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Torsion classes in the cohomology of KHT Shimura's varieties

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A particular case of Bergeron-Venkatesh's conjecture predicts that torsion classes in the cohomology of Shimura's varieties are rather rare. According to this and for Kottwitz-Harris-Taylor type of Shimura's varieties, we first associate to each such torsion class an infinity of irreducible automorphic representations in characteristic zero, which are pairwise non isomorphic and weakly congruent in the sense of [14]. Then, using completed cohomology, we construct torsion classes in regular weight.

Introduction

Let $F = EF^+$ be a CM field and B/F a central division algebra of dimension d^2 equipped with an involution of second kind: we can then define a group of similitudes $G/\mathbb{Q}$ as explained in §1.2. We denote $X_{I,\eta} \to \operatorname{Spec} F$ the Shimura variety of Kottwitz-Harris-Taylor type associated to some unitary group $G/\mathbb{Q}$ and an open compact subgroup I. For a fixed prime number l, consider the set $\operatorname{Spl}(I)$ of places v of F over a prime number $p \neq l$ such that

- $p = uu^c$ is split in the quadratic imaginary extension E/Q,
- $G(\mathbb{Q}_p)$ is split, i.e. of the following shape $\mathbb{Q}_p^\times \times \prod_{w|u} (B_w^{op})^\times$,
- the local component at p of I, is maximal,
- $v|u$ and $B_v^\times \simeq GL_d(F_v)$.

Given an irreducible algebraic representation ξ of G which gives a $\overline{\mathbb{Z}}_l$-local system V_ξ over $X_{I,\bar\eta}$, if we believe in the general conjectures of [1], and as the defect equals 0, asymptotically as the level I increases, the torsion cohomology classes in $H^i(X_{I,\bar\eta}, V_\xi[d-1])$ are rather rare. In this direction, the main theorem of [7] gives a criterion to cancel this torsion which rests on the modulo l Satake parameters at any $v \in \operatorname{Spl}(I)$: put another way for a torsion cohomology class to exist, the associated set of modulo l Satake's parameters should, at any place $v \in \operatorname{Spl}(I)$, contain a subset of the shape $\{\alpha, q_v\alpha\}$ where q_v is the cardinal of the residual field at v. In this paper we first proceed this result, see corollary 2.9.

Theorem. *Let $\mathfrak{m}$ be a maximal ideal of some unramified Hecke algebra, associated to some non trivial torsion classes in the cohomology of $X_{I,\bar\eta}$ with coefficients in $V_\xi[d-1]$: we denote by i the greatest integer such that the torsion of $H^{-i}(X_{I,\bar\eta}, V_\xi[d-1])_\mathfrak{m}$ is non trivial. There exists then a set*

$$\Big\{ \Pi(v) : \ v \in \operatorname{Spl}(I) \Big\}$$

of irreducible automorphic ξ-cohomological representations such that for all $w \in \operatorname{Spl}(I)$ distinct from v, the local component at w of $\Pi(v)$ is unramified, its modulo l Satake parameters being given by $\mathfrak{m}$. On the other hand, $\Pi(v)$ is ramified at v and more precisely, it's isomorphic to a representation of the following shape

$$\operatorname{St}_{i+2}(\chi_{w,0}) \times \chi_{w,1} \times \cdots \times \chi_{w,d-i-2}$$

where $\chi_{w,0}, \cdots, \chi_{w,d-i-2}$ are unramified characters of F_w.

So for $v \neq w \in \operatorname{Spl}(I)$, the representations $\Pi(v)$ and $\Pi(w)$ are not isomorphic but are weakly congruent in the sense of §3 [14], i.e. they share the same modulo l Satake parameters at each place of $\mathrm{Spl}(I) - \{v, w\}$. We can then see this result as some arithmetic application to the existence of torsion cohomology classes which suggest trying to construct such classes. In §3, we investigate this question with the help of the notion of completed cohomology

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) := \varprojlim_n \varinjlim_{I_l} H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}/\lambda^n}[d-1]).$$

As they are independent of the choice of the weight ξ,

- by taking ξ the trivial representation and using the Hochschild-Serre spectral sequence computing, starting from the completed cohomology, the cohomology at a finite level, we can show that for each diviseur s of d, the free quotient of $\widetilde{H}^{1-s}_{I^l}$ is non trivial provided that the level I^l outside l is small enough: to be precise, here, we just prove an imprecise version of this fact, see proposition 3.2.
- Then taking ξ regular and as the free quotient of the finite cohomology outside the middle degree are trivial, we observe that, for each divisor s of d, we can find torsion classes in level I_l so that, when I_l increase, they organize themselves in some torsion free class in the completed cohomology.

1. Notations and backgroung

1.1. Induced representation

Consider a local fiel K with its absolute value $|-|$: let q denote the cardinal of its residual field. For a representation π of $GL_d(K)$ and $n \in \frac{1}{2}\mathbb{Z}$, set

$$\pi\{n\} := \pi \otimes q^{-n \,\mathrm{val} \circ \det}.$$

1.1.1. Notations. *For π_1 and π_2 representations of respectively $GL_{n_1}(K)$ and $GL_{n_2}(K)$, $\pi_1 \times \pi_2$, we will denote by*

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1,n_1+n_2}(K)}^{GL_{n_1+n_2}(K)} \pi_1\{\frac{n_2}{2}\} \otimes \pi_2\{-\frac{n_1}{2}\},$$

the normalized parabolic induced representation where for any sequence $\underline{r} = (0 < r_1 < r_2 < \cdots < r_k = d)$, we write $P_{\underline{r}}$ for the standard parabolic subgroup of GL_d with Levi

$$GL_{r_1} \times GL_{r_2-r_1} \times \cdots \times GL_{r_k-r_{k-1}}.$$

Remind that an irreducible representation is called supercuspidal if it's not a subquotient of some proper parabolic induced representation.

1.1.2. Definition. *(see [15] §9 and [4] §1.4) Let g be a divisor of $d = sg$ and π an irreducible cuspidal representation of $GL_g(K)$. The induced representation*

$$\pi\{\frac{1-s}{2}\} \times \pi\{\frac{3-s}{2}\} \times \cdots \times \pi\{\frac{s-1}{2}\}$$

holds an unique irreducible quotient (resp. subspace) denoted $\mathrm{St}_s(\pi)$ *(resp.* $\mathrm{Speh}_s(\pi)$*); it's a generalized Steinberg (resp. Speh) representation.*

Remark: from a galoisian point of view through the local Langlands correspondence, the representation matches to the direct sum $\sigma(\frac{1-s}{2}) \oplus \cdots \oplus \sigma(\frac{s-1}{2})$ where σ matches to π. *More generally* for π any irreducible representation of $GL_g(K)$ associated to σ by the local Langlands correspondence, we will denote $\mathrm{Speh}_s(\pi)$ the representation of $GL_{sg}(K)$ matching, through the local Langlands correspondence, $\sigma(\frac{1-s}{2}) \oplus \cdots \oplus \sigma(\frac{s-1}{2})$.

1.1.3. Definition. *A smooth $\overline{\mathbb{Q}}_l$-representation of finite length π of $GL_d(K)$ is said* entire *if there exist a finite extension $E/\mathbb{Q}_l$ contained in $\overline{\mathbb{Q}}_l$, with ring of integers $\mathcal{O}_E$, and a $\mathcal{O}_E$-representation L of $GL_d(K)$, which is a free $\mathcal{O}_E$-module, such that $\overline{\mathbb{Q}}_l \otimes_{\mathcal{O}_E} L \simeq \pi$ and L is a $\mathcal{O}_E$ $GL_n(K)$-module of finite type. Let κ_E the residual field of $\mathcal{O}_E$, we say that*

$$\overline{\mathbb{F}}_l \otimes_{\kappa_E} \kappa_E \otimes_{\mathcal{O}_E} L$$

is the modulo l reduction of L.

Remark: the *Brauer-Nesbitt principle* asserts that the semi-simplification of $\overline{\mathbb{F}}_l \otimes_{\mathcal{O}_E} L$ is a finite length $\overline{\mathbb{F}}_l$-representation of $GL_d(K)$ which is independent of the choice of L. Its image in the Grothendieck group will be denoted $r_l(\pi)$ and called the *modulo l reduction of π*.
Example: from [13] V.9.2 or [8] §2.2.3, we know that the modulo l reduction of $\mathrm{Speh}_s(\pi)$ is irreducible.

1.2. Geometry of KHT Shimura's varieties

Let $F = F^+E$ be a CM field where $E/\mathbb{Q}$ is quadratic imaginary and $F^+/\mathbb{Q}$ totally real with a fixed real embedding $\tau : F^+ \hookrightarrow \mathbb{R}$. For a place v of F, we will denote

- F_v the completion of F at v,
- $\mathcal{O}_v$ the ring of integers of F_v,
- ϖ_v a uniformizer,
- q_v the cardinal of the residual field $\kappa(v) = \mathcal{O}_v/(\varpi_v)$.

Let B be a division algebra with center F, of dimension d^2 such that at every place x of F, either B_x is split or a local division algebra. Further we assume B provided with an involution of second kind $*$ such that $*_{|F}$ is the complexe conjugation. For any $\beta \in B^{*=-1}$, denote $\sharp_\beta$ the involution $x \mapsto x^{\sharp_\beta} = \beta x^* \beta^{-1}$ and $G/\mathbb{Q}$ the group of similitudes, denoted G_τ in [10], defined for every $\mathbb{Q}$-algebra R by

$$G(R) \simeq \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^{\sharp_\beta} = \lambda\}$$

with $B^{op} = B \otimes_{F,c} F$. If x is a place of $\mathbb{Q}$ split $x = yy^c$ in E then

$$G(\mathbb{Q}_x) \simeq (B_y^{op})^\times \times \mathbb{Q}_x^\times \simeq \mathbb{Q}_x^\times \times \prod_{z_i} (B_{z_i}^{op})^\times, \tag{1.2.4}$$

where, identifying places of F^+ over x with places of F over y, $x = \prod_i z_i$ in F^+.

Convention: for $x = yy^c$ a place of $\mathbb{Q}$ split in E and z a place of F over y as before, we shall make throughout the text, the following abuse of notation by denoting $G(F_z)$ in place of the factor $(B_z^{op})^\times$ in the formula (1.2.4) as well as

$$G(\mathbb{A}^z) := G(\mathbb{A}^x) \times \big(\mathbb{Q}_x^\times \times \prod_{z_i \neq z} (B_{z_i}^{op})^\times\big).$$

In [10], the author justify the existence of some G like before such that moreover

- if x is a place of $\mathbb{Q}$ non split in E then $G(\mathbb{Q}_x)$ is quasi split;
- the invariants of $G(\mathbb{R})$ are $(1, d-1)$ for the embedding τ and $(0, d)$ for the others.

As in [10] bottom of page 90, a compact open subgroup U of $G(\mathbb{A}^\infty)$ is said *small enough* if there exists a place x such that the projection from U^v to $G(\mathbb{Q}_x)$ does not contain any element of finite order except identity.

1.2.5. Notation. *Denote $\mathcal{I}$ the set of open compact subgroup small enough of $G(\mathbb{A}^\infty)$. For $I \in \mathcal{I}$, write $X_{I,\eta} \longrightarrow \operatorname{Spec} F$ the associated Shimura variety of Kottwitz-Harris-Taylor type.*

From now on, we fix a prime number l unramified in E.

1.2.6. Definition. *Define* Spl *the set of places v of F such that $p_v := v_{|\mathbb{Q}} \neq l$ is split in E and $B_v^\times \simeq GL_d(F_v)$. For each $I \in \mathcal{I}$, write* Spl(I) *the subset of* Spl *of places which doesn't divide the level I.*

Remark: for every $v \in \mathrm{Spl}$, the variety $X_{I,\eta}$ have a projective model $X_{I,v}$ over $\operatorname{Spec} \mathcal{O}_v$ with special fiber X_{I,s_v}. For I going through $\mathcal{I}$, the projective system $(X_{I,v})_{I\in\mathcal{I}}$ is naturally equipped with an action of $G(\mathbb{A}^\infty) \times \mathbb{Z}$ such that w_v in the Weil group W_v of F_v acts by $-\deg(w_v) \in \mathbb{Z}$, where $\deg = \mathrm{val} \circ \mathrm{Art}^{-1}$ and $\mathrm{Art}^{-1} : W_v^{ab} \simeq F_v^\times$ is Artin's isomorphism which sends geometric Frobenius to uniformizers.

1.2.7. Notations. *(see [3] §1.3) For $I \in \mathcal{I}$, the Newton stratification of the geometric special fiber $X_{I,\bar{s}_v}$ is denoted*

$$X_{I,\bar{s}_v} =: X^{\geq 1}_{I,\bar{s}_v} \supset X^{\geq 2}_{I,\bar{s}_v} \supset \cdots \supset X^{\geq d}_{I,\bar{s}_v}$$

where $X^{=h}_{I,\bar{s}_v} := X^{\geq h}_{I,\bar{s}_v} - X^{\geq h+1}_{I,\bar{s}_v}$ is an affine scheme[1]*, smooth of pure dimension $d-h$ built up by the geometric points whose connected part of its Barsotti-Tate group is of rank h. For each $1 \leq h < d$, write*

$$i_{h+1} : X^{\geq h+1}_{I,\bar{s}_v} \hookrightarrow X^{\geq h}_{I,\bar{s}_v}, \quad j^{\geq h} : X^{=h}_{I,\bar{s}_v} \hookrightarrow X^{\geq h}_{I,\bar{s}_v}.$$

1.3. Cohomology groups over $\overline{\mathbb{Q}}_l$

Let us begin with some known facts about irreducible algebraic representations of G, see for example [10] p.97. Let $\sigma_0 : E \hookrightarrow \overline{\mathbb{Q}}_l$ be a fixed embedding and et write Φ the set of embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_l$ whose restriction to E equals σ_0. There exists then an explicit bijection between irreducible algebraic representations ξ of G over $\overline{\mathbb{Q}}_l$ and $(d+1)$-uple $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ where $a_0 \in \mathbb{Z}$ and for all $\sigma \in \Phi$, we have $\overrightarrow{a_\sigma} = (a_{\sigma,1} \leq \cdots \leq a_{\sigma,d})$.

For $K \subset \overline{\mathbb{Q}}_l$ a finite extension of $\mathbb{Q}_l$ such that the representation $\iota^{-1} \circ \xi$ of highest weight $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$, is defined over K, write $W_{\xi,K}$ the space of this representation and $W_{\xi,\mathcal{O}}$ a stable lattice under the action of the maximal

[1] see for example [11]

open compact subgroup $G(\mathbb{Z}_l)$, where $\mathcal{O}$ is the ring of integers of K with uniformizer λ.
Remark: if ξ is supposed to be l-small, in the sense that for all $\sigma \in \Phi$ and all $1 \leq i < j \leq n$ we have $0 \leq a_{\tau,j} - a_{\tau,i} < l$, then such a stable lattice is unique up to a homothety.

1.3.8. Notation. *We will denote $V_{\xi,\mathcal{O}/\lambda^n}$ the local system on $X_{\mathcal{I}}$ as well as*

$$V_{\xi,\mathcal{O}} = \varprojlim_{n} V_{\xi,\mathcal{O}/\lambda^n} \quad \textit{and} \quad V_{\xi,K} = V_{\xi,\mathcal{O}} \otimes_{\mathcal{O}} K.$$

For $\overline{\mathbb{Z}}_l$ and $\overline{\mathbb{Q}}_l$ version, we will write respectively $V_{\xi,\overline{\mathbb{Z}}_l}$ and $V_{\xi,\overline{\mathbb{Q}}_l}$. We'll add the symbol ξ on a sheaf to indicate its torsion by $V_{\xi,\overline{\mathbb{Z}}_l}$: for example $HT_\xi(\pi_v, \Pi_t) := HT(\pi_v, \Pi_t) \otimes V_{\xi,\overline{\mathbb{Z}}_l}$.

Remark: the representation ξ is said *regular* if its parameter $\big(a_0, (\overrightarrow{a_\sigma})_{\sigma\in\Phi}\big)$ verify for all $\sigma \in \Phi$ that $a_{\sigma,1} < \cdots < a_{\sigma,d}$.

1.3.9. Definition. *An irreducible automorphic representation Π is said ξ-cohomological if there exists an integer i such that*

$$H^i\big((\mathrm{Lie}\ G(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}, U, \Pi_\infty \otimes \xi^\vee\big) \neq (0),$$

where U is a maximal open compact subgroup modulo the center of $G(\mathbb{R})$.

For Π an automorphic irreducible representation ξ-cohomological of $G(\mathbb{A})$, then, see for example lemma 3.2 of [6], for each $v \in \mathrm{Spl}$, the local component Π_v is isomorphic to some $\mathrm{Speh}_s(\pi_v)$ where π_v is an irreducible non degenerate representation and $s \geq 1$ an integer which is independent of the place $v \in \mathrm{Spl}$.

1.3.10. Definition. *The integer s mentioned above is called the degeneracy depth of Π.*

From now on, we fix $v \in \mathrm{Spl}$.

1.3.11. Notation. *For π_v an irreducible cuspidal representation of $GL_g(F_v)$, write*

$$s_\xi(\pi_v)$$

for the biggest integer s such that there exists an automorphic ξ-cohomological representation Π such that its local component at v is isomorphic to some

$\mathrm{Speh}_s(\pi'_v)\times ?$ *where* π'_v *is inertially equivalent to* π_v *and ? is an unknown representation of* $GL_{d-sg_{-1}}(F_v)$ *we don't try to precise.*

1.3.12. Notation. *For* $1 \leq h \leq d$*, let us denote* $\mathcal{I}_v$ *the set of open compact subgroups of the following shape*

$$U_v(\underline{m}) := U_v(\underline{m}^v) \times K_v(m_1),$$

where $K_v(m_1) = \ker\big(GL_d(\mathcal{O}_v) \longrightarrow GL_d(\mathcal{O}_v/(\varpi_v^{m_1}))\big)$*. The notation* $[H^i(h,\xi)]$ *(resp.* $[H^i_!(h,\xi)]$*) means the image of*

$$\lim_{\substack{\longrightarrow \\ I\in\mathcal{I}_v}} H^i(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Q}}_l}[d-h]) \qquad \textit{resp.} \ \lim_{\substack{\longrightarrow \\ I\in\mathcal{I}_v}} H^i_c(X^{=h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Q}}_l}[d-h])$$

in the Grothendieck group $\mathrm{Groth}(v)$ *of admissible representation of* $G(\mathbb{A}^{\infty,v}) \times GL_d(F_v) \times \mathbb{Z}$*.*

Remark: recall that the action of $\sigma \in W_v$ on these $GL_d(F_v) \times \mathbb{Z}$-modules is given by those of $-\deg \sigma \in \mathbb{Z}$.

1.3.13. Notation. *For* $\Pi^{\infty,v}$ *an irreducible representation of* $G(\mathbb{A}^{\infty,v})$*, let* $\mathrm{Groth}(h)\{\Pi^{\infty,v}\}$ *denote the subgroup of* $\mathrm{Groth}(v)$ *generated by irreducible of the shape* $\Pi^{\infty,v} \otimes \Psi_v \otimes \zeta$ *where* Ψ_v *(resp.* ζ*) is any irreducible representation of* $GL_d(F_v)$ *(resp. of* $\mathbb{Z}$*). We will denote then*

$$[H^i(h,\xi)]\{\Pi^{\infty,v}\}$$

the projection of $[H^i(h,\xi)]$ *on this direct factor.*

We write

$$[H^i(h,\xi)]\{\Pi^{\infty,v}\} = \Pi^{\infty,v} \otimes \Big(\sum_{\Psi_v,\xi} m_{\Psi_v,\zeta}(\Pi^{\infty,v})\Psi_v \otimes \zeta\Big),$$

where Ψ_v (resp. ξ) goes through irreducible admissible representations of $GL_d(F_v)$, (resp. of $\mathbb{Z}$ which can be considered as an unramified representation of W_v).

1.3.14. Proposition. *Let* Π *be an automorphic irreducible tempered representation* ξ*-cohomological.*

(i) *For all* $h = 1, \cdots, d$ *and all* $i \neq 0$,

$$[H^i(h,\xi)]\{\Pi^{\infty,v}\} \qquad \textit{and} \qquad [H^i_!(h,\xi)]\{\Pi^{\infty,v}\}$$

are trivial.

(ii) *If* $[H^0(h,\xi)]\{\Pi^{\infty,v}\}$ *(resp.* $[H^0_!(h,\xi)]\{\Pi^{\infty,v}\}$*) has non trivial invariants under the action of* $GL_d(\mathcal{O}_v)$ *then the local component* Π_v *of* Π *at* v *is isomorphic to a representation of the following shape*

$$\mathrm{St}_r(\chi_{v,0}) \times \chi_{v,1} \times \cdots \times \chi_{v,r}$$

where $\chi_{v,0}, \cdots, \chi_{v,t}$ *are unramified characters and* $r = h$ *(resp.* $r \geq h$*).*

Proof. (i) This is exactly proposition 1.3.9 of [7].

(ii) The result for $H^i(h,\xi)$ is a particular case of proposition 3.6 of [6] (which proposition follows directly from proposition 3.6.1 of [4]) for the constant local system, i.e. when π_v is the trivial representation and $s = 1$.

Concerning the cohomology with compact supports, we can use either proposition 3.12 of [6] or the description, given by corollary 5.4.1 of [3], of this extension by zero in terms of local systems on Newton strata with indices $h' \geq h$. □

1.3.15. Proposition. *(see [4] theorem 4.3.1) Let* Π *be an automorphic irreducible representation* ξ*-cohomological with depth of degeneracy* $s > 1$*. Then for* ξ *the trivial character,* $[H^{1-s}_!(h,\xi)]\{\Pi^{\infty,v}\}$ *is non trivial.*

Remark: if ξ is a regular parameter then the depth of degeneracy of any irreducible automorphic representation ξ-cohomological is necessary equal to 1. In particular theorem 4.3.1 of [4] is compatible with the classical result saying that for a regular ξ, the cohomology of the Shimura variety X_I with coefficients in $V_{\xi,\overline{\mathbb{Q}}_l}$, is concentrated in middle degree.

1.4. Hecke's algebras

1.4.16. Notation. *For* $I \in \mathcal{I}$ *a finite level, write*

$$\mathbb{T}_I := \overline{\mathbb{Z}}_l\big[T_{w,i} : \ w \in \mathrm{Spl}(I) \ \textit{and} \ i = 1, \cdots, d\big],$$

the Hecke algebra associated to $\mathrm{Spl}(I)$ *where* $T_{w,i}$ *is the characteristic function of*

$$GL_d(\mathcal{O}_w)\,\mathrm{diag}(\overbrace{\varpi_w,\cdots,\varpi_w}^{i},\overbrace{1,\cdots,1}^{d-i})GL_d(\mathcal{O}_w) \subset GL_d(F_w).$$

Consider a fixed maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$. For every $w \in \mathrm{Spl}(I)$, we denote

$$P_{\mathfrak{m},w}(X) := \sum_{i=0}^{d} (-1)^i q_w^{\frac{i(i-1)}{2}} \overline{T_{w,i}} X^{d-i} \in \overline{\mathbb{F}}_l[X]$$

the Hecke polynomial and

$$S_{\mathfrak{m}}(w) := \big\{ \lambda \in \mathbb{T}_I/\mathfrak{m} \simeq \overline{\mathbb{F}}_l \text{ such that } P_{\mathfrak{m},w}(\lambda)=0 \big\},$$

the multi-set of modulo l Satake's parameters at w associated to $\mathfrak{m}$. With the previous notations, the image $\overline{T_{w,i}}$ of $T_{w,i}$ inside $\mathbb{T}_I/\mathfrak{m}$ can be written

$$\overline{T_{w,i}} = q_w^{\frac{i(1-i)}{2}} \sigma_i(\lambda_1,\cdots,\lambda_d)$$

where we write $S_{\mathfrak{m}}(w)=\{\lambda_1,\cdots,\lambda_d\}$ and where the σ_i are the elementary symmetric functions.

1.4.17. Notation. *We will denote* $\mathfrak{m}^\vee$ *the maximal ideal of* $\mathbb{T}_I$ *defined by*

$$T_{w,i} \in \mathbb{T}_I \mapsto q_w^{\frac{i(1-i)}{2}} \sigma_i(\lambda_1^{-1},\cdots,\lambda_d^{-1}) \in \overline{\mathbb{F}}_l.$$

2. Automorphic congruences

Consider from now on a fixed place $v \in \mathrm{Spl}(I)$.

2.1. Definition. *A* $\mathbb{T}_I$*-module* M *is said to verify property* ***(P)****, if it has a finite filtration*

$$(0)=\mathrm{Fil}^0(M) \subset \mathrm{Fil}^1(M) \cdots \subset \mathrm{Fil}^r(M)=M$$

such that for every $k=1,\cdots,r$*, there exists*

- *an automorphic irreducible entire representation* Π_k *of* $G(\mathbb{A})$*, which appears in the cohomology of* $(X_{I,\overline{\eta}_v})_{I\in\mathcal{I}}$ *with coefficients in* $V_{\xi,\overline{\mathbb{Q}}_l}$ *and such that its local component* $\Pi_{k,v}$ *is ramified, i.e.* $(\Pi_{k,v})^{GL_d(\mathcal{O}_v)}=(0)$*;*

- *an unramified entire irreducible representation* $\widetilde{\Pi}_{k,v}$ *of* $GL_d(F_v)$ *with the same cuspidal support than* $\Pi_{k,v}$ *and*
- *a stable* $\mathbb{T}_I$*-lattice* Γ *of* $(\Pi_k^{\infty,v})^{I^v} \otimes \widetilde{\Pi}_{k,v}^{GL_d(\mathcal{O}_v)}$ *such that*
  - *either* $\mathrm{gr}^k(M)$ *is free, isomorphic to* Γ,
  - *either* $\mathrm{gr}^k(M)$ *is torsion and equals to some subquotient of* Γ/Γ' *where* $\Gamma' \subset \Gamma$ *is another stable* $\mathbb{T}_I$*-lattice.*

We will say that $\mathrm{gr}^k(M)$ *is of type* i *if moreover* $\Pi_{k,v}$ *looks like*

$$\mathrm{St}_i(\chi) \times \chi_1 \times \cdots \times \chi_{d-i}$$

where $\chi, \chi_1, \cdots, \chi_{d-i}$ *are unramified characters.*

Remark: property **(P)** is by definition stable through extensions and subquotients: replacing condition ξ-cohomological by $\xi^\vee$-cohomological, it is also stable by duality.

2.2. Lemma. *Consider* $h \geq 1$ *and* M *an irreducible subquotient of*

$$H_c^0(X_{I,\bar{s}_v}^{=h}, V_{\xi,\overline{\mathbb{Q}}_l}[d-h]) \quad \textit{resp. of } H^0(X_{I,\bar{s}_v}^{\geq h}, V_{\xi,\overline{\mathbb{Q}}_l}[d-h]),$$

then

- *either* M *verify property* ***(P)*** *and then is of type* h *or* $h+1$ *(resp. of type* h*),*
- *either* M *is not a subquotient of* $H^0(X_{I,\bar{s}_v}^{\geq h+1}, V_{\xi,\overline{\mathbb{Q}}_l}[d-h-1])$.

Proof. The result follows from explicit computations of these $\overline{\mathbb{Q}}_l$- cohomology groups with infinite level given in [4]: the reader can see a presentation of them at §3.3 (resp. §3.2) of [6]. Precisely for Π^∞ an irreducible representation of $G(\mathbb{A}^\infty)$, the isotypic component

$$\varinjlim_{I \in \mathcal{I}} H_c^{d-h}(X_{\mathcal{I},\bar{s}_v}^{=h}, V_\xi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)\{\Pi^{\infty,v}\}, \quad \text{resp. } \varinjlim_{I \in \mathcal{I}} H_c^{d-h}(X_{\mathcal{I},\bar{s}_v}^{=h}, V_\xi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l)\{\Pi^{\infty,v}\}$$

is zero if Π^∞ is not the component outside ∞ of an automorphic ξ-cohomological representation Π. Otherwise, we distinguish three cases according to the local component Π_v of Π at v:

(i) $\Pi_v \simeq \mathrm{St}_r(\chi_v) \times \pi'_v$ with $h \leq r \leq d$,

(ii) $\Pi_v \simeq \mathrm{Speh}_r(\chi_v) \times \pi'_v$ with $h \leq r \leq d$,

(iii) Π_v is not of the two previous shape,

where χ_v is an unramified character of $F_v^\times$ and π'_v is an irreducible admissible unramified representation of $GL_{d-h}(F_v)$. Then this isotypic component, as a $GL_d(F_v) \times \mathbb{Z}$-representation is of the following shape:

- in case (i) we obtain $\big(\mathrm{Speh}_h(\chi\{\frac{h-r}{2}\}) \times \mathrm{St}_{r-h}(\chi\{\frac{h}{2}\})\big) \times \pi'_v \otimes \Xi^{\frac{r-h}{2}}$ (resp. zero if $r \neq h$ and otherwise $\mathrm{St}_h(\chi_v)$);
- zero in the case (ii) if $r \neq h$ and otherwise $\mathrm{Speh}_h(\chi_v) \times \pi'_v$.
- Finally in case (iii), the obtained $GL_d(F_v)$-representation won't have non trivial invariants under $GL_d(\mathcal{O}_v)$.

Thus taking invariants under I and because v doesn't divide I,

- case (i): we obtain a $\mathbb{T}_I$-module verifying property **(P)** which is of type h ou $h+1$ according $r=h$ or $h+1$.
- case (ii): the obtained $\mathbb{T}_I$-module is not a subquotient of $H^{d-h-1}(X^{\geq h+1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Q}}_l})$,
- and case (iii): as it doesn't have non trivial invariants under $GL_d(\mathcal{O}_v)$, we obtain nothing else than zero.

□

From now on we assume that there exists i such that the torsion of $H^i(X_{I,\bar{\eta}_v}, V_\xi)$ is non trivial and we fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ such that the torsion of $H^i(X_{I,\bar{\eta}_v}, V_\xi)_\mathfrak{m}$ is non trivial. Let $1 \leq h \leq d$ be maximal such that there exists i for which the torsion subspace $H^i(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor}$ of $H^i(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_\mathfrak{m}$ is non reduced to zero. Notice that

- since the dimension of $X^{\geq d}_{I,\bar{s}_v}$ equals zero then we have $h < d$;
- by the smooth base change theorem $H^i(X_{I,\bar{\eta}_v}, V_\xi) \simeq H^i(X^{\geq 1}_{I,\bar{s}_v}, V_\xi)$ so that $h \geq 1$.

2.3. Lemma. *With the previous notations, $d-h$ is the smallest indice i such that $H^i(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor} \neq (0)$. Then every irreducible non trivial submodule of $H^{d-h}(X^{\geq h}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor}$ verifies property **(P)** being of type $h+1$.*

Proof. Consider the following short exact sequence of perverse sheaves

$$(2.4)\quad 0 \to i_{h+1,*}V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h+1}_{I,\bar{s}_v}}[d-h-1] \longrightarrow j^{\geq h}_! j^{\geq h,*} V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h}_{I,\bar{s}_v}}[d-h] \longrightarrow V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h}_{I,\bar{s}_v}}[d-h] \to 0.$$

Indeed as the strata $X^{\geq h}_{I,\bar{s}_v}$ are smooth and $j^{\geq h}$ is affine, the three terms of this exact sequence are perverse and even free in the sense of the natural torsion theory from the linear $\overline{\mathbb{Z}}_l$-linear structure, see [5] §1.1-1.3.

Moreover from Artin's theorem, see for example theorem 4.1.1 of [2], using the affiness of $X^{=h}_{I,\bar{s}_v}$, we deduce that

$$H^i(X^{\geq h}_{I,\bar{s}_v}, j^{\geq h}_! j^{\geq h,*} V_{\xi,\overline{\mathbb{Z}}_l,|X^{\geq h}_{I,\bar{s}_v}}[d-h])$$

is zero for every $i<0$ and without torsion for $i=0$, so that for $i>0$, we have

(2.5)
$$0 \to H^{-i-1}(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^{-i}(X^{\geq h+1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1]) \to 0,$$

and for $i=0$,

$$(2.6)\quad 0 \to H^{-1}(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X^{\geq h+1}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h-1]) \longrightarrow \\ H^0(X^{\geq h}_{I,\bar{s}_v}, j^{\geq h}_! j^{\geq h,*} V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \longrightarrow H^0(X^{\geq h}_{I,\bar{s}_v}, V_{\xi,\overline{\mathbb{Z}}_l}[d-h]) \to \cdots$$

Thus if the torsion of $H^i(X^{\geq h}_{I,\bar{s}_v}, V_\xi[d-h])$ is non trivial then $i \geq 0$ and thanks to Grothendieck-Verdier duality, the smallest such indice is necessary $i=0$. Furthermore the torsion of $H^0(X^{\geq h}_{I,\bar{s}_v}, V_\xi[d-h])$ raises both into $H^0_c(X^{=h}_{I,\bar{s}_v}, V_\xi[d-h])$ and $H^0(X^{\geq h+1}_{I,\bar{s}_v}, V_\xi[d-h])$, which are both free. Thus by the previous lemma, the torsion of $H^0(X^{\geq h}_{I,\bar{s}_v}, V_\xi[d-h])_{\mathfrak{m}}$ verifies property **(P)** being of type $h+1$. □

2.7. Lemma. *With previous notations, for all $1 \leq h' \leq h$, the greatest i such that the torsion of $H^{-i}(X^{\geq h'}_{I,\bar{s}_v}, V_\xi)_{\mathfrak{m},tor}$ is non zero, equals $h-h'$. Moreover this torsion verifies property **(P)** being of type $h+1$.*

Proof. We argue by induction on h' from h to 1. The case $h'=h$ follows directly from the previous lemma so that we suppose the result true up to $h'+1$ and consider the cas of h'. Resume the spectral sequences (2.4) with h'. Then the result follows from (2.5) and the induction hypothesis. □

Using the smooth base change theorem, the case $h'=1$ of the previous lemma, then gives the following proposition.

2.8. Proposition. *Let i be maximal, if it exists, such that the torsion of $H^{d-1-i}(X_{I,\bar{\eta}_v}, V_\xi)_{\mathfrak{m}}$ is non zero. Then it verifies property **(P)** being of type $i+2$.*

2.9. Corollary. *Consider a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_I$ and i maximal, if it exists, such that the torsion of $H^{-i}(X_{I,\bar\eta}, V_\xi)_\mathfrak{m}$ is non zero. Then there exists a set $\{\Pi(v) :\ v \in \mathrm{Spl}(I)\}$ of irreducible automorphic ξ-cohomological representations such that*

- *for any $w \in \mathrm{Spl}(I)$ different of v, the local component at w of $\Pi(v)$ is unramified with modulo l Satake's parameters given by $S_\mathfrak{m}$;*
- *the local component $\Pi(v)_v$ of Π at v is isomorphic to a representation of the following shape*

$$\mathrm{St}_{i+2}(\chi_{v,0}) \times \chi_{v,1} \times \cdots \times \chi_{v,d-i-2},$$

where $\chi_{v,0}, \cdots, \chi_{v,d-i-2}$ are unramified characters of F_w.

Proof. Consider an irreducible $\mathbb{T}_I$-submodule M of $H^{-i}(X_{I,\bar\eta}, V_\xi)_{\mathfrak{m},tor}$. For any place $v \in \mathrm{Spl}(I)$, thanks to the smooth base change theorem, we have

$$H^{-i}(X_{I,\bar\eta}, V_\xi[d-1])_\mathfrak{m} \simeq H^{-i}(X^{\geq 1}_{I,\bar s_v}, V_\xi[d-1])_\mathfrak{m}.$$

From the previous proposition, this module M verifies property **(P)** being of type $i+2$ so that it exists an automorphic irreducible ξ-cohomological representation $\Pi(v)$ verifying the required properties. □

3. Completed cohomology and torsion classes

Given a level $I^l \in \mathcal{I}$ maximal at l, recall that the completed cohomology groups are

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}/\lambda^n}) := \varinjlim_{I_l} H^i(X_{I^lI_l}, V_{\xi,\mathcal{O}/\lambda^n}[d-1])$$

and

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) := \varprojlim_{n} \widetilde{H^i}_{I^l}(V_{\xi,\mathcal{O}/\lambda^n}),$$

where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_l$ on which the representation ξ is defined.

3.1. Notation. *When $\xi = 1$ is the trivial representation, we will denote*

$$\widetilde{H}^i_{I^l} := \widetilde{H}^i_{I^l}(V_{1,\mathcal{O}}) \otimes_\mathcal{O} \overline{\mathbb{Z}}_l.$$

Remark: for n fixed, there exists an open compact subgroup $I_l(n)$ such that, using the notations below 1.3.8, every $I_l \subset I_l(n)$ acts trivialy on $W_{\xi,\mathcal{O}} \otimes_\mathcal{O}$

$\mathcal{O}/\lambda^n$. We then deduce that the completed cohomology groups don't depend of the choice of ξ in the sense where, see theorem 2.2.17 of [9]:

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_l \simeq \widetilde{H}^i_{I^l} \otimes W_\xi$$

where $G(\mathbb{Q}_l)$ acts diagonally on the right side.

Scholze, see [12] proposition IV.2.2, has showed that the $\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}})$ are trivial for all $i > 0$. In our situation we can prove that for all divisor s of d, there are non zero for $i = 1 - s$: the argument is quite simple but it uses some particular results about entire notions of intermediate extension of Harris-Taylor's local systems. As we don't really need such precision, we only prove the following property.

3.2. Proposition. *For each divisor s of $d = sg$ and for a level I^l outside l small enough, there exists $i \leq 1 - s$ such that $\widetilde{H}^i_{I^l} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$ has, as a $GL_d(F_v)$-representation, an irreducible quotient with degeneracy depth equals to s.*

Proof. Recall the Hochschild-Serre spectral sequence allowing to compute the cohomology at finite levvel from completed one

$$E_2^{i,j} = H^i(I_l, \widetilde{H}^j_{I_l} \otimes V_\xi) \Rightarrow H^{i+j}(X_{I^l I_l}, V_\xi[d-1]). \tag{3.3}$$

Let $v \in \mathrm{Spl}(I^l)$ be a fixed place over some prime number $p \neq l$. Consider then a divisor s of $d = sg$ and an automorphic representation Π which is cohomological relatively to a algebraic representation ξ of G and such that its local component at the place v is isomorphic to $\mathrm{Speh}_s(\pi_v)$ where π_v is an irreducible cuspidal representation of $GL_g(F_v)$. As before we choose a finite level I^l outside l so that Π has non trivial invariants vectors under I^l. According to [4], the Π^∞-isotypic factor of the $\overline{\mathbb{Q}}_l$-cohomology group of indice $1 - s$ is non trivial for $I = I^l I_l$ with I_l small enough. The result then follows from the spectral sequence (3.3). □

a) Consider now a regular algebraic representation ξ so that the $\overline{\mathbb{Q}}_l$-cohomology of X_I with coefficients in V_ξ, is concentrated in middle degree. With the notations of the previous proposition, we deduce that for all divisor s of d, there exist

- $i \leq 1 - s$,
- a finite level I^l outside l such that
- for all $n \geq 1$, there exists $m \geq n$ an on open compact subgroup I_l small enough,

for which $H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}/\lambda^m}[d-1])$ has a class of exactly λ^n-torsion so that these classes give, when m increases, a torsion free class generating an automorphic representation Π with depth of degeneracy equals to s. Moreover from proposition 3.2, for $\mathfrak{m}$ a maximal ideal of $\mathbb{T}_I$ associated to Π, there exists a set $\{\Pi(v);\ v \in \mathrm{Spl}(I)\}$ such that the properties of corollary 2.9 hold, in particular these $\Pi(v)$ are non isomorphic and weakly congruent in twos.

b) We can also exploit the isomorphism of theorem 2.2.17 of [9]

$$\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}}) \otimes_{\mathcal{O}} \overline{\mathbb{Z}}_l \simeq \widetilde{H}^i_{I^l} \otimes V_{\xi,\overline{\mathbb{Z}}_l} \tag{3.4}$$

where $G(\mathbb{Q}_l)$ acts diagonally on the right member. We already saw that for each divisor s of $d = sg$, there exists $i \leq 1-s$ such that the free quotient of $\widetilde{H}^{1-s}_{I^l}$ has an irreducible quotient Π with depth of degeneracy s. Consider then a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{I^l}$ associated to such a Π. Thus for any irreducible algebraic representation ξ non necessarily trivial, from (3.4), we have $\widetilde{H}^i_{I^l}(V_{\xi,\mathcal{O}})_{\mathfrak{m}} \neq (0)$ so that for every I_l small enough and for all n, we have $H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}/\lambda^n[d-1])_{\mathfrak{m}} \neq (0)$. In particular

(i) either these cohomology classes organize themselves when n increases, to generate a free quotient of $H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1])_{\mathfrak{m}}$,

(ii) or the torsion of $H^i(X_{I^l I_l}, V_{\xi,\mathcal{O}}[d-1])_{\mathfrak{m}}$ is non zero.

- In case (i), we obtain a ξ-cohomological automorphic representation Π' which is weakly congruent to Π, which means its modulo l Satake's parameters at every place $v \in \mathrm{Spl}(I)$ not over l, are given by $\mathfrak{m}$ and coincide so to these of Π.

- In case (ii) and, see §2, as the torsion cohomology classes raise in characteristic zero, we obtain in the same way an ξ-cohomological automorphic representation Π' whose modulo l Satake's parameters outside $I \cup \{v\}$ are given by $\mathfrak{m}$.

So in all cases we succeed to construct weakly automorphic congruences between representations of different weight.
Remark: note that Π' obtained in case (ii) is tempered but is not in case (i).

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