p-STABILIZATION IN HIGHER DIMENSION

by

Boyer Pascal

Abstract. — Using l-adic completed cohomology in the context of Shimura varieties of Kottwitz-Harris-Taylor type attached to some fixed similitude group G, we prove, allowing to increase the levet at l, some new automorphic congruences between any degenerate automorphic representation with a non degenerate one of the same weight.

Contents

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1. Introduction

In his proof of the converse to Herbrand theorem, Ribet constructed congruences between modular forms. More precisely, let E_χ be the Eisenstein series whose constant term is the special value $L(\chi, -1)$; if it's divisible by some prime λ of $\overline{\mathbb{Q}}$ over l, then he showed the existence of some cuspidal Hecke eigenform whose Hecke eigenvalues are congruent to those of E_χ modulo λ. The associated Galois representation is then irreducible but its reduction modulo λ is not and can be chosen non semisimple, thus giving rise to a non split extension of one dimensional Galois modules over $\overline{\mathbb{F}}_l$ which can be interpreted as a non zero element in the χ-component of the class group of $\mathbb{Q}(\mu_l)$.

In the automorphic setting, one starts from an automorphic representation π of some Levi subgroup M of G (eg $M = GL_1 \times GL_1$ inside $G = GL_2$) and induces it to some automorphic representation Π of $G(\mathbb{A})$. Then if some special value of $L(\pi)$ is divisible by λ, one can try to construct some automorphic representation Π' whose Satake parameters are congruent to those of Π. If we can attach Galois representations to Π and Π', then we obtain as in Ribet situation, some non-split extension of Galois modules which we can interprets as lying in some appropriate Selmer group attached to π. This strategy have been studied by several authors: Mazur, Wiles, Bellache-Chenevier, Skinner-Urban, Brown, Berger, Klosin. One of the main difficulty is the construction of Π'.

In this paper, using completed cohomology, we propose to give a relative flexible way to construct automorphic congruences between tempered and non tempered automorphic representations of the discrete spectrum of $GL_d(\mathbb{A}_F)$ for some CM field F that verify the following properties: they are in the image of the correspondence of Jacquet-Langlands described in [**8**] theorem VI.1.1, cohomological relatively to the same parameter and autodual, so that they correspond thanks to [**8**] theorem VI.2.1 to cohomological automorphic representations of some similitude group G with signatures $(1, d-1), (0, d), \cdots, (0, d)$, see §2.1.

To state the main result, recall that two automorphic representations are said weakly congruent, if their Satake parameters are congruent at each place where these two representations are unramified, cf. 3.2.3.

Theorem. — *Start from an irreducible automorphic $\overline{\mathbb{Q}}_l$-representation Π of $G(\mathbb{A})$ of some fixed weight ξ, with non trivial invariants under some* *open compact subgroup I of $G(\mathbb{A})$, with degeneracy depth $s > 1$, cf. definition 2.2.4. There exists then an irreducible automorphic representation Π' of $G(\mathbb{A})$ of the same weight ξ, weakly congruent to Π modulo l such that*

- Π' *is tempered, i.e. with degeneracy depth* 1,
- Π' *is of level I' such that $(I')^l = I^l$.*

Remark: The completed cohomology was essentially introduced by Emerton in order to prove the principle of local-global compatibility of some expected p-adic Langlands correspondence for reductive groups as he proved it for $GL_2(\mathbb{Q}_p)$ in [**6**]. The completed cohomology groups encode in particular, besides torsion, all the automorphic congruences between automorphic representations.

Recall first some of the constructions of automorphic congruences already obtained by studying torsion classes in the usual l-adic etale cohomology groups of Shimura varieties of Kottwitz-Harris-Taylor type.

- First, see corollary 2.9 of [**5**], to each non trivial torsion cohomology class of level I, we can associate a collection $\{\Pi(v) : v \in \mathrm{Spl}(I)\}$ indexed by some set of places of the CM field used to defined G, of non isomorphic weakly congruent irreducible automorphic representations unramified outiside $I \cup \{v\}$ and ramified at v, each of them being tempered and so of degeneracy depth 1.
- In section 3 of [**5**] and for some regular weight ξ, we constructed such torsion classes so that we obtained the previous automorphic congruences in regular weight between tempered automorphic representations. Moreover we proved that each of these tempered representation is weakly congruent to some degenerate automorphic representation [(1)] of trivial weight.
- More generally at the end of [**5**], we are able to construct automorphic congruences between representations of different weight but without further informations about their degeneracy depth.

Here as we allow to increase the level at the prime l, the result may be understood in the same spirit as the ordinary p-stabilization of the Eisenstein series $E_k^*(z) := E_k(z) - p^{k-1}E_k(pz)$ of the classical Eisenstein series $E_k(z)$ for $SL_2(\mathbb{Z})$, which become cuspidal at one of the two cusp for $\Gamma_0(p)$. If we think about Ribet's proof of Herbrand theorem, to be able to produce non trivial elements in some Selmer groups, we would

1. So its degeneracy depth is > 1.

need to give a criterion to ensure $I'_l = I_l$ in terms of special values of the L-function associated to Π.

2. Background

2.1. Geometry of KHT Shimura varieties. — Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is quadratic imaginary and $F^+/\mathbb{Q}$ totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we will denote

- F_v the completion of F at v,
- $\mathcal{O}_v$ the ring of integers of F_v,
- ϖ_v a uniformizer,
- q_v the cardinal of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F, of dimension d^2 such that at every place x of F, either B_x is split or a local division algebra and suppose B provided with an involution of second kind $*$ such that $*_{|F}$ is the complex conjugation. For any $\beta \in B^{*=-1}$, denote $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and $G/\mathbb{Q}$ the group of similitudes, denoted G_τ in [**8**], defined for every $\mathbb{Q}$-algebra R by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$ split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \qquad (2.1.1)$$

where, identifying places of F^+ over x with places of F over y, $x = \prod_i z_i$ in F^+.

Convention: for $x = yy^c$ a place of $\mathbb{Q}$ split in E and z a place of F over y as before, we shall make throughout the text, the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (2.1.1).

In [**8**], the author justify the existence of some G like before such that moreover

- if x is a place of $\mathbb{Q}$ non split in E then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [**8**] bottom of page 90, a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

2.1.2. Notation. — *Denote $\mathcal{I}$ the set of open compact subgroup small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $X_{I,\eta} \longrightarrow \operatorname{Spec} F$ the associated Shimura variety of Kottwitz-Harris-Taylor type.*

2.1.3. Definition. — *Define* Spl *the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write* Spl(I) *the subset of* Spl *of places which doesn't divide the level I.*

In the sequel, v will denote a place of F in Spl. For such a place v the scheme $X_{I,\eta}$ has a projective model $X_{I,v}$ over $\operatorname{Spec} \mathcal{O}_v$ with special fiber X_{I,s_v}. For I going through $\mathcal{I}$, the projective system $(X_{I,v})_{I\in\mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \operatorname{val} \circ \operatorname{Art}^{-1}$ and $\operatorname{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is Artin isomorphism which sends geometric Frobenius to uniformizers.

2.1.4. Notations. — *(see* [**1**] *§1.3) For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber $X_{I,\bar{s}_v}$ is denoted*

$$X_{I,\bar{s}_v} =: X^{\geq 1}_{I,\bar{s}_v} \supset X^{\geq 2}_{I,\bar{s}_v} \supset \cdots \supset X^{\geq d}_{I,\bar{s}_v}$$

where $X^{=h}_{I,\bar{s}_v} := X^{\geq h}_{I,\bar{s}_v} - X^{\geq h+1}_{I,\bar{s}_v}$ is an affine scheme(2)*, smooth of pure dimension $d-h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank h. For each $1 \leq h < d$, write*

$$i_h : X^{\geq h}_{I,\bar{s}_v} \hookrightarrow X^{\geq 1}_{I,\bar{s}_v}, \quad j^{\geq h} : X^{=h}_{I,\bar{s}_v} \hookrightarrow X^{\geq h}_{I,\bar{s}_v},$$

and $j^{=h} = i_h \circ j^{\geq h}$.

2.2. Cohomology groups over $\overline{\mathbb{Q}}_l$. — From now on, we fix a prime number l unramified in E and suppose that for every place v of F considered after, its restriction $v_{|\mathbb{Q}}$ is not equal to l. Let us first recall some known facts about irreducible algebraic representations of G, see for example [**8**] p.97. Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and et write Φ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists then an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leq \cdots \leq a_{\sigma,d})$.

For $K \subset \overline{\mathbb{Q}}_l$ a finite extension of $\mathbb{Q}_l$ such that the representation $\iota^{-1} \circ \xi$ of highest weight $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$, is defined over K, write $W_{\xi,K}$ the space of this representation and $W_{\xi,\mathcal{O}}$ a stable lattice under the action of the

2. see for example [**9**]

maximal open compact subgroup $G(\mathbb{Z}_l)$, where $\mathcal{O}$ is the ring of integers of K with uniformizer λ.
Remark: if ξ is supposed to be l-small, in the sense that for all $\sigma \in \Phi$ and all $1 \leq i < j \leq n$ we have $0 \leq a_{\tau,j} - a_{\tau,i} < l$, then such a stable lattice is unique up to a homothety.

2.2.1. Notation. — *We will denote $V_{\xi,\mathcal{O}/\lambda^n}$ the local system on $X_{\mathcal{I}}$ as well as*

$$V_{\xi,\mathcal{O}} = \lim_{\overleftarrow{n}} V_{\xi,\mathcal{O}/\lambda^n} \quad \textit{and} \quad V_{\xi,K} = V_{\xi,\mathcal{O}} \otimes_{\mathcal{O}} K.$$

For $\overline{\mathbb{Z}}_l$ and $\overline{\mathbb{Q}}_l$ version, we will write respectively $V_{\xi,\overline{\mathbb{Z}}_l}$ and $V_{\xi,\overline{\mathbb{Q}}_l}$.

Remark: the representation ξ is said *regular* if its parameter $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ verifies for all $\sigma \in \Phi$ that $a_{\sigma,1} < \cdots < a_{\sigma,d}$.

2.2.2. Definition. — *An irreducible automorphic representation Π is said ξ-cohomological if there exists an integer i such that*

$$H^i\big((\mathrm{Lie}\ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$.

2.2.3. Notation. — *For π_v an irreducible admissible cuspidal representation of $GL_g(F_v)$ and $n \in \frac{1}{2}\mathbb{Z}$, set $\pi_v\{n\} := \pi_v \otimes q^{-n \,\mathrm{val} \circ \det}$. Define then the Steinberg representation $\mathrm{St}_s(\pi_v)$ (resp. the Speh representation $\mathrm{Speh}_s(\pi_v)$) of $GL_{sg}(F_v)$, as the unique irreducible quotient (resp. subspace) of the standard parabolic induced representation $\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\}$.*

A non degenerate irreducible representation of $GL_d(F_v)$ can be written as a full parabolic induced representations $\mathrm{St}_{t_1}(\pi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\pi_{v,r})$. For a place v such that $G(F_v) \simeq GL_d(F_v)$ in the sense of our previous convention, the local component Π_v of Π at v is isomorphic to some $\mathrm{Speh}_s(\pi_v)$ where π_v is an irreducible non degenerate representation, $s \geq 1$ is an integer and $\mathrm{Speh}_s(\pi_v)$ is the Langlands quotient of the parabolic induced representation $\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\}$. In terms of the Langlands correspondence, $\mathrm{Speh}_s(\pi_v)$ corresponds to $\sigma \oplus \sigma(1) \oplus \cdots \oplus \sigma(s-1)$ where σ is the representation of $\mathrm{Gal}(\bar{F}/F)$ associated to π_v by the local Langlands correspondence.

2.2.4. Definition. — *(cf. [**10**]) For Π an automorphic irreducible representation ξ-cohomological of $G(\mathbb{A})$, then, see for example lemma 3.2 of [**3**], there exists an integer s called the degeneracy depth of Π, such* *that through the correspondence of Jacquet-Langlands and base change, its associated representation of $GL_d(\mathbb{A}_\mathbb{Q})$ is isobaric of the following form*

$$\mu|\det|^{\frac{1-s}{2}} \boxplus \mu|\det|^{\frac{3-s}{2}} \boxplus \cdots \boxplus \mu|\det|^{\frac{s-1}{2}}$$

where μ is an irreducible cuspidal representation of $GL_{d/s}(\mathbb{A}_\mathbb{Q})$.

Remark: if ξ is a regular parameter then the depth of degeneracy of any irreducible automorphic representation ξ-cohomological is necessary equal to 1. In particular theorem 4.3.1 of [**2**] is compatible with the classical result saying that for a regular ξ, the cohomology of the Shimura variety X_I with coefficients in $V_{\xi,\overline{\mathbb{Q}}_l}$, is concentrated in middle degree.

2.2.5. Notation. — *For any finite level I^l outside the place l, we denote for every $1 \leq h \leq d$:*

$$H^i_{I^l,\xi}(c,h) = \varinjlim_{I_l} H^i_c(X^{=h}_{I^l I_l,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]),$$

$$H^i_{I^l,\xi}(*,h) = \varinjlim_{I_l} H^i(X^{=h}_{I^l I_l,\bar{s}_v}, j^{=h}_* V_{\xi,\overline{\mathbb{Z}}_l}[d-h]),$$

and

$$H^i_{I^l,\xi}(h) = \varinjlim_{I_l} H^i_c(X^{\geq h}_{I^l I_l,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]),$$

where the limit is taken over all open compact subgroup I_l of $G(\mathbb{Q}_l)$.

Over $\overline{\mathbb{Q}}_l$, we can described these cohomology groups by taking the invariants under I^l of

$$H^i_\xi(c,h) = \varinjlim_{I^l \in \mathcal{I}} H^i_{I^l,\xi}(c,h) \quad \text{and} \quad H^i_\xi(h) = \varinjlim_{I^l \in \mathcal{I}} H^i_{I^l,\xi}(h),$$

which are representations of $G(\mathbb{A}^{\infty,v})$ explicitly described in [**2**].

2.2.6. Proposition. — [**2**] *§3.6 or* [**3**] *§3.2*
Let $1 \leq h \leq d-1$ and Π^∞ be an irreducible subquotient of $H^i_\xi(c,h) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ for some $i > 0$. Then there exists a unique automorphic representation ξ-cohomological $\widetilde{\Pi}$ such that $\widetilde{\Pi}^{\infty,v} \simeq \Pi^{\infty,v}$ and is of degeneracy depth $s = h + i$.

2.2.7. Proposition. — [**3**] *§3.2*
Let $1 \leq h \leq d-1$ and Π^∞ be an irreducible subquotient of $H^i_\xi(h) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$. Then there exists a unique automorphic representation ξ-cohomological $\widetilde{\Pi}$ such that $\widetilde{\Pi}^{\infty,v} \simeq \Pi^{\infty,v}$. Moreover

– *if* $\widetilde{\Pi}$ *is tempered, i.e. of degeneracy depth* 1*, then* $i=0$.
– *Otherwise, its degeneracy depth* s *is* $\geq h$ *and* $i \equiv s-h \mod 2$ *with* $h-s \leq i \leq s-h$.

2.3. Completed cohomology. — Given a level $I^l \in \mathcal{I}$ maximal at l, recall that the completed cohomology groups are

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}/\lambda^n}) := \varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}/\lambda^n}[d-1])$$

and

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) := \varprojlim_{n} \widetilde{H^i}_{I^l}(V_{\xi,\mathcal{O}/\lambda^n}),$$

where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_l$ on which the representation ξ is defined.

2.3.1. Notation. — *When* $\xi = 1$ *is the trivial representation, we will denote*

$$\widetilde{H}^i_{I^l} := \widetilde{H}^i_{I^l}(V_{1,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_l.$$

Remark: for n fixed, there exists an open compact subgroup $I_l(n)$ such that, using the notations below 2.2.1, every $I_l \subset I_l(n)$ acts trivialy on $W_{\xi,\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}/\lambda^n$. We then deduce that the completed cohomology groups doesn't depend of the choice of ξ in the sense where, see theorem 2.2.17 of [**7**]:

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_l \simeq \widetilde{H}^i_{I^l} \otimes W_\xi$$

where $G(\mathbb{Q}_l)$ acts diagonally on the right side.
Remark: the choice of the " tame " level I^l is harmless in the sense that, cf. [**7**] theorem 0.1 (ii), for any $I^l \subset J^l$, we can recover $\widetilde{H}^i_{J_l}$ from $\widetilde{H}^i_{J_l}$ by taking invariants under J^l/I^l,

$$\widetilde{H}^i_{J_l} = \big(\widetilde{H}^i_{I_l}\big)^{J_l/I_l}.$$

To recover the cohomology at finite level from completed cohomology groups, on has to use the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(I_l, \widetilde{H}^j_{I_l} \otimes V_\xi) \Rightarrow H^{i+j}(X_{I^l I_l}, V_\xi[d-1]). \tag{2.3.2}$$

Consider also

$$\widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}}) = \varprojlim_{n}\Big(\varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1]) \ / \ \lambda^n H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1])\Big)$$

the l-adic completion of $H^i_{I^l}(V_{\xi,\mathcal{O}}) := \varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1])$ and the l-adic Tate module of $H^i_{I^l}(V_{\xi,\mathcal{O}})$

$$T_l H^i_{I^l}(V_{\xi,\mathcal{O}}) := \varprojlim_n H^i_{I^l}(V_{\xi,\mathcal{O}})[\lambda^n].$$

Note that the l-adic completion kills the l-divisible part while the l-adic Tate module knows only about torsion. Recall then the short exact sequence

$$0 \to \widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \longrightarrow \widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \longrightarrow T_l H^{i+1}_{I^l}(V_{\xi,\mathcal{O}}) \to 0. \tag{2.3.3}$$

3. Cohomology of Harris-Taylor perverse sheaves

We want to study $\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_l$ through the short exact sequence (2.3.3) so that we have to understand both the inductive limit of the free part of the $H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1])$ when I_l describe the subgroup of $G(\mathbb{Q}_l)$ and the Tate module $T_p H^{i+1}_{I^l}(V_{\xi,\mathcal{O}})$. The strategy is to use the smooth base change theorem at a place v not above l with a level I^l such that its local component at v is maximal, i.e. $I_v = GL_d(\mathcal{O}_v)$.

3.1. Completed cohomology groups.

— To use similar notations as in previous work, we introduce the following.

3.1.1. Notation. — *For $1 \leq h \leq d$, let denote $\mathcal{F}(\mathbb{1}, h)$ the trivial shifted local system $\overline{\mathbb{Z}}_l[d-h]$ over $X^{=h}_{I,\bar{s}_v}$. We will also use $\mathcal{F}_\xi(\mathbb{1}, h) := \mathcal{F}(\mathbb{1}, h) \otimes V_\xi$.*

Recall that for any finite level $I \in \mathcal{I}$ such that $I_v = GL_d(\mathcal{O}_v)$ is maximal, $X^{\geq h}_{I,\bar{s}_v}$ is smooth of dimension $d-h$. In particular the shifted local system $\mathcal{F}(\mathbb{1}, h)[d-h]$ is perverse and its intermediate extension

$$ {}^p j^{=h}_{!*} \mathcal{F}(\mathbb{1}_v, h) \simeq {}^{p+} j^{=h}_{!*} \mathcal{F}(\mathbb{1}_v, h), \tag{3.1.2}$$

where $p+$ is the t-structure obtained by Grothendieck-Verdier duality, from the classical one, named p, is simply isomorphic to the trivial shifted local system on $X^{\geq h}_{I,\bar{s}_v}$. The trivial short exact sequence

$$0 \to (\overline{\mathbb{Z}}_l)_{X^{=h}_{I,\bar{s}_v}} \longrightarrow (\overline{\mathbb{Z}}_l)_{X^{\geq h}_{I,\bar{s}_v}} \longrightarrow (\overline{\mathbb{Z}}_l)_{X^{\geq h+1}_{I,\bar{s}_v}} \to 0$$

can be written in terms of perverse sheaves as

$$0 \to {}^p j_{!*}^{=h+1} \mathcal{F}_\xi(\mathbb{1}_v, h+1) \longrightarrow j_!^{=h} \mathcal{F}_\xi(\mathbb{1}_v, h) \longrightarrow {}^p j_{!*}^{=h} \mathcal{F}_\xi(\mathbb{1}_v, h) \to 0. \quad (3.1.3)$$

or dually

$$0 \to {}^p j_{!*}^{=h} \mathcal{F}_\xi(\mathbb{1}_v, h) \longrightarrow j_*^{=h} \mathcal{F}_\xi(\mathbb{1}_v, h) \longrightarrow {}^p j_{!*}^{=h+1} \mathcal{F}_\xi(\mathbb{1}_v, h+1) \to 0. \quad (3.1.4)$$

3.1.5. Notation. — *For $1 \leq h \leq d$, we denote*

- $\widehat{H}^i_{I_l,\xi}(h)$ *the l-adic completion of $H^i_{I^l,\xi}(h)$,*
- $T_l H^i_{I^l,\xi}(h) = \lim\limits_{\substack{\leftarrow \\ n}} H^i_{I^l,\xi}(h)[l^n]$ *its l-adic Tate module and*
- $\widetilde{H}^i_{I^l}(h)$ *its completed cohomology group.*

3.1.6. Proposition. — *Let I^l a open compact subgroup of $G(\mathbb{A}^{\infty,l})$ maximal at v. Then for any $2 \leq h \leq d$ and for any $i > 0$, the cohomology groups $H^i_{I^l,\xi}(h)$ are divisible and free. For $h = 1$, they are divisible and free for any $i > 1$ and divisible for $i = 1$.*

Proof. — From the smooth base change theorem, we have $H^i(X_{I^l I_l,\bar{s}_v}, V_{\xi,\mathbb{Z}_l}) \simeq H^i(X_{I^l I_l,\bar{\eta}_v}, V_{\xi,\mathbb{Z}_l})$. From corollary IV.2.2 of [**11**], we know that the

- $\widetilde{H}^i(V_{\xi,\overline{\mathbb{Z}}_l})$ are trivial for all $i > d-1$, so that from (2.3.3), we can deduce that the p-adic completion $\widehat{H}^i_{I^l}(V_{\xi,\mathcal{O}})$ of $H^i_{I^l}(V_{\xi,\mathcal{O}})$ is trivial, that is $H^i_{I^l}(V_{\xi,\mathcal{O}})$ is divisible for every $i > 0$;
- for every $i > 0$, the Tate module $T_p H^{i+1}_{I^l}(V_{\xi,\mathcal{O}})$ is trivial so that for every $i > 1$, the divisible module $H^i_{I^l}(V_{\xi,\mathcal{O}})$ is also free.

Then we argue by induction on h: assume the proposition is true for $h-1$. Recall that $X^{=h}_{I^l I_l,\bar{s}_v}$ is an affine scheme, so that the cohomology groups $H^i(X_{I^l I_l,\bar{s}_v}, j_*^{=h}\mathcal{F}_\xi(\mathbb{1}_v, h))$ are trivial for $i > 0$. The long exact sequence of cohomology groups associated to the short exact sequence (3.1.4), gives for every $i > 0$:

$$H^i(X_{I^l I_l}, {}^p j_{!*}^{=h+1} \mathcal{F}_\xi(\mathbb{1}_v, h+1)) \simeq H^{i+1}(X_{I^l I_l}, {}^p j_{!*}^{=h} \mathcal{F}_\xi(\mathbb{1}_v, h)),$$

which allows us to conclude par induction. $\square$

3.2. Automorphic congruences. — We want now to understand the $\widetilde{H}^i_{I^l}(\chi_v,h)\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$ for $i<0$. For this we first need some notations about Hecke algebras. Let $\mathrm{Unr}(I)$ be the union of

– places $q\neq l$ of $\mathbb{Q}$ inert in E not below a place of Bad and where I_q is maximal,
– and places $w\in\mathrm{Spl}(I)$.

3.2.1. Notation. — *For $I\in\mathcal{I}$ a finite level, write*

$$\mathbb{T}_I:=\prod_{x\in\mathrm{Unr}(I)}\mathbb{T}_x$$

where for x a place of $\mathbb{Q}$ (resp. $x\in\mathrm{Spl}(I)$), $\mathbb{T}_x$ is the unramified Hecke algebra of $G(\mathbb{Q}_x)$ (resp. of $GL_d(F_x)$) over $\overline{\mathbb{Z}}_l$.

Example: for $w\in\mathrm{Spl}(I)$, we have

$$\mathbb{T}_w=\overline{\mathbb{Z}}_l\big[T_{w,i}:\ i=1,\cdots,d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w,\cdots,\varpi_w}^{i},\overbrace{1,\cdots,1}^{d-i})GL_d(\mathcal{O}_w)\subset GL_d(F_w).$$

More generally, the Satake isomorphism identifies $\mathbb{T}_x$ with $\overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ where

– T_x is a split torus,
– W_x is the spherical Weyl group
– and $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.

Consider a fixed maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ and for every $x\in\mathrm{Unr}(I)$ let denote $S_{\mathfrak{m}}(x)$ be the multi-set [3] of modulo l Satake parameters at x associated to $\mathfrak{m}$.
Example: for every $w\in\mathrm{Spl}(I)$, the multi-set of Satake parameters at w corresponds to the roots of the Hecke polynomial

$$P_{\mathfrak{m},w}(X):=\sum_{i=0}^{d}(-1)^i q_w^{\frac{i(i-1)}{2}}\overline{T_{w,i}}X^{d-i}\in\overline{\mathbb{F}}_l[X]$$

i.e. $S_{\mathfrak{m}}(w):=\big\{\lambda\in\mathbb{T}_I/\mathfrak{m}\simeq\overline{\mathbb{F}}_l$ such that $P_{\mathfrak{m},w}(\lambda)=0\big\}$. For a maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_{I^l}\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l$, we also have the multi-set of Satake parameters

$$S_{\widetilde{\mathfrak{m}}}(w):=\big\{\lambda\in\mathbb{T}_I\otimes_{\overline{\mathbb{Z}}_l}\overline{\mathbb{Q}}_l/\widetilde{\mathfrak{m}}\simeq\overline{\mathbb{Q}}_l\text{ such that }P_{\widetilde{\mathfrak{m},w}}(\lambda)=0\big\}.$$

3. A multi-set is a set with multiplicities.

3.2.2. Notation. — *Let* Π *be an irreducible automorphic representation of* $G(\mathbb{A})$ *of level* I *which means here, that* Π *has non trivial invariants under* I *and for every* $x \in \mathrm{Unr}(I)$*, then* Π_x *is unramified. Then* Π *defines a maximal ideal* $\widetilde{\mathfrak{m}}(\Pi)$ *of* $\mathbb{T}_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$*.*

Let Π_1 and Π_2 be two such automorphic representations both defined over some finite extension K of $\mathbb{Q}_l$: let then ϖ_K be a uniformiser of the ring of integers $\mathcal{O}_K$ of K. For an open compact subgroup I of $G(\mathbb{A})$ such that both Π_1 and Π_2 are of level I, we have their set of Satake parameters $S_{\widetilde{\mathfrak{m}}(\Pi_1)}(w)$ and $S_{\widetilde{\mathfrak{m}}(\Pi_2)}(w)$.

3.2.3. Definition. — *We then say that two such automorphic representations* Π_1 *and* Π_2 *are weakly congruent modulo* l*, if modulo* ϖ_K *and for almost all place* $w \in \mathrm{Unr}(I)$*, the two multi-sets* $S_{\widetilde{\mathfrak{m}}(\Pi_1)}(w)$ *and* $S_{\widetilde{\mathfrak{m}}(\Pi_2)}(w)$ *are the same.*

Remark: Π_1 and Π_2 are weakly congruent modulo l if and only if $\widetilde{\mathfrak{m}}(\Pi_1)$ and $\widetilde{\mathfrak{m}}(\Pi_2)$ are both contained in the same maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$.

Recall that for an irreducible automorphic cohomological representation Π with degeneracy depth equals to s, associated to the maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ where $\Pi^{I^l} \neq (0)$, then for any $w \in \mathrm{Spl}(I)$, the set $S_{\widetilde{\mathfrak{m}}}(w)$

- can be written as a disjoint union of g segments $\{\alpha, q_w\alpha, \cdots, q_w^{s-1}\alpha\}$ of length s,
- and does not contain any segment of length strictly greater than s.

We will say that $\widetilde{\mathfrak{m}}$ is of degeneracy depth s.

3.3. Proof of the main result. — Fix now an irreducible automorphic $\overline{\mathbb{Q}}_l$-representation Π of $G(\mathbb{A})$ of some fixed weight ξ, with level I and degeneracy depth $s > 1$. Let $\widetilde{\mathfrak{m}}(\Pi)$ be the prime ideal of $\mathbb{T}_I$ associated to Π and $\mathfrak{m}$ the maximal ideal of $\mathbb{T}_I$ containing $\widetilde{\mathfrak{m}}(\Pi)$.

3.3.1 — *Hypothesis*: through all this section, we then suppose by absurdity, that the conclusion of the theorem in the introduction is false, i.e. there is no tempered irreducible representation Π' of weight ξ which is weakly congruent to Π, and with non trivial invariants under $I' = I^l I'_l$ whatever is the choice of I_l.

3.3.2. Proposition. — *Under the previous hypothesis, for all* $1 \leq t \leq d$ *and every* i*, the localized cohomology groups* $H^i_{I^l,\xi}(h)_{\mathfrak{m}}$ *are free.*

Proof. — We follow, in the spirit, the proof of the main theorem 4.7 of [**4**]. We use the short exact sequence (3.1.3). Over $\overline{\mathbb{Q}}_l$, the only non trivial map between

$$H^i(X_{I^lI_l,\bar{s}_v}, {}^p j^{=t+1}_{!*}\mathcal{F}_\xi(\mathbb{1}_v, t+1)) \longrightarrow H^i(X_{I^lI_l,\bar{s}_v}, j^{=h}_{!}\mathcal{F}_\xi(\mathbb{1}_v, h)) \quad (3.3.2)$$

is for $i = 0$ and corresponds to tempered representations. In particular, under our hypothesis, all these maps are trivial after $\mathfrak{m}$-localisation.

Argue then by induction on t from d to 1. The case $t = d$ is trivial as $X^{\geq d}_{I^lI_l,\bar{s}_v}$ is 0-dimensional. Suppose now the result true for $t > t_0$. As $X^{=t_0}_{I^lI_l,\bar{s}_v}$ is affine, then $H^i(X_{I^lI_l,\bar{s}_v}, j^{=h}_{!}\mathcal{F}_\xi(\mathbb{1}_v, h))$ is trivial for every $i < 0$ and free for $i = 0$. In particular the result is true for every $i < 0$ and for $i = 0$, it's true because the map in (3.3.2) for $i = 0$ and $t = t_0$, is, after $\mathfrak{m}$-localisation, trivial. We then deduce the result for $i > 0$, using Grothendieck-Verdier duality and the isomorphism (3.1.2). □

3.3.3. Definition. — *Let $s_0 \geq s$ maximal such that there exists an irreducible automorphic representation Π' of weight ξ weakly congruent to Π and with level of the form $I' = I^l I'_l$ for some compact open subgroup I'_l of $G(\mathbb{Q}_l)$.*

3.3.4. Corollary. — *Under our hypothesis, $\widetilde{H}^i_{I^l}(t)_\mathfrak{m} \simeq \widehat{H}^i_{I^l}(t)_\mathfrak{m}$ verify the following properties:*

- *It's trivial for every $t > s_0$.*
- *For $t \leq s_0$, if it's non trivial then $i \geq t - s_0$, and it's non trivial for $i = t - s_0$.*

Remark: recall from proposition 3.1.6, that these completed cohomology groups are trivial for $i > 0$.

Proof. — The isomorphism follows from the previous proposition and the short exact sequence (2.3.3). If $t > s_0$, the result follows from the fact that for every i, we have $H^i_{I^l,\xi}(t)_\mathfrak{m} = (0)$. For $t = s_0$, the same argument tells us that $\widetilde{H}^i_{I^l}(s_0)_\mathfrak{m} = (0)$ for all $i \neq 0$. If $\widetilde{H}^0_{I^l}(s_0)_\mathfrak{m}$ were trivial, then from the spectral sequence (2.3.2), then all the $H^i(X^{s_0}_{I^lI_l}, V_{\xi,\overline{\mathbb{Z}}_l})_\mathfrak{m}$ would be trivial, which is not the case for $i = 0$.

Suppose now the result true for $t > t_0$. Recall from the previous proof, that the long exact sequence associated to (3.1.3), gives for all $i < 0$:

$$H^i(X_{I^lI_l,\bar{s}_v}, {}^p j^{=t}_{!*}\mathcal{F}_\xi(\mathbb{1}_v, t))_\mathfrak{m} \simeq H^{i+1}(X_{I^lI_l,\bar{s}_v}, {}^p j^{=t+1}_{!*}\mathcal{F}_\xi(\mathbb{1}_v, t+1))_\mathfrak{m},$$

so we conclude by induction. □

Then we look at the cohomology groups $H^i_{I^l,\xi}(*,s_0-1)_{\mathfrak{m}}$ through the short exact sequence (3.1.4) where we recall that, thanks to our absurd hypothesis, the $H^i_{I^l,\xi}(h)_{\mathfrak{m}}$ are free. In particular we have

$$\cdots \to H^0_{I^l,\xi}(s_0)_{\mathfrak{m}} \longrightarrow H^1_{I^l,\xi}(s_0-1)_{\mathfrak{m}} \longrightarrow H^1_{I^l,\xi}(*,s_0-1)_{\mathfrak{m}} \to \cdots$$

where, as over $\overline{\mathbb{Q}}_l$,

$$H^0_{I^l,\xi}(s_0)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \hookrightarrow H^1_{I^l,\xi}(s_0-1)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$$

the first map $H^0_{I^l,\xi}(s_0)_{\mathfrak{m}} \longrightarrow H^1_{I^l,\xi}(s_0-1)_{\mathfrak{m}}$ is injective. But we show that

- $H^0_{I^l,\xi}(s_0)_{\mathfrak{m}}$ is not divisible, cf. previous corollary;
- $H^1_{I^l,\xi}(s_0-1)_{\mathfrak{m}}$ is divisible, cf. proposition 3.1.6.

We then deduce that the torsion of $H^1_{I^l,\xi}(*,s_0-1)_{\mathfrak{m}}$ is non trivial which is impossible because, as $X^{=s_0-1}_{I^lI_l,\bar{s}_v}$ is affine, then

$$H^i(X^{\geq s_0-1}_{I^lI_l,\bar{s}_v}, j^{\geq s_0-1}_* \mathcal{F}_\xi(\mathbb{1}, s_0-1)) = (0) \quad \forall i>0.$$

We then conclude that our previous hypothesis was absurd and so the main theorem is proved.

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Boyer Pascal, Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS, UMR 7539, F-93430, Villetaneuse (France), PerCoLaTor: ANR-14-CE25
E-mail : `boyer@math.univ-paris13.fr`