

Novel discrete $W^{\alpha,2}(L^2)$ -a priori estimate of a SUSHI scheme for a fractional partial differential equation ¹

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Aim of the presentation

The aim of this talk is to provide a discrete $W^{\alpha,2}(L^2)$ -*a priori* estimate of a SUSHI scheme applied to a fractional partial differential equation. Such $W^{\alpha,2}(L^2)$ -*a priori* estimate would lead to an error estimate in a $W^{\alpha,2}(L^2)$ -discrete seminorm, i.e. an estimate for the error between a discrete fractional derivative of the approximate solution and the fractional order derivative of the exact solution in an $L^2(L^2)$ -discrete norm.



Plan of this presentation

- 1 References
- 2 Equation to be solved
- 3 Introduction: Finite Volume methods from Admissible to Nonconforming meshes (SUSHI scheme)
- 4 Finite Volume scheme for a time fractional partial differential equation
- 5 Some known continuous and discrete a-priori estimates
- 6 Statement of the main result
- 7 Conclusion and some Perspectives



Some references Part I

- Benkhaldoun, F., Bradji, A.: An $L^\infty(H^1)$ -error estimate for gradient schemes applied to time fractional diffusion equations. FVCA X., 177–185, Springer Proc. Math. Stat., 432, Springer, Cham, 2023.
- A. Bradji. A new analysis for the convergence of the gradient discretization method for multidimensional time fractional diffusion and diffusion-wave equations. Comput. Math. Appl., 79/2, 500–520 (2020).
- A. Bradji. Notes on the convergence order of gradient schemes for time fractional differential equations. C. R. Math. Acad. Sci. Paris 356/4 (2018).



Some references Part II

- Bradji, A, Fuhrmann. J.: Some abstract error estimates of a finite volume scheme for a nonstationary heat equation on general nonconforming multidimensional spatial meshes. Appl. Math. 58 /1, 1–38 (2013).
- R. Eymard, T. Gallouët, R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes (SUSHI). IMA J. Numer. Anal. 30/4 (2010).
- K. Sakamoto, M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. J. Math. Anal. Appl. 382/1, 426–447 (2011).



Equation to be solved

Equation

We consider the following time fractional diffusion equation:

$$\partial_t^\alpha u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times (0, T), \quad (1)$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, $T > 0$, and f is a given function. Here the operator ∂_t^α is the Caputo derivative defined by:

$$\partial_t^\alpha u(\mathbf{x}, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial u(\mathbf{x}, s)}{\partial s} ds, \quad 0 < \alpha < 1. \quad (2)$$

Initial condition

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Homogeneous Dirichlet boundary

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T).$$



What about time fractional diffusion equation?

Some physics

Fractional differential equations have been successfully used in the modeling of many different processes and systems. They are used, for instance, to describe anomalous transport in disordered semiconductors, penetration of light beam through a turbulent medium, transport of resonance radiation in plasma, blinking fluorescence of quantum dots, penetration and acceleration of cosmic ray in the Galaxy, and large-scale statistical Cosmography. We refer to the monograph Uchaikin (Fractional Derivatives for Physicists and Engineers, Springer-Verlag Heidelberg, 2013) where we find many details.



Introduction: Finite Volume from Admissible meshes to Nonconforming meshes

Finite volume methods are numerical methods approximating different types of Partial Differential Equations (PDEs). They are based on three principle ideas:

- Subdivision of the spatial domain into subsets called **Control Volumes**.
- Integration of the equation to be solved over the **Control Volumes**.
- Approximation of the derivatives appearing after integration.



¹We mean here the "pure" finite volume methods and not finite volume-element methods

Introduction (suite): Finite Volume from Admissible meshes to Nonconforming meshes

Finite Volume methods passed by three steps:

First step

Finite Volume methods using Admissible meshes.

Second step

SUSHI method.



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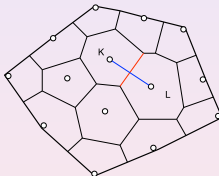


Introduction (suite): Finite Volume methods on admissible meshes

Definition

Let $\mathcal{T}$ be an Admissible Mesh in the sense of Eymard et al. (Handbook, 2000).

$K \in \mathcal{T}$ are the control volumes and σ are the edges of the control volumes K .



$$T_{K,L} = m_{K,L} / d_{K,L}$$

Figure: transmissivity between K and L : $\mathcal{T}_\sigma = T_{K|L} = \frac{m_{K,L}}{d_{K,L}}$



Introduction (suite): Finite Volume methods on admissible meshes

Main properties of Admissible mesh:

- 1 Convexity of the Control Volumes.
- 2 The orthogonality property: the $(\mathbf{x}_K \mathbf{x}_L)$ is orthogonal to the common edge σ between the control volumes K and L .



Introduction (suite): Finite Volume methods on admissible meshes

Model to be solved:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega \quad \text{and} \quad u(\mathbf{x}) = 0, \mathbf{x} \in \partial\Omega. \quad (3)$$

Principles of Finite Volume scheme:

- 1 Integration on each control volume K : $-\int_K \Delta u(\mathbf{x}) d\mathbf{x} = \int_K f(\mathbf{x}) d\mathbf{x},$
- 2 Integration by Parts gives : $-\int_{\partial K} \nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d\gamma(\mathbf{x}) = \int_K f(\mathbf{x}) d\mathbf{x}$
- 3 Summing on the lines of K : $-\sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \nabla u(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) d\gamma(\mathbf{x}) = \int_K f(\mathbf{x}) d\mathbf{x}$



Introduction (suite): Finite Volume methods on admissible meshes

Approximate Finite Volume Solution $u_{\mathcal{T}} = (u_K)_K$

$$-\sum_{\sigma \in \mathcal{E}_K} \frac{m(\sigma)}{d_K|L|} (u_L - u_K) = \int_K f(x) dx. \quad (4)$$

Matrix Form

$$\mathcal{A}^T u_{\mathcal{T}} = f_{\mathcal{T}}.$$



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Introduction (suite): Finite Volume methods on admissible meshes

Theorem

Let $\mathcal{X}(\mathcal{T})$: functions which are constant on each control volume K . Let $e_{\mathcal{T}} \in \mathcal{X}(\mathcal{T})$ be defined by $e_K = u(\mathbf{x}_K) - u_K$ for any $K \in \mathcal{T}$. Assume that the exact solution u satisfies $u \in C^2(\overline{\Omega})$. Then the following convergence results hold:

1 H_0^1 -error estimate

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch\|u\|_{2,\overline{\Omega}}, \quad (5)$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the H_1^0 -norm $\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = \sum_{\sigma=K|L \in \mathcal{E}} \frac{m(\sigma)}{d_{\sigma}} (u_L - u_K)^2$.

2 L^2 -error estimate:

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch\|u\|_{2,\overline{\Omega}}. \quad (6)$$



Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Definition (New mesh of Eymard *et al.*, IMAJNA 2010)

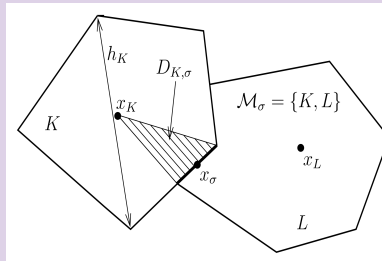


Figure: Notations for two neighbouring control volumes in $d = 2$

Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Main properties of this new mesh:

- 1 (mesh defined at any space dimension): $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$
- 2 (orthogonality property is not required): the orthogonality property is not required in this new mesh. But, additional discrete unknowns are required.
- 3 (convexity): the classical admissible mesh should satisfy that the control volumes are convex, whereas the convexity property is not required in this new mesh.



Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI scheme

Principles of discretization for Poisson's equation:

- 1 **Discrete unknowns**: the space of solution as well as the space of test functions are in

$$\mathcal{X}_{\mathcal{D},0} = \{((v_K)_{K \in \mathcal{M}}, (v_\sigma)_{\sigma \in \mathcal{E}}), v_K, v_\sigma \in \mathbb{R}, v_\sigma = 0, \forall \sigma \in \mathcal{E}_{\text{ext}}\}$$

- 2 **Discretization of the gradient**: the discretization of ∇ can be performed using a stabilized discrete gradient denoted by $\nabla_{\mathcal{D}}$, see Eymard et *al.* (IMAJNA, 2010):

- 1 The discrete gradient $\nabla_{\mathcal{D}}$ is stable
- 2 The discrete gradient $\nabla_{\mathcal{D}}$ is consistent.



Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Weak formulation for Poisson's equation: Find $u \in H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \quad \forall v \in H_0^1(\Omega). \quad (7)$$

SUSHI (Scheme Using stabilized Hybrid Interfaces) for Poisson's equation: Find $u_{\mathcal{D}} \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\int_{\Omega} \nabla_{\mathcal{D}} u_{\mathcal{D}}(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}. \quad (8)$$



Introduction (suite): Finite Volume methods using nonconforming grids, SUSHI

Theorem

Assume that the exact solution u satisfies $u \in C^2(\overline{\Omega})$. Then the following convergence result hold:

1 H_0^1 -error estimate

$$\|\nabla u - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^2(\Omega)^d} \leq Ch \|u\|_{2, \overline{\Omega}}. \quad (9)$$

2 L^2 -error estimate:

$$\|u - \Pi_{\mathcal{M}} u_{\mathcal{D}}\|_{L^2(\Omega)} \leq Ch \|u\|_{2, \overline{\Omega}}. \quad (10)$$



Principles of the discretization

Discretization in time

We define $k = T/(M + 1)$.

$$\partial_t^\alpha u(x, t_{n+1}) = \sum_{j=0}^n k \lambda_j^{n+1} \partial^1 u(x, t_{j+1}) + \mathbb{T}_1^{n+1}(x), \quad (11)$$

where

$$\partial^1 \varphi^{j+1} = \frac{v^{j+1} - v^j}{k} \quad \text{and} \quad \lambda_j^{n+1} = \frac{(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}}{k^\alpha \Gamma(2-\alpha)} \quad (12)$$

and

$$|\mathbb{T}_1^{n+1}| \leq Ck^{2-\alpha}. \quad (13)$$



Principles of the discretization (suite)

Discretization in space

We use SUSHI scheme



Formulation of scheme

The finite volume scheme can then be defined as:

- Discretization of initial condition: Find $u_{\mathcal{D}}^0 \in \mathcal{X}_{\mathcal{D},0}$ such that

$$\left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^0, \nabla_{\mathcal{D}} v \right)_{(\mathbb{L}^2(\Omega))^d} = - \left(\Delta u^0, \Pi_{\mathcal{M}} v \right)_{\mathbb{L}^2(\Omega)}, \quad \forall v \in \mathcal{X}_{\mathcal{D},0}, \quad (14)$$

- Discretization of the fractional heat equation: and for any $n \in \llbracket 0, M \rrbracket$, find $u_{\mathcal{D}}^{n+1} \in \mathcal{X}_{\mathcal{D},0}$ such that, for all $v \in \mathcal{X}_{\mathcal{D},0}$

$$\begin{aligned} \sum_{j=0}^n \lambda_j^{n+1} \left(\Pi_{\mathcal{M}} \partial^1 u_{\mathcal{D}}^{j+1}, v \right)_{\mathbb{L}^2(\Omega)} &+ \left(\nabla_{\mathcal{D}} u_{\mathcal{D}}^{n+1}, \nabla_{\mathcal{D}} v \right)_{(\mathbb{L}^2(\Omega))^d} \\ &= \left(f(t_{n+1}), v \right)_{\mathbb{L}^2(\Omega)}. \end{aligned} \quad (15)$$



Convergence result

Theorem (An $L^\infty(L^2)$ -error estimate)

$$\|u(t_n) - \Pi_{\mathcal{M}} u_{\mathcal{D}}^n\|_{\mathbb{L}^2(\Omega)} \leq C \left(\frac{CT^{1-\alpha}}{\Gamma(2-\alpha)} h_{\mathcal{D}} + \frac{\alpha}{6\Gamma(1-\alpha)} k^{2-\alpha} \right). \quad (16)$$



Some known Continuous and Discrete a-priori estimates in space

$L^\infty(H_0^1)$ - a priori estimates

- Continuous $L^\infty(H_0^1)$ - a priori estimates. Sakamoto and Yamamoto (JMAA, 2011, Theorem 2.4.-Pages 430–431) proved that if $u^0 \equiv 0$ and f is smooth, e.g. $f \in C^1(L^2)$

$$\|u\|_{C(H^2)} \leq C\|f\|_{C^1(L^2)}. \quad (17)$$

- Discrete $L^\infty(H_0^1)$ - a priori estimates. Bradji (Comput. Math. Appl., 2020) proved that if $u^0 \equiv 0$

$$\max_{n=0}^{N+1} \|\nabla_{\mathcal{D}} u_{\mathcal{D}}^n\|_{(L^2)^d} \leq C \max_{n=0}^{N+1} \|f(t_n)\|_{L^2}. \quad (18)$$



Some known Continuous and Discrete a-priori estimates in time

$W^{\alpha,2}(L^2)$ - a priori estimates

- Continuous $W^{\alpha,2}(L^2)$ - a priori estimates. Sakamoto and Yamamoto (JMAA, 2011, (2.5)-Page 429) proved that if $u^0 \equiv 0$

$$\|\partial_t^\alpha u\|_{L^2(L^2)} \leq C \|f\|_{L^2(L^2)}. \quad (19)$$

- Discrete $W^{\alpha,2}(L^2)$ - a priori estimates. Bankaldoun and Bradji (ICNAAM 2026) proved that if $u^0 \equiv 0$

$$\left(\sum_{n=0}^N k \left\| \sum_{j=0}^n k \lambda_j^{n+1} \partial^1 \eta^{j+1} \right\|^2 \right)^{\frac{1}{2}} \leq C \left(\left\| \sum_{n=0}^N k \|f(t_n)\|^2 \right\|^{\frac{1}{2}} \right). \quad (20)$$



Conclusion and Perspectives

Conclusion

A Finite Volume method combined the L1-formula on uniform temporal mesh are used to establish a fully discrete scheme. A new discrete $W^{\alpha,2}(L^2)$ - a priori estimate is proved.

First perspective

Use this result to prove $W^{\alpha,2}(L^2)$ -error estimate.

Second perspective

Extension to other approximation for the Caputo derivative, e.g. L2-formula (instead of L1-formula).

Third perspective

Gradient schemes framework for space fractional diffusion equations



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