfrak{m},v}$ is trivial, which means that the modulo l reduction of $\pi_{\widetilde{\mathfrak{m}},v}$ is unramified, then our statement says that there exists a lifting $\widetilde{\mathfrak{m}}'$ of the same level as $\pi_{\widetilde{\mathfrak{m}}}$ outside v and with $\pi_{\widetilde{\mathfrak{m}}',v}$ unramified. This is exactly the statement of *the classical Mazur's principle.*
- For $d \geqslant 2$ with $\widetilde{\mathfrak{m}}$ such that $\pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_d(\chi_v)$, the multiplicity free hypothesis is equivalent to asking the order of $q_v \in \mathbb{F}_l^\times$ to be greater than d. For $\underline{\bar{d}_{\mathfrak{m},v}} = (t_1 \geqslant t_2 \geqslant \cdots \geqslant t_r)$ the result then insures the existence of $\widetilde{\mathfrak{m}}' \subset \mathfrak{m}$ with $\pi_{\widetilde{\mathfrak{m}}'}$ of the same level as $\pi_{\widetilde{\mathfrak{m}}}$ outside v and such that
$$\pi_{\widetilde{\mathfrak{m}}',v} \simeq \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r}).$$
For any partition $\underline{d}$ of d, we denote by $\underline{d}^*$ its dual partition where the lines of $\underline{d}^*$ are the columns of $\underline{d}$. Then $\pi_{\widetilde{\mathfrak{m}}',v}$ has non trivial vectors invariant under $I_{\underline{d_{\widetilde{\mathfrak{m}},v}}^*}(\mathcal{O}_v)$ the parahoric subgroup associated to the dual partition of $\underline{d_{\widetilde{\mathfrak{m}},v}}$, cf. (5.6). In the case where
$$\underline{\bar{d}_{\mathfrak{m},v}} = (1 \geqslant \cdots \geqslant 1),$$
$\pi_{\widetilde{\mathfrak{m}}',v}$ is thus unramified. Moreover $\{\widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$ is then of order at least d.
- Using the usual automorphic cyclic base change, one can formulate a statement for automorphic representations π of GL_d with $\pi^\vee \simeq \pi^c$.

The strategy rests on the study of the cohomology of KHT Shimura vatieties. More precisely, for any open compact subgroup I of $G(\mathbb{A}^\infty)$, let denote by $\mathrm{Sh}_{I,v} \to \operatorname{Spec} \mathcal{O}_v$ the Kottwitz-Harris-Taylor Shimura variety with level I, cf. definition 2.4, where $\mathcal{O}_v$ is the ring of integers of the local field F_v of F at v. If one believe that Tate conjecture is true, then in the sense of definition 5.1, $\mathfrak{m}$ should be KHT-typic which implies that $\rho_{\mathfrak{m}}$ appear in the cohomology of $\mathrm{Sh}_{I,v}$ localized at $\mathfrak{m}$, cf. proposition 5.3.

1.3. Definition. — *(cf. the introduction of* [**15**]*)*
We say that $\mathfrak{m}$ *is KHT-free if the cohomology groups of the Kottwitz-Harris-Taylor Shimura variety of notation 2.4, localized at* $\mathfrak{m}$*, are free.*

From [**9**], any of the following properties ensure KHT-freeness of $\mathfrak{m}$.

(1) There exists a place $w_1 \notin S$ of F above a prime p_1 splits in E, such that the multi-set $S_{\mathfrak{m}}(w_1)$ of roots of the characteristic polynomial $P_{\mathfrak{m},w_1}(X)$ of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{w_1})$, does not contain any sub-multi-set of the shape $\{\alpha, q_{w_1}\alpha\}$ where q_{w_1} is the order of the residue field of F at w_1. This hypothesis is called *generic* in [**16**].
(2) When $[F(\exp(2i\pi/l)) : F] > d$, if we suppose the following property to be true, cf. [**9**] 4.17. If $\theta : G_F \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ is an irreducible continuous representation such that for all place $w \notin S$ above a prime $x \in \mathbb{Z}$ split in E, then $P_{\mathfrak{m},w}(\theta(\mathrm{Frob}_w)) = 0$ (resp. $P_{\mathfrak{m}^\vee,w}(\theta(\mathrm{Frob}_w)) = 0$) implies that θ is equivalent to $\overline{\rho}_{\mathfrak{m}}$ (resp. $\overline{\rho}_{\mathfrak{m}^\vee}$), where $\mathfrak{m}^\vee$ is the maximal ideal of $\mathbb{T}_\xi^S$ associated to the dual mutliset of Satake parameters, cf. [**9**] notation 4.4. In [**17**], the authors proved that the previous property is verified in each of the following cases:
   - either $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K where K/F is a cyclic galoisian extension;
   - or $l \geqslant d$ and $SL_d(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^\times GL_d(k)$ for some subfield $k \subset \overline{\mathbb{F}}_l$.
(3) In [**14**], we proved $\mathfrak{m}$ to be KHT-free as soon as $\overline{\rho}_{\mathfrak{m}}$ is irreducible and $[F(\exp(2i\pi/l)) : F] > d$.

By Cebotarev's theorem, the hypothesis $[F(\exp(2i\pi/l)) : F] > d$ allows to pick places v of F such that the order of q_v modulo l is greater than d. In theorem 5.5 we then add the assumptions that $\mathfrak{m}$ is both KHT-free and KHT-typic. If one believe in Tate's conjecture then $\mathfrak{m}$ should always be KHT-typic, cf. proposition 5.3.

The two main ingredients of the proof are:

– first we show, cf. proposition 3.1, that the filtration given by the monodromy operator at v, of the middle cohomology group of the geometric generic fiber of $\mathrm{Sh}_{I,v}$, is strict, i.e. also gives a filtration of the modulo l cohomology group.

– We construct a $\overline{\mathbb{Z}}_l$-structure of the monodromy operator on the nearby cycle perverse sheaf which coincides with the usual monodromy operator over $\overline{\mathbb{Q}}_l$ and such that $\overline{\mathbb{F}}_l$ its order of nilpotency is d.

From this two points, we infer both

– a nilpotent monodromy operator $\overline{N}^{coho}_{\mathfrak{m},v}$ acting on the $\overline{\mathbb{F}}_l$-cohomology group in middle degree of $\mathrm{Sh}_{I,v}$;
– a $\overline{\mathbb{Z}}_l$-monodromy operator $N_{\mathfrak{m},v}$ on $\rho_{\mathfrak{m}}$,

such that the following **main observation** occurs (cf. corollary 4.2): *Suppose that* $\mathfrak{m}$ *is KHT-free and KHT-typic in the sense of definition 5.1, then the (multi)-set of Jordan's blocks of* $N_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ *is the union over* $\{\widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$*, of the (multi)-set of Jordan's block of* $N_{\widetilde{\mathfrak{m}},v}$.

The existence of various liftings of $\overline{\rho}_{\mathfrak{m}}$ with different levels at v produce constraints on the partition $\overline{\underline{d}}_{\mathfrak{m},v}$ given by the Jordan blocks of the modulo l monodromy operator, which have to be smaller than all the $\underline{d}_{\widetilde{\mathfrak{m}},v}$. We can then see our result as a reciprocal statement.
Remark. Concerning the link with the results of [**13**], we refer the reader to §6 where we consider **the case** $\overline{\rho}_{\mathfrak{m}}$ **reducible.**

To see that situations as in the main results really exist, we can follow the strategy of Ribet in [**23**], where he considers an absolutely irreducible representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_l),$$

which is modular of level N, meaning that it arises from a cusp form of weight 2 and trivial character on $\Gamma_0(N)$. Then for a prime $p \nmid lN$ such that

$$\mathrm{Tr}\ \overline{\rho}(\mathrm{Frob}_p) \equiv \pm(p+1) \mod l,$$

with $p+1 \not\equiv 0 \mod l$, he proves that $\overline{\rho}$ also arises from a modular form of level pN which is p-new, i.e. the automorphic representation associated to this modular form has a local component at p which is isomorphic to the Steinberg representation of $GL_2(\mathbb{Q}_p)$.

In [**26**] Sorensen generalizes this level raising congruences in higher dimension for a connected reductive group G over a totally real field F^+ such that G_∞ is compact. One might also look at [**1**] theorem 1.1 and theorem 4.1, for the case of automorphic representations of unitary type of GL_{2n}. More generally it seems that level raising is more or less settle, cf. theorem 5.1.5, or theorem 4.4.1 of [**2**].

We also need to be convinced that such degeneration of the monodromy when passing modulo l, could also appears when $\mathfrak{m}$ is supposed to be KHT-free. For example, as pointed out to me, the two following newforms which can be found at https://www.pnas.org/content/pnas/94/21/11143.full.pdf

- $f(q) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots \in S_2(\Gamma_0(11))$,
- $g(q) = q + q^2 + 2q^3 - q^4 - 2q^5 + 2q^6 - q^7 + \cdots \in S_2(\Gamma_0(77))$,

give Galois representations ρ_f and ρ_g which are congruent modulo 3. In Serre's 1972 Inventiones paper it is shown that the image of this modulo 3 representation is $GL_2(\mathbb{F}_3)$ so that $\overline{\rho}_f$ is irreducible. Note then that the 3-adic representation ρ_g has non trivial monodromy at 7 and $\overline{\rho}_f$ is unramified at 7 with $7 \not\equiv -1 \mod 3$. Consider then a real quadratic field F^+ in which 7 splits and which is linearly disjoint over $\mathbb{Q}$ with the fixed field of $\mathrm{Ker}(\overline{\rho}_g)$. Let D/F^+ a quaternion algebra which is non split t one real place and one place above 11. The base change of g to F^+ gives a cohomological automorphic representation of $GL_2(\mathbb{A}_{F^+})$ whose 3-adic Galois representation appears in the cohomology of a Shimura curve attached to D: the 3-adic monodromy at a place dividing 7 is then non trivial while the modulo 3 monodromy is. Base changing to $F = F^+E$ for some suitable quadractic imaginary field E, we then obtain an example of a maximal ideal $\mathfrak{m}$ which is KHT-free and with degeneration of the monodromy at v.

2. KHT-Shimura varieties and its nearby cycles

Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary and F^+ totally real. Let B/F be a central division algebra with dimension d^2 with an involution of second kind $*$. For $\beta \in B^{*=-1}$, consider the similitude group $G/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra R by

$$G(R) := \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\},$$

with $B^{op} = B \otimes_{F,c} F$ where $c = *_{|F}$ is the complex conjugation and $\sharp_\beta$ is the involution $x \mapsto x^{\sharp\beta} := \beta x^* \beta^{-1}$. Following [**21**], we assume from now on that $G(\mathbb{R})$ has signatures $(1, d-1), (0, d), \cdots, (0, d)$.

2.1. Definition. — *Let* Spl *be the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$.*

We now suppose that $p = uu^c$ splits in E so that

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$$

where w describes the places of F above u and we fix a place $v \in \mathrm{Spl}$ dividing p.

2.2. Definition. — *For a finite set S of places of $\mathbb{Q}$ containing the places where G is ramified, denote by $\mathbb{T}_{abs}^S := \prod_{x \notin S} \mathbb{T}_{x,abs}$ the abstract unramified Hecke algebra where $\mathbb{T}_{x,abs} \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.*

Example. For $x = uu^c$ split in E we have

$$\mathbb{T}_{x,abs} = \prod_{w\ |u} \overline{\mathbb{Z}}_l\big[T_{w,i} :\ i = 1, \cdots, d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w, \cdots, \varpi_w}^{i}, \overbrace{1, \cdots, 1}^{d-i})GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

We then denote by $\mathcal{I}$ the set of open compact subgroups

$$U^p(m_1, \cdots, m_r) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^{r} \mathrm{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times)$$

where U^p is any small enough open compact subgroup of $G(\mathbb{A}^{p,\infty})$ and $\mathcal{O}_{B_{v_i}}$ is the maximal order of B_{v_i} with maximal ideal $\mathcal{P}_{v_i}$ and where $v = v_1, \cdots, v_r$ are the places of F above u with $p = uu^c$.

2.3. Notation. — *For $I = U^p(m_1, \cdots, m_r) \in \mathcal{I}$, we will denote by $I^v(n) := U^p(n, m_2, \cdots, m_r)$. We also denote by* $\mathrm{Spl}(I)$ *the subset of* Spl *of places which does not divide the level I.*

2.4. Notation. — *As defined in* [**21**]*, attached to each $I \in \mathcal{I}$ is a Shimura variety called of KHT-type and denoted by*

$$\mathrm{Sh}_{I,v} \longrightarrow \mathrm{Spec}\,\mathcal{O}_v$$

where $\mathcal{O}_v$ denote the ring of integers of the completion F_v of F at v.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ for the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists then, cf. [**21**] p.97, an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leqslant \cdots \leqslant a_{\sigma,d})$. We then denote by

$$V_{\xi,\overline{\mathbb{Z}}_l}$$

the associated $\overline{\mathbb{Z}}_l$-local system on $\mathrm{Sh}_{I,v}$.

2.5. Notation. — *Let $\mathbb{T}^S_\xi$ be the image of $\mathbb{T}^S_{abs}$ inside*

$$\bigoplus_{i=0}^{2d-2} \lim_{\substack{\longrightarrow \\ I}} H^i_{free}(\mathrm{Sh}_{I,\bar\eta_v}, V_{\xi,\overline{\mathbb{Z}}_l})$$

where the limit concerned the ideals I which are maximal at each places outside S, and $\mathrm{Sh}_{I,\bar\eta_v}$ is the geometric generic fiber of $\mathrm{Sh}_{I,v}$.

Remark. H^i_{free} is the free quotient of the cohomology group H^i. From the main result of [**9**], the torsion classes of any of the $H^i(\mathrm{Sh}_{I,\bar\eta_v}, V_{\xi,\overline{\mathbb{Z}}_l})$ raise in characteristic zero, so one can erase the index *free* in the previous notation.

To each maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}^S_\xi[1/l]$, or equivalently a minimal prime of $\mathbb{T}^S_\xi$, which we now supposed to be non-Eisenstein, is associated an irreducible automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ which is ξ-cohomological, i.e. there exists an integer i such that

$$H^i\big((\mathrm{Lie}\ G(\mathbb{R}))\otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$.

2.6. Notation. — *Let denote by $\mathrm{Scusp}_v(\widetilde{\mathfrak{m}})$, the supercuspidal support of its local component at v, denoted $\Pi_{\widetilde{\mathfrak{m}},v}$. Note*[(1)] *that the modulo l reduction of $\mathrm{Scusp}_v(\widetilde{\mathfrak{m}})$ is independent of the choice of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$: we denote it $\mathrm{Scusp}_v(\mathfrak{m})$.*

Recall that the geometric special fiber $\mathrm{Sh}_{I,\bar s_v}$ of $\mathrm{Sh}_{I,v}$, is equipped with the Newton stratification

$$\mathrm{Sh}^{\geqslant d}_{I,\bar s_v} \subset \mathrm{Sh}^{\geqslant d-1}_{I,\bar s_v} \subset \cdots \subset \mathrm{Sh}^{\geqslant 1}_{I,\bar s_v} = \mathrm{Sh}_{I,\bar s_v},$$

(1) It follows, through the Langlands correspondence, from Cebotarev's theorem and the fact that a semi-simple representation is determined, up to isomorphism, by its characteristic polynomials.

where for $1 \leqslant h \leqslant d$, $\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v}$ (resp. $\mathrm{Sh}^{=h}_{I,\bar{s}_v}$) is the closed (resp. the open) Newton stratum of height h and of pure dimension $d-h$, defined as the sub-scheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to h (resp. equal to h).

Moreover for $1 \leqslant h < d$, the Newton stratum $\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d-h}(F_v)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely, cf. [**3**] §10.4, this means that there exists a closed sub-scheme $\mathrm{Sh}^{=h}_{I,\bar{s}_v,\overline{1_h}}$ stabilized by the Hecke action of $P_{h,d-h}(F_v)$ and such that

$$\varprojlim_n \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v} \simeq \big(\varprojlim_n \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v,\overline{1_h}}\big) \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

2.7. Notation. — *For a representation* π_v *of* $GL_d(F_v)$ *with coefficients either* $\overline{\mathbb{Q}}_l$ *or* $\overline{\mathbb{F}}_l$*, and* $n \in \frac{1}{2}\mathbb{Z}$*, we set* $\pi_v\{n\} := \pi_v \otimes \nu^n$ *where* $\nu(g) := q_v^{-\operatorname{val}\det(g)}$*. Recall that the normalized induction of two representations* $\pi_{v,1}$ *and* $\pi_{v,2}$ *of respectively* $GL_{n_1}(F_v)$ *and* $GL_{n_2}(F_v)$ *is*

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\{\frac{n_2}{2}\} \otimes \pi_{v,2}\{-\frac{n_1}{2}\},$$

and we define inductively

$$\pi_1 \times \cdots \times \pi_s := (\pi_1 \times \cdots \times \pi_{s-1}) \times \pi_s \simeq \pi_1 \times (\pi_2 \times \cdots \times \pi_s).$$

Recall that a representation π_v of $GL_d(F_v)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. a subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true over $\overline{\mathbb{F}}_l$. For example the modulo l reduction of an irreducible $\overline{\mathbb{Q}}_l$-representation is still cuspidal but not necessary supercuspidal, its supercuspidal support being a Zelevinsky segment.

2.8. Definition. — *(see* [**27**] *§9 and* [**5**] *§1.4) Let* g *be a divisor of* $d = sg$ *and* π_v *an irreducible cuspidal* $\overline{\mathbb{Q}}_l$*-representation of* $GL_g(F_v)$*. The induced representation*

$$\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\} \tag{2.8}$$

holds a unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi_v)$ *(resp.* $\mathrm{Speh}_s(\pi_v)$*); it is a generalized Steinberg (resp. Speh) representation.*

Remark. For χ_v a character, $\mathrm{Speh}_s(\chi_v)$ is the character $\chi_v \circ \det$ of $GL_s(F_v)$.

Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ and fix $t \geqslant 1$ such that $tg \leqslant d$. Thanks to Igusa varieties, Harris and Taylor constructed a local system on $\mathrm{Sh}^{=tg}_{\mathcal{I},\bar{s}_v,\overline{1_{tg}}}$

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1_{tg}}}$$

where

- $\pi_v[t]_D$ is the representation of $D^\times_{v,tg}$ which is the image of the contragredient of $\mathrm{St}_t(\pi_v)$ by the Jacquet-Langlands correspondence,
- $D_{v,tg}$ is the central division algebra over F_v with invariant tg,
- with maximal order denoted by $\mathcal{D}_{v,tg}$
- and with $(\pi_v[t]_D)_{|\mathcal{D}^\times_{v,tg}} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with $\rho_{v,i}$ irreductible.

The Hecke action of $P_{tg,d-tg}(F_v)$ is then given through its quotient

$$P_{tg,d-tg}(F_v) \twoheadrightarrow GL_{tg}(F_v) \times GL_{d-tg}(F_v) \twoheadrightarrow GL_{d-tg}(F_v) \times \mathbb{Z},$$

where $GL_{tg}(F_v) \times GL_{d-tg}(F_v)$ is the Levi quotient of the parabolic $P_{tg,d-tg}(F_v)$ and the second map is given by the valuation of the determinant map $GL_{tg}(F_v) \twoheadrightarrow \mathbb{Z}$. These local systems have stable $\overline{\mathbb{Z}}_l$-lattices and we will write simply $\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}}$ for any $\overline{\mathbb{Z}}_l$-stable lattice that we do not want to specify.

2.9. Notations. — *For Π_t any $\overline{\mathbb{Q}}_l$-representation of $GL_{tg}(F_v)$, and $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce*

$$\widetilde{HT}_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t) := \mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t) := \Big(\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}\Big) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

where W_v is the Weil group at v, is given

- *by the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, where $\deg : W_v \longrightarrow \mathbb{Z}$ sends geometric Frobenius to 1,*

– *and the action of* $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ *on* $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t) := \widetilde{HT}_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t)[d-tg],$$

and the perverse sheaf

$$P_{\overline{\mathbb{Q}}_l}(t,\pi_v)_{\overline{1_{tg}}} := j^{=tg}_{\overline{1_{tg}},!*} HT_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \mathrm{St}_t(\pi_v)) \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t)$ *and* $P_{\overline{\mathbb{Q}}_l}(t,\pi_v)$, *where*

$$j^{=tg} = i^{tg} \circ j^{\geqslant tg} : \mathrm{Sh}^{=tg}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant tg}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}_{\mathcal{I},\bar{s}_v}$$

and $\mathbb{L}$ *is the local Langlands correspondence composed by contragredient.*

We will also denote by $HT_{\overline{\mathbb{Q}}_l,\xi}(\pi_v, \Pi_t) := HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t) \otimes V_\xi$ *and similarly for the other notations as for example* $P_{\overline{\mathbb{Q}}_l,\xi}(t,\pi_v) := P_{\overline{\mathbb{Q}}_l}(t,\pi_v) \otimes V_\xi$.

Finally we denote by $\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v,\neq 1} := \mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v} \setminus \mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v,\bar{1}_h}$ *and*

$$j^{=h}_{\neq 1} : \mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v,\neq 1} \hookrightarrow \mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}_{\mathcal{I},\bar{s}_v}.$$

Remarks:

- We will simply denote by $P(t,\pi_v)$ any $\overline{\mathbb{Z}}_l$-lattice of $P_{\overline{\mathbb{Q}}_l}(t,\pi_v)$ that we do not want to precise except that it is stable under the various actions. We will use a similar convention for the other sheaves introduced before. When considering $\overline{\mathbb{F}}_l$-coefficients, we will put $\overline{\mathbb{F}}_l$ in place of $\overline{\mathbb{Q}}_l$ in the notations.
- Recall that π'_v is said inertially equivalent to π_v, and we write $\pi_v \sim_i \pi'_v$, if there exists a character $\zeta : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \mathrm{val} \circ \det)$. We denote by e_{π_v} the order of the inertial class of π_v.
- Note, cf. [**4**] 2.1.4, that $P_{\overline{\mathbb{Q}}_l}(t,\pi_v)$ depends only on the inertial class of π_v and
$$P_{\overline{\mathbb{Q}}_l}(t,\pi_v) = e_{\pi_v} \mathcal{P}_{\overline{\mathbb{Q}}_l}(t,\pi_v)$$
where $\mathcal{P}_{\overline{\mathbb{Q}}_l}(t,\pi_v)$ is an irreducible perverse sheaf.
- Over $\overline{\mathbb{Z}}_l$, we also have the $p+$-perverse structure which is dual to the usual p-structure.

2.10. Notation. — *Let denote by*

$$e_v(l) = \begin{cases} l, & \textit{if } q_v \equiv 1 \mod l \\ \textit{the order of } q_v \textit{ modulo } l, & \textit{otherwise} \end{cases}$$

Remark. For a character[(2)] χ_v and when $t < e_v(l)$, up to homothety there is only one stable $\overline{\mathbb{Z}}_l$-stable lattice of $\mathcal{L}(\pi_v[t]_D)$. From the description of the modulo l reduction of $\mathrm{St}_t(\chi_v)$ in [**6**], the same is then true for $\mathcal{P}(t, \chi_v)$.

2.11. Notation. — *Let denote by*

$$\Psi_v := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])(\frac{d-1}{2})$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ *of* $\mathrm{Sh}_{I,v}$.

In cite [**10**] proposition 3.1.3, we proved the following splitting

$$\Psi_v \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_\varrho,$$

where $\mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ is the set of inertial equivalence classes of irreducible $\overline{\mathbb{F}}_l$-supercuspidal of $GL_g(F_v)$ and where

$$\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} \Psi_{\pi_v},$$

where $\mathrm{Cusp}(\varrho)$ is the set of inertial equivalence classes of irreducible $\overline{\mathbb{Q}}_l$-cuspidal representations which modulo l reduction is inertially equivalent to ϱ.

We now fix a $\overline{\mathbb{F}}_l$*-character* ϱ. Following the constructions of [**7**] §2.3,

- using the adjonction $j_!j^* \to \mathrm{Id}$, we can first define a filtration

$$\mathrm{Fil}^1_!(\Psi_\varrho) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^d_!(\Psi_\varrho) = \Psi_\varrho,$$

  where $\mathrm{Fil}^h_!(\Psi_\varrho)$ is the saturated image of $j^{=h}_! j^{=h,*}\Psi_\varrho \longrightarrow \Psi_\varrho$. The graded parts $\mathrm{gr}^k_!(\Psi_\varrho)$ are free perverse sheaves. Over $\overline{\mathbb{Q}}_l$, this filtration coincides with the iterated kernel of N_v, i.e. $\mathrm{Fil}^k_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq \mathrm{Ker}(N_v^k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. We also write $\mathrm{coFil}^k_!(\Psi_\varrho) := \Psi_\varrho / \mathrm{Fil}^k_!(\Psi_\varrho)$.
- For every $1 \leqslant h < d$, we have a short exact sequence of free Hecke-perverse sheaves

$$0 \to j^{=h}_{\neq 1,!} j^{=h,*}_{\neq 1} \mathrm{gr}^h_!(\Psi_\varrho) \longrightarrow \mathrm{gr}^h_!(\Psi_\varrho) \longrightarrow {}^p j^{=h}_{\overline{1_h},!*} j^{=h,*}_{\overline{1_h}} \mathrm{gr}^h_!(\Psi_\varrho) \to 0, \quad (2.11)$$

(2)For a general supercuspidal representation π_v whose modulo l reduction ϱ is still supercuspidal, the same is true if $t < m(\varrho)$ where $m(\varrho)$ is either the order of the Zelevinsky line of ϱ if it is > 1, otherwise $m(\varrho) := l$.

where

$$j^{=h,*}\mathrm{gr}_!^h(\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l \simeq \bigoplus_{\chi_v\in\mathrm{Cusp}(\varrho)} HT_{\overline{\mathbb{Q}}_l}(\chi_v,\mathrm{St}_h(\chi_v))(\frac{1-h}{2}).$$

– Dually using the adjunction $\mathrm{Id}\to j_*j^*$, we can define a cofiltration

$$\mathrm{coFil}_*^d(\Psi_\varrho)\twoheadrightarrow\cdots\twoheadrightarrow\mathrm{coFil}_*^1(\Psi_\varrho),$$

where $\mathrm{coFil}_*^h(\Psi_\varrho)$ is the saturated coimage of $\Psi_\varrho\longrightarrow j_*^{=h}j^{=h,*}\Psi_\varrho$. The graded parts $\mathrm{gr}_*^h(\Psi_\varrho)$ are also given by

$$0\to {}^p j_{1,!*}^{=h} j_1^{=h,*}\mathrm{gr}_*^h(\Psi_\varrho)\longrightarrow \mathrm{gr}_*^h(\Psi_\varrho)\longrightarrow {}^p j_{\neq 1,!*}^{=h} j_{\neq 1}^{=h,*}\mathrm{gr}_*^h(\Psi_\varrho)\to 0,$$

where

$$j_{\neq 1}^{=h,*}\mathrm{gr}_*^h(\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l \simeq \bigoplus_{\chi_v\in\mathrm{Cusp}(\varrho)} HT_{\overline{\mathbb{Q}}_l}(\chi_v,\mathrm{St}_h(\chi_v))(\frac{h-1}{2}).$$

– We can also refine the previous filtration to obtain $\mathrm{Fill}^\bullet(\Psi_\varrho)$ whose graded parts $\mathrm{grr}^r(\Psi_v)$ are free $\overline{\mathbb{Z}}_l$-perverse sheaves, cf. [**7**] §1, such that

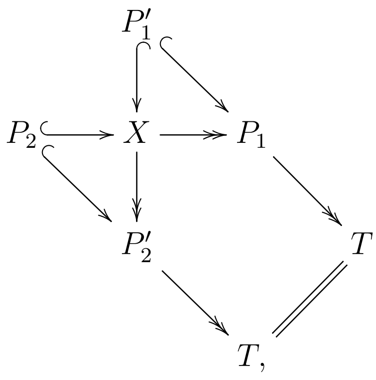
$$\mathrm{grr}^r(\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l \simeq \bigoplus_{\chi_v\in\mathrm{Cusp}(\varrho)} {}^p j_{!*}^{=tg} HT(\chi_v,\mathrm{St}_t(\chi_v))(\frac{1-t+2\delta}{2})$$

for some $0\leqslant\delta\leqslant t-1$.

Remark. As the order of q_v modulo l is $>d$ then $\mathrm{Cusp}(\varrho)$ contains only characters so that, cf. (3.2), the Harris-Taylor local systems have only one intermediate extension, i.e.

$${}^p j_{\overline{1_h},!*}^{=h} j_{\overline{1_h}}^{=h,*}\mathrm{gr}_!^h(\Psi_\varrho)\simeq {}^{p+} j_{\overline{1_h},!*}^{=h} j_{\overline{1_h}}^{=h,*}\mathrm{gr}_!^h(\Psi_\varrho).$$

Exchange basic step: to go from filtration to another, one can repeat the following process to exchange the order of appearance of two consecutive subquotient:



where

- P_1 and P_2 are two consecutive subquotient in a given filtration and X is the subquotient gathering them as a subquotient of this filtration.
- Over $\overline{\mathbb{Q}}_l$, the extension $X \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is split, so that on can write X as an extension of P'_2 by P'_1 with $P'_1 \hookrightarrow P_1$ and $P_2 \hookrightarrow P'_2$ have the same cokernel T, a perverse sheaf of torsion.

Remark. In the particular case when P_1 and P_2 are intermediate extensions of local systems living on different strata such that the two associated intermediate extensions for the p and $p+$ t-structure are isomorphic, then T is necessary zero and X is then split over $\overline{\mathbb{Z}}_l$.

Repeating exchange basic steps, on can then pass from $\mathrm{Fil}^\bullet_*(\Psi_\varrho)$ to $\mathrm{Fil}^\bullet_!(\Psi_\varrho)$.

2.12. Lemma. — *The socle (resp. the cosocle) of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is, up to multiplicities, $j^{=d}_{!*} HT(\varrho, \mathrm{St}_d(\varrho))(\frac{d-1}{2})$ (resp. $j^{=d}_{!*} HT(\varrho, \mathrm{St}_d(\varrho))(\frac{1-d}{2})$).*

Proof. — The result follows quite immediately from [**10**] where we described the sheaves of cohomology of the $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ as the modulo l reduction of its $\overline{\mathbb{Z}}_l$-cohomology sheaves. From this computation we deduce that the socle of $\mathrm{gr}^h_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, up to multiplicities, is $j^{=d}_{!*} HT_{\overline{\mathbb{F}}_l}(\varrho, \mathrm{St}_d(\varrho))(\frac{d+1-2h}{2})$. From the filtration $\mathrm{Fil}^\bullet_!(\Psi_\varrho)$, we then deduce that the socle of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$

- contains $j^{=d}_{!*} HT(\varrho, \mathrm{St}_d(\varrho))(\frac{d-1}{2})$,
- and if it contains something else, without considering multiplicities, it has to be $j^{=d}_{!*} HT(\varrho, \mathrm{St}_d(\varrho))(\frac{d-1-2\delta}{2})$ for $1 \leqslant \delta < d$.

We argue by contradiction by assuming that $j_{!*}^{=d}HT(\varrho,\mathrm{St}_d(\varrho))(\frac{d-1-2\delta}{2})$ for some $1\leqslant\delta<d$, belongs to the socle. By duality then $j_{!*}^{=d}HT(\varrho,\mathrm{St}_d(\varrho))(\frac{1-d+2\delta}{2})$ belongs to the cosocle so that $HT(\varrho,\mathrm{St}_d(\varrho))(\frac{d-1-2\delta}{2})$ is a subquotient of $h^0\Psi_\varrho\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$ which contradicts the main result of [**10**].

For the cosocle, we conclude by duality. □

Remark. The same arguments, with more precautions, allow to show that non split extensions inside $\Psi_\varrho\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ between Harris-Taylor perverse sheaves remains non split in $\Psi_\varrho\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$.

When dealing with sheaves, there is no need to introduce the local system $V_{\xi,\overline{\mathbb{Z}}_l}$ because it suffices to add $\otimes_{\overline{\mathbb{Z}}_l}V_{\xi,\overline{\mathbb{Z}}_l}$ to the formulas. We now consider a fixed local system $V_{\xi,\overline{\mathbb{Z}}_l}$ and, following previous notations, we write $\Psi_{\varrho,\xi}:=\Psi_\varrho\otimes_{\overline{\mathbb{Z}}_l}V_{\xi,\overline{\mathbb{Z}}_l}$. We then have a spectral sequence

$$E_1^{p,q}=H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v},\mathrm{grr}^{-p}(\Psi_{v,\xi}))\Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Z}}_l}). \tag{2.13}$$

As pointed out in [**13**], if for some $\mathfrak{m}$ the spectral sequence is concentrated in middle degree, i.e. $E_{1,\mathfrak{m}}^{p,q}=0$ for $p+q\neq d-1$, and all the $E_{1,\mathfrak{m}}^{p,d-1-p}$ are free, then, for $l>d$, the action of the monodromy operator $N_{v,\mathfrak{m}}^{coho}$ on $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v},V_{\xi,\overline{\mathbb{Q}}_l})_\mathfrak{m}$ comes from the action of N_v on $\Psi_v\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$. The aim of the next section is to prove that this property remains true over $\overline{\mathbb{F}}_l$.

3. A saturated filtration of the cohomology

The aim of this section is the following proposition.

3.1. Proposition. — *Consider a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\xi^S$ such that:*
- *$\overline{\rho}_\mathfrak{m}$ is irreducible;*
- *$\mathfrak{m}$ is KHT-free;*
- *the restriction $\rho_{\mathfrak{m},v}$ to the decomposition group at v is, in the Grothendieck group and up to the action of N_v, the direct sum of character. We moreover suppose that the set $S_v(\mathfrak{m})$ of modulo l eigenvalues of $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_v)$ does not contain any subset of the form $\{\lambda,q_v\lambda,\cdots,q_v^{e_v(l)-1}\lambda\}$, where $e_v(l)$ is defined in 2.10.*

Then the $E_{1,\mathfrak{m}}^{p,q}$ are torsion free and trivial for $p+q\neq d-1$.

Note that, as (2.13) after localization at $\mathfrak{m}$, degenerates at E_1 over $\mathbb{Q}_l$, then the spectral sequence gives us a saturated filtration of

$H^{d-1}(\mathrm{Sh}_{I,\bar\eta_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$. The proof uses Grothendieck-Verdier duality which explains we need that the p and $p+$ intermediate extensions of Harris-Taylor local systems $HT(\pi_v, \mathrm{St}_t(\pi_v))$ to be isomorphic. In [**10**], for a irreducible cuspidal representation π_v of $GL_g(F_v)$ and $1 \leqslant t \leqslant d/g$ such that the modulo l reduction of $\mathrm{St}_t(\pi_v)$ remains irreducible, we proved that

$$ {}^p j^{=tg}_{!*} HT(\pi_v, \Pi_h) \simeq {}^{p+} j^{=tg}_{!*} HT(\pi_v, \Pi_h). \tag{3.2}$$

The proof is rather difficult but almost obvious in the case where π_v is a character in which case the condition is that $t < e_v(l)$ which is clearly true with the hypothesis $e_v(l) > d$.

Proof. — As $\mathfrak{m}$ is supposed to be KHT-free, then all the $E^n_{\infty,\mathfrak{m}}$ are free. Moreover, as $\overline{\rho}_{\mathfrak{m}}$ is irreducible, then, cf. [**5**] §3.6, the $E^{p,q}_{1,\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are all zero if $p+q \neq d-1$. As by hypothesis $\rho_{\mathfrak{m},v}$ is made of characters, we can consider direct factors Ψ_ϱ for $\varrho \in \mathrm{Scusp}_v(\mathfrak{m})$ a character. Moreover as $e_v(l) > d$, in $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ we have only to deal with characters χ_v so that, by (3.2), the p and $p+$ intermediate extensions coincide.

3.3. Proposition. — *(cf.* [**10**] *§2.3) We have the following equivariant resolution*

$$0 \to j^{=d}_! HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{h-d}{2}\}) \times \mathrm{Speh}_{d-h}(\chi_v\{h/2\})) \otimes \Xi^{\frac{d-h}{2}} \longrightarrow \cdots$$
$$\longrightarrow j^{=h+1}_! HT(\chi_v, \mathrm{St}_h(\chi_v(-1/2)) \times \chi_v\{h/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow$$
$$j^{=h}_! HT(\chi_v, \mathrm{St}_h(\chi_v)) \longrightarrow {}^p j^{=h}_{!*} HT(\chi_v, \mathrm{St}_h(\chi_v)) \to 0. \tag{3.4}$$

Note that

- as this resolution is equivalent to the computation of the sheaves cohomology groups of ${}^p j^{=h}_{!*} HT(\chi_v, \mathrm{St}_h(\chi_v))$ as explained for example in [**10**] proposition B.1.5 of appendice B, then, over $\overline{\mathbb{Q}}_l$, it follows from the main results of [**4**].
- Over $\overline{\mathbb{Z}}_l$, as every terms are free perverse sheaves, then all the maps are necessary strict.
- This resolution, for a a general supercuspidal representation with supercuspidal modulo l reduction, is one of the main result of [**10**] §2.3. However the case of a character χ_v as above, is almost obvious. Indeed as the strata $\mathrm{Sh}^{\geqslant h}_{I^v,\bar s_v,1}$ are smooth, then the constant sheaf, up to shift, is perverse and so equals to the intermediate extension of the constant sheaf, shifted by $d-h$, on $\mathrm{Sh}^{=h}_{I^v,\bar s_v,1}$. In particular its

sheaves cohomology groups are well known so that the resolution is completely obvious for ${}^p j^{=h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \mathrm{St}_h(\chi_v))$ if one remember that $\mathrm{Speh}_i(\chi_v)$ is just the character $\chi_v \circ \det$ of $GL_i(F_v)$.

The stated resolution is then simply the induced version of the resolution of ${}^p j^{=h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \mathrm{St}_h(\chi_v))$: recall that a direct sum of intermediate extensions is still an intermediate extension.

By adjunction property, the map

$$j_!^{=h+\delta} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2}$$
$$\longrightarrow j_!^{=h+\delta-1} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \quad (3.5)$$

is given by

$$HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \longrightarrow$$
$$j^{=h+\delta,*}({}^p i^{h+\delta,!}(j_!^{=h+\delta-1} HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}})) \quad (3.6)$$

To compute this last term we use the resolution (3.4). Precisely denote by $\mathcal{H} := HT(\chi_v, \mathrm{St}_h(\chi\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$, and write the previous resolution as follows

$$0 \to K \longrightarrow j_!^{=h+\delta} \mathcal{H}' \longrightarrow Q \to 0,$$
$$0 \to Q \longrightarrow j_!^{=h+\delta-1} \mathcal{H} \longrightarrow {}^p j_{!*}^{=h+\delta-1} \mathcal{H} \to 0,$$

with

$$\mathcal{H}' := HT\Big(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2}.$$

As the support of K is contained in $\mathrm{Sh}^{\geqslant h+\delta+1}_{I,\bar{s}_v}$ then ${}^p i^{h+\delta,!} K = K$ and $j^{=h+\delta,*}({}^p i^{h+\delta,!} K)$ is zero. Moreover ${}^p i^{h+\delta,!}({}^p j_{!*}^{=h+\delta-1} \mathcal{H})$ is zero by construction of the intermediate extension. We then deduce that

$$j^{=h+\delta,*}({}^p i^{h+\delta,!}(j_!^{=h+\delta-1} HT(\chi_v, \mathrm{St}_t(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}))$$
$$\simeq HT\Big(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\})$$
$$\times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2} \quad (3.7)$$

3.8 — Fact. In particular, up to homothety, the map (3.7), and so those of (3.6), is unique. Finally as the maps of (3.4) are strict, the given maps (3.5) are uniquely determined, that is, if we forget the infinitesimal parts, these maps are independent of the chosen t in (3.4).

For every $1 \leqslant h \leqslant d$, let denote by $i(h)$ the smallest index i such that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v)))_{\mathfrak{m}}$ has non trivial torsion: if it does not exist then we set $i(h) = +\infty$. By duality, as ${}^p j_{!*} = {}^{p+} j_{!*}$ for Harris-Taylor local systems associated to characters, note that when $i(h)$ is finite then $i(h) \leqslant 0$. Suppose by absurdity there exists h with $i(h)$ finite and denote h_0 the biggest such h.

3.9. Lemma. — *For $1 \leqslant h \leqslant h_0$ then $i(h) = h - h_0$.*

Remark. A similar result is proved in [**9**] when the level is maximal at v.

Proof. — a) We first prove that for every $h_0 < h \leqslant d$, the cohomology groups of $j_!^{=h} HT(\chi_v, \Pi_h)$ are torsion free. Consider the following strict filtration in the category of free perverse sheaves

$$(0) = \mathrm{Fil}^{-1-d}(\chi_v, h) \hookrightarrow\!\!\!\!\!+ \mathrm{Fil}^{-d}(\chi_v, h) \hookrightarrow\!\!\!\!\!+ \cdots \hookrightarrow\!\!\!\!\!+ \mathrm{Fil}^{-h}(\chi_v, h) = j_!^{=h} HT(\chi_v, \Pi_h) \quad (3.9)$$

where the symbol $\hookrightarrow\!\!\!\!\!+$ means a strict monomorphism, with graded parts

$$\mathrm{gr}^{-k}(\chi_v, h) \simeq {}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}).$$

Over $\overline{\mathbb{Q}}_l$, the result is proved in [**4**] §4.3. From [**7**] such a filtration can be constructed over $\overline{\mathbb{Z}}_l$ up to the fact that the graduate parts are only known to verify

$$\begin{aligned} {}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}) \hookrightarrow\!\!\!\!\!\rightarrow \mathrm{gr}^{-k}(\chi_v, h) \\ \hookrightarrow\!\!\!\!\!\rightarrow {}^{p+} j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}), \end{aligned}$$

and we can conclude thanks to (3.2). The associated spectral sequence localized at $\mathfrak{m}$, is then concentrated in middle degree and torsion free which gives the claim.

b) Before watching the cases $h \leqslant h_0$, note that the spectral sequence associated to (3.4) for $h = h_0 + 1$, has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the maps

$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+\delta} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+\delta-1} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}} \quad (3.10)$$

are saturated, i.e. their cokernel are free $\overline{\mathbb{Z}}_l$-modules. Then from the previous fact stressed after (3.7), this property remains true when we consider the associated spectral sequence for $1 \leqslant h' \leqslant h_0$.

c) Consider now $h = h_0$ and the spectral sequence associated to (3.4) where

$$E_2^{p,q} = H^{p+2q}(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+q} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v(-q/2)) \times \mathrm{Speh}_q(\chi_v\{h/2\})) \otimes \Xi^{\frac{q}{2}})_{\mathfrak{m}} \quad (3.11)$$

By definition of h_0, we know that some of the $E_\infty^{p,-p}$ should have a non trivial torsion subspace. We saw that

- the contributions from the deeper strata are torsion free and
- $H^i(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \Pi_{h_0}))_{\mathfrak{m}}$ are zero for $i < 0$ and is torsion free for $i = 0$, whatever is Π_{h_0}.
- Then there should exist a non strict map $d_1^{p,q}$. But, we have just seen that it can not be maps between deeper strata.
- Finally, using the previous points, the only possibility is that the cokernel of

$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0+1} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v\{\frac{-1}{2}\}) \times \chi_v\{h_0/2\})) \otimes \Xi^{1/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v))\big)_{\mathfrak{m}} \quad (3.12)$$

  has a non trivial torsion subspace.

In particular we have $i(h_0) = 0$.

d) Finally using the fact 2.18 and the previous points, for any $1 \leqslant h \leqslant h_0$, in the spectral sequence (3.11)

- by point a), $E_2^{p,q}$ is torsion free for $q \geqslant h_0 - h + 1$ and so it is zero if $p + 2q \neq 0$;

- by affiness of the open strata, cf. [**9**] theorem 1.8, $E_2^{p,q}$ is zero for $p+2q<0$ and torsion free for $p+2q=0$;
- by point b), the maps $d_2^{p,q}$ are saturated for $q \geqslant h_0-h+2$;
- by point c), $d_2^{-2(h_0-h+1),h_0-h+1}$ has a cokernel with a non trivial torsion subspace.
- Moreover, over $\overline{\mathbb{Q}}_l$, the spectral sequence degenerates at E_3 and $E_3^{p,q}=0$ if $(p,q)\neq(0,0)$.

We then deduce that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v,\Pi_h))_{\mathfrak{m}}$ is zero for $i<h-h_0$ and for $i=h-h_0$ it has a non trivial torsion subspace. □

Consider now the filtration of stratification of Ψ_ϱ constructed using the adjunction morphisms $j_!^{=h} j^{=h,*}$ as in [**7**]

$$\mathrm{Fil}_!^1(\Psi_\varrho) \lhook\joinrel\relbar\joinrel\rightarrow \mathrm{Fil}_!^2(\Psi_\varrho) \lhook\joinrel\relbar\joinrel\rightarrow \cdots \lhook\joinrel\relbar\joinrel\rightarrow \mathrm{Fil}_!^d(\Psi_\varrho) \tag{3.13}$$

where $\mathrm{Fil}_!^h(\Psi_\varrho)$ is the saturated image of $j_!^{=h} j^{=h,*}\Psi_\varrho \longrightarrow \Psi_\varrho$.
Remark. Recall that the filtration $\mathrm{Fill}^\bullet$ is a refinement of $\mathrm{Fil}_!^\bullet$ as one can see it in the next proposition.

For our fixed χ_v, let denote $\mathrm{Fil}^1_{!,\chi_v}(\Psi) \lhook\joinrel\relbar\joinrel\rightarrow \mathrm{Fil}_!^1(\Psi_\varrho)$ such that $\mathrm{Fil}^1_{!,\chi_v}(\Psi)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l \simeq \mathrm{Fil}_!^1(\Psi_{\chi_v})$ where Ψ_{χ_v} is the direct factor of $\Psi\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ associated to χ_v, cf. [**7**].

3.14. Proposition. — *(cf.* [**10**] *3.3.5) We have the following resolution of* $\mathrm{gr}^h_{!,\chi_v}(\Psi)$

$$\begin{aligned}0\to j_!^{=d} HT(\chi_v, LT_{h,d}(\chi_v))\otimes L_g(\chi_v(\frac{d-h}{2}))\longrightarrow \\ j_!^{=d-1} HT(\chi_v, LT_{h,d-1}(\chi_v))\otimes L_g(\chi_v(\frac{d-h-1}{2}))\longrightarrow \\ \cdots\longrightarrow j_!^{=h} HT(\chi_v,\mathrm{St}_h(\chi_v))\otimes\mathbb{L}(\chi_v)\longrightarrow \mathrm{gr}^h_{!,\chi_v}(\Psi)\to 0,\end{aligned} \tag{3.15}$$

where

- $LT_{h,h+\delta}(\chi_v)\hookrightarrow \mathrm{St}_h(\chi_v\{-\delta/2\})\times\mathrm{Speh}_\delta(\chi_v\{h/2\})$, *is the only irreducible sub-space of this induced representation,*
- *and* $\mathbb{L}$ *is the local Langlands correspondence composed by contragredient.*

Remarks:

- As explained after proposition 3.3, it amounts to describe the germs of the $\overline{\mathbb{Z}}_l$-sheaf cohomology of $\mathrm{gr}^h_{!,\chi_v}(\Psi_{v,\xi})$. Over $\overline{\mathbb{Q}}_l$, the resolution (3.15) is then proved in [**4**].

– Over $\overline{\mathbb{Z}}_l$, it is proved in full generality in [**10**] for every irreducible supercuspidal representation π_v in place of χ_v. It amounts to prove that the germs of the sheaf cohomology of $\mathrm{gr}^1_{!,\chi_v}(\Psi_{v,\xi})$ are free. The case of a character is however much more simple. Indeed consider then the torsion part of the cokernel of one of these maps. Note that, thanks to (3.2), such a cokernel must have non trivial invariants under the action the Iwahori sub-group at v. We then work at Iwahori level at v. As said above, it amounts to understand the germs of the $\overline{\mathbb{Z}}_l$-sheaf cohomology of $\mathrm{gr}^h_{!,\chi_v}(\Psi)$ which are described, cf. [**18**], by the cohomology of the Lubin-Tate tower. By the comparison theorem of Faltings-Fargues, cf. [**19**], one is reduced to compute the cohomology of the Drinfeld tower in Iwahori level which is already done in [**24**]: we then note that there are all free $\overline{\mathbb{Z}}_l$-modules.

We can then apply the previous arguments a)-d) above, for $h \leqslant h_0$ (resp. $h > h_0$) the torsion of $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{gr}^h_{!,\chi_v}(\Psi_{v,\xi}))_{\mathfrak{m}}$ is trivial for any $i \leqslant h - h_0$ (resp. for all i) and the free parts are concentrated for $i = 0$. Using then the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ would have non trivial torsion which is false as $\mathfrak{m}$ is supposed to be KHT-free. □

4. Local behavior of monodromy over $\overline{\mathbb{F}}_l$

Recall that ϱ is a fixed $\overline{\mathbb{F}}_l$-character and Ψ_ϱ is the associated direct factor of Ψ_v. Over $\overline{\mathbb{Q}}_l$, the monodromy operator define a nilpotent morphism $N_{\varrho,\overline{\mathbb{Q}}_l} : \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \longrightarrow \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ compatible with the filtration $\mathrm{Fil}^\bullet_!(\Psi_\varrho)$ in the sense that $\mathrm{Fil}^h_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ coincides with the kernel of $N^h_{\varrho,\overline{\mathbb{Q}}_l}$. The aim of this section is to construct a $\overline{\mathbb{Z}}_l$-version N_ϱ of $N_{\varrho,\overline{\mathbb{Q}}_l}$ such that $\mathrm{Fil}^h_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ coincides with the kernel of $N^h_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

First step: consider

$$0 \to \mathrm{Fil}^1_!(\Psi_\varrho) \longrightarrow \Psi_\varrho \longrightarrow \mathrm{coFil}^1_!(\Psi_\varrho) \to 0,$$

and the following long exact sequence

$$0 \to \mathrm{Hom}(\mathrm{coFil}^1_!(\Psi_\varrho), \Psi_\varrho) \longrightarrow \mathrm{Hom}(\Psi_\varrho, \Psi_\varrho) \longrightarrow \mathrm{Hom}(\mathrm{Fil}^1_!(\Psi_\varrho), \Psi_\varrho) \longrightarrow \cdots$$

where Hom is taken in the category of equivariant Hecke perverse sheaves.(3) Note that $N_{\varrho,\overline{\mathbb{Q}}_l} \in \mathrm{Hom}(\Psi_\varrho,\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ comes from $\mathrm{Hom}(\mathrm{coFil}^1_!(\Psi_\varrho),\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$, so that we focus on $\mathrm{Hom}(\mathrm{coFil}^1_!(\Psi_\varrho),\Psi_\varrho)$. From

$$0\to \mathrm{gr}^2_!(\Psi_\varrho)\longrightarrow \mathrm{coFil}^1_!(\Psi_\varrho)\longrightarrow \mathrm{coFil}^2_!(\Psi_\varrho)\to 0,$$

we obtain

$$\begin{aligned}0\to \mathrm{Hom}(\mathrm{coFil}^2_!(\Psi_\varrho),\Psi_\varrho)\longrightarrow \mathrm{Hom}(\mathrm{coFil}^1_!,\Psi_\varrho)\longrightarrow&\\ \mathrm{Hom}(\mathrm{gr}^2_!(\Psi_\varrho),\Psi_\varrho)\longrightarrow \mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho),\Psi_\varrho))\longrightarrow\cdots\end{aligned}$$

Note then that $N_{\varrho,\overline{\mathbb{Q}}_l}\in \mathrm{Hom}(\mathrm{coFil}^1_!,\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ does not belong to the image of $\mathrm{Hom}(\mathrm{coFil}^2_!(\Psi_\varrho),\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$.

4.1. Lemma. — *The $\overline{\mathbb{Z}}_l$-module $\mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho),\Psi_\varrho))$ is torsion free.*

Proof. — Let $\alpha\in\mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho),\Psi_\varrho))$ which is killed by some power of l and let

$$0\to\Psi_\varrho\longrightarrow P\longrightarrow \mathrm{coFil}^2_!(\Psi_\varrho)\to 0,$$

be the extension defined by α. Applying $\otimes^{\mathbb{L}}_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$, this short exact sequence split so that P can be written

$$0\to \widetilde{\mathrm{coFil}}^2_!(\Psi_\varrho)\longrightarrow P\longrightarrow \widetilde{\Psi}_\varrho\to 0,$$

so that composing through P we obtain

$$0\to \widetilde{\mathrm{coFil}}^2_!(\Psi_\varrho)\longrightarrow \mathrm{coFil}^2_!(\Psi_\varrho)\longrightarrow T\to 0,$$

$$0\to\Psi_\varrho\longrightarrow\widetilde{\Psi}_\varrho\longrightarrow T\to 0,$$

for the same torsion perverse sheaf T appearing as cokernel of the two previous maps. By tensoring with $\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l$, we then obtain

$$\begin{aligned}0\to {}^p h^{-1}(T\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l)\longrightarrow \widetilde{\mathrm{coFil}}^2_!(\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l&\\ \longrightarrow \mathrm{coFil}^2_!(\Psi_\varrho)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l\longrightarrow {}^p h^0(T\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l)\to 0,\end{aligned}$$

and

$$\begin{aligned}0\to {}^p h^{-1}(T\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l)\longrightarrow \Psi_\varrho\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l&\\ \longrightarrow \widetilde{\Psi}_\varrho\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l\longrightarrow {}^p h^0(T\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{F}}_l)\to 0,\end{aligned}$$

(3) We do not ask the map to be Galois equivariant as $N_{\varrho,\overline{\mathbb{Q}}_l}$ is not.

where we denote by ${}^p h^\bullet K$ the p-perverse cohomology complexes of K. As by lemma 2.12, the socle of $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is, up to multiplicity, $j^{=d}_{!*} HT_{\overline{\mathbb{F}}_l}(\varrho, \mathrm{St}_d(\varrho))(\frac{d-1}{2})$ then it has to be a constituant of ${}^p h^{-1}(T \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. But, using that the order of q_v modulo l is strictly greater than d, $j^{=d}_{!*} HT_{\overline{\mathbb{F}}_l}(\varrho, \mathrm{St}_d(\varrho))(\frac{d-1}{2})$ is not a constituant of $\mathrm{coFil}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. We then deduce that T is forced to be zero, which means that the extension P is split, i.e. $\alpha \in \mathrm{Ext}^1(\mathrm{coFil}^2_!(\Psi_\varrho), \Psi_\varrho)$ is zero. □

We are then led to study

$$\mathrm{Hom}(\mathrm{gr}^2_!(\Psi_\varrho), \Psi_\varrho) \simeq \mathrm{Hom}(\mathrm{gr}^2_!(\Psi_\varrho), \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)))$$

where

$$0 \to \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \longrightarrow \mathrm{Fil}^1_!(\Psi_\varrho) \longrightarrow \mathrm{coFil}^1_*(\mathrm{Fil}^1_!(\Psi_\varrho) \to 0.$$

Note that, up to an unramified Galois twist, $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ and the cosocle of $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ (resp. $\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho)) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$) is some multiple, depending on the lattice of $j^{=2,*}\mathrm{gr}^2_!(\Psi_\varrho)$ (resp. $j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho))$, of $j^{=2}_{!*} HT(\varrho, \mathrm{St}_2(\varrho))$. Consider then

$$0 \to P_- \longrightarrow \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho)) \longrightarrow {}^p j^{=2}_{!*} j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho)) \to 0,$$

where P_- has support in $\mathrm{Sh}^{\geqslant 3}_v$. Then using as before long exact sequence, we note that

$$\mathrm{Hom}(\mathrm{gr}^2_!(\Psi_\varrho), \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho))) \simeq \mathrm{Hom}({}^p j^{=2}_{!*} j^{=2,*}\mathrm{gr}^2_!(\Psi_\varrho), {}^p j^{=2}_{!*} j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho))).$$

In particular if the two local systems ${}^p j^{=2}_{!*} j^{=2,*}\mathrm{gr}^2_!(\Psi_\varrho)$ and ${}^p j^{=2}_{!*} j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho))$ were isomorphic, then there exist a element in $\mathrm{Hom}(\mathrm{gr}^2_!(\Psi_\varrho), \mathrm{Fil}^1_*(\mathrm{gr}^1_1(\Psi_\varrho)))$ which is an isomorphism. This element then gives us a $\overline{\mathbb{Z}}_l$-morphism $N_v \in \mathrm{Hom}(\Psi_\varrho, \Psi_\varrho)$ so that $\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ is in the image.

Second step: we want to prove that the local systems $j^{=2,*}\mathrm{gr}^2_!(\Psi_\varrho)$ and $j^{=2,*}\,\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ are isomorphic. Consider first the following situation: let $\mathcal{L}_k$ and $\mathcal{L}_{k+1}$ be $\overline{\mathbb{Z}}_l$-local systems on a scheme X such that:

– $\mathrm{gr}_{k+1,\overline{\mathbb{Q}}_l} := (\mathcal{L}_{k+1}/\mathcal{L}_k) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is irreducible and we introduce

$$\begin{array}{ccc} \mathrm{gr}_{k+1} & \hookrightarrow & \mathrm{gr}_{k+1,\overline{\mathbb{Q}}_l} \\ \downarrow & & \downarrow \\ \mathcal{L}_{k+1} & \hookrightarrow & \mathcal{L}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{array}$$

We moreover suppose that the $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ are also irreducible so the various stable lattices of gr_{k+1} are homothetic.

– $\mathcal{L}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathcal{L}_k \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \oplus \mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$.

We then have

$$0 \to \mathcal{L}_k \oplus \mathrm{gr}_{k+1} \longrightarrow \mathcal{L}_{k+1} \longrightarrow T \to 0,$$

where T is torsion and can be viewed as a quotient

$$\mathcal{L}_k \hookrightarrow \mathcal{L}'_k \twoheadrightarrow T, \quad \mathrm{gr}_{k+1} \hookrightarrow \mathrm{gr}'_{k+1} \twoheadrightarrow T,$$

with, cf. also the exchange basic step of §2

$$\mathcal{L}_k \hookrightarrow \mathcal{L}_{k+1} \twoheadrightarrow \mathrm{gr}'_{k+1}, \qquad \mathrm{gr}_{k+1} \hookrightarrow \mathcal{L}_{k+1} \twoheadrightarrow \mathcal{L}'_k.$$

As $\mathrm{gr}_{k+1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is irreducible, then $\mathrm{gr}_{k+1} \hookrightarrow \mathrm{gr}'_{k+1}$ is given by multiplication by l^δ and the extension is characterized by this δ.

Consider then the $\overline{\mathbb{Z}}_l$-local system $\mathcal{L} := j^{=1,*}\Psi_\varrho$ and recall that

$$\mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{i=1}^r HT_{\overline{\mathbb{Q}}_l}(\chi_{v,i}, \chi_{v,i}),$$

where we fix any numbering of $\mathrm{Cusp}(\varrho) = \{\chi_{v,1}, \cdots, \chi_{v,r}\}$. For $k = 1, \cdots, r$, we introduce

$$\begin{array}{ccc} \mathcal{L}_k & \dashrightarrow & \bigoplus_{i=1}^k HT(\chi_{v,i}, \chi_{v,i}) \\ \downarrow & & \downarrow \\ \mathcal{L} & \hookrightarrow & \mathcal{L} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l. \end{array}$$

Let denote by T_{k+1} the torsion local system such that

$$0 \to \mathcal{L}_k \oplus \mathrm{gr}_{k+1} \longrightarrow \mathcal{L}_{k+1} \longrightarrow T_{k+1} \to 0,$$

where $\mathrm{gr}_{k+1} := \mathcal{L}_{k+1}/\mathcal{L}_k$, as above. We can apply the previous remark and denote by δ_k the power of l which define the homothety $\mathrm{gr}_{k+1} \hookrightarrow \mathrm{gr}'_{k+1} \twoheadrightarrow T_{k+1}$. The set of δ_k for $k = 1, \cdots, r$ is then a numerical data to characterize $\mathcal{L}$ inside $j^{=1,*}\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$.

(i) From the main result of [**10**], cf. its introduction paragraph, $\mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho))$ is obtained as follows

$$0 \to \mathrm{Fil}^1_*(\mathrm{gr}^1_!(\Psi_\varrho)) \longrightarrow j^{=1}_{\neq 1,!} j^{=1,*}_{\neq 1} \Psi_\varrho \longrightarrow {}^p j^{=1}_{\neq 1,!*} j^{=1,*}_{\neq 1} \Psi_\varrho \to 0,$$

so that $j^{=2,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho)) \simeq {}^ph^{-1}i^{2,*p}j^{=1}_{\neq 1,!*}j^{=1,*}_{\neq 1}\Psi_\varrho$. With the previous notations, we have

$$0 \to {}^pj^{=1}_{\neq 1,!*}\mathcal{L}_k \longrightarrow {}^pj^{=1}_{\neq 1,!*}\mathcal{L}_{k+1} \longrightarrow {}^pj^{=1}_{\neq 1,!*}T_{k+1} \to 0,$$

from which we obtain the following description of $j^{=2,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho))$:

- there exists local systems $\mathcal{L}_k^+$ for $k = 1, \cdots, r$ so that $\mathcal{L}_k^+ \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{i=1}^k HT_{\overline{\mathbb{Q}}_l}(\chi_{v,i}, \operatorname{St}_2(\chi_{v,i})(-1/2)$;
- with $\operatorname{gr}^+_{k+1}$ defined, as before, with

$$0 \to \mathcal{L}_k^+ \oplus \operatorname{gr}^+_{k+1} \longrightarrow \mathcal{L}^+_{k+1} \longrightarrow T_{k+1} \to 0,$$

  where T_{k+1} is killed by $l^{\delta_{k+1}}$.

Finally $j^{=2,*}\operatorname{Fil}^1_*(\operatorname{gr}^1_!(\Psi_\varrho))$ is described with the same numerical data $\{\delta_k : k = 1, \cdots, r\}$ as $j^{=1,*}\Psi_\varrho$.

(ii) The same arguments apply with $j^{=1}_{\neq 1,*}j^{=1,*}_{\neq 1}\Psi_\varrho$ so that the local system $\widetilde{\mathcal{L}} := {}^{p+}h^1i^{2,*p+}j^{=1}_{\neq 1,!*}j^{=1,*}_{\neq 1}\Psi_\varrho$ is also characterized by the same numerical data $\{\delta_k : k = 1, \cdots, r\}$ except that $\widetilde{\mathcal{L}}$ can not directly be identified to $j^{=2,*}\operatorname{gr}^2_!(\Psi_\varrho)$. Indeed we are interested in the lattice of $\bigoplus_{\chi_v\in\operatorname{Cusp}(\varrho)} j^{=2}_{!*}HT_{\overline{\mathbb{Q}}_l}(\chi_v, \operatorname{St}_2(\chi_v))(-1/2)$ given by $\Psi_\varrho/\operatorname{Fil}^1_!(\Psi_\varrho)$. But by now the previous lattice of $P_1 := {}^pj^{=2}_{!*}\widetilde{\mathcal{L}}$ described by $\{\delta_k : k = 1, \cdots r\}$ is obtained using a filtration where P_1 appears as the socle of the perverse sheaf Q defined as follows:

$$0 \to {}^pj^{=1}_{!*}j^{=1,*}\operatorname{coFil}^1_*(\Psi_\varrho) \longrightarrow \operatorname{coFil}^1_*(\Psi_\varrho) \longrightarrow Q \to 0.$$

As explained in §2, we have to use basic exchange steps as many times as needed to move P_1 until it appears as the cosocle of $\operatorname{Fil}^2_!(\Psi_\varrho) \hookrightarrow \Psi_\varrho$, cf. the discussion before lemma 2.12.

Note then that all the perverse sheaves which are exchanged with P_1 during this process, are lattice of $j^{=h}_{!*}HT_{\overline{\mathbb{Q}}_l}(\chi_v, \operatorname{St}_h(\chi_v))(\frac{1-h+\delta}{2})$ with $h \geqslant 3$. As explained in the remark after the definition of the exchange basic step, as ${}^pj^{=2}_{!*}HT(\chi_v, \operatorname{St}_2(\chi_v)) \simeq {}^{p+}j^{=2}_{!*}HT(\chi_v, \operatorname{St}_2(\chi_v))$, for all these exchange, we have $T = 0$ and P_1 remains unchanged during all the basic exchange steps.

Third step: at this stage we constructed a $\overline{\mathbb{Z}}_l$-version of the monodromy operator

$$\begin{array}{ccc} N_\varrho : \Psi_\varrho & \longrightarrow & \Psi_\varrho \\ \searrow & & \nearrow \\ & \Psi_\varrho / \mathrm{Fil}^1_!(\Psi_\varrho) & \end{array}$$

such that the kernel of

$$N_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l : \big(\Psi_\varrho / \mathrm{Fil}^1_!(\Psi_\varrho)\big) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \longrightarrow \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l,$$

does not contain any irreducible subquotient of $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

Recall that we suppose the order of q_v modulo l to be $> d$, so that the irreducible constituants of $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ are disjoint from that of $\big(\Psi_\varrho / \mathrm{Fil}\, 2_!(\Psi_\varrho)\big) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

Moreover, arguing as in lemma 2.12, we see that the socle of $\big(\Psi_\varrho / \mathrm{Fil}^1_!(\Psi_\varrho)\big) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is up to multiplicity $j^{=d}_{!*} HT_{\overline{\mathbb{F}}_l}(\varrho, \mathrm{St}_d(\varrho))(\frac{d-3}{2})$ which is a constituant of $\mathrm{gr}^2_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

From the previous facts, we then deduce that the kernel of $N_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l : \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \longrightarrow \Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is reduced to $\mathrm{Fil}^1_!(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ and is so the modulo l reduction of the kernel of N_v.

4.2. Corollary. — *Under the hypothesis of the proposition 3.1 on* $\mathfrak{m}$*, the action of* N_ϱ *on* Ψ_ϱ *defined above for every* $\overline{\mathbb{F}}_l$*-character* ϱ*, induces a nilpotent monodromy operator* $N^{coho}_{\mathfrak{m},v}$ *on* $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_\mathfrak{m}$ *such that the (multi)-set of Jordan's blocks of* $\overline{N}^{coho}_{\mathfrak{m},v} := N^{coho}_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ *acting on* $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$*, is the disjoint union under* $\{\widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$*, of the (multi)-sets of Jordan's blocks of* $N_{\widetilde{\mathfrak{m}},v}$.

Proof. — Recall first that the (multi)-set of Jordan's blocks of $N^{coho}_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ (resp. $\overline{N}^{coho}_{\mathfrak{m},v}$) is given by the collection of the dimensions $e_{\mathfrak{m},v}(r)$ (resp. $\overline{e}_{\mathfrak{m},v}(r)$) of $\mathrm{Ker}(N^{coho}_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)^r$ (resp. $\mathrm{Ker}\, \overline{N}^{coho}_{\mathfrak{m},v}$) for $r \geqslant 1$: the columns of the Young diagram associated to $N^{coho}_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$, are of length $e_{\mathfrak{m},v}(r+1) - e_{\mathfrak{m},v}(r)$ for $r \geqslant 0$.

- Proposition 3.1 gives us that the $\overline{\mathbb{F}}_l$-spectral sequence of nearby cycles degenerates at E_1;
- while from above, we know that $\overline{e}_{\mathfrak{m},v}(r)$ is the sum over the $\overline{\mathbb{F}}_l$-characters ϱ of the $\overline{\mathbb{F}}_l$-dimension of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^r_!(\Psi_{\varrho,\xi}))_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

– But this dimension is also the $\overline{\mathbb{Q}}_l$- dimension of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^r_!(\Psi_{\varrho,\xi}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ and so equals to $e_{\mathfrak{m},v}(r)$.

The result then follows from

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^r_!(\Psi_{\varrho,\xi}))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\widetilde{\mathfrak{m}} \subset \mathfrak{m}} H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^r_!(\Psi_{\varrho,\xi}))_{\widetilde{\mathfrak{m}}},$$

and the fact that $N_{\widetilde{\mathfrak{m}},v} = N^{coho}_{\mathfrak{m},v} \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m}}} \mathbb{T}^S_{\xi,\widetilde{\mathfrak{m}}}$. □

5. Mazur's principle

5.1. Definition. — *(cf. [**25**] §5) We say that $\mathfrak{m}$ is KHT-typic if, as a $\mathbb{T}^S_{\xi,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,*

$$H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}} \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m}}} \rho_{\mathfrak{m}},$$

for some $\mathbb{T}^S_{\xi,\mathfrak{m}}$-module $\sigma_{\mathfrak{m}}$ on which $\mathrm{Gal}_{F,S}$ acts trivially and

$$\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}^S_{\xi,\mathfrak{m}})$$

is the stable lattice of $\bigoplus_{\widetilde{\mathfrak{m}} \subset \mathfrak{m}} \rho_{\widetilde{\mathfrak{m}}}$ introduced in the introduction.

Remark. When $\mathfrak{m}$ is both KHT-typic and verifies the hypothesis of proposition 3.1, then $N^{coho}_{\mathfrak{m},v}$ induces a monodromy operator $N_{\mathfrak{m},v}$ on $\rho_{\mathfrak{m}}$.

As explained in [**21**], the $\overline{\mathbb{Q}}_l$-cohomology of $\mathrm{Sh}_{I,\bar{\eta}}$ can be written as

$$H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}}, V_{\xi,\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \simeq \bigoplus_{\pi \in \mathcal{A}_{\xi,I}(\mathfrak{m})} (\pi^\infty)^I \otimes V(\pi^\infty),$$

where

- $\mathcal{A}_{\xi,I}(\mathfrak{m})$ is the set of equivalence classes of automorphic representations of $G(\mathbb{A})$ with non trivial I-invariants and such that its modulo l Satake's parameters outside S are prescribed by $\mathfrak{m}$,
- and $V(\pi^\infty)$ is a representation of $\mathrm{Gal}_{F,S}$.

As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible, then as explained in chapter VI of [**21**], if $V(\pi^\infty)$ is non zero, then π is a weak transfer of a ξ-cohomological automorphic representation (Π, ψ) of $GL_d(\mathbb{A}_F) \times \mathbb{A}_F^\times$ with $\Pi^\vee \simeq \Pi^c$ where c is the complex conjugation. Attached to such a Π is a global Galois representation $\rho_{\Pi,l} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ which is irreducible.

5.2. Theorem. — *(cf.* [**20**] *theorem 2.20)*
If $\rho_{\Pi,l}$ is strongly irreducible, meaning it remains irreducible when it is restricted to any finite index subgroup, then $V(\pi^\infty)$ is a semi-simple representation of $\mathrm{Gal}_{F,S}$.

Remark. The Tate conjecture predicts that $V(\pi^\infty)$ is always semi-simple.

5.3. Proposition. — *We suppose that for all $\pi \in \mathcal{A}_{\xi,I}(\mathfrak{m})$, the Galois representation $V(\pi^\infty)$ is semi-simple. Then $\mathfrak{m}$ is KHT-typic.*

Proof. — By proposition 5.4 of [**25**] it suffices to deal with $\overline{\mathbb{Q}}_l$-coefficients. From [**21**] proposition VII.1.8 and the semi-simplicity hypothesis, then $V(\pi^\infty) \simeq \widetilde{R}_\xi(\pi)^{\oplus n(\pi)}$ where $\widetilde{R}_\xi(\pi)$ is of dimension d. We then write

$$(\pi^\infty)^I \otimes_{\overline{\mathbb{Q}}_l} R_\xi(\pi) \simeq (\pi^\infty)^I \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l}} (\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

and $(\pi^\infty)^I \otimes_{\overline{\mathbb{Q}}_l} V(\pi^\infty) \simeq ((\pi^\infty)^I)^{\oplus n(\pi)} \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l}} (\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l})^d$ and finally

$$H^{d-1}(\mathrm{Sh}_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Q}}_l})_\mathfrak{m} \simeq M \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l}} (\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

with $M \simeq \bigoplus_{\pi\in\mathcal{A}_{\xi,I}(\mathfrak{m})}((\pi^\infty)^I)^{\oplus n(\pi)}$. The result then follows from [**21**] theorem VII.1.9 which insures that $R_\xi(\pi) \simeq \rho_{\widetilde{\mathfrak{m}}}$, if $\widetilde{\mathfrak{m}}$ is the prime ideal associated to π, □

5.4. Definition. — *Let $\overline{\rho}(\mathfrak{m})_\bullet$ be the filtration of $\rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ defined by its iterated socle, that is*
- *$\overline{\rho}(\mathfrak{m})_0$ is the socle of $\rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$*
- *and for $i \geqslant 1$, $\overline{\rho}(\mathfrak{m})_i/\overline{\rho}(\mathfrak{m})_{i-1}$ is the socle of $(\rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)/\overline{\rho}(\mathfrak{m})_{i-1}$.*

The depth of $\mathfrak{m}$ is then the length of this filtration.

5.5. Theorem. — *(**Mazur's principle**) Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^S_{\xi,\overline{\mathbb{Z}}_l}$ such that:*
- *$\overline{\rho}_\mathfrak{m}$ is absolutely irreducible and its restriction to the decomposition group at v is, up to the action of the monodromy operator, a direct sum of characters;*
- *$\mathfrak{m}$ is KHT-free and KHT-typic.*

Let $\underline{\bar{d}_{\mathfrak{m},v}} = (t_1 \geqslant \cdots \geqslant t_r)$ be the partition of d given by the Jordan blocks of $\overline{N}_{\mathfrak{m},v}$. Then there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that

$$\rho_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{Sp}_{t_1}(\chi_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{t_r}(\chi_{v,r}),$$

where $\chi_{v,i}$ are non isomorphic characters.

For $\underline{d}$ a partition of d, its associated parahoric subgroup is

$$I_{\underline{d}}(\mathcal{O}_v) = \mathrm{Ker}(GL_d(\mathcal{O}_v) \longrightarrow P_{\underline{d}}(\kappa(v))) \qquad (5.6)$$

where $P_{\underline{d}}$ is the standard parabolic subgroup associated to $\underline{d}$. The dominance order $(d_1 \geqslant \cdots d_r) \leqslant (e_1 \geqslant \cdots \geqslant e_s)$ is the given by

$$\forall i \geqslant 1 : \sum_{i=1}^{k} d_i \leqslant \sum_{i=1}^{k} e_i.$$

Recall, cf. lemma 1.1.7 of [**13**] that $\Pi_v \simeq \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r})$ has non trivial invariant vectors upon $I_{\underline{d}}(\mathcal{O}_v)$ if and only if $\underline{d}$ is smaller than the dual partition $\underline{e}^*$ of $\underline{e} := (t_1 \geqslant \cdots \geqslant t_r)$ whose lines are the rows of $\underline{e}$. The theorem then says that $\Pi_{\widetilde{\mathfrak{m}},v}$ has non trivial invariant vectors upon $I_{\underline{\bar{d}_{\mathfrak{m},v}}^*}(\mathcal{O}_v)$.

Proof. — We will consider three different types of situations at v:

- infinite and we then denote by $H(I^v(\infty))_{\mathfrak{m}} := H^0(\mathrm{Sh}_{I^v(\infty),\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ and $\overline{H}(I^v(\infty))_{\mathfrak{m}} := H(I^v(\infty))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$;
- $H(I^v, \underline{d})_{\mathfrak{m}} := H^0(\mathrm{Sh}_{I^v I_{\underline{d}}(\mathcal{O}_v),\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ and $\overline{H}(I^v, \underline{d})_{\mathfrak{m}} := H(I^v, \underline{d})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$;
- $H(I^v, h)_{\mathfrak{m}} := J_{P_h}^{GL_d}\big(H(I^v(\infty))_{\mathfrak{m}}\big)^{GL_h(\mathcal{O}_v)}$ and $\overline{H}(I^v, h)_{\mathfrak{m}} := H(I^v, h)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$; where $J_{P_h}^{GL_d}$ is the Jacquet functor associated to the parabolic $P_h(F_v)$.

We also denote by $\mathbb{T}_{\xi,\mathfrak{m}}(I^v, \underline{d})$ the image of $\mathbb{T}_{abs}^S$ inside $H(I^v, \underline{d})_{\mathfrak{m}}$. Let $\underline{d}$ be minimal for the previous dominance order, such that $H(I^v, \underline{d}^*)_{\mathfrak{m}} \neq 0$ and consider now a broken row in $\overline{T}_{\mathfrak{m},v}$ that is $\overline{\lambda} \in \overline{\mathbb{F}}_l$ such that $q_v\overline{\lambda}$ does not belong to the same line with $\overline{\lambda}$ in the labelled Young diagram of $\overline{N}_{\mathfrak{m},v}$.

5.7. Lemma. — *There exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ with $H(I^v, \underline{d}^*)_{\widetilde{\mathfrak{m}}} \neq (0)$, such that inside $T_{\widetilde{\mathfrak{m}},v}$, the liftings*(4) *$\lambda_1$ and λ_2 of $q_v\overline{\lambda}$ and $\overline{\lambda}$, do not belong to the same line, i.e. $\lambda_1/\lambda_2 \neq q_v^{\pm 1}$.*

Proof. — Let denote by $\rho_{\mathfrak{m}}(\underline{d}) := \rho_{\mathfrak{m}} \otimes_{\mathbb{T}_\xi^S} \mathbb{T}_\xi(I^v, \underline{d}^*)$ and $N_{\mathfrak{m},v}(\underline{d}) := N_{\mathfrak{m},v} \otimes_{\mathbb{T}_\xi^S} \mathbb{T}_\xi(I^v, \underline{d}^*)$. We consider then the eigenspaces $V_I(\overline{\lambda})$ and $V(\overline{\lambda})$ for the eigenvalue $\overline{\lambda}$ of the action of Frob_v, respectively, on the $\overline{\mathbb{F}}_l$-vector spaces

$$\mathrm{Im}\, N_{\mathfrak{m},v}(\underline{d}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \subset \rho_{\mathfrak{m}}(\underline{d}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

(4) cf. the multiplicity free hypothesis

Note that, as the eigenvalues of Frob_v acting on $\rho_{\mathfrak{m}} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l$ are supposed to be pairwise distinct, then

$$\dim_{\overline{\mathbb{F}}_l} V(\overline{\lambda}) = \sharp\{\widetilde{\mathfrak{m}} \subset \mathfrak{m};, H(I^v, \underline{d}^*)_{\widetilde{\mathfrak{m}}} \neq (0)\}.$$

Note that if the conclusion of the lemma were not true, then $\dim_{\overline{\mathbb{F}}_l} V_I(\overline{\lambda}) = \dim_{\overline{\mathbb{F}}_l} V(\overline{\lambda})$. Indeed this is true over $\overline{\mathbb{Q}}_l$ and from corollary 4.2, we know that the image of $N^{coho}_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is the modulo l reduction of the image of $N_{\mathfrak{m},v}$, which gives us the previous equality.

Consider now the previous filtration $\overline{\rho}(\mathfrak{m}, \underline{d})_i$ of $\rho_{\mathfrak{m}}(\underline{d}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, where all the graded parts $\mathrm{gr}_i(\mathfrak{m}, \underline{d})$ are a direct sum of $\overline{\rho}_{\mathfrak{m}}$. If we want $\dim_{\overline{\mathbb{F}}_l} V_I(\overline{\lambda})$ to be equal to the number $\sharp\{\widetilde{\mathfrak{m}} \subset \mathfrak{m};, H(I^v, \underline{d}^*)_{\widetilde{\mathfrak{m}}} \neq (0)\}$ of irreducible subquotients of $\rho_{\mathfrak{m}}(\underline{d}) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, then Frob_v should induces an isomorphism $V(q_v\overline{\lambda}) \to V_I(\overline{\lambda})$. But note that $V(q_v\overline{\lambda})$ intersects $\overline{\rho}(\mathfrak{m}, \underline{d})_0$ and, as $q_v\overline{\lambda} \to \overline{\lambda}$ is broken in $\overline{T}_{\mathfrak{m},v}$, and the eigenvalues are all distinct, then Frob_v acts trivially on $V(q_v\overline{\lambda}) \cap \overline{\rho}(\mathfrak{m})^0$ so that $\dim_{\overline{\mathbb{F}}_l} V_I(\overline{\lambda}) < \dim_{\overline{\mathbb{F}}_l} V(q_v\overline{\lambda}) = \sharp\{\widetilde{\mathfrak{m}} \subset \mathfrak{m};, H(I^v, \underline{d}^*)_{\widetilde{\mathfrak{m}}} \neq (0)\}$. □

End of the proof: recall that $\underline{d}$ was taken minimal such that $H(I^v, \underline{d}^*)_{\mathfrak{m}} \neq 0$. By freeness of the cohomology groups, there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ with $H_{\overline{\mathbb{Q}}_l}(I^v, \underline{d}^*)_{\mathfrak{m}} \neq 0$, so that $\underline{d}_{\widetilde{\mathfrak{m}},v}$ is greater that $\underline{d}$ and finally, by minimality of $\underline{d}$, we have $\underline{d} = \underline{d}_{\widetilde{\mathfrak{m}},v}$, and it remains to prove that they are equal to $\underline{d}_{\mathfrak{m},v}$. Note that $\underline{d}_{\widetilde{\mathfrak{m}},v}$ is obtained from $\underline{d}_{\mathfrak{m},v}$ by glueing some of its line so that if it were note equal to $\underline{d}_{\mathfrak{m},v}$, the previous lemma would tell us that $\underline{d}$ were not maximal.

□

5.8. Proposition. — *With the previous notations, for every* $0 \leqslant i \leqslant r-1$ *where* r *is the depth of* $\mathfrak{m}$, *then* $\mathrm{gr}^i(\mathfrak{m})$ *is irreducible and then isomorphic to* $\overline{\rho}_{\mathfrak{m}}$.

Proof. — Automorphic representation $\pi_{\widetilde{\mathfrak{m}}}$ in level $I^v(\infty)$ having a cuspidal support at v made of characters, are in finite numbers so that $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \mathbb{T}_{\mathfrak{m},\xi}$ is defined over a noetherian ring R inside $\prod_{\widetilde{\mathfrak{m}} \subset \mathfrak{m}} K_{\widetilde{\mathfrak{m}}}$ where the finite extension $K_{\widetilde{\mathfrak{m}}}/\mathbb{Q}_l$ is the field of definition of $\rho_{\widetilde{\mathfrak{m}}}$. We then denote by $\kappa = R/\mathfrak{m}$ and we still denote by $\overline{\rho}_{\mathfrak{m}}$ the associated κ-vector space with its action of the Galois group.

By construction $\mathrm{gr}^i(\mathfrak{m})$ is then a semi-simple $\mathrm{Gal}_{F,S}$-module with underlying representation space a free rank $\mathbb{T}^{S,(i)}_{\xi,\mathfrak{m}} \otimes_R \kappa$-module $V^{(i)}$ where

$\mathbb{T}^S_{\xi,\mathfrak{m}} \twoheadrightarrow \mathbb{T}^{S,(i)}_{\xi,\mathfrak{m}}$. We then have a $\mathbb{T}^{S,(i)}_{\mathfrak{m},\xi}[\mathrm{Gal}_{F,S}]$-equivariant isomorphism

$$\overline{\rho}_{\mathfrak{m}} \otimes_R \kappa \simeq \overline{\rho}_{\mathfrak{m}} \otimes_\kappa \mathrm{Hom}_{\mathrm{Gal}_{F,S}}(\overline{\rho}_{\mathfrak{m}}, V),$$

where $\mathrm{Gal}_{F,S}$ acts on the first factor, and $\mathbb{T}^{S,(i)}_{\mathfrak{m},\xi}$ on the second. As a deformation of $\overline{\rho}_{\mathfrak{m}}$ over the local κ-algebra $\mathbb{T}^{S,(i)}_{\xi,\mathfrak{m}} \otimes_R \kappa$, it is given by a map $R_{\overline{\rho}_{\mathfrak{m}}} \longrightarrow \mathbb{T}^{S,(i)}_{\xi,\mathfrak{m}} \otimes_R \kappa$ where $R_{\overline{\rho}_{\mathfrak{m}}}$ is the usual deformation ring. The previous isomorphism implies that this last map has to factor through the residue field of $R_{\overline{\rho}_{\mathfrak{m}}}$ which is κ. Moreover from $R = T$ theorem, we have a surjection

$$R_{\overline{\rho}_{\mathfrak{m}}} \twoheadrightarrow T^S_{\xi,\mathfrak{m}} \twoheadrightarrow T^{S,(i)}_{\xi,\mathfrak{m}}$$

so that $T^{S,(i)}_{\xi,\mathfrak{m}} \simeq \kappa$. By Nakayama's theorem, $T^{S,(i)}_{\xi,\mathfrak{m}}$ is then a local ring corresponding to an unique $\widetilde{\mathfrak{m}}$. □

The set of partitions $\underline{d}_{\widetilde{\mathfrak{m}},v}$ for various $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, could be used to obtain informations about the depth of $\mathfrak{m}$. Consider for example the following situation:

- $S_v(\mathfrak{m}) = \{\alpha, q_v\alpha, \cdots, q_v^{d-1}\alpha\}$;
- $\overline{N}_{\mathfrak{m},v}$ is zero;
- there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_d(\chi_v)$ for some character χ_v.

5.9. Lemma. — *With the hypothesis of 5.5 and the three above assumptions, then the depth of* $\mathfrak{m}$ *is greater than* d.

Proof. — By construction each of the $\overline{\rho}_i(\mathfrak{m})$ is a direct sum of copies of $\overline{\rho}_{\mathfrak{m}}$ so that the nilpotent monodromy operator $\overline{N}_{\mathfrak{m},v}$ acts trivially. We then deduce that $\overline{N}_{\mathfrak{m},v}$ sends $\overline{\rho}(\mathfrak{m})_i$ onto $\overline{\rho}(\mathfrak{m})_{i-1}$. Our last hypothesis then implies that $\overline{N}^{d-1}_{\mathfrak{m},v} \neq 0$ so that the depth of $\mathfrak{m}$ should be greater than d. □

More generally, consider

- r maximal such that there exists α with $\{\alpha, q_v\alpha, \cdots, q_v^{r-1}\alpha\} \subset S_v(\mathfrak{m})$. We also denote by $e_0, \cdots, e_{r-1}$ the associated eigenvectors of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$.
- Let denote by $i_0 = 0 < i_1 < \cdots < i_k \leqslant r-1$ the indexes i such that $e_i \in \mathrm{Ker}\, \overline{N}_{\mathfrak{m},v}$.
- We moreover assume the existence of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}}} \simeq \mathrm{St}_r(\chi_v) \times ?$ where χ_v is a character of $F_v^\times$ such that $\chi_v(\varpi_v) = \alpha$ and where ? means a irreducible representation we do not want to precise.

5.10. Lemma. — *With the hypothesis of 5.5 and the three above assumptions, then the depth of* $\mathfrak{m}$ *is strictly greater than* k.

Proof. — The existence of $\widetilde{\mathfrak{m}}$ implies that there exists an eigenvector f_{r-1} of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ for the eigenvalue $q_v^{r-1}\alpha$ such that $(\overline{N}_{\mathfrak{m},v}^{coho})^{r-1}(f_{r-1}) \neq 0$. We first introduce the following notations:

– i such that $f_{r-1} \in \overline{\rho}(\mathfrak{m})_i$;
– for $\leqslant j \leqslant r-1$, let $f_j = N_{\mathfrak{m},v}^{r-1-j}(f_{r-1})$.

Through $\overline{\rho}(\mathfrak{m})_i \twoheadrightarrow \overline{\rho}(\mathfrak{m})_i / \overline{\rho}(\mathfrak{m})_{i-1}$, the image of f_{i_k} by $\overline{N}_{\mathfrak{m},v}$ is zero so that $f_{i_k-1} \in \overline{\rho}(\mathfrak{m})_{i-1}$. As $S_v(\mathfrak{m})$ is supposed to be multiplicity free then the image of f_{i_k-1} in $\overline{\rho}(\mathfrak{m})_{i-1}/\overline{\rho}(\mathfrak{m})_{i-2} \simeq \overline{\rho}_{\mathfrak{m}}^{\bigoplus m_{i-1}}$ belongs to the space generated by the e_{i_k-1} in each of the copies of $\overline{\rho}_{\mathfrak{m}}$. We can then repeat the previous observation so that the image of $f_{i_{k-1}-1} \in \overline{\rho}(\mathfrak{m})_{i-2}$ and that finally the depth of $\mathfrak{m}$ should be greater than k. □

6. The reducible case

In [**13**], $N_{\mathfrak{m},v}^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ was wrongly identified with the direct sum of $\overline{N}_{\mathfrak{m},v}$, so that [**13**] proposition 3.1.12 is false. The aim of loc. cit. was to give conditions on $\underline{\overline{d_{\mathfrak{m},v}}}$ so that the torsion of the cohomology of the KHT-Shimura variety is non trivial, and then play with the level in order to obtain level lowering.

In this section we want to resume the strategy of [**13**] and give a correct statement about level lowering. From the main result of [**14**], we then have to consider the case where $\overline{\rho}_{\mathfrak{m}}$ is reducible. In [**13**] the irreducibility hypothesis for $\overline{\rho}_{\mathfrak{m}}$ was used to insure the free quotients of the various cohomology groups to be concentrated in middle degree. To keep this property we then make the following hypothesis.

Hypothesis: We now consider the case where

– $\overline{\rho}_{\mathfrak{m}}$ is reducible
– and the semi-simplification of its restriction to the decomposition group at v, which we write $\bigoplus_{i=1}^r \mathrm{Sp}_{s_i}(\chi_{v,i})$ without taking into account the monodromy operator, verifies the following property where $\mathrm{Sp}_{s_i}(\chi_{v_i}) = \chi_{v,i}(\frac{1-s_i}{2}) \oplus \cdots \chi_{v,i}(\frac{s_i-1}{2})$ as before:
  - the $\chi_{v,i}(\frac{1-s_i+2t_i}{2})$ for $i = 1, \cdots, r$ and $0 \leqslant t_i \leqslant s_i - 1$, are pairwise distinct, i.e. $\chi_{v,i}(\frac{1-s_i+2t_i}{2})$ is not isomorphic to any of the $\chi_{v,j}(\frac{1-s_j+2t_j}{2})$ whatever are t_i and t_j verifying the previous

inequality. Taking $i=j$, we obtain in particular that, for all $i=1,\cdots,r$, the order of q_v modulo l is strictly greater than s_i.
- Finally we impose that there exists $i \neq j$ with $s_i \neq s_j$.

Recall that if $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ appears in the cohomology of our KHT Shimura variety such that $\rho_{\widetilde{\mathfrak{m}}}$ is reducible then $T_{\widetilde{\mathfrak{m}},v}$ is a rectangle. It is in particular excluded from the previous hypothesis, i.e. for every $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ then $\rho_{\widetilde{\mathfrak{m}}}$ is irreducible. In particular the free quotients of the cohomology localized at $\mathfrak{m}$ are still concentrated in middle degree but there might be non trivial torsion classes. Note also that the modulo l reduction of $\rho_{\widetilde{\mathfrak{m}}}$ depends on $\widetilde{\mathfrak{m}}$ but also of the chosen stable lattice inside it.

As now KHT-typicness is no more verified, we consider, for a partition $\underline{d}$ of d, a $\mathrm{Gal}_{F,S}$-equivariant filtration of the free quotient $H(I^v,\underline{d}^*)_{\mathfrak{m},free}$ of $H(I^v,\underline{d}^*)_{\mathfrak{m}}$, so that each graded part is a stable lattice of some $\rho_{\widetilde{\mathfrak{m}}}$ with $\pi_{\widetilde{\mathfrak{m}},v}^{I_{\underline{d}^*}(\mathcal{O}_v)} \neq (0)$. The modulo l reductions of each of these lattices, give then partitions of d associated to the Jordan blocks of the monodromy operator at v and we denote by $\underline{d_{\mathfrak{m},v}(\underline{d})}$ the minimal one.

Remark. Note that, a priori, $\underline{d_{\mathfrak{m},v}(\underline{d})}$ might depend on the choice of the filtration of $H(I^v,\underline{d}^*)_{\mathfrak{m},free}$.

6.1. Definition. — *(cf.* [**13**] *§1.1 and definition 1.3.1)*
For $\underline{d}$ a partition of d, we denote by $\underline{d}^{(1)}$ the partition such that its Young diagram is obtain from those of $\underline{d}$ by deleting its first column.

We then say that $\mathfrak{m}$ is degraded relatively to $\underline{d}$ if $\underline{d}^{(1)}=(d_1 \geqslant d_2 \geqslant \cdots)$ is not contained in $\underline{d_{\mathfrak{m},v}(\underline{d})}=(t_1 \geqslant t_2 \geqslant \cdots)$ i.e. there exists $i \geqslant 1$ such that $d_i \geqslant t_i$.

6.2. Proposition. — *Let $\underline{d}$ minimal such that $H(I^v,\underline{d}^*)_{\mathfrak{m},free} \neq (0)$ and suppose that $\mathfrak{m}$ is degraded relatively to $\underline{d}$, then for every $w \in \mathrm{Spl}(I)\backslash\{v\}$, there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ with non zero invariant vectors upon $I(w)$ defined as follows:*

- *outside v and w it coincides with $I^{v,w}$;*
- *at w it is of parahoric type for a partition $(t,1,\cdots,1)$ for some $t \leqslant d$;*
- *at v, $I(w)_v=I_v(\underline{e}^*)$ with $\underline{e}<\underline{d}$.*

Proof. — By hypothesis $\underline{d_{\mathfrak{m},v}(\underline{d})}$ is strictly smaller than $\underline{d}$. We first want to prove that in the spectral sequence which degenerates in E_1 over $\overline{\mathbb{Q}}_l$,

$$E_{1,\mathfrak{m}}^{p,q} = H^{p+q}(\mathrm{Sh}_{I^v I_{\underline{d}^*}(\mathcal{O}_v),\bar{s}_v}, \mathrm{grr}^{-p}(\Psi_{v,\xi}))_{\mathfrak{m}} \Rightarrow H^{p+q}(\mathrm{Sh}_{I^v I_{\underline{d}^*}(\mathcal{O}_v),\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}},$$

some of the $E_{1,\mathfrak{m}}^{p,q}$ for $p+q \in \{0,1\}$ are not torsion free. We argue by contradiction. From lemma 3.9, there is then no nontrivial torsion classes in the initial terms of the spectral sequence. We follows the proof of theorem 5.5. First the conclusion of lemma 5.7 is still true. Indeed

- consider a $\mathrm{Gal}_{F,S}$-equivariant filtration $\mathrm{Fil}^\bullet$ of $H(I^v, \underline{d}^*)_{\mathfrak{m},free}$ such that the graded parts gr^k are free and irreducible after inverting l.
- Modulo l we then obtain a filtration with graded parts $\overline{\mathrm{gr}}^k$ isomorphic to $\Gamma/l\Gamma$ where Γ is some stable lattice of some $\rho_{\widetilde{\mathfrak{m}}}$: we then denote by $\underline{d}(k)$ the associated partition given by the monodromy operator.
- Let k be minimal such that in the labelled Young diagram of $\underline{d}(k)$, there exists $q_v\bar{\lambda}$ and $\bar{\lambda}$ which are not in the same line: by hypothesis such a k exists.
- Let denote by $V_k(\bar{\lambda})$ the eigenspace of Frob_v acting on $\overline{\mathrm{Fil}}_k$ for the eigenvalue $\bar{\lambda}$. Note then that the dimension of $\overline{N}_{\mathfrak{m},v}^{coho} V_k(q_v\bar{\lambda})$ is strictly less than $\dim V_k(\bar{\lambda})$ and this inequality remains true for every $k' \geqslant k$.
- The conclusion is then similar to those in the proof of 5.7.

If there were no torsion, then following the proof in the previous section, by minimality of $\underline{d}$, we would obtain $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\underline{d} = \underline{d_{\widetilde{\mathfrak{m}},v}}$ would be equal to $\underline{d_{\mathfrak{m},v}(\underline{d})}$ which is not the case as $\mathfrak{m}$ is supposed to be degraded relatively to $\underline{d}$.

So we know that at least one of the $E_{1,\mathfrak{m}}^{p,q}$ for $p+q \in \{0,1\}$, has non trivial torsion classes. We now take up the arguments of [**13**] §3.3:

- There should exists h_0 such that the cokernel of (3.12) has non trivial torsion. In particular, cf. [**5**], there should exists an automorphic representation Π, irreducible, ξ-cohomological with

$$\Pi_v \simeq \mathrm{St}_{t_0+1}(\chi_{v,0}) \times \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r}),$$

  where $\chi_{v,i}$ are characters of $F_v^\times$ and $t_1, \cdots, t_r$ are integers we do not want to precise at this point;
- to have non trivial torsion in level $I^v I_v(\underline{d}^*)$, Π^v should have non trivial invariant vectors under I^v and the partition associated to $(t_0, 1, t_1, \cdots, t_r)$ should be less than $\underline{d}$.

– From lemma 3.9, then $H^{1-t_0}(\mathrm{Sh}_{I^vI_v(\underline{e}^*),\bar{s}_v}, {}^p j^{=1}_{!*} HT(\chi_v,\chi_v))_{\mathfrak{m}}$ has non trivial torsion subspace where $\underline{e}^*$ is the dual of $(\overbrace{1,\cdots,1}^{t_0+1},t_1,\cdots,t_r)$.
– If $t_0 \geqslant 2$ then as $(\overbrace{1,\cdots,1}^{t_0+1},t_1,\cdots,t_r) < (t_0,1,t_1,\cdots,t_r) \leqslant \underline{d}$. We then consider the spectral sequence (2.13) in level $I^vI_v(\underline{e}^*)$ so that the initial terms are all torsion, some of then being non zero. The proof of proposition 3.1 then gives us that E^i_∞ are then torsion and non zero for some i.

We now have to deal with the case where $t_0 = 1$ and $\underline{e} = \underline{d}$. As in the proof of [**13**] lemma 3.3.3, we then consider $\mathrm{coFil}^1_!(\Psi_{v,\xi}) := \Psi_{v,\xi}/\mathrm{Fil}^1_!(\Psi_{v,\xi})$ and its $\overline{\mathbb{Z}}_l$-exhaustive filtration where now $\mathcal{P}(1,\chi_v)$ does not appear anymore so that the spectral sequence associated to this filtration degenerates as before in E_1 with torsion free initial terms. In particular as in the proof of corollary 4.2, the dimensions $e_{\mathfrak{m},v}(r)$ (resp. $\overline{e}_{\mathfrak{m},v}(r)$) of $N^r_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ (resp. $(\overline{N}^{coho}_{\mathfrak{m},v})^r$) for $r \geqslant 2$ coincide while for $r = 1$ we have $\overline{e}_{\mathfrak{m},v}(1) = e_{\mathfrak{m},v}(1) + \delta$ for some $0 \leqslant \delta \leqslant e_{\mathfrak{m},v}(2) - e_{\mathfrak{m},v}(1)$.

The labelled Young's diagrams of $\overline{N}^{coho}_{\mathfrak{m},v}$ are then obtained from those of $\underline{d}$ allowing to untie its first column. We then argue as before to conclude that $\underline{d}^{(1)}$ has to be contained in every labelled Young's diagram of the modulo l reduction, relatively to lattices given by the cohomology, of $\rho_{\widetilde{\mathfrak{m}},v}$ and so in particular $\underline{d}^{(1)}$ is contained in $\underline{d_{\mathfrak{m},v}}(\underline{d})$ telling that $\mathfrak{m}$ is not degraded.

Finally, we deduce that there exists some non trivial torsion classes in some of the $H^i(\mathrm{Sh}_{I^vI_v(\underline{d}'),\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$. From the main result of [**11**], up to increase the level at some extra place $w \in \mathrm{Spl}(I)\backslash\{v\}$ where the level become parahoric for $(t,1\cdots,1)$ for some $t \leqslant d$, we can lift $\mathfrak{m}$ with level $I_v(\underline{e}^*)$ at v.

□

7. Automorphic congruences

As in [**8**], we can use the freeness of the cohomology groups of the Harris-Taylor perverse sheaves, to produce automorphic congruences. Consider then $\mathfrak{m}$ verifying the hypothesis of proposition 3.1 so that the $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j^{=h}_{!*} HT_\xi(\chi_v,h))_{\mathfrak{m}}$ are free and concentrated in degree $i = 0$

with

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v, h))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_{\xi,\overline{\mathbb{F}}_l}(r_l(\chi_v), h))_{\mathfrak{m}} \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi'_v, h))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \quad (7.1)$$

whatever is χ'_v such that the modulo l reduction $r_l(\chi'_v)$ of χ'_v is isomorphic to those of χ_v. Recall then from [**5**], the description of the $\overline{\mathbb{Q}}_l$-cohomology groups of ${}^p j_{!*}^{=h} HT_\xi(\chi_v, h)$ localized at $\mathfrak{m}$.

7.2. Proposition. — *(cf.* [**5**] *§3.6 with*[5] $s = 1$*)*
For χ_v an unitary character of $F_v^\times$, then, for $1 \leqslant h \leqslant d$, as a $\mathbb{T}^S_{\mathfrak{m}}[GL_d(F_v)]$-module, we have

$$\lim_{\overrightarrow{I_v}} H^0(\mathrm{Sh}_{I^v I_v,\bar{s}_v}, {}^p j_{!*}^{=h} HT_{\xi,\overline{\mathbb{Q}}_l}(\chi_v, h))_{\mathfrak{m}} \simeq \bigoplus_{\Pi \in \mathcal{A}(I,\xi,h,\chi_v,\mathfrak{m})} m(\Pi)(\Pi^{\infty,v})^{I^v} \otimes \Pi_v,$$

where

- *$\mathcal{A}(I, h, \chi_v, \mathfrak{m})$ is the set of irreducible ξ-cohomological automorphic representations Π of $G(\mathbb{A})$ with non zero invariants under I^v with modulo l Satake's parameters prescribed by $\mathfrak{m}$,*
- *such that Π_v is of the following shape*

$$\Pi_v \simeq \mathrm{St}_h(\chi_v) \times \Psi_v$$

  where Ψ_v is a representation of $GL_{d-h}(F_v)$,
- *and $m(\Pi)$ is the multiplicity of Π in the space of automorphic forms.*

Remark. We write the local component Π_v of $\Pi \in \mathcal{A}(I, \xi, h, \chi_v, \mathfrak{m})$ as

$$\Pi_v \simeq \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r}) \times \Psi'_v,$$

where

- the $\chi_{v,i}$ are inertially equivalent characters,
- Ψ'_v is an irreducible representation of $GL_{d-\sum_{i=1}^r t_i}(F_v)$ whose cuspidal support, made of character by hypothesis, does not contain a character inertially equivalent to $\chi_{v,1}$.

(5) As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, the integer s of [**5**] §3.6 is necessary equal to 1.

Then Π contributes k times in the isomorphism of the previous proposition, where $k = \sharp\{1 \leqslant i \leqslant r \text{ such that } t_i = h\}$.

We are now in the same situation as in [**8**] where we prove that the conjecture 5.4.3 implies the conjecture 5.2.1 and the translation in terms of automorphic congruences explained at the end of §5.2 The situation here is much more simple as $s = 1$.

7.3. Corollary. — *Let Π be an irreducible automorphic representation of $G(\mathbb{A})$ which is ξ-comological of level K and such that*

- *its modulo l Satake's parameters are given by $\mathfrak{m}$,*
- *and its local component Π_v at v is isomorphic to $\Pi_v \simeq \mathrm{St}_h(\chi_v) \times \Psi_v$, where χ_v is a characters and Ψ_v is an irreducible representation of $GL_{d-h}(F_v)$.*

Consider then any character χ'_v of $F_v^\times$ which is congruent to χ_v modulo l. Then there exists an irreducible automorphic representation Π' of $G(\mathbb{A})$ which is ξ-cohomological of the same level K and such that

- *its modulo l Satake's parameters are given by $\mathfrak{m}$,*
- *its local component at v is of the following shape*

$$\Pi'_v \simeq \mathrm{St}_h(\chi'_v) \times \Psi'_v.$$

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