$\overline{\mathbb{F}}_l$-GALOIS SEMI-SIMPLICITY AND LEVEL RAISING

by

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Abstract. — For a maximal ideal $\mathfrak{m}$ of some anemic Hecke algebra $\mathbb{T}^S$, associated to an irreducible Galois $\overline{\mathbb{F}}_l$-representation of dimension d, on can also define a Galois $\mathbb{T}^S_{\mathfrak{m}}$-representation $\rho_{\mathfrak{m}}$. The length of $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is equal to the number of prime ideals $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ and we try to translate some of the properties of $\{\widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$ into those of $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. For example the level raising (or lowering) property is encoded by the non semi-simplicity of $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$.

Contents

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1. Introduction

Let $F = EF^+$ be a finite CM extension of $\mathbb{Q}$ with $E/\mathbb{Q}$ an imaginary quadratic field and F^+ totally real. Consider then a similitude group $G/\mathbb{Q}$ as in §2 and a place v of F above a prime number $p = uu^c$ split in E and such that G is split at p with $G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times GL_d(F_v) \times \prod_{v \neq w|u} (B_w^{op})^\times$, cf. §2. For any finite set $S \ni v$ of places of F, let $\mathbb{T}^S$ be the anemic Hecke algebra and, for ξ an algebraic representation of $G(\mathbb{Q})$, we denote by $\mathbb{T}_\xi^S$ the quotient of $\mathbb{T}^S$ of ξ-cohomological Satake's parameters. For any maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\xi^S$, and for a prime ideal $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, we denote by

$$\rho_{\widetilde{\mathfrak{m}}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

the Galois $\overline{\mathbb{Q}}_l$-representation associated to $\widetilde{\mathfrak{m}}$, cf. [**21**], where $\mathrm{Gal}_{F,S}$ is the Galois group of the maximal extension of F which is unramified outside S. By Cebotarev's density theorem and the fact that a semi-simple representation is determined, up to isomorphism, by characteristic polynomials, then the semi-simple class $\overline{\rho}_{\mathfrak{m}}$ of the reduction modulo l of $\rho_{\widetilde{\mathfrak{m}}}$ depends only of the maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\xi^S$ containing $\widetilde{\mathfrak{m}}$.

Main assumptions: *we now suppose that $l \geqslant d$ and*

- $\overline{\rho}_{\mathfrak{m}}$ *is absolutely irreducible so that for every $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, $\rho_{\widetilde{\mathfrak{m}}}$ has, up to homothety, only one stable $\overline{\mathbb{Z}}_l$-lattice.*
- *The set $S_v(\mathfrak{m})$ does not contain any subset of the form $\{\lambda, q_v\lambda, \cdots, q_v^{e_v(l)-1}\lambda\}$ where $e_v(l)$ is equal to either the order of $q_v \in \mathbb{F}_l^\times$ if it is different from 1 or l otherwise, cf. notation 2.10.*

By classical arguments due to Carayol, we can then define a representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}_{\xi,\mathfrak{m}}^S),$$

interpolating the $\rho_{\widetilde{\mathfrak{m}}}$ for all $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$. By Cebotarev such a $\rho_{\mathfrak{m}}$ is, up to isomorphism, uniquely determined and by construction

$$\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\widetilde{\mathfrak{m}} \subset \mathfrak{m}} \rho_{\widetilde{\mathfrak{m}}}$$

is semi-simple.

Main result: *Under the previous assumptions, if $\mathfrak{m}$ has the level raising (or lowering) property then $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is not semi-simple, cf. theorem 5.4*

Remark. In 5.4 we suppose that $\mathfrak{m}$ is moreover both KHT-free and KHT-typic. If one believe in Tate conjecture then $\mathfrak{m}$ should always be KHT-typic, cf. proposition 5.3. Moreover in [**11**], we prove that KHT-freeness is implied by the previous main assumptions.
Remark. In the end of §5, we moreover prove some results on the depth of $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ in the sense of definition 5.5 which roughly speaking says that if you are able to find $\widetilde{\mathfrak{m}}_1$ and $\widetilde{\mathfrak{m}}_2$ such that their monodromy at v are really different, then $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is far from being semi-simple.

The strategy rests on the study of the cohomology of KHT Shimura vatieties. More precisely, for any open compact subgroup I of $G(\mathbb{A}^\infty)$, let denote by $\mathrm{Sh}_{I,v} \rightarrow \mathrm{Spec}\,\mathcal{O}_v$ the Kottwitz-Harris-Taylor Shimura variety with level I, cf. definition 2.4, where $\mathcal{O}_v$ is the ring of integers of the local field F_v of F at v. If one believe that Tate conjecture is true, then $\rho_{\mathfrak{m}}$ should appear in the cohomology of $\mathrm{Sh}_{I,v}$ localized at $\mathfrak{m}$, cf. proposition 5.3. If moreover we choose such $\mathfrak{m}$ so that these localized cohomology groups are torsion free concentrated in middle degree, then $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ also appears in the $\overline{\mathbb{F}}_l$-cohomology of $\mathrm{Sh}_{I,v}$. The idea is then to analyse the action of the nilpotent monodromy operator $\overline{N}^{coho}_{\mathfrak{m},v}$ acting on the $\overline{\mathbb{F}}_l$-cohomology group in middle degree of $\mathrm{Sh}_{I,v}$. More precisely for $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, we denote by

$$\underline{d_{\widetilde{\mathfrak{m}},v}} = \Big(n_1(\widetilde{\mathfrak{m}}) \geqslant n_2(\widetilde{\mathfrak{m}}) \geqslant \cdots \geqslant n_r(\widetilde{\mathfrak{m}}) \Big)$$

the partition of $d = n_1(\widetilde{\mathfrak{m}}) + \cdots n_r(\widetilde{\mathfrak{m}})$ corresponding to the restriction $\rho_{\widetilde{\mathfrak{m}},v}$ of $\rho_{\widetilde{\mathfrak{m}}}$ to the decomposition group at v:

$$\rho_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{Sp}_{n_1(\widetilde{\mathfrak{m}})}(\rho_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{n_r(\widetilde{\mathfrak{m}})}(\rho_{v,r}),$$

where the $\rho_{v,i}$ are supposed to be characters, and

$$\mathrm{Sp}_{n_i(\widetilde{\mathfrak{m}})}(\rho_{v,i}) = \rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})}{2}) \oplus \rho_{v,i}(\frac{3-n_i(\widetilde{\mathfrak{m}})}{2}) \oplus \cdots \oplus \rho_{v,i}(\frac{n_i(\widetilde{\mathfrak{m}})-1}{2}),$$

where $N_{\widetilde{\mathfrak{m}},v}$ induces isomorphisms $\rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})+2\delta}{2}) \longrightarrow \rho_{v,i}(\frac{1-n_i(\widetilde{\mathfrak{m}})+2(\delta+1)}{2})$ for $0 \leqslant \delta < n_i(\widetilde{\mathfrak{m}}) - 1$ and is trivial on $\rho_{v,i}(\frac{n_i(\widetilde{\mathfrak{m}})-1}{2})$.

As the order of unipotency of the monodromy operator is trivially $\leqslant d$, for $l \geqslant d$ we can define its logarithm in $\overline{\mathbb{F}}_l$ and so define the modulo l nilpotent monodromy operator $\overline{N}_{\mathfrak{m},v}$ associated to $\overline{\rho}_{\mathfrak{m}}$ at the place v: recall that as $\overline{\rho}_{\mathfrak{m}}$ is supposed to be irreducible, each of the $\rho_{\widetilde{\mathfrak{m}}}$ has, up to homothety, a unique stable $\overline{\mathbb{Z}}_l$-lattice so that $\overline{N}_{\mathfrak{m},v}$ is well defined and does not depend on the choice of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$.

1.1. Definition. — *(cf. the introduction of* [**12**]*)*
We say that $\mathfrak{m}$ *is KHT-free if the cohomology groups of the Kottwitz-Harris-Taylor Shimura variety of notation 2.4, localized at* $\mathfrak{m}$*, are free.*

From [**8**], any of the following properties ensure KHT-freeness of $\mathfrak{m}$.

(1) There exists a place $w_1 \notin S$ of F above a prime p_1 splits in E, such that the multi-set $S_{\mathfrak{m}}(w_1)$ of roots of the characteristic polynomial $P_{\mathfrak{m},w_1}(X)$ of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_{w_1})$, does not contain any sub-multi-set of the shape $\{\alpha, q_{w_1}\alpha\}$ where q_{w_1} is the order of the residue field of F at w_1. This hypothesis is called *generic* in [**13**].
(2) When $[F(\exp(2i\pi/l)) : F] > d$, if we suppose the following property to be true, cf. [**8**] 4.17. If $\theta : G_F \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ is an irreducible continuous representation such that for all place $w \notin S$ above a prime $x \in \mathbb{Z}$ split in E, then $P_{\mathfrak{m},w}(\theta(\mathrm{Frob}_w)) = 0$ (resp. $P_{\mathfrak{m}^\vee,w}(\theta(\mathrm{Frob}_w)) = 0$) implies that θ is equivalent to $\overline{\rho}_{\mathfrak{m}}$ (resp. $\overline{\rho}_{\mathfrak{m}^\vee}$), where $\mathfrak{m}^\vee$ is the maximal ideal of $\mathbb{T}_S$ associated to the dual mutliset of Satake parameters, cf. [**8**] notation 4.4. In [**16**], the authors proved that the previous property is verified in each of the following cases:
   - either $\overline{\rho}_{\mathfrak{m}}$ is induced from a character of G_K where K/F is a cyclic galoisian extension;
   - or $l \geqslant d$ and $SL_d(k) \subset \overline{\rho}_{\mathfrak{m}}(G_F) \subset \overline{\mathbb{F}}_l^\times GL_d(k)$ for some subfield $k \subset \overline{\mathbb{F}}_l$.

Main observation: *Suppose that* $\mathfrak{m}$ *is KHT-free and KHT-typic in the sense of definition 5.1, then the order of nilpotency of* $\overline{N}^{coho}_{\mathfrak{m},v}$ *is equal to that of* $N_{\mathfrak{m},v}$ *acting on* $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ *and is given, cf. corollary 4.3, by the formula* $\max\{n_1(\widetilde{\mathfrak{m}}), \widetilde{\mathfrak{m}} \subset \mathfrak{m}\}$.

The idea is then to produce $\widetilde{\mathfrak{m}}_1$ and $\widetilde{\mathfrak{m}}_2$ in $\mathfrak{m}$ such that $n_1(\widetilde{\mathfrak{m}}_1) > n_1(\widetilde{\mathfrak{m}}_2)$, so that the order of nilpotency of $\overline{N}_{\mathfrak{m},v}$ is $\leqslant n_1(\widetilde{\mathfrak{m}}_2)$. If $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ were semi-simple, and so isomorphic to copies of $\overline{\rho}_{\mathfrak{m}}$, then the order of nilpotency of $\overline{N}^{coho}_{\mathfrak{m},v}$ would be $\leqslant n_1(\widetilde{\mathfrak{m}}_2)$, which contradicts the main observation.

1.2. Definition. — *A maximal ideal* $\mathfrak{m}$ *is said with indefinite maximal nilpotency at* v *if there exists* $\widetilde{\mathfrak{m}}_1$ *and* $\widetilde{\mathfrak{m}}_2$ *in* $\mathfrak{m}$ *such that* $n_1(\widetilde{\mathfrak{m}}_1) > n_1(\widetilde{\mathfrak{m}}_2)$.

To see that $\mathfrak{m}$ is with indefinite maximal nilpotency at v, we can follow the strategy of Ribet in [**25**], where he considers an absolutely irreducible representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\overline{\mathbb{F}}_l),$$

which is modular of level N, meaning that it arises from a cusp form of weight 2 and trivial character on $\Gamma_0(N)$. Then for a prime $p \nmid lN$ such that

$$\mathrm{Tr}\ \overline{\rho}(\mathrm{Frob}_p) \equiv \pm(p+1) \mod l,$$

with $p+1 \not\equiv 0 \mod l$, he proves that $\overline{\rho}$ also arises from a modular form of level pN which is p-new, i.e. the automorphic representation associated to this modular form has a local component at p which is isomorphic to the Steinberg representation of $GL_2(\mathbb{Q}_p)$. Then the associated maximal ideal $\mathfrak{m}$ is with indefinite maximal nilpotency at p.

In [**28**] Sorensen generalizes this level raising congruences in higher dimension for a connected reductive group G over a totally real field F^+ such that G_∞ is compact. One might also look at [**1**] theorem 1.1 and theorem 4.1, for the case of automorphic representations of unitary type of GL_{2n}.

A maximal ideal $\mathfrak{m}$ *which is KHT-free an with indefinite maximal nilpotency* provided by the anonymous referee whom we thanks to allow us to reproduce it here. The two following newforms which can be found at https://www.pnas.org/content/pnas/94/21/11143.full.pdf

- $f(q) = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \cdots \in S_2(\Gamma_0(11))$,
- $g(q) = q + q^2 + 2q^3 - q^4 - 2q^5 + 2q^6 - q^7 + \cdots \in S_2(\Gamma_0(77))$,

give Galois representations ρ_f and ρ_g which are congruent modulo 3. In Serre's 1972 Inventiones paper it is shown that the image of this modulo 3 representation is $GL_2(\mathbb{F}_3)$ so that $\overline{\rho}_f$ is irreducible. Note then that the 3-adic representation ρ_g has non trivial monodromy at 7 and $\overline{\rho}_f$ is unramified at 7 with $7 \not\equiv -1 \mod 3$. Consider then a real quadratic field F^+ in which 7 splits and which is linearly disjoint over $\mathbb{Q}$ with the fixed field of $\mathrm{Ker}(\overline{\rho}_g)$. Let D/F^+ a quaternion algebra which is non split t one real place and one place above 11. The base change of g to F^+ gives a cohomological automorphic representation of $GL_2(\mathbb{A}_{F^+})$ whose 3-adic Galois representation appears in the cohomology of a Shimura curve attached to D: the 3-adic monodromy at a place dividing 7 is then non trivial while the modulo 3 monodromy is. Base changing to $F = F^+E$ for some suitable quadractic imaginary field E, we then obtain an example of a maximal ideal $\mathfrak{m}$ which is KHT-free and with indefinite maximal nilpotency at v.

Another way to produce $\mathfrak{m}$ which is KHT-free and with indefinite maximal nilpotency at a fixed place v, is to start with an irreducible Galois representation $\rho : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ such that

- there exists p' split in E such that there exists a place v' in F above p' such that the set $S_{v'}(\rho)$ of modulo l eigenvalues of $\rho(\mathrm{Frob}_{v'})$ does not contain any subset of the form $\{\alpha, q_{v'}\alpha\}$.
- There exists $p \neq p'$ split in E and a place v above p such that the partition $\underline{d}_v(\rho) = (n_1 \geqslant \cdots \geqslant n_r)$ associated to the monodromy operator at v verifies the following property. There exists α such that $\{\alpha, q_v\alpha, \cdots, q_v^{n_1}\alpha\}$ is contained in the set $S_v(\rho)$ of modulo l eigenvalues of $\rho(\mathrm{Frob}_v)$.

Then the existence of a lift ρ' of $\overline{\rho}$ such that $\underline{d}_v(\rho') = (n'_1 \geqslant \cdots \geqslant n'_{r'})$ verifies $n'_1 > n_1$ is insured by the next proposition which relies on a conjectural Ihara's lemma in higher dimension proposed by Clozel, Harris and Taylor, for compact unitary groups. In [**12**] we proved some instances of this conjecture implied by a non compact version of Ihara's lemma for $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$. In the last section, we first prove that Ihara's lemma for compact unitary group implies Ihara's lemma for KHT unitary groups and the following result.

1.3. Proposition. — *Suppose that this higher dimensional version of Ihara's lemma is true and that $q_v \equiv 1 \mod l$. Then there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\underline{d}_{\widetilde{\mathfrak{m}},v}$ is as maximal as possible, i.e.*

$$\rho_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{Sp}_{n_1}(\rho_{v,1}) \oplus \cdots \oplus \mathrm{Sp}_{n_r}(\rho_{v,r})$$

such that the eigenvalues of the $\rho_{v,i}(\mathrm{Frob}_v)$ for $i = 1, \cdots, r$, are pairwise distincts modulo l.

2. The monodromy operator acting on the cohomology

Let $F = F^+E$ be a CM field with $E/\mathbb{Q}$ quadratic imaginary and F^+ totally real. Let B/F be a central division algebra with dimension d^2 with an involution of second kind $*$. For $\beta \in B^{*=-1}$, consider the similitude group $G/\mathbb{Q}$ defined for any $\mathbb{Q}$-algebra R by

$$G(R) := \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\},$$

with $B^{op} = B \otimes_{F,c} F$ where $c = *_{|F}$ is the complex conjugation and $\sharp_\beta$ is the involution $x \mapsto x^{\sharp\beta} := \beta x^* \beta^{-1}$. Following [**21**], we assume from now on that $G(\mathbb{R})$ has signatures $(1, d-1), (0, d), \cdots, (0, d)$.

2.1. Definition. — *Let* Spl *be the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$.*

We now suppose that $p = uu^c$ splits in E so that

$$G(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$$

where w describes the places of F above u and we fix a place $v \in \mathrm{Spl}$ dividing p.

2.2. Definition. — *For a finite set S of places of $\mathbb{Q}$ containing the places where G is ramified, denote by $\mathbb{T}_{abs}^S := \prod_{x \notin S} \mathbb{T}_{x,abs}$ the abstract unramified Hecke algebra where $\mathbb{T}_{x,abs} \simeq \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$ for T_x a split torus, W_x the spherical Weyl group and $X^{un}(T_x)$ is the set of $\overline{\mathbb{Z}}_l$-unramified characters of T_x.*

Example. For $x = uu^c$ split in E we have

$$\mathbb{T}_{x,abs} = \prod_{w|u} \overline{\mathbb{Z}}_l\big[T_{w,i} : \ i = 1, \cdots, d\big],$$

where $T_{w,i}$ is the characteristic function of

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w, \cdots, \varpi_w}^{i}, \overbrace{1, \cdots, 1}^{d-i})GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

We then denote by $\mathcal{I}$ the set of open compact subgroups

$$U^p(m_1, \cdots, m_r) = U^p \times \mathbb{Z}_p^\times \times \prod_{i=1}^r \mathrm{Ker}(\mathcal{O}_{B_{v_i}}^\times \longrightarrow (\mathcal{O}_{B_{v_i}}/\mathcal{P}_{v_i}^{m_i})^\times)$$

where U^p is any small enough open compact subgroup of $G(\mathbb{A}^{p,\infty})$ and $\mathcal{O}_{B_{v_i}}$ is the maximal order of B_{v_i} with maximal ideal $\mathcal{P}_{v_i}$ and where $v = v_1, \cdots, v_r$ are the places of F above u with $p = uu^c$.

2.3. Notation. — *For $I = U^p(m_1, \cdots, m_r) \in \mathcal{I}$, we will denote by $I^v(n) := U^p(n, m_2, \cdots, m_r)$. We also denote by* $\mathrm{Spl}(I)$ *the subset of* Spl *of places which does not divide the level I.*

2.4. Notation. — *As defined in* [**21**], *attached to each $I \in \mathcal{I}$ is a Shimura variety called of KHT-type and denoted by*

$$\mathrm{Sh}_{I,v} \longrightarrow \mathrm{Spec}\, \mathcal{O}_v$$

where $\mathcal{O}_v$ denote the ring of integers of the completion F_v of F at v.

Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and write Φ for the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists then, cf. [**21**] p.97, an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leqslant \cdots \leqslant a_{\sigma,d})$. We then denote by

$$V_{\xi,\overline{\mathbb{Z}}_l}$$

the associated $\overline{\mathbb{Z}}_l$-local system on $\mathrm{Sh}_{I,v}$.

2.5. Notation. — *Let $\mathbb{T}^S_\xi$ be the image of $\mathbb{T}^S_{abs}$ inside*

$$\bigoplus_{i=0}^{2d-2} \varinjlim_I H^i_{free}(\mathrm{Sh}_{I,\bar\eta_v}, V_{\xi,\overline{\mathbb{Z}}_l})$$

where the limit concerned the ideals I which are maximal at each places outside S, and $\mathrm{Sh}_{I,\bar\eta_v}$ is the geometric generic fiber of $\mathrm{Sh}_{I,v}$.

Remark. Above H^i_{free} is the free quotient of the cohomology group H^i. From the main result of [**8**], the torsion classes of any of the $H^i(\mathrm{Sh}_{I,\bar\eta_v}, V_{\xi,\overline{\mathbb{Z}}_l})$ raise in characteristic zero, so one can erase the index *free* in the previous notation.

To each maximal ideal $\widetilde{\mathfrak{m}}$ of $\mathbb{T}^S_\xi[1/l]$, or equivalently a minimal prime of $\mathbb{T}^S_\xi$, which we now supposed to be non-Eisenstein, is associated an irreducible automorphic representation $\Pi_{\widetilde{\mathfrak{m}}}$ which is ξ-cohomological, i.e. there exists an integer i such that

$$H^i\big((\mathrm{Lie}\ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$.

2.6. Notation. — *Let denote by $\mathrm{Scusp}_v(\widetilde{\mathfrak{m}})$, the supercuspidal support of its local component at v, denoted $\Pi_{\widetilde{\mathfrak{m}},v}$. Note*[(1)] *that the modulo l reduction of $\mathrm{Scusp}_v(\widetilde{\mathfrak{m}})$ is independent of the choice of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$: we denote it $\mathrm{Scusp}_v(\mathfrak{m})$.*

Recall that the geometric special fiber $\mathrm{Sh}_{I,\bar s_v}$ of $\mathrm{Sh}_{I,v}$, is equipped with the Newton stratification

$$\mathrm{Sh}^{\geqslant d}_{I,\bar s_v} \subset \mathrm{Sh}^{\geqslant d-1}_{I,\bar s_v} \subset \cdots \subset \mathrm{Sh}^{\geqslant 1}_{I,\bar s_v} = \mathrm{Sh}_{I,\bar s_v},$$

(1) It follows, through the Langlands correspondence, from Cebotarev's theorem and the fact that a semi-simple representation is determined, up to isomorphism, by its characteristic polynomials.

where for $1 \leqslant h \leqslant d$, $\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v}$ (resp. $\mathrm{Sh}^{=h}_{I,\bar{s}_v}$) is the closed (resp. the open) Newton stratum of height h and of pure dimension $d-h$, defined as the sub-scheme where the connected component of the universal Barsotti-Tate group is of rank greater or equal to h (resp. equal to h).

Moreover for $1 \leqslant h < d$, the Newton stratum $\mathrm{Sh}^{=h}_{\mathcal{I},\bar{s}_v}$ is geometrically induced under the action of the parabolic subgroup $P_{h,d-h}(F_v)$, defined as the stabilizer of the first h vectors of the canonical basis of F_v^d. Concretely, cf. [**2**] §10.4, this means that there exists a closed sub-scheme $\mathrm{Sh}^{=h}_{I,\bar{s}_v,\overline{1_h}}$ stabilized by the Hecke action of $P_{h,d-h}(F_v)$ and such that

$$\varprojlim_n \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v} \simeq \big(\varprojlim_n \mathrm{Sh}^{=h}_{I^v(n),\bar{s}_v,\overline{1_h}}\big) \times_{P_{h,d-h}(F_v)} GL_d(F_v).$$

2.7. Notation. — *For a representation π_v of $GL_d(F_v)$ with coefficients either $\overline{\mathbb{Q}}_l$ or $\overline{\mathbb{F}}_l$, and $n \in \frac{1}{2}\mathbb{Z}$, we set $\pi_v\{n\} := \pi_v \otimes \nu^n$ where $\nu(g) := q_v^{-\operatorname{val}\det(g)}$. Recall that the normalized induction of two representations $\pi_{v,1}$ and $\pi_{v,2}$ of respectively $GL_{n_1}(F_v)$ and $GL_{n_2}(F_v)$ is*

$$\pi_1 \times \pi_2 := \operatorname{ind}_{P_{n_1,n_2}(F_v)}^{GL_{n_1+n_2}(F_v)} \pi_{v,1}\{\frac{n_2}{2}\} \otimes \pi_{v,2}\{-\frac{n_1}{2}\},$$

and we define inductively

$$\pi_1 \times \cdots \times \pi_s := (\pi_1 \times \cdots \times \pi_{s-1}) \times \pi_s \simeq \pi_1 \times (\pi_2 \times \cdots \times \pi_s).$$

Recall that a representation π_v of $GL_d(F_v)$ is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. a subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero then these two notions coincides, but this is no more true over $\overline{\mathbb{F}}_l$. For example the modulo l reduction of an irreducible $\overline{\mathbb{Q}}_l$-representation is still cuspidal but not necessary supercuspidal, its supercuspidal support being a Zelevinsky segment.

2.8. Definition. — *(see [**30**] §9 and [**4**] §1.4) Let g be a divisor of $d = sg$ and π_v an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$. The induced representation*

$$\pi_v\{\frac{1-s}{2}\} \times \pi_v\{\frac{3-s}{2}\} \times \cdots \times \pi_v\{\frac{s-1}{2}\} \tag{2.8}$$

holds a unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi_v)$ (resp. $\mathrm{Speh}_s(\pi_v)$); it is a generalized Steinberg (resp. Speh) representation.

Remark. For χ_v a character, $\mathrm{Speh}_s(\chi_v)$ is the character $\chi_v \circ \det$ of $GL_s(F_v)$.

Let π_v be an irreducible cuspidal $\overline{\mathbb{Q}}_l$-representation of $GL_g(F_v)$ and fix $t \geqslant 1$ such that $tg \leqslant d$. Thanks to Igusa varieties, Harris and Taylor constructed a local system on $\mathrm{Sh}^{=tg}_{\mathcal{I},\bar{s}_v,\overline{1_{tg}}}$

$$\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} = \bigoplus_{i=1}^{e_{\pi_v}} \mathcal{L}_{\overline{\mathbb{Q}}_l}(\rho_{v,i})_{\overline{1_{tg}}}$$

where

- $\pi_v[t]_D$ is the representation of $D^\times_{v,tg}$ which is the image of the contragredient of $\mathrm{St}_t(\pi_v)$ by the Jacquet-Langlands correspondence,
- $D_{v,tg}$ is the central division algebra over F_v with invariant tg,
- with maximal order denoted by $\mathcal{D}_{v,tg}$
- and with $(\pi_v[t]_D)_{|\mathcal{D}^\times_{v,tg}} = \bigoplus_{i=1}^{e_{\pi_v}} \rho_{v,i}$ with $\rho_{v,i}$ irreductible.

The Hecke action of $P_{tg,d-tg}(F_v)$ is then given through its quotient

$$P_{tg,d-tg}(F_v) \twoheadrightarrow GL_{tg}(F_v) \times GL_{d-tg}(F_v) \twoheadrightarrow GL_{d-tg}(F_v) \times \mathbb{Z},$$

where $GL_{tg}(F_v) \times GL_{d-tg}(F_v)$ is the Levi quotient of the parabolic $P_{tg,d-tg}(F_v)$ and the second map is given by the valuation of the determinant map $GL_{tg}(F_v) \twoheadrightarrow \mathbb{Z}$. These local systems have stable $\overline{\mathbb{Z}}_l$-lattices and we will write simply $\mathcal{L}(\pi_v[t]_D)_{\overline{1_{tg}}}$ for any $\overline{\mathbb{Z}}_l$-stable lattice that we do not want to specify.

2.9. Notations. — *For Π_t any $\overline{\mathbb{Q}}_l$-representation of $GL_{tg}(F_v)$, and $\Xi : \frac{1}{2}\mathbb{Z} \longrightarrow \overline{\mathbb{Z}}_l^\times$ defined by $\Xi(\frac{1}{2}) = q^{1/2}$, we introduce*

$$\widetilde{HT}_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v,\Pi_t) := \mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}$$

and its induced version

$$\widetilde{HT}_{\overline{\mathbb{Q}}_l}(\pi_v,\Pi_t) := \Big(\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Pi_t \otimes \Xi^{\frac{tg-d}{2}}\Big) \times_{P_{tg,d-tg}(F_v)} GL_d(F_v),$$

where the unipotent radical of $P_{tg,d-tg}(F_v)$ acts trivially and the action of

$$(g^{\infty,v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times P_{tg,d-tg}(F_v) \times W_v$$

where W_v is the Weil group at v, is given

- *by the action of g_v^c on Π_t and $\deg(\sigma_v) \in \mathbb{Z}$ on $\Xi^{\frac{tg-d}{2}}$, where $\deg : W_v \longrightarrow \mathbb{Z}$ sends geometric Frobenius to 1,*

– *and the action of* $(g^{\infty,v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty,v}) \times GL_{d-tg}(F_v) \times \mathbb{Z}$ *on* $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\pi_v[t]_D)_{\overline{1_{tg}}} \otimes \Xi^{\frac{tg-d}{2}}$.

We also introduce

$$HT_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t) := \widetilde{HT}_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \Pi_t)[d-tg],$$

and the perverse sheaf

$$P_{\overline{\mathbb{Q}}_l}(t, \pi_v)_{\overline{1_{tg}}} := j^{=tg}_{\overline{1_{tg}},!*} HT_{\overline{\mathbb{Q}}_l,\overline{1_{tg}}}(\pi_v, \mathrm{St}_t(\pi_v)) \otimes \mathbb{L}(\pi_v),$$

and their induced version, $HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t)$ *and* $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$, *where*

$$j^{=tg} = i^{tg} \circ j^{\geqslant tg} : \mathrm{Sh}^{=tg}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}^{\geqslant tg}_{\mathcal{I},\bar{s}_v} \hookrightarrow \mathrm{Sh}_{\mathcal{I},\bar{s}_v}$$

and $\mathbb{L}$ *is the local Langlands correspondence composed by contragredient.*

We will also denote by $HT_{\overline{\mathbb{Q}}_l,\xi}(\pi_v, \Pi_t) := HT_{\overline{\mathbb{Q}}_l}(\pi_v, \Pi_t) \otimes V_\xi$ *and similarly for the other notations as for example* $P_{\overline{\mathbb{Q}}_l,\xi}(t, \pi_v) := P_{\overline{\mathbb{Q}}_l}(t, \pi_v) \otimes V_\xi$.

Remarks:

– We will simply denote by $P(t, \pi_v)$ any $\overline{\mathbb{Z}}_l$-lattice of $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ that we do not want to precise except that it is stable under the various actions. We will use a similar convention for the other sheaves introduced before. When considering $\overline{\mathbb{F}}_l$-coefficients, we will put $\overline{\mathbb{F}}_l$ in place of $\overline{\mathbb{Q}}_l$ in the notations.
– Recall that π'_v is said inertially equivalent to π_v, and we write $\pi_v \sim_i \pi'_v$, if there exists a character $\zeta : \mathbb{Z} \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that $\pi'_v \simeq \pi_v \otimes (\zeta \circ \mathrm{val} \circ \det)$. We denote by e_{π_v} the order of the inertial class of π_v.
– Note, cf. [**3**] 2.1.4, that $P_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ depends only on the inertial class of π_v and
$$P_{\overline{\mathbb{Q}}_l}(t, \pi_v) = e_{\pi_v} \mathcal{P}_{\overline{\mathbb{Q}}_l}(t, \pi_v)$$
where $\mathcal{P}_{\overline{\mathbb{Q}}_l}(t, \pi_v)$ is an irreducible perverse sheaf.
– Over $\overline{\mathbb{Z}}_l$, we also have the $p+$-perverse structure which is dual to the usual p-structure.

2.10. Notation. — *Let denote by*

$$e_v(l) = \begin{cases} l, & \textit{if } q_v \equiv 1 \mod l \\ \textit{the order of } q_v \textit{ modulo } l, & \textit{otherwise} \end{cases}$$

Remark. For a character(2) χ_v and when $t < e_v(l)$, up to homothety there is only one stable $\overline{\mathbb{Z}}_l$-stable lattice of $\mathcal{L}(\pi_v[t]_D)$. From the description of the modulo l reduction of $\mathrm{St}_t(\chi_v)$ in [**5**], the same is then true for $\mathcal{P}(t,\chi_v)$.

2.11. Notation. — *Let denote by*

$$\Psi_v := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])(\frac{d-1}{2})$$

the nearby cycles autodual free perverse sheaf on the geometric special fiber $\mathrm{Sh}_{I,\bar{s}_v}$ *of* $\mathrm{Sh}_{I,v}$.

Following the constructions of [**6**] §2.3, we can then define a $\overline{\mathbb{Z}}_l$-filtration $\mathrm{Fil}^\bullet(\Psi_v)$ whose graded parts $\mathrm{gr}^r(\Psi_v)$ are free $\overline{\mathbb{Z}}_l$-perverse sheaves, cf. [**6**] §1, of the following shape

$$\begin{aligned}{}^p j^{=tg}_{!*} HT(\pi_v, \mathrm{St}_t(\pi_v))(\frac{1-t+2\delta}{2}) \hookrightarrow\!\!\!\!\!\rightarrow \mathrm{gr}^r(\Psi_v) \\ \hookrightarrow\!\!\!\!\!\rightarrow {}^{p+} j^{=tg}_{!*} HT(\pi_v, \mathrm{St}_t(\pi_v))(\frac{1-t+2\delta}{2})\end{aligned}$$

for some $0 \leqslant \delta \leqslant t-1$, where $\hookrightarrow\!\!\!\!\!\rightarrow$ means a bimorphism(3), that is both a epimorphism and a monomorphism, and where the lattice $HT(\pi_v, \mathrm{St}_t(\pi_v))$ depends of the construction in the general situation but we will see that, with our hypothesis, up to isomorphism, there is only one such stable lattice.

Remarks:

- In [**9**], we prove that if you always use the adjunction maps $j^{=h}_! j^{=h,*} \to \mathrm{Id}$ (resp. $\mathrm{Id} \to j^{=h}_* j^{=h,*}$) then all the previous graded parts are isomorphic to p-intermerdiate (resp. $p+$) extensions. In our situation this issue disappear thanks to lemma 3.2.
- We can easily arrange the filtration so that it is compatible with the nilpotent monodromy operator N_v, i.e. so that for any r the image of $\mathrm{Fil}^r(\Psi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ under N_v is some $\mathrm{Fil}^{\phi(r)}(\Psi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ for some decreasing function ϕ.
- When dealing with sheaves, there is no need to introduce the local system $V_{\xi,\overline{\mathbb{Z}}_l}$ because it suffices to add $\otimes_{\overline{\mathbb{Z}}_l} V_{\xi,\overline{\mathbb{Z}}_l}$ to the formulas.

(2)For a general supercuspidal representation π_v whose modulo l reduction ϱ is still supercuspidal, the same is true if $t < m(\varrho)$ where $m(\varrho)$ is either the order of the Zelevinsky line of ϱ if it is > 1, otherwise $m(\varrho) := l$.

(3)by [**6**] corollary 1.4.6, the p and $p+$ intermediate extensions are free

We now consider a fixed local system $V_{\xi,\overline{\mathbb{Z}}_l}$ and, following previous notations, we write $\Psi_{\xi,v} := \Psi_v \otimes_{\overline{\mathbb{Z}}_l} V_{\xi,\overline{\mathbb{Z}}_l}$. We then have a spectral sequence

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{gr}^{-p}(\Psi_{\xi,v})) \Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l}). \tag{2.12}$$

As pointed out in [**10**], if for some $\mathfrak{m}$ the spectral sequence is concentrated in middle degree, i.e. $E_{1,\mathfrak{m}}^{p,q} = 0$ for $p+q \neq d-1$, and all the $E_{1,\mathfrak{m}}^{p,d-1-p}$ are free, then, for $l > d$, the action of the monodromy operator $N_{v,\mathfrak{m}}^{coho}$ on $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ comes from the action of N_v on Ψ_v.

3. A saturated filtration of the cohomology

The aim of this section is the following proposition.

3.1. Proposition. — *Consider a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_\xi^S$ such that:*
- *$\overline{\rho}_{\mathfrak{m}}$ is irreducible;*
- *$\mathfrak{m}$ is KHT-free;*
- *the set $S_v(\mathfrak{m})$ of modulo l eigenvalues of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ does not contain any subset of the form $\{\lambda, q_v\lambda, \cdots, q_v^{e_v(l)-1}\lambda\}$, where $e_v(l)$ is defined in 2.10.*

Then the $E_{1,\mathfrak{m}}^{p,q}$ are torsion free and trivial for $p+q \neq d-1$.

Note that, as (2.12) after localization at $\mathfrak{m}$, degenerates at E_1 over $\mathbb{Q}_l$, then the spectral sequence gives us a saturated filtration of $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$. As the proof uses Grothendieck-Verdier duality and we need to understand the difference between the p and $p+$ intermediate extensions of Harris-Taylor local systems $HT(\pi_v, \mathrm{St}_t(\pi_v))$.

3.2. Lemma. — *With the previous notations, we have an isomorphism*

$${}^p j^{\geqslant h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \Pi_h) \simeq {}^{p+} j^{\geqslant h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \Pi_h).$$

Proof. — By definition we have

$$HT_{\overline{1_h}}(\chi_v, \Pi_h)[h-d] = (\overline{\mathbb{Z}}_l)_{|\,\mathrm{Sh}^{=h}_{I,\bar{s}_v,\overline{1_h}}} \otimes \Pi_h,$$

where the action of the fondamental group goes through the character χ_v. As, cf. [**23**], $\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v,\overline{1_h}}$ is smooth over $\mathrm{Spec}\,\overline{\mathbb{F}}_p$, then $P := (\overline{\mathbb{Z}}_l)_{|\,\mathrm{Sh}^{\geqslant h}_{I,\bar{s}_v,\overline{1_h}}} \otimes \Pi_h$ is perverse for the two t-structures with

$$i^{h\leqslant +1,*}_{\overline{1_h}} P \in {}^p\mathcal{D}^{<0} \text{ and } i^{h\leqslant +1,!}_{\overline{1_h}} P \in {}^{p+}\mathcal{D}^{\geqslant 1}.$$

□

Proof. — (of proposition 3.1)
As $\mathfrak{m}$ is supposed to be KHT-free, then all the $E^n_{\infty,\mathfrak{m}}$ are free. Moreover, as $\overline{\rho}_{\mathfrak{m}}$ is irreducible, then, cf. [**4**] §3.6, the $E^{p,q}_{1,\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ are all zero if $p+q \neq d-1$. Recall, cf. [**9**] proposition 3.1.3, that we have the following splitting

$$\Psi_v \simeq \bigoplus_{g=1}^{d} \bigoplus_{\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)} \Psi_\varrho$$

where $\mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$ is the set of inertial equivalence classes of irreducible $\overline{\mathbb{F}}_l$-supercuspidal representations of $GL_g(F_v)$, with

$$\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \bigoplus_{\chi_v \equiv \varrho \mod l} \Psi_{\chi_v},$$

with the property that the irreducible constituents of Ψ_{χ_v} are exactly the perverse Harris-Taylor sheaf $\mathcal{P}(h,\chi_v)(\frac{1-h+2k}{2})$ for $1 \leqslant h \leqslant d$ and $0 \leqslant k < h$. Note that for every $\varrho \in \mathrm{Scusp}_{\overline{\mathbb{F}}_l}(g)$, the cohomology groups of Ψ_ϱ are torsion free.
Remark. When computing the $\mathfrak{m}$-localized cohomology groups, we are only concerned with $\varrho \in \mathrm{Scusp}_v(\mathfrak{m})$ which are characters. Moreover as $e_v(l) > d$, in $\Psi_\varrho \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ we have only to deal with characters χ_v so that, by the previous lemma, the p and $p+$ intermediate extensions coincide.

3.3. Proposition. — *We have the following equivariant resolution*

$$\begin{aligned} 0 \to j^{=d}_! HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{h-d}{2}\}) \times \mathrm{Speh}_{d-h}(\chi_v\{h/2\})) \otimes \Xi^{\frac{d-h}{2}} \longrightarrow \cdots \\ \longrightarrow j^{=h+1}_! HT(\chi_v, \mathrm{St}_h(\chi_v(-1/2)) \times \chi_v\{h/2\}) \otimes \Xi^{\frac{1}{2}} \longrightarrow \\ j^{=h}_! HT(\chi_v, \mathrm{St}_h(\chi_v)) \longrightarrow {}^p j^{=h}_{!*} HT(\chi_v, \mathrm{St}_h(\chi_v)) \to 0. \quad (3.4) \end{aligned}$$

Note that

- as this resolution is equivalent to the computation of the sheaves cohomology groups of ${}^p j^{=h}_{!*} HT(\chi_v, \mathrm{St}_h(\chi_v))$ as explained for example in [**9**] proposition B.1.5 of appendice B, then, over $\overline{\mathbb{Q}}_l$, it follows from the main results of [**3**].
- Over $\overline{\mathbb{Z}}_l$, as every terms are free perverse sheaves, then all the maps are necessary strict.
- This resolution, for a a general supercuspidal representation with supercuspidal modulo l reduction, is one of the main result of [**9**] §2.3.

Proof. — For the case of a character χ_v as above, the argument is almost obvious. Indeed as the strata $\mathrm{Sh}^{\geqslant h}_{I^v,\bar{s}_v,1}$ are smooth, then, cf. the proof of the lemma 3.2, the constant sheaf, up to shift, is perverse and so equals to the intermediate extension of the constant sheaf, shifted by $d-h$, on $\mathrm{Sh}^{=h}_{I^v,\bar{s}_v,1}$. In particular its sheaves cohomology groups are well known so that the resolution is completely obvious for ${}^p j^{=h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \mathrm{St}_h(\chi_v))$ if one remember that $\mathrm{Speh}_i(\chi_v)$ is just the character $\chi_v \circ \det$ of $GL_i(F_v)$.

The stated resolution is then simply the induced version of the resolution of ${}^p j^{=h}_{\overline{1_h},!*} HT_{\overline{1_h}}(\chi_v, \mathrm{St}_h(\chi_v))$: recall that a direct sum of intermediate extensions is still an intermediate extension. □

By adjunction property, the map

$$\begin{aligned} & j^{=h+\delta}_! HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \\ \longrightarrow\ & j^{=h+\delta-1}_! HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \end{aligned} \tag{3.5}$$

is given by

$$\begin{aligned} & HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \longrightarrow \\ & j^{=h+\delta,*}({}^p i^{h+\delta,!}(j^{=h+\delta-1}_! HT(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}})) \end{aligned} \tag{3.6}$$

To compute this last term we use the resolution (3.4). Precisely denote by $\mathcal{H} := HT(\chi_v, \mathrm{St}_h(\chi\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$, and write the previous resolution as follows

$$0 \to K \longrightarrow j^{=h+\delta}_! \mathcal{H}' \longrightarrow Q \to 0,$$

$$0 \to Q \longrightarrow j^{=h+\delta-1}_! \mathcal{H} \longrightarrow {}^p j^{=h+\delta-1}_{!*} \mathcal{H} \to 0,$$

with

$$\mathcal{H}' := HT\Big(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2}.$$

As the support of K is contained in $\mathrm{Sh}_{I,\bar{s}_v}^{\geqslant h+\delta+1}$ then ${}^pi^{h+\delta,!}K = K$ and $j^{=h+\delta,*}({}^pi^{h+\delta,!}K)$ is zero. Moreover ${}^pi^{h+\delta,!}({}^pj_{!*}^{=h+\delta-1}\mathcal{H})$ is zero by construction of the intermediate extension. We then deduce that

$$\begin{aligned} j^{=h+\delta,*}\big({}^pi^{h+\delta,!}(j_!^{=h+\delta-1}HT(\chi_v, \mathrm{St}_t(\chi_v\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}})\big) \\ \simeq HT\Big(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \\ \times \big(\mathrm{Speh}_{\delta-1}(\chi_v\{-1/2\}) \times \chi_v\{\frac{\delta-1}{2}\}\big)\{h/2\}\Big) \otimes \Xi^{\delta/2} \quad (3.7) \end{aligned}$$

3.8 — Fact. In particular, up to homothety, the map (3.7), and so those of (3.6), is unique. Finally as the maps of (3.4) are strict, the given maps (3.5) are uniquely determined, that is, if we forget the infinitesimal parts, these maps are independent of the chosen t in (3.4).

For every $1 \leqslant h \leqslant d$, let denote by $i(h)$ the smallest index i such that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^pj_{!*}^{=h}HT(\chi_v, \mathrm{St}_h(\chi_v)))_{\mathfrak{m}}$ has non trivial torsion: if it does not exist then we set $i(h) = +\infty$. By duality, as ${}^pj_{!*} = {}^{p+}j_{!*}$ for Harris-Taylor local systems associated to characters, note that when $i(h)$ is finite then $i(h) \leqslant 0$. Suppose by absurdity there exists h with $i(h)$ finite and denote h_0 the biggest such h.

3.9. Lemma. — *For $1 \leqslant h \leqslant h_0$ then $i(h) = h - h_0$.*

Remark. A similar result is proved in [**8**] when the level is maximal at v.

Proof. — a) We first prove that for every $h_0 < h \leqslant d$, the cohomology groups of $j_!^{=h}HT(\chi_v, \Pi_h)$ are torsion free. Consider the following strict filtration in the category of free perverse sheaves

$$\begin{aligned} (0) = \mathrm{Fil}^{-1-d}(\chi_v, h) \hookrightarrow\!\!\!\!\!+ \; \mathrm{Fil}^{-d}(\chi_v, h) \hookrightarrow\!\!\!\!\!+ \; \cdots \\ \hookrightarrow\!\!\!\!\!+ \; \mathrm{Fil}^{-h}(\chi_v, h) = j_!^{=h}HT(\chi_v, \Pi_h) \end{aligned}$$

where the symbol $\hookrightarrow\!\!\!\!\!+$ means a strict monomorphism, with graded parts

$$\mathrm{gr}^{-k}(\chi_v, h) \simeq {}^pj_{!*}^{=k}HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}).$$

Over $\overline{\mathbb{Q}}_l$, the result is proved in [**3**] §4.3. From [**6**] such a filtration can be constructed over $\overline{\mathbb{Z}}_l$ up to the fact that the graduate parts are only known to verify

$$
{}^p j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{gr}^{-k}(\chi_v, h) \\
\hookrightarrow\!\!\!\!\rightarrow {}^{p+} j_{!*}^{=k} HT(\chi_v, \Pi_h\{\frac{h-k}{2}\} \otimes \mathrm{St}_{k-h}(\chi_v\{h/2\}))(\frac{h-k}{2}),
$$

and we can conclude thanks to lemma 3.2. The associated spectral sequence localized at $\mathfrak{m}$, is then concentrated in middle degree and torsion free which gives the claim.

b) Before watching the cases $h \leqslant h_0$, note that the spectral sequence associated to (3.4) for $h = h_0 + 1$, has all its E_1 terms torsion free and degenerates at its E_2 terms. As by hypothesis the aims of this spectral sequence is free and equals to only one E_2 terms, we deduce that all the maps

$$
H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+\delta} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2}\big)_{\mathfrak{m}} \\
\longrightarrow \\
H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+\delta-1} HT_\xi(\chi_v, \mathrm{St}_h(\chi_v\{\frac{1-\delta}{2}\}) \\
\times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}\big)_{\mathfrak{m}} \quad (3.10)
$$

are saturated, i.e. their cokernel are free $\overline{\mathbb{Z}}_l$-modules. Then from the previous fact stressed after (3.7), this property remains true when we consider the associated spectral sequence for $1 \leqslant h' \leqslant h_0$.

c) Consider now $h = h_0$ and the spectral sequence associated to (3.4) where

$$
E_2^{p,q} = H^{p+2q}(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h+q} \\
HT_\xi(\chi_v, \mathrm{St}_h(\chi_v(-q/2)) \times \mathrm{Speh}_q(\chi_v\{h/2\})) \otimes \Xi^{\frac{q}{2}})_{\mathfrak{m}} \quad (3.11)
$$

By definition of h_0, we know that some of the $E_\infty^{p,-p}$ should have a non trivial torsion subspace. We saw that

- the contributions from the deeper strata are torsion free and
- $H^i(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \Pi_{h_0}))_{\mathfrak{m}}$ are zero for $i < 0$ and is torsion free for $i = 0$, whatever is Π_{h_0}.
- Then there should exist a non strict map $d_1^{p,q}$. But, we have just seen that it can not be maps between deeper strata.

– Finally, using the previous points, the only possibility is that the cokernel of

$$H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0+1} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v\{\frac{-1}{2}\}) \times \chi_v\{h_0/2\})) \otimes \Xi^{1/2}\big)_{\mathfrak{m}} \longrightarrow H^0\big(\mathrm{Sh}_{I,\bar{s}_v}, j_!^{=h_0} HT_\xi(\chi_v, \mathrm{St}_{h_0}(\chi_v))\big)_{\mathfrak{m}} \quad (3.12)$$

has a non trivial torsion subspace.

In particular we have $i(h_0) = 0$.

d) Finally using the fact 2.18 and the previous points, for any $1 \leqslant h \leqslant h_0$, in the spectral sequence (3.11)

– by point a), $E_2^{p,q}$ is torsion free for $q \geqslant h_0 - h + 1$ and so it is zero if $p + 2q \neq 0$;
– by affiness of the open strata, cf. [**8**] theorem 1.8, $E_2^{p,q}$ is zero for $p + 2q < 0$ and torsion free for $p + 2q = 0$;
– by point b), the maps $d_2^{p,q}$ are saturated for $q \geqslant h_0 - h + 2$;
– by point c), $d_2^{-2(h_0-h+1),h_0-h+1}$ has a cokernel with a non trivial torsion subspace.
– Moreover, over $\overline{\mathbb{Q}}_l$, the spectral sequence degenerates at E_3 and $E_3^{p,q} = 0$ if $(p, q) \neq (0, 0)$.

We then deduce that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j_{!*}^{=h} HT_\xi(\chi_v, \Pi_h))_{\mathfrak{m}}$ is zero for $i < h - h_0$ and for $i = h - h_0$ it has a non trivial torsion subspace. □

Consider now the filtration of stratification of Ψ_ϱ constructed using the adjunction morphisms $j_!^{=h} j^{=h,*}$ as in [**6**]

$$\mathrm{Fil}_!^1(\Psi_\varrho) \lhook\joinrel\longrightarrow \mathrm{Fil}_!^2(\Psi_\varrho) \lhook\joinrel\longrightarrow \cdots \lhook\joinrel\longrightarrow \mathrm{Fil}_!^d(\Psi_\varrho) \quad (3.13)$$

where $\mathrm{Fil}_!^h(\Psi_\varrho)$ is the saturated image of $j_!^{=h} j^{=h,*}\Psi_\varrho \longrightarrow \Psi_\varrho$. For our fixed χ_v, let denote $\mathrm{Fil}_{!,\chi_v}^1(\Psi) \lhook\joinrel\longrightarrow \mathrm{Fil}_!^1(\Psi_\varrho)$ such that $\mathrm{Fil}_{!,\chi_v}^1(\Psi) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq \mathrm{Fil}_!^1(\Psi_{\chi_v})$ where Ψ_{χ_v} is the direct factor of $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ associated to χ_v, cf. [**6**].

3.14. Proposition. — *We have the following resolution of* $\mathrm{Fil}^1_{!,\chi_v}(\Psi)$

$$
\begin{aligned}
0 \to j_!^{=d} HT(\chi_v, \mathrm{Speh}_d(\chi_v)) \otimes \mathbb{L}(\chi_v(\frac{d-1}{2})) \longrightarrow \\
j_!^{=d-1} HT(\chi_v, \mathrm{Speh}_{d-1}(\chi_v)) \otimes \mathbb{L}(\chi_v(\frac{d-2}{2})) \longrightarrow \\
\cdots \longrightarrow j_!^{=1} HT(\chi_v, \chi_v) \otimes \mathbb{L}(\chi_v) \longrightarrow \mathrm{Fil}^1_{!,\chi_v}(\Psi_v) \to 0, \quad (3.15)
\end{aligned}
$$

where we recall that $\mathbb{L}$ *is the local Langlands correspondence composed by contragredient.*

Remarks:

- As explained after proposition 3.3, it amounts to describe the germs of the $\overline{\mathbb{Z}}_l$-sheaf cohomology of $\mathrm{Fil}^1_{!,\chi_v}(\Psi_{v,\xi})$. Over $\overline{\mathbb{Q}}_l$, the resolution (3.15) is then proved in [**3**].
- Over $\overline{\mathbb{Z}}_l$, it is proved in full generality in [**9**] for every irreducible supercuspidal representation π_v in place of χ_v. It amounts to prove that the germs of the sheaf cohomology of $\mathrm{Fil}^1_{!,\chi_v}(\Psi_{v,\xi})$ are free.

Proof. — We then just need to verify that every map is strict. Consider then the torsion part of the cokernel of one of these maps. Note that, thanks to lemma 3.2, such a cokernel must have non trivial invariants under the action the Iwahori sub-group at v. We then work at Iwahori level at v. As said above, it amounts to understand the germs of the $\overline{\mathbb{Z}}_l$-sheaf cohomology of $\mathrm{Fil}^1_{!,\chi_v}(\Psi)$ which are described, cf. [**17**], by the cohomology of the Lubin-Tate tower. By the comparison theorem of Faltings-Fargues, cf. [**18**], one is reduced to compute the cohomology of the Drinfeld tower in Iwahori level which is already done in [**26**]: we then note that there are all free $\overline{\mathbb{Z}}_l$-modules. □

We can then apply the previous arguments a)-d) above, so that $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^1_{!,\chi_v}(\Psi_{v,\xi})_{\mathfrak{m}}$ has non trivial torsion for $i = 1 - h_0$ and its free quotient is zero for $i \neq 0$.

Consider now the other graded parts. We also have a similar resolution

$$
\begin{aligned}
0 \to j_!^{=d} HT(\chi_v, LT_{h,d}(\chi_v)) \otimes L_g(\chi_v(\frac{d-h}{2})) \longrightarrow \\
j_!^{=d-1} HT(\chi_v, LT_{h,d-1}(\chi_v)) \otimes L_g(\chi_v(\frac{d-h-1}{2})) \longrightarrow \\
\cdots \longrightarrow j_!^{=h} HT(\chi_v, \mathrm{St}_h(\chi_v)) \otimes L_g(\chi_v) \longrightarrow \mathrm{Fil}^h_{!,\chi_v}(\Psi) \to 0, \quad (3.16)
\end{aligned}
$$

where

$$LT_{h,h+\delta}(\chi_v) \hookrightarrow \mathrm{St}_h(\chi_v\{-\delta/2\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\}),$$

is the only irreducible sub-space of this induced representation. By the same arguments, for $h \leqslant h_0$ (resp. $h > h_0$) the torsion of $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathrm{Fil}^h_{!,\chi_v}(\Psi_{v,\xi}))_{\mathfrak{m}}$ is trivial for any $i \leqslant h - h_0$ (resp. for all i) and the free parts are concentrated for $i = 0$. Using then the spectral sequence associated to the previous filtration, we can then conclude that $H^{1-t_0}(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{v,\xi})_{\mathfrak{m}}$ would have non trivial torsion which is false as $\mathfrak{m}$ is supposed to be KHT-free.

□

4. Local behavior of monodromy over $\overline{\mathbb{F}}_l$

Let ϱ be a $\overline{\mathbb{F}}_l$-character. In [**6**] §2.3, we explained how to construct a filtration $\mathrm{Fill}^\bullet(\Psi_\varrho)$ of Ψ_ϱ, called exhaustive, whose graded parts $\mathrm{grr}^r(\Psi_\varrho)$ are free $\overline{\mathbb{Z}}_l$-perverse sheaves of the following shape

$$\mathrm{grr}^r(\Psi_\varrho) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \simeq {}^p j^{=h}_{!*} HT_{\overline{\mathbb{Q}}_l}(\chi_v, \mathrm{St}_h(\chi_v))(\frac{1-h+2\delta}{2})$$

for some $0 \leqslant \delta \leqslant t-1$ and χ_v a $\overline{\mathbb{Q}}_l$-character whose modulo l reduction is inertially equivalent to ϱ. In order to compute the cohomology groups localized at $\mathfrak{m}$ of $\mathrm{Sh}_{I,v}$, by hypothesis on $\mathfrak{m}$, we are only concerned with the cases where $1 \leqslant h < e_v(l)$. For a character χ_v with modulo l reduction inertially equivalent to ϱ, and for $2 \leqslant h \leqslant d$, consider $r_{\chi_v,\pm}(h)$ such that $\mathrm{grr}^{r_{\chi_v,\pm}(h)}(\Psi_\varrho)$ is isomorphic to $\mathcal{P}(h,\chi_v)(\pm\frac{1-h}{2})$. Then N_v^{h-1} induces a map

$$\begin{tikzcd}
\mathrm{Fill}^{r_{\chi_v,+}(h)}(\Psi_\varrho) \arrow[r, "N_v^{h-1}"] \arrow[d] & \mathrm{Fill}^{r_{\chi_v,-}(h)}(\Psi_\varrho) \arrow[d, two heads] \\
\mathrm{Fill}^{r_{\chi_v,+}(h)}(\Psi_\varrho) \arrow[r, "N(\chi_v,h)"] & \mathrm{grr}^{r_{\chi_v,-}(h)}(\Psi_\varrho),
\end{tikzcd}$$

so that $N(\chi_v,h) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ is an isomorphism. In this section we want to prove that this property remains true modulo l.

4.1. Proposition. — *Suppose $e_v(l) > t$, then the morphism on $\mathrm{Fill}^{r_+(h)}(\Psi_\varrho)/\mathrm{Fill}^{r_-(h)-1}(\Psi_\varrho)$ induced by the monodromy operator N_v, is strict.*

Proof. — Recall first that the filtration $\mathrm{Fill}^\bullet(\Psi_\varrho)$ is constructed so that it is compatible with the action of N_v in the sense that, over $\overline{\mathbb{Q}}_l$, the image of $\mathrm{Fill}^r(\Psi_\varrho)$ is some $\mathrm{Fill}^{\phi(r)}(\Psi_\varrho)$. We then have to prove that for every $r_-(h) \leqslant r \leqslant r_+(h)$, we have a p-epimorphism $N_v : \mathrm{Fill}^r(\Psi_\varrho) \twoheadrightarrow \mathrm{Fill}^{\phi(r)}(\Psi_\varrho)$ which is clearly equivalent to prove that for every $1 \leqslant h' \leqslant h$, then N_v induces a isomorphism

$$
{}^p j_{!*}^{=h'} HT(\chi_v, \mathrm{St}_{h'}(\chi_v))(\frac{1-h'+2\delta}{2}) \longrightarrow {}^p j_{!*}^{=h'} HT(\chi_v, \mathrm{St}_{h'}(\chi_v))(\frac{1-h'+2(\delta-1)}{2}), \quad (4.2)
$$

for every $1 \leqslant \delta < h'$, where each of these two perverse sheaf is given by the graded parts $\mathrm{grr}^r(\Psi_\varrho)$ and $\mathrm{grr}^{\phi(r)}(\Psi_\varrho)$.

Recall that under the hypothesis that $e_v(q) > h \geqslant h'$, then the reduction modulo l of $\chi_v[h']_D \otimes L_g(\chi_v) \otimes \mathrm{St}_{h'}(\chi_v)$ is irreducible so that there exists, up to homothety, an unique stable lattice of $HT(\chi_v, \mathrm{St}_h(\chi_v))$ which means that to prove (4.2) is a isomorphism, it suffices to prove its reduction modulo l is non zero. To do so, it suffices to work in the Iwahori level and use the arguments of [**10**] §3.1 where the monodromy action is, thanks to Rapoport-Zink cf. [**22**] 3.6.13, described explicitly and is of maximal nilpotency. □

4.3. Corollary. — *Under the hypothesis of the proposition 3.1 on $\mathfrak{m}$, the order of nilpotency of the monodromy operator $\overline{N}_{\mathfrak{m},v}^{coho}$ is equal to the maximal of the order of nilpotency of $N_{\widetilde{\mathfrak{m}},v}$ where $\widetilde{\mathfrak{m}}$ described the set of prime ideals contained in $\mathfrak{m}$.*

Proof. — For any character ϱ, the previous filtration of Ψ_ϱ induces a filtration of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_\varrho \otimes V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ which is strict thanks to proposition 3.1. Moreover the action of $N_{\mathfrak{m},v}^{coho}$ is given by its action on Ψ_ϱ. In particular its order of nilpotency is less that the maximal number r so that $\{\lambda, q_v\lambda, \cdots, q_v^{r-1}\lambda\} \subset S_v(\mathfrak{m})$ where $\lambda = \varrho(\mathrm{Frob}_v)$ up to multiplication by q_v^k with $k \in \frac{1}{2}\mathbb{Z}$. Moreover as r is supposed to be strictly less than $e_v(l)$, then the previous proposition tells us that the image of $(N_{\mathfrak{m},v}^{coho})^{r-1}$ inside $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_\varrho \otimes V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ is strict, i.e. the cokernel is free.

Then after taking $\otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$, we then obtain a filtration of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_\varrho \otimes V_{\xi,\overline{\mathbb{F}}_l})_{\mathfrak{m}}$ such that the $(\overline{N}_{\mathfrak{m},v}^{coho})^{r-1}$ is non zero.

The result then follows from the decomposition of $H^0(\mathrm{Sh}_{I,\bar\eta_v}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ as direct sum of the $H^0(\mathrm{Sh}_{I,\bar s_v}, \Psi_\varrho \otimes V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}}$ where ϱ describe the set of inertial equivalence classes of $\overline{\mathbb{F}}_l$-characters of $F_v^\times$. □

5. Proof of the main result

5.1. Definition. — *(cf. [**27**] §5) We say that $\mathfrak{m}$ is KHT-typic if, as a $\mathbb{T}_{S,\mathfrak{m}}[\mathrm{Gal}_{F,S}]$-module,*

$$H^{d-1}(\mathrm{Sh}_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Z}}_l})_{\mathfrak{m}} \simeq \sigma_{\mathfrak{m}} \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m}}} \rho_{\mathfrak{m}},$$

for some $\mathbb{T}^S_{\xi,\mathfrak{m}}$-module $\sigma_{\mathfrak{m}}$ on which $\mathrm{Gal}_{F,S}$ acts trivially and

$$\rho_{\mathfrak{m}} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\mathbb{T}^S_{\xi,\mathfrak{m}})$$

is the stable lattice of $\bigoplus_{\widetilde{\mathfrak{m}}\subset\mathfrak{m}} \rho_{\widetilde{\mathfrak{m}}}$ introduced in the introduction.

As explained in [**21**], the $\overline{\mathbb{Q}}_l$-cohomology of $\mathrm{Sh}_{I,\bar\eta}$ can be written as

$$H^{d-1}(\mathrm{Sh}_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Q}}_l})_{\mathfrak{m}} \simeq \bigoplus_{\pi\in\mathcal{A}_{\xi,I}(\mathfrak{m})} (\pi^\infty)^I \otimes V(\pi^\infty),$$

where

- $\mathcal{A}_{\xi,I}(\mathfrak{m})$ is the set of equivalence classes of automorphic representations of $G(\mathbb{A})$ with non trivial I-invariants and such that its modulo l Satake's parameters outside S are prescribed by $\mathfrak{m}$,
- and $V(\pi^\infty)$ is a representation of $\mathrm{Gal}_{F,S}$.

As $\overline{\rho}_{\mathfrak{m}}$ is supposed to be absolutely irreducible, then as explained in chapter VI of [**21**], if $V(\pi^\infty)$ is non zero, then π is a weak transfer of a ξ-cohomological automorphic representation (Π,ψ) of $GL_d(\mathbb{A}_F) \times \mathbb{A}_F^\times$ with $\Pi^\vee \simeq \Pi^c$ where c is the complex conjugation. Attached to such a Π is a global Galois representation $\rho_{\Pi,l} : \mathrm{Gal}_{F,S} \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ which is irreducible.

5.2. Theorem. — *(cf. [**20**] theorem 2.20)*
If $\rho_{\Pi,l}$ is strongly irreducible, meaning it remains irreducible when it is restricted to any finite index subgroup, then $V(\pi^\infty)$ is a semi-simple representation of $\mathrm{Gal}_{F,S}$.

Remark. The Tate conjecture predicts that $V(\pi^\infty)$ is always semi-simple.

5.3. Proposition. — *We suppose that for all $\pi \in \mathcal{A}_{\xi,I}(\mathfrak{m})$, the Galois representation $V(\pi^\infty)$ is semi-simple. Then $\mathfrak{m}$ is KHT-typic.*

Proof. — By proposition 5.4 of [**27**] it suffices to deal with $\overline{\mathbb{Q}}_l$-coefficients. From [**21**] proposition VII.1.8 and the semi-simplicity hypothesis, then $V(\pi^\infty) \simeq \widetilde{R}_\xi(\pi)^{\bigoplus n(\pi)}$ where $\widetilde{R}_\xi(\pi)$ is of dimension d. We then write

$$(\pi^\infty)^I \otimes_{\overline{\mathbb{Q}}_l} R_\xi(\pi) \simeq (\pi^\infty)^I \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l}} (\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

and $(\pi^\infty)^I \otimes_{\overline{\mathbb{Q}}_l} V(\pi^\infty) \simeq ((\pi^\infty)^I)^{\bigoplus n(\pi)} \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l}} (\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l})^d$ and finally

$$H^{d-1}(\mathrm{Sh}_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Q}}_l})_\mathfrak{m} \simeq M \otimes_{\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l}} (\mathbb{T}^S_{\xi,\mathfrak{m},\overline{\mathbb{Q}}_l})^d,$$

with $M \simeq \bigoplus_{\pi\in\mathcal{A}_{\xi,I}(\mathfrak{m})}((\pi^\infty)^I)^{\bigoplus n(\pi)}$. The result then follows from [**21**] theorem VII.1.9 which insures that $R_\xi(\pi) \simeq \rho_{\widetilde{\mathfrak{m}}}$, if $\widetilde{\mathfrak{m}}$ is the prime ideal associated to π, □

5.4. Theorem. — *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^S_{\xi,\overline{\mathbb{Z}}_l}$ such that:*

- *$\overline{\rho}_\mathfrak{m}$ is absolutely irreducible;*
- *$\mathfrak{m}$ is KHT-free and with indefinite maximal nilpotency at v;*
- *$H^{d-1}(\mathrm{Sh}_{I,\bar\eta}, V_{\xi,\overline{\mathbb{Q}}_l})_\mathfrak{m}$ is Galois semi-simple.*

Then $\rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is not semi-simple.

Proof. — By hypothesis there exists $\widetilde{\mathfrak{m}}_1$ and $\widetilde{\mathfrak{m}}_2$ such that $n_1(\widetilde{\mathfrak{m}}_1) > n_1(\widetilde{\mathfrak{m}}_2)$. Then we saw that the order of nilpotency of $\overline{N}^{coho}_{\mathfrak{m},v}$ is $\geqslant n_1(\widetilde{\mathfrak{m}}_1)$ while those of $\overline{N}_{\mathfrak{m},v}$ on $\overline{\rho}_\mathfrak{m}$ is $\leqslant n_1(\widetilde{\mathfrak{m}}_2)$. If $\rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ were semi-simple and so isomorphic to $\overline{\rho}_\mathfrak{m}^{\bigoplus k}$ then the order of nilpotency of $N_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ would be $\leqslant n_1(\widetilde{\mathfrak{m}}_2)$.

Moreover from our hypothesis and the previous proposition, we know that $\mathfrak{m}$ is KHT-typic and so the order of nilpotency of $\overline{N}^{coho}_{\mathfrak{m},v}$ should be equal to those of $N_{\mathfrak{m},v} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ and so $\leqslant n_1(\widetilde{\mathfrak{m}}_2)$, which is incompatible with the fact it is $\geqslant n_1(\widetilde{\mathfrak{m}}_1)$. □

5.5. Definition. — *Let define the depth of $\mathfrak{m}$ as follows.*

- *We first denote by $\overline{\rho}(\mathfrak{m})^0 := \rho_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$,*
- *and, inductively for any $i \geqslant 1$, we denote by*
  - *$\overline{\rho}_i(\mathfrak{m})$ the socle of $\overline{\rho}(\mathfrak{m})^{i-1}$,*
  - *$\overline{\rho}(\mathfrak{m})^i := \overline{\rho}(\mathfrak{m})^{i-1}/\overline{\rho}_i(\mathfrak{m})$,*
  - *and $\rho(\mathfrak{m})_i$ the kernel of $\rho(\mathfrak{m})^0 \twoheadrightarrow \rho(\mathfrak{m})^i$.*

The depth of $\mathfrak{m}$ is then the smaller r such that $\overline{\rho}^r(\mathfrak{m})$ is zero.

The set of partitions $\underline{d}_{\widetilde{\mathfrak{m}},v}$ for various $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$, could be used to obtain informations about the depth of $\mathfrak{m}$. Consider for example the following situation:

- $S_v(\mathfrak{m}) = \{\alpha, q_v\alpha, \cdots, q_v^{d-1}\alpha\}$;
- $\overline{N}_{\mathfrak{m},v}$ is zero;
- there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_d(\chi_v)$ for some character χ_v.

5.6. Lemma. — *With the hypothesis of 5.4 and the three above assumptions, then the depth of* $\mathfrak{m}$ *is greater than d.*

Proof. — By construction each of the $\overline{\rho}_i(\mathfrak{m})$ is a direct sum of copies of $\overline{\rho}_{\mathfrak{m}}$ so that the nilpotent monodromy operator $\overline{N}_{\mathfrak{m},v}$ acts trivially. We then deduce that $\overline{N}_{\mathfrak{m},v}$ sends $\overline{\rho}(\mathfrak{m})_i$ onto $\overline{\rho}(\mathfrak{m})_{i-1}$. Our last hypothesis then implies that $\overline{N}_{\mathfrak{m},v}^{d-1} \neq 0$ so that the depth of $\mathfrak{m}$ should be greater than d. □

As explained in §6, the existence of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_d(\chi_v)$, should be given by the higher dimension of Ihara's lemma. To deal with more general situations it is convenient to suppose that the order of q_v modulo l is strictly greater than d and $S_v(\mathfrak{m})$ to be multiplicity free. Consider then

- α and r maximal such that $\{\alpha, q_v\alpha, \cdots, q_v^{r-1}\alpha\} \subset S_v(\mathfrak{m})$. We also denote by $e_0, \cdots, e_{r-1}$ the associated eigenvectors of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$.
- Let denote by $i_0 = 0 < i_1 < \cdots < i_k \leqslant r-1$ the index i such that $e_i \in \mathrm{Ker}\, \overline{N}_{\mathfrak{m},v}$.
- We moreover assume the existence of $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}}} \simeq \mathrm{St}_r(\chi_v) \times ?$ where χ_v is a character of $F_v^\times$ such that $\chi_v(\varpi_v) = \alpha$ and where ? means a irreducible representation we. do not want to precise.

5.7. Lemma. — *With the hypothesis of 5.4 and the three above assumptions, then the depth of* $\mathfrak{m}$ *is strictly greater than k.*

Proof. — The existence of $\widetilde{\mathfrak{m}}$ implies that there exists an eigenvector f_{r-1} of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ for the eigenvalue $q_v^{r-1}\alpha$ such that $\overline{N}_{\mathfrak{m},v}^{r-1}(f_r) \neq 0$. We first introduce the following notations:

- i such that $f_r \in \overline{\rho}(\mathfrak{m})^i$;
- for $\leqslant j \leqslant r-1$, let $f_j = N_{\mathfrak{m},v}^{r-1-j}(f_{r-1})$.

As in $\overline{\rho}(\mathfrak{m})_i \twoheadrightarrow \overline{\rho}_i(\mathfrak{m})$ the image of f_{i_k} is zero then $f_{i_k} \in \overline{\rho}(\mathfrak{m})_{i-1}$. As $S_v(\mathfrak{m})$ is supposed to be multiplicity free then the image of f_{i_k} in $\overline{\rho}^{i-1}(\mathfrak{m}) \simeq$

$\overline{\rho}_{\mathfrak{m}}^{\bigoplus m_{i-1}}$ belongs to the space generated by the e_{i_k-1} in each of the copies of $\overline{\rho}_{\mathfrak{m}}$. We can then repeat the previous observation so that the image of $f_{i_{k-1}}$ is zero by $\overline{\rho}(\mathfrak{m})_{i-1} \twoheadrightarrow \overline{\rho}_{i-1}(\mathfrak{m})$ and that finally the depth of $\mathfrak{m}$ should be greater than k. $\square$

6. Ihara's lemma and level raising

We now want to explain how, using Ihara's lemma in higher dimension, to construct $\mathfrak{m}$ with indefinite maximal monodromy at a fixed place v. In the Taylor-Wiles method, Ihara's lemma is the key ingredient to extend a $R = T$ property from the minimal case to a non minimal one. In higher dimension, Clozel, Harris and Taylor in their first proof of Sato-Tate theorem [**14**], proposed a generalization which involves, for example, a similitude group $\overline{G}$ over $\mathbb{Q}$ verifying the following hypothesis.

- there exists a prime number $p' = u'(u')^c$ decomposed in E, such that $p' \neq p$ and $\overline{G}(\mathbb{A}^{\infty,p'})$ is isomorphic to our previous $G(\mathbb{A}^{\infty,p'})$,
- the associated unitary group of $\overline{G}(\mathbb{R})$ is compact and
- $\overline{G}(\mathbb{Q}_{p'}) \simeq \mathbb{Q}_{p'}^\times \times \prod_{w'|u'} (\overline{B}_{w'}^{op})^\times$ where w' describe the places of F above u',
- where there exists a place $v'|u'$ such that $\overline{B}_{w'} \simeq B_{w'}$ for all $w'|u'$ distinct from v' and $\overline{B}_{v'} \simeq GL_d(F_{v'})$ while $B_{v'}$ is a division algebra with invariant $1/d$.

With the previous notations, consider a finite set S of places of F and a maximal ideal $\mathfrak{m}$ of the anemic Hecke algebra $\mathbb{T}_\xi^S$ such that $\overline{\rho}_{\mathfrak{m}}$ is absolutely irreducible.

6.1. Conjecture. — *Let $\overline{U}$ be an open compact subgroup of $\overline{G}(\mathbb{A})$ unramified outside S and let $\bar{\pi}$ be an irreducible sub-representation of $\mathcal{C}^\infty(\overline{G}(\mathbb{Q})\backslash\overline{G}(\mathbb{A})/\overline{U}^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$, where $\overline{U} = \overline{U}_v\overline{U}^v$. Then its local component $\bar{\pi}_v$ at v is generic.*

More precisely the conjecture is supposed to be true for all similitude groups $\overline{G}/\mathbb{Q}$ such that the associated unitary group of $\overline{G}(\mathbb{R})$ is compact. We can also formulate a similar conjecture for a similitude group G as in §2 to define a KHT Shimura variety. We could then hope that any irreducible sub-space of

$$\varinjlim_n H^{d-1}(\mathrm{Sh}_{I^v(n),\bar{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}},$$

as a representation of $GL_d(F_v)$, is necessary generic. In [**12**] §2.3, we prove that if this KHT version of Ihara's conjecture would be true then the Clozel-Harris-Taylor usual version would also be true. We now first want to prove a reciprocal statement for G and $\overline{G}$ as above.

6.2. Proposition. — *Take $l > d$ such that $p' \equiv 1 \mod l$ and consider a maximal ideal $\mathfrak{m}$ which is KHT-free. Then the Ihara's conjecture for $\overline{G}$ implies the Ihara's conjecture for G.*

Proof. — Thanks to [**24**] theorem 6.23, we have a rigid-analytic uniformization of $\mathrm{Sh}_{I,\bar\eta_{v'}}$ as

$$\overline{G}(\mathbb{Q})\backslash\Big(\breve{\Omega}_d \times G(\mathbb{A}^{\infty,p'}) \times \prod_{w|u',\ w\neq v'} (B_w^{op})^\times\Big)/I^{v'},$$

where $\breve{\Omega}_d$ is the Drinfeld rigid-analytic space of GL_d. By [**19**] §4.5.3, this uniformization allows us to compute the cohomology of $\mathrm{Sh}_{I,\bar\eta_{v'}}$ through a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^p_{GL_d(F_{v'})}\Big(H_c^{2d-q-2}(\breve{\Omega}_d, \overline{\mathbb{F}}_l)(d-1), \overline{\mathcal{A}}^{I^{v'}}\Big) \Rightarrow H^{p+q}(\mathrm{Sh}_{I,\bar\eta_{v'}}, \overline{\mathbb{F}}_l),$$

where $I_{v'}$ is supposed to be the inversible group of the maximal order of $B_{v'}$.

As $l > d$ and $p' \equiv 1 \mod l$, we are then in the banal case where the theory of $\overline{\mathbb{F}}_l$-representations and those over $\overline{\mathbb{Q}}_l$ are the same, cf. [**29**]. Recall, cf. [**26**], that the $H^i(\breve{\Omega}_d, \overline{\mathbb{Z}}_l)$ are torsion free so that

$$H^i(\breve{\Omega}_d, \overline{\mathbb{F}}_l) \simeq H^i(\breve{\Omega}_d, \overline{\mathbb{Z}}_l) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

We then localize the previous spectral sequence at $\mathfrak{m}$ so that, $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l = (0)$ if $p+q \neq d-1$: in particular it degenerates at E_1. Indeed as a $GL_d(F_{v'})$-representation, $H^{2d-2-q}(\breve{\Omega}_d, \overline{\mathbb{Q}}_l)$ is isomorphic to $LT_{\mathbb{1}_{v'}}(d,q)$ defined as the unique irreducible sub-space of $\mathrm{St}_{q+1}(\mathbb{1}_{v'}(\frac{q-d}{2})) \times \mathrm{Speh}_{d-q-1}(\mathbb{1}_{v'}(\frac{q}{2}))$ so that, as $\mathfrak{m}$ is not pseudo Eisenstein, then the only irreducible automorphic representations Π of $\overline{G}(\mathbb{A})$ giving a non zero term in the spectral sequence are those for which $\Pi_{v'} \simeq \mathrm{St}_d(\chi_v)$ with χ_v a character inertially equivalent to the trivial one. Then it is well-known, cf. [**15**] theorem 1.3 for example, that for $0 \leqslant q \leqslant d-1$, the only $E_{1,\mathfrak{m}}^{p,q} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ which is non zero, is for $p = d-1-q$ with $p, q \geqslant 0$.

By the previous remark, the same is then true over $\overline{\mathbb{F}}_l$ so that we have a filtration of $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_{v'}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ with graded parts the $E_{1,\mathfrak{m}}^{d-1-q,q}$. By hypothesis each of these $E_{1,\mathfrak{m}}^{d-1-q,q}$ satisfies the Ihara property at the place v, so that the same is true for $H^{d-1}(\mathrm{Sh}_{I,\bar{\eta}_{v'}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$.

□

Consider now a maximal ideal $\mathfrak{m}$ of the anemic Hecke algebra $\mathbb{T}_\xi^S$ which is KHT-free and such that $\overline{\rho}_{\mathfrak{m}}$ is irreducible. We moreover suppose that $q_v \equiv 1 \mod l$ with $l > d$, and that $S_v(\mathfrak{m})$ is made of characters. Let then $\varrho \in S_v(\mathfrak{m})$ and we denote by s its multiplicity. Here is the automorphic version of proposition 1.3.

6.3. Proposition. — *If Ihara's lemma is true for G, then there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ such that $\Pi_{\widetilde{\mathfrak{m}},v} \simeq \mathrm{St}_s(\chi_v) \times \Psi_v$ where $\chi_v \equiv \varrho \mod l$, and Ψ_v is a irreducible representation of $GL_{d-s}(F_v)$ whose modulo l reduction has a supercuspidal support disjoint from*[4] *ϱ.*

Proof. — The strategy is to suppose that the conclusion is false and then prove that Ihara's lemma is then not verified. Let then $t < s$ be maximal such that there exists $\widetilde{\mathfrak{m}} \subset \mathfrak{m}$ with $\Pi_{\widetilde{\mathfrak{m}}} \simeq \mathrm{St}_t(\chi_v) \times \Gamma_v$ such that $\chi_v \equiv \varrho \mod l$. Then, by proposition 7.2, for all $h > t$, the $H^i(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, h))_{\mathfrak{m}}$ are zero and the spectral sequence associated to $\mathrm{Fill}^\bullet(\Psi_\varrho)$ gives

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \hookrightarrow H^0(\mathrm{Sh}_{I,\bar{s}_v}, \Psi_{\varrho,\xi})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l.$$

The idea is then to construct an irreducible $\overline{\mathbb{F}}_l[GL_d(F_v)]$ sub-module of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ which is not generic. By proposition 7.2, $\Pi_{\widetilde{\mathfrak{m}}}$ is a $\overline{\mathbb{Q}}_l[GL_d(F_v)]$ sub-module of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}}$ so that, by taking a saturated lattice, $\Pi_{\widetilde{\mathfrak{m}}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is a $\overline{\mathbb{F}}_l[GL_d(F_v)]$ sub-module of $H^0(\mathrm{Sh}_{I,\bar{s}_v}, \mathcal{P}_\xi(\chi_v, t))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$. Using the fact that the strata are induced and $s \leqslant d < l$ so that $\mathrm{St}_t(\chi_v) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ is irreducible isomorphic to $\mathrm{St}_t(\varrho)$, we are then reduced to prove that $\mathrm{St}_t(\varrho) \times (\Gamma_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$ has an irreducible sub-space which is not generic, whatever is the stable lattice of Γ_v taken to compute $(\Gamma_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$.

As $\mathfrak{m}$ is not pseudo-Eisenstein and $S_v(\mathfrak{m})$ is made of characters, then Γ_v is of the following shape $\mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r})$ where, as we supposed $t < s$, at least one of the characters $\chi_{v,i}$ is congruent to ϱ modulo l. Take then an irreducible sub-space $\bar{\pi}$ of $(\Gamma_v \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l)$. If $\bar{\pi}$ is not generic we are

[4] Recall that as $q_v \equiv 1 \mod l$, then the $\overline{\mathbb{F}}_l$-Zelevinsky line of ϱ is $\{\varrho\}$.

done, otherwise $\bar{\pi} \simeq \mathrm{St}_{s-t}(\varrho) \times \bar{\psi}$ where the supercuspidal support of $\bar{\psi}$ does not contain ϱ. We then conclude by noting that $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ has a non generic subspace. Indeed $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ is the modulo l reduction of $\mathrm{St}_t(\chi_v(\frac{t-s}{2})) \times \mathrm{St}_{s-t}(\chi_v(\frac{t}{2}))$ which has an unique irreducible subspace which is non generic, and we conclude as an irreducible non generic representation does not have any irreducible generic subquotient. □

Remark. In the previous proof, note that $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ is also the modulo l reduction of $\mathrm{St}_t(\chi_v(-\frac{t-s}{2})) \times \mathrm{St}_{s-t}(\chi_v(-\frac{t}{2}))$ which has an unique irreducible sub-space which is $\mathrm{St}_s(\chi_v)$, so that $\mathrm{St}_t(\varrho) \times \mathrm{St}_{s-t}(\varrho)$ is the direct sum of a generic representation with a non generic one.

7. Automorphic congruences

As in [**7**], we can use the freeness of the cohomology groups of the Harris-Taylor perverse sheaves, to produce automorphic congruences. Consider then $\mathfrak{m}$ verifying the hypothesis of proposition 3.1 so that the $H^i(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j^{=h}_{!*} HT_\xi(\chi_v,h))_\mathfrak{m}$ are free and concentrated in degree $i=0$ with

$$H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j^{=h}_{!*} HT_\xi(\chi_v,h))_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j^{=h}_{!*} HT_{\xi,\overline{\mathbb{F}}_l}(r_l(\chi_v),h))_\mathfrak{m} \simeq H^0(\mathrm{Sh}_{I,\bar{s}_v}, {}^p j^{=h}_{!*} HT_\xi(\chi'_v,h))_\mathfrak{m} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l, \quad (7.1)$$

whatever is χ'_v such that the modulo l reduction $r_l(\chi'_v)$ of χ'_v is isomorphic to those of χ_v. Recall then from [**4**], the description of the $\overline{\mathbb{Q}}_l$-cohomology groups of ${}^p j^{=h}_{!*} HT_\xi(\chi_v,h)$ localized at $\mathfrak{m}$.

7.2. Proposition. — *(cf.* [**4**] *§3.6 with*[5] $s=1$*)*
For χ_v an unitary character of $F_v^\times$, then, for $1 \leqslant h \leqslant d$, as a $\mathbb{T}^S_\mathfrak{m}[GL_d(F_v)]$-module, we have

$$\lim_{\overrightarrow{I_v}} H^0(\mathrm{Sh}_{I^vI_v,\bar{s}_v}, {}^p j^{=h}_{!*} HT_{\xi,\overline{\mathbb{Q}}_l}(\chi_v,h))_\mathfrak{m} \simeq \bigoplus_{\Pi \in \mathcal{A}(I,\xi,h,\chi_v,\mathfrak{m})} m(\Pi)(\Pi^{\infty,v})^{I^v} \otimes \Pi_v,$$

where

(5) As $\overline{\rho}_\mathfrak{m}$ is supposed to be irreducible, the integer s of [**4**] §3.6 is necessary equal to 1.

- *$\mathcal{A}(I, h, \chi_v, \mathfrak{m})$ is the set of irreducible ξ-cohomological automorphic representations Π of $G(\mathbb{A})$ with non zero invariants under I^v with modulo l Satake's parameters prescribed by $\mathfrak{m}$,*
- *such that Π_v is of the following shape*

$$\Pi_v \simeq \mathrm{St}_h(\chi_v) \times \Psi_v$$

  where Ψ_v is a representation of $GL_{d-h}(F_v)$,
- *and $m(\Pi)$ is the multiplicity of Π in the space of automorphic forms.*

Remark. We write the local component Π_v of $\Pi \in \mathcal{A}(I, \xi, h, \chi_v, \mathfrak{m})$ as

$$\Pi_v \simeq \mathrm{St}_{t_1}(\chi_{v,1}) \times \cdots \times \mathrm{St}_{t_r}(\chi_{v,r}) \times \Psi'_v,$$

where

- the $\chi_{v,i}$ are inertially equivalent characters,
- Ψ'_v is an irreducible representation of $GL_{d-\sum_{i=1}^r t_i}(F_v)$ whose cuspidal support, made of character by hypothesis, does not contain a character inertially equivalent to $\chi_{v,1}$.

Then Π contributes k times in the isomorphism of the previous proposition, where $k = \sharp\{1 \leqslant i \leqslant r \text{ such that } t_i = h\}$.

We are now in the same situation as in [**7**] where we prove that the conjecture 5.4.3 implies the conjecture 5.2.1 and the translation in terms of automorphic congruences explained at the end of §5.2 The situation here is much more simple as $s = 1$.

7.3. Corollary. — *Let Π be an irreducible automorphic representation of $G(\mathbb{A})$ which is ξ-comological of level K and such that*

- *its modulo l Satake's parameters are given by $\mathfrak{m}$,*
- *and its local component Π_v at v is isomorphic to $\Pi_v \simeq \mathrm{St}_h(\chi_v) \times \Psi_v$, where χ_v is a characters and Ψ_v is an irreducible representation of $GL_{d-h}(F_v)$.*

Consider then any character χ'_v of $F_v^\times$ which is congruent to χ_v modulo l. Then there exists an irreducible automorphic representation Π' of $G(\mathbb{A})$ which is ξ-cohomological of the same level K and such that

- *its modulo l Satake's parameters are given by $\mathfrak{m}$,*
- *its local component at v is of the following shape*

$$\Pi'_v \simeq \mathrm{St}_h(\chi'_v) \times \Psi'_v.$$

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