Overview on some numerical methods

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Aim of the presentation

We give an overview on some existing numerical methods applied to approximate Differential and Partial Differential Equations.

Plan of this presentation

1. Importance of the Numerical Methods and their Analysis.
2. Overview on Finite Difference Methods
3. Overview on Finite Element Methods
4. Overview on Mixed Finite Element Methods
5. Overview on Finite Volume Methods (Standard and SUSHI)
6. Moving to an Abstract setting: Gradient Discretization Method (A framework of the convergence and analysis of a large class of numerical methods).

Importance of the Numerical Methods and their Analysis

In several situations when solve Applied Mathematical Problems, some (or all) the following steps are followed:

- Real (Physical) Phenomenon. Example: Falling body, Propagation of Heat in a body.
- Modeling: writing the Physical Problem under the Mathematical forms, i.e. relations, equations, ...
  - For Falling body, we find Newton's law of universal gravitation: $F = mg$, where F is the force exerted on a mass m by the Earth's gravitational field of strength g
  - For Propagation of Heat in a body, we find the Heat equation: $u_t - \Delta u = f$.

Importance of the Numerical Methods and their Analysis (Suite)

- Mathematical study. In this step, we prove for instance the existence, uniqueness, and well-posedness of the PDEs (Partial Differential Equations) modeling physical phenomena.
- Numerical Approximation: there are many numerical methods which allow to approximate different problems, Finite Difference, Finite Element, Finite Volumes methods.
- Algorithms and Programming.
- Simulations on machines.

Overview on Finite Difference Methods: References

- S. Godounov and V. Riabenki: Schémas aux Différences, Editions Mir, Moscow, (French), 1977.
- R. D. Richtmyer and K. W. Morton: Difference Methods for Initial Value Problems, Reprint of the 2nd Ed., 1967, Krieger Publishing Company, Melbourne, FL, 1994.
- R.D. Richtmyer: Principles of advanced mathematical physics. Vol. I., Texts and Monographs in Physics. Berlin-Heidelberg-New York: Springer-Verlag. XV, 1978. Explains several models in Physics along with basic background.
- Murray R. Spiegel: Vector Analysis and an Introduction to Tensor Analysis. Schaum's Outline Series. New York etc.: McGraw-Hill Book Comp., 1959. Calculus for function with several variables.
- P. Wesseling: Principles of Computational Fluid Dynamics. Springer Series in Computational Mathematics. 29. Berlin: Springer, 2000. Explains several Numerical Methods.

Some basic Notations and Definitions: Gradient

Definition of the gradient ∇

- If $u = u(\boldsymbol{x})$ (Function with one variable). In this case

$$\nabla u(\boldsymbol{x}) = u_{\boldsymbol{x}}(\boldsymbol{x}) = u'(\boldsymbol{x}).$$

- If $u = u(x, y)$ (Function with two variables). In this case

$$\nabla u = \begin{pmatrix} u_x = \frac{\partial u}{\partial x} \\ u_y = \frac{\partial u}{\partial y} \end{pmatrix}.$$

- If $u = u(x, y, z)$ (Function with three variables). In this case

$$\nabla u = \begin{pmatrix} u_x = \frac{\partial u}{\partial x} \\ u_y = \frac{\partial u}{\partial y} \\ u_z = \frac{\partial u}{\partial z} \end{pmatrix}.$$

Some basic Notations and Definitions: Laplace

Definition of the Laplace operator Δ

- If $u = u(\boldsymbol{x})$ (Function with one variable). In this case

$$\Delta u(\boldsymbol{x}) = u_{\boldsymbol{xx}}(\boldsymbol{x}) = u''(\boldsymbol{x}).$$

- If $u = u(x, y)$ (Function with two variables). In this case

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

- If $u = u(x, y, z)$ (Function with three variables). In this case

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Some basic Notations and Definitions: Divergence

Definition of the Divergence operator $\mathrm{div} = \nabla\cdot$

- If $u = u(\boldsymbol{x})$ (Function with one variable). In this case

$$\mathrm{div} u(\boldsymbol{x}) = u_{\boldsymbol{x}}(\boldsymbol{x}) = u'(\boldsymbol{x}).$$

- If $u = u(x, y)$ (Function with two variables). In this case

$$\mathrm{div} u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.$$

- If $u = u(x, y, z)$ (Function with three variables). In this case

$$\mathrm{div} u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}.$$

Relation between Laplace and Divergence

Relation between Laplace and Divergence

$$\Delta u = \operatorname{div} \nabla u. \tag{1}$$

Some basic Notations and Definitions (Suite)

Definition of a Differential Equation (or also Ordinary Differential Equation-ODE)

It is a relation (equation) between an unknown function and its derivatives.

Definition of a Partial Differential Equation-PDE

It is a relation (equation) between an unknown function, depending on several variables, and its partial derivatives.

Overview on Finite Differences methods: Principles

Model

Let us consider the Differential (or Partial Differential) equation as:

$$\mathcal{L}u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{2}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ and $\mathcal{L}$ is a differential operator.

Principle of Finite Differences methods

The basic ingredient of the Finite Difference methods is to choose some points belonging to Ω (this process is called Discretization of the domain) and these points are called Mesh Points, on which we approximate the derivatives which appear in the operator $\mathcal{L}$ by difference quotients.

An example

As an example, we approximate $u'(\boldsymbol{x}_i)$, where $\boldsymbol{x}_i$ is a mesh point, by the quotient

$$(u(\boldsymbol{x}_{i+1}) - u(\boldsymbol{x}_i))/(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i) \approx u'(\boldsymbol{x}_i).$$

Overview on Finite Differences methods: Example

An example of Differential Equation to be solved

Let us consider the one dimensional stationary heat equation:

$$-u_{xx}(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in (0,1) \quad \text{and} \quad u(0) = u(1) = 0. \tag{3}$$

Discretization of the domain $(0,1)$

We consider the uniform mesh $\boldsymbol{x}_i = ih$, $i \in \{0, \ldots, N\}$, with $N \in \mathbb{N} \setminus \{0\}$ and $h = 1/N$.

Discretisation of the equation

Replacing $\boldsymbol{x}$ by $\boldsymbol{x}_i$ in (3), we get

$$u_{xx}(\boldsymbol{x}_i) = f(\boldsymbol{x}_i), \quad \forall i \in \{0, \ldots, N\}. \tag{4}$$

Overview on Finite Differences methods: Example (suite)

- We consider the following approximation for $u_{xx}(\boldsymbol{x}_i)$

$$\frac{u(\boldsymbol{x}_{i+1}) - 2u(\boldsymbol{x}_i) + u(\boldsymbol{x}_{i-1})}{h^2} \approx u_{xx}(\boldsymbol{x}_i). \tag{5}$$

- Using a convenient Taylor expansion

$$\left| \frac{u(\boldsymbol{x}_{i+1}) - 2u(\boldsymbol{x}_i) + u(\boldsymbol{x}_{i-1})}{h^2} - u_{xx}(\boldsymbol{x}_i) \right| \leq Ch^2 |u_{xx}|_{\mathcal{C}([0,1])}. \tag{6}$$

- Formulation of the scheme: the discrete unknowns are the finite set $\{u_i : \quad i = 1, \ldots, N-1\}$

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(\boldsymbol{x}_i), \quad \forall i \in \{1, \ldots, N-1\} \tag{7}$$

with

$$u_0 = u_N = 0. \tag{8}$$

Overview on Finite Differences methods: Useful comments

What serve scheme (7)–(8)?

The discrete unknown values $\{u_i : \quad i = 1, \ldots, N-1\}$ of are expected to approximate values $\{u(\boldsymbol{x}_i) : \quad i = 1, \ldots, N-1\}$.

How to compute the unknowns of scheme (7)–(8)?

The scheme yields a linear system:

$$AU = F, \tag{9}$$

where

- U is the unknown vector and F is given (RHS of the system).
- A is a square symmetric tridiagonal matrix with $N-1$ lines.
- The vectors U and F are given by $U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix}$ and $F = \begin{pmatrix} f(\boldsymbol{x}_1) \\ f(\boldsymbol{x}_2) \\ \vdots \\ f(\boldsymbol{x}_{N-1}) \end{pmatrix}$.

Overview on Finite Differences methods: Useful comments

Scheme (7)–(8) is very known

The scheme (7)–(8) is called the Three Points Centered Finite Difference Scheme.

To justify the convergence of FD schemes

The main tool to prove the convergence of Finite Difference schemes is to justify two properties: To prove the convergence of finite difference schemes, we justify two properties

- Consistency
- Stability

Overview on Finite Element Methods: References

- P. G. Ciarlet: The Finite Element Method for Elliptic Problems. Classics in Applied Mathematics, 40. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2002).
- A. Quarteroni and A. Valli: Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics 23. Berlin: Springer. (2008).
- P.-A. Raviart, J.-M. Thomas: Introduction à l'Analyse Numérique des Equations aux Dérivées Partielles. Mathématiques Appliquées pour la maîtrise. Dunod, 2004.
- V. Thomée: Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Second Edition, Berlin (2006).

Overview on Mixed Finite Element Methods: Principles

Let us consider the Differential (or Partial Differential) equation as:

$$\mathcal{L}u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{10}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with some convenient boundary conditions (Dirichlet, Neumann, Robin,...)

Principle of Finite Element methods

The basic ingredient of the Finite Element methods is the approximation of an equivalent weak Formulation for the original problem. For instance such weak formulation can be given as: Find $u \in \mathcal{H}$ such that

$$a(u, v) = \mathcal{F}(v), \quad \forall v \in \mathcal{H}. \tag{11}$$

The space $\mathcal{H}$ is given for instance using the so-called Sobolov spaces.

Overview on Finite Element Methods: Example

Example of FEM

Let us consider the above one dimensional problem (3).

- The weak formulation is given by: Find $u \in H_0^1(0,1)$ such that

$$\int_0^1 u_{\boldsymbol{x}}(\boldsymbol{x}) v_{\boldsymbol{x}}(\boldsymbol{x}) d\boldsymbol{x} = \int_0^1 f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(0,1). \tag{12}$$

- The Finite Element space is an approximation of $H_0^1(0,1)$ which can be for instance the space of piecewise linear functions:

$$\mathcal{V}_h = \left\{ v \in \mathcal{C}[0,1] : \quad v|_{[\boldsymbol{x}_i, \boldsymbol{x}_{i+1}]} \in \mathcal{P}_1 \right\} \tag{13}$$

  where $0 = \boldsymbol{x}_0 < \boldsymbol{x}_1 < \boldsymbol{x}_2 < \ldots < \boldsymbol{x}_N = 1$.

- Finite element scheme: Find $u_h \in \mathcal{V}_h$ such that

$$\int_0^1 (u_h)_{\boldsymbol{x}} (\boldsymbol{x}) \, (v_h)_{\boldsymbol{x}} (\boldsymbol{x}) d\boldsymbol{x} = \int_0^1 f(\boldsymbol{x}) v_h(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{V}_h. \tag{14}$$

Overview on Finite Element methods: Main method to prove the convergence

Main tools to prove the convergence of finite element schemes

- We compare the error between the approximate solution and exact solution and the error between the the exact solution and its interpolation.
- We determine an estimate for the error between the the exact solution and its interpolation.

Overview on Mixed Finite Element Methods: References

- P. G. Ciarlet: The Finite Element Method for Elliptic Problems. Classics in Applied Mathematics, 40. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2002).
- A. Quarteroni and A. Valli: Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics 23. Berlin: Springer. (2008).
- P.-A. Raviart, J. M. Thomas: A mixed finite element method for 2nd order elliptic problems. Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), pp. 292–315.
- V. Thomée: Galerkin Finite Element Methods for Parabolic Problems. Springer-Verlag, Second Edition, Berlin (2006).

Overview on Mixed Finite Element Methods: Principles

Main idea of Mixed Finite Element Methods

We introduce two variables, one called velocity and the other called pressure. This lead to a system of equations whose the order of derivatives in each equation is less than of the original equation.

Overview on Mixed Finite Element Methods: Principles

Model Equation: Poisson Equation

Heat equation:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{15}$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, f is given function.

Homogeneous Dirichlet boundary

$$u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{16}$$

General principles of MFEMs

First step: Writing the problem as:

$$p = -\nabla u \quad \text{and} \quad \mathrm{div} p = f. \tag{17}$$

Second step: Weak formulation for (17)

$$(p, \psi)_{L^2(\Omega)} - (u, \mathrm{div}\psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in H_{\mathrm{div}}(\Omega) \tag{18}$$

and

$$(\mathrm{div} p, \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in L^2(\Omega), \tag{19}$$

$H_{\mathrm{div}}(\Omega)$ is the space defined by

$$H_{\mathrm{div}}(\Omega) = \{\xi \in \left(L^2(\Omega)\right)^d : \quad \mathrm{div}\xi \in L^2(\Omega)\}.$$

Approximation of MFE: Principles

We have two finite element spaces:

Spaces involved in the approximation of the weak formulation

- We approximate the space of velocity $H_{\text{div}}(\Omega)$
- We approximate the space of pressure $L^2(\Omega)$.

Well used spaces

Raviart-Thomas finite element spaces.

Overview on Finite Volume Methods: References

- Eymard, Robert; Gallouët, Thierry; Herbin, Raphaèle: Finite volume methods. Handbook of numerical analysis, Vol. VII, 713–1020, Handb. Numer. Anal., VII, North-Holland, Amsterdam, 2000.
- Eymard, Robert; Gallouët, Thierry; Herbin, Raphaèle: Finite volume methods. Handbook of numerical analysis. hal-02100732 , version 2 (12-08-2019).
- Eymard, Robert; Gallouët, Thierry; Herbin, Raphaèle: Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: A scheme using stabilization and hybrid interfaces. IMAJNA, 2010.
- Randall J. LeVeque: Finite Volume Methods for Hyperbolic Problems. Cambridge University Press, 2012.

Overview on Finite Volume Methods: Principles

Fundamental highlight on the method

The finite volume method is a discretization method which is well suited for the numerical simulation of various types (elliptic, parabolic or hyperbolic, for instance) of conservation laws; it has been extensively used in several engineering fields, such as fluid mechanics, heat and mass transfer or petroleum engineering.

Main ingredients of the method

- We subdivide the domain into subsets called control volumes.
- Integration of the equation to be solved on each control volume.
- Integration by parts to transfer the integration on the control volumes to integration on the interfaces across the control volumes.
- When the control volumes satisfy a condition called transmissibility, the approximation of the integrals along the interfaces becomes easier.

Overview on Finite Volume Methods: Principles

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- When the control volumes satisfy a condition called transmissibility, the approximation of the integrals along the interfaces becomes easier.

Overview on Finite Volume Methods: One Dimensional Example

Let us consider the one dimensional stationary heat problem (3). We derive a finite volume scheme.

Definition

An admissible mesh of $(0, 1)$, denoted by $\mathcal{T}$, is given by a family of subsets (called control volumes) $(K_i)_{i=1,\ldots,N}$, $N \in \mathbb{N}^\star$ with $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ and $\boldsymbol{x}_{1/2} = 0$ and $\boldsymbol{x}_{N+1/2} = 1$.

We assume in addition that there is a family $(\boldsymbol{x}_i)_{i=1,\ldots,N}$ such that $\boldsymbol{x}_i \in K_i$. For the convenience of approximation, we set $\boldsymbol{x}_0 = 0$ and $\boldsymbol{x}_{N+1} = 1$.

We set

$$h_i = \text{meas}(K_i) = \boldsymbol{x}_{i+\frac{1}{2}} - \boldsymbol{x}_{i-\frac{1}{2}}, \quad \text{for } i \in \{1, \ldots, N\},$$

$$h_{i+\frac{1}{2}} = \boldsymbol{x}_{i+1} - \boldsymbol{x}_i,\ i = 0, \ldots, N \quad \text{and} \quad \text{size}(\mathcal{T}) = h = \max\{h_i,\ i = 1, \ldots, N\}.$$

Overview on Finite Volume Methods: One Dimensional Example-Derivation of the scheme

- Integrating the equation over the control volume $K_i = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ yields

$$u(\boldsymbol{x}_{i-\frac{1}{2}}) - u(\boldsymbol{x}_{i+\frac{1}{2}}) = \int_{K_i} f(\boldsymbol{x}) d\boldsymbol{x}. \tag{20}$$

- The approximation of $u(\boldsymbol{x}_{i+\frac{1}{2}})$ is given by $\left(u(\boldsymbol{x}_{i+1}) - u(\boldsymbol{x}_i)\right)/h_{i+\frac{1}{2}}$.
- We therefore have, thanks to the previous two items

$$\left(u(\boldsymbol{x}_i) - u(\boldsymbol{x}_{i-1})\right)/h_{i-\frac{1}{2}} - \left(u(\boldsymbol{x}_{i+1}) - u(\boldsymbol{x}_i)\right)/h_{i+\frac{1}{2}} \approx \int_{K_i} f(\boldsymbol{x}) d\boldsymbol{x}. \tag{21}$$

- The scheme is given by:

$$\left(u_i - u_{i-1}\right)/h_{i-\frac{1}{2}} - \left(u_{i+1} - u_i\right)/h_{i+\frac{1}{2}} = \int_{K_i} f(\boldsymbol{x}) d\boldsymbol{x}, \quad i \in \{1, \ldots, N\} \tag{22}$$

with $u_0 = u_{N+1} = 0$.

Overview on Finite Volume Methods: One Dimensional Example-Nice Remark on FDM and FVM

Nice comment between FDM and FVM

As we have remarked that with respect to Finite Differences method, we gain a derivative in Finite Volume method. Indeed, in the case of the one dimensional stationary heat equation $-u'' = f$, we have to approximate the second derivative when we are dealing with Finite Differences method and only the first derivative when we are dealing with Finite Volume method.

Finite Volume methods in several space dimensions on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K.

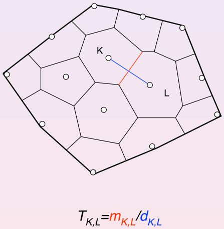

Figure: transmissivity between K and L: $\mathcal{T}_\sigma = \mathcal{T}_{K|L} = \frac{m_{K,L}}{d_{K,L}}$

Finite Volume methods on admissible meshes (Standard FVM)

Main properties of Admissible mesh:

1. Convexity of the Control Volumes.
2. The orthogonality property: the $(\boldsymbol{x}_K\boldsymbol{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L.

Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \quad \text{and} \quad u(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{23}$$

Principles of Finite Volume scheme:

1. Integration on each control volume K: $-\int_K \Delta u(\boldsymbol{x}) d\boldsymbol{x} = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$,
2. Integration by Parts gives: $-\int_{\partial K} \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$
3. Summing on the lines of K: $-\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x}) = \int_K f(\boldsymbol{x}) d\boldsymbol{x}$

Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma\in\mathcal{E}_K}\frac{\mathrm{m}(\sigma)}{d_{K|L}}(u_L - u_K) = \int_K f(\boldsymbol{x})d\boldsymbol{x}. \tag{24}$$

Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

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Finite Volume methods on admissible meshes

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Matrix Form

$$\mathcal{A}^{\mathcal{T}} u_{\mathcal{T}} = f_{\mathcal{T}}.$$

Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K. Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\boldsymbol{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence results hold:

1. H_0^1-*error estimate*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \le Ch\|u\|_{2,\overline{\Omega}}, \tag{25}$$

where $\|\cdot\|_{1,\mathcal{T}}$ *is the* H_1^0-*norm* $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma = K|L \in \mathcal{E}} \frac{\mathrm{m}(\sigma)}{d_\sigma}(u_L - u_K)^2$.

2. L^2-*error estimate:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{26}$$

Finite Volume methods on admissible meshes: three useful remarks

First remark: Conservativity of the numerical fluxes

It means that fluxes, that is the numerical flux is conserved from one discretization cell to its neighbour. This last feature makes the finite volume method quite attractive when modelling problems for which the flux is of importance, such as in fluid mechanics, semi-conductor device simulation, heat and mass transfer...
For the approximation given above

$$-\frac{\mathrm{m}(\sigma)}{d_{K|L}}(u(\boldsymbol{x}_L) - u(\boldsymbol{x}_K)) \approx -\int_\sigma \nabla u(\boldsymbol{x}) \cdot \mathbf{n}(\boldsymbol{x}) d\gamma(\boldsymbol{x})$$

If we denote by the numerical flux $F_{K,L} = -\dfrac{\mathrm{m}(\sigma)}{d_{K|L}}(u(\boldsymbol{x}_L) - u(\boldsymbol{x}_K))$, then the following **Conservativity** property holds

$$F_{K,L} = -F_{L,K}.$$

Finite Volume methods on admissible meshes: two useful remarks (Suite)

Second remark: Consistency of the approximation of the Flux

In contrast of Finite Difference Methods, in which we approximate directly the derivatives and quantities of the equation to be solved, in Finite Volume Methods we approximate these terms after integration. In the case of the Poisson's problem, we have to approximate the Flux which yields the **Consistency** of the approximation of the Flux.

Third remark: FVM is different from FDM and FEM

The Finite Volume Method is quite different from (but sometimes related to) the Finite Difference Method and the Finite Element Method, see details in Handbook of Eymard et *al.* (2000).

Finite Volume methods using nonconforming grids, SUSHI scheme

Abbreviation

SUSHI: Scheme Using Stabilization and Hybrid Interfaces.

Definition (New mesh of Eymard et *al.*, IMAJNA 2010)

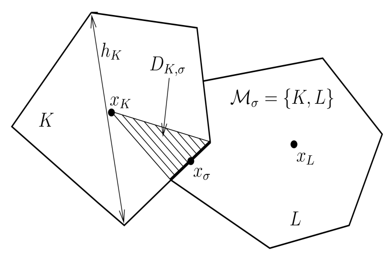

Figure: Notations for two neighbouring control volumes in $d = 2$

Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

1. (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$
2. (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
3. (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.

Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for the Poisson's problem:

1. Discrete unknowns: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{\left((v_K)_{K\in\mathcal{M}}\,,\ (v_\sigma)_{\sigma\in\mathcal{E}}\right),\ \ v_K, v_\sigma \in \mathbb{R},\ \ v_\sigma = 0,\ \forall\sigma\in\mathcal{E}_{\rm ext}\}$$

2. Discretization of the gradient: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

   1. The discrete gradient $\nabla_{\mathcal{D}}$ is stable
   2. The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.

Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in H_0^1(\Omega). \tag{27}$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_\Omega \nabla_{\mathcal{D}} u_{\mathcal{D}}(\boldsymbol{x}) \cdot \nabla_{\mathcal{D}} v(\boldsymbol{x}) d\boldsymbol{x} = \int_\Omega f(\boldsymbol{x}) v(\boldsymbol{x}) d\boldsymbol{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{28}$$

Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in \mathcal{C}^2(\overline{\Omega})$. Then the following convergence result hold:

1. H_0^1*-error estimate*

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{29}$$

2. L^2*-error estimate:*

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \le Ch\|u\|_{2,\overline{\Omega}}. \tag{30}$$

Overview on the Gradient Discretization Method: References

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Overview on the Gradient Discretization Method: What is GDM simply ?

What is GDM?

Is a framework for the convergence and analysis of a large class of the numerical methods.

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Overview on the Gradient Discretization Method: Definition

Definition (Definition of a generic approximate gradient discretization, Droniou et al. (Springer book, 2018))

Let Ω be an open domain of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$. An approximate gradient discretization $\mathcal{D}$ is defined by $\mathcal{D} = (\mathcal{X}_{\mathcal{D},0}, h_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where

1. The set of discrete unknowns $\mathcal{X}_{\mathcal{D},0}$ is a finite dimensional vector space on $\mathbb{R}$.
2. The space step $h_{\mathcal{D}} \in (0, +\infty)$ is a positive real number.
3. The linear mapping $\Pi_{\mathcal{D}} : \mathcal{X}_{\mathcal{D},0} \to L^2(\Omega)$ is the reconstruction of the approximate function.
4. The mapping $\nabla_{\mathcal{D}} : \mathcal{X}_{\mathcal{D},0} \to L^2(\Omega)^d$ is the reconstruction of the gradient of the function; it must be chosen such that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $\mathcal{X}_{\mathcal{D},0}$.

Overview on the Gradient Discretization Method: Additional parameters

Definition (Additional hypotheses on the approximate gradient discretization)

- The **coercivity** of the discretization is measured through the the constant $C_{\mathcal{D}}$ given by:

$$C_{\mathcal{D}} = \max_{v \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}}. \tag{31}$$

- The **strong consistency**: $S_{\mathcal{D}} : H_0^1(\Omega) \to [0, +\infty)$ defined by, for all $\varphi \in H_0^1(\Omega)$

$$S_{\mathcal{D}}(\varphi) = \min_{v \in \mathcal{X}_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|^2_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|^2_{L^2(\Omega)^d} \right)^{\frac{1}{2}}. \tag{32}$$

- The **dual consistency**: For all $\varphi \in H_{\mathrm{div}}(\Omega)$, $W_{\mathcal{D}}(\varphi)$ is given by

$$\max_{u \in \mathcal{X}_{\mathcal{D},0} \setminus \{0\}} \frac{1}{\|\nabla_{\mathcal{D}} u\|_{L^2(\Omega)^d}} \left| \int_{\Omega} \left(\nabla_{\mathcal{D}} u(\boldsymbol{x}) \cdot \varphi(\boldsymbol{x}) + \Pi_{\mathcal{D}} u(\boldsymbol{x}) \mathrm{div} \varphi(\boldsymbol{x}) \right) d\boldsymbol{x} \right|.$$

Overview on the Gradient Discretization Method: A typical example-Conforming Finite Element Method

FE is a GDM

Let $\{\mathcal{T}_h; h > 0\}$ be a family of shape regular and quasi-uniform triangulations of the domain Ω. Let $\mathcal{V}^h$ be the standard finite element space of continuous, piecewise polynomial functions of degree less or equal $l \in \mathbb{N} \setminus \{0\}$ and we denote by $\mathcal{V}_0^h = \mathcal{V}^h \cap H_0^1(\Omega)$.

Assume that $\mathcal{V}_0^h$ is spanned by the usual basis functions $\varphi_1, \ldots, \varphi_M$. The space $\mathcal{X}_{\mathcal{D},0}$ can be $\mathbb{R}^M$ and for any $(u_1, \ldots, u_M) \in \mathcal{X}_{\mathcal{D},0}$, we define $\Pi_{\mathcal{D}} u = \sum_{i=1}^M u_i \varphi \in \mathcal{V}_0^h \subset H_0^1(\Omega)$ and $\nabla_{\mathcal{D}} u = \sum_{i=1}^M u_i \nabla \varphi = \nabla \Pi_{\mathcal{D}} u$. Using the Poincaré inequality, we have for all $u \in \mathcal{X}_{\mathcal{D},0}$, $\|\Pi_{\mathcal{D}} u\|_{\mathbb{L}^2(\Omega)} \le C(\Omega) \|\nabla_{\mathcal{D}} u\|_{\mathbb{L}^2(\Omega)}$.

Conditions of GDM are well satisfied by FE

Therefore, the assumption (31) of Definition 6 holds with constant $C_{\mathcal{D}}$ only depending on Ω. In addition to this, we have $W_{\mathcal{D}}(\varphi) = 0$, for all $\varphi \in H_{\mathrm{div}}(\Omega)$, and $S_{\mathcal{D}}(\varphi)$ is bounded above by (up to a multiplicative constant independent of the mesh) $h^l |\varphi|_{l+1,\Omega}$, for all $\varphi \in H^{l+1}(\Omega)$.

Overview on the Gradient Discretization Method: An example of application

- Let us consider the Poisson problem described above in (23).
- The weak formulation for this problem is: Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)^d} = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega).$$

- The gradient scheme applied to the Poisson's problem is: Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$(\nabla_{\mathcal{D}} u_{\mathcal{D}}, \nabla_{\mathcal{D}} v)_{L^2(\Omega)^d} = (f, v)_{L^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \tag{33}$$

Overview on the Gradient Discretization Method: Convergence

On the convergence

The convergence is well detailed in the above References, see for instance Eymard et *al.* (ESAIM-2012).