

A fully discrete mixed finite element scheme for a generalized Bingham flow type equation ¹

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Talk in ICNAAM 2026

September 15–September 21 2026, Virtual Talk, Crete, Greece



¹Supported by MCS team (LAGA Laboratory) of the “Université Sorbonne- Paris Nord”

Aim of the presentation

The aim of this talk is to establish a fully discrete Mixed Finite Element scheme for a generalized Bingham flow type equation (which is Non-Linear Parabolic) along with a convergence analysis when the exact solution is smooth.



Plan of this presentation

- 1 References
- 2 Problem to be solved and some of Physics about.
- 3 An overview on Mixed Finite Element methods
- 4 Formulation of a MFE scheme
- 5 Statement of the convergence
- 6 Conclusion and Perspectives



Some references Part I

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- W. Aboussi, F. Benkhaldoun, and A. Bradji: SUSHI for a Bingham flow type problem. Numerical methods and applications, 1–13, Lecture Notes in Comput. Sci., 13858, Springer, Cham, 2023.
- F. Benkhaldoun and A. Bradji: Novel analysis approach for the convergence of a second order time accurate mixed finite element scheme for parabolic equations. Comput. Math. Appl. 133, 85–103 (2023).
- F. Benkhaldoun and A. Bradji: Two new error estimates of a fully discrete primal-dual mixed finite element scheme for parabolic equations in any space dimension. Results Math. 76/4, Paper No. 182, 25 Pages (2021).



Some references Part II

- A. Bradji: An analysis for the convergence order of gradient schemes for semilinear parabolic equations. *Comput. Math. Appl.* 72/5, 1287–1304 (2016).
- E. J. Dean, R. Glowinski, and G. Guidoboni: Review on the numerical simulation of Bingham visco-plastic flow: Old and new results. *Journal of Non-Newtonian Fluid Mechanics* 142/1-3, 36–62 (2007).
- A. Quarteroni and A. Valli: Numerical Approximation of Partial Differential Equations. Springer Series in Computational Mathematics 23. Berlin: Springer. (2008).
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Equation to be solved

Equation

We consider the following nonlinear parabolic problem, for $(\mathbf{x}, t) \in \Omega \times (0, T)$

$$u_t(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) - \nabla \cdot \mathcal{F}(\nabla u)(\mathbf{x}, t) + \mathcal{G}(\nabla u)(\mathbf{x}, t) = 0, \quad (1)$$

where Ω is an open bounded polyhedral subset in $\mathbb{R}^d$, with $d \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $T > 0$, and $\mathcal{F}$ and $\mathcal{G}$ are functions defined on $\mathbb{R}^d$ into respectively $\mathbb{R}^d$ and $\mathbb{R}$

Initial condition

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \quad (2)$$

Homogeneous Dirichlet boundary

$$u(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T). \quad (3)$$



Related to

$$\rho \partial_t u_\varepsilon - \mu \Delta u_\varepsilon - g \nabla \cdot \left(\frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \right) = C, \quad (4)$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$. The positive parameters μ and g denote respectively the viscosity and plasticity yield, see Dean, Glowinski, and G. Guidoboni (2007). C is the pressure drop per unit length (it is a function of t , only, and possibly a constant). The equation (4) is a regularization for the simplified Bingham flow equation in cylinders introduced in Dean, Glowinski, and G. Guidoboni (2007).



Assumptions on the data

Assumptions on the data

The analysis of this paper is based on the following assumptions. The functions u^0 , $\mathcal{F}$, and $\mathcal{G}$ are assumed to satisfy:

1. $\mathcal{F} \in \mathcal{C}^2(\mathbb{R}^d)$. We set $\mathcal{F} = (F_1, \dots, F_d)$, we assume that

$$F_1, \dots, F_d, \text{ their first and second derivatives are bounded.} \quad (5)$$

2. The function $\mathcal{G} \in \mathcal{C}^1(\mathbb{R}^d)$ and satisfies

$$\mathcal{G} \text{ and their partial first derivatives are bounded.} \quad (6)$$

3. The initial data function u^0 satisfies $u^0 \in \mathcal{C}^2(\overline{\Omega})$.



Primal MFEMs

Model Equation: Poisson Equation

Heat equation:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} \in \Omega, \quad (7)$$

where $\Omega \subset \mathbb{R}^d$ is an open domain of $\mathbb{R}^d$, f is given function.

Homogeneous Dirichlet boundary

$$u(\mathbf{x}) = 0, \mathbf{x} \in \partial\Omega. \quad (8)$$



General principles of MFEM

First step: Writing the problem as:

$$p = -\nabla u. \quad (9)$$

and

$$\operatorname{div} p = f. \quad (10)$$

Second step: Weak formulation for (9)–(10)

We have at least two possible weak formulations:

- 1 Primal Weak Formulation.
- 2 Dual (or also Primal Dual) Weak Formulation



Primal Weak Formulation

Weak Formulation of Primal MFEMs

Find $(p, u) \in L^2(\Omega)^d \times H_0^1(\Omega)$ such that

$$(p, \tau)_{L^2(\Omega)} + (\nabla u, \tau)_{L^2(\Omega)} = 0, \quad \forall \tau \in L^2(\Omega)^d \quad (11)$$

and

$$-(p, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega). \quad (12)$$



Dual Weak Formulation

Weak Formulation of Dual MFEMs

Find $(p, u) \in H_{\text{div}}(\Omega) \times L^2(\Omega)$ such that

$$(p, \psi)_{L^2(\Omega)} - (u, \text{div} \psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in H_{\text{div}}(\Omega) \quad (13)$$

and

$$(\text{div} p, \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in L^2(\Omega), \quad (14)$$

$H_{\text{div}}(\Omega)$ is the space defined by

$$H_{\text{div}}(\Omega) = \left\{ \xi \in \left(L^2(\Omega) \right)^d : \text{div} \xi \in L^2(\Omega) \right\}.$$



Finite Element spaces

Finite Element spaces

Let $V_h^{\text{div}} \subset H_{\text{div}}(\Omega)$ and $W_h \subset L^2(\Omega)$ be two finite dimensional spaces such that:

- **Compatibility condition (also known as the inf – sup-condition).** There exists $\beta^* > 0$ independent of h such that, for all $q \in W_h$

$$\sup_{w \in V_h^{\text{div}} \setminus \{0\}} \frac{1}{\|w\|_{H_{\text{div}}(\Omega)}} \int_{\Omega} q(\mathbf{x}) \text{div} w(\mathbf{x}) d\mathbf{x} \geq \beta^* \|q\|_{L^2(\Omega)}. \quad (15)$$

- The subspace G_h of V_h^{div} given by $G_h = \{w \in V_h^{\text{div}} : \int_{\Omega} q(\mathbf{x}) \text{div} w(\mathbf{x}) d\mathbf{x} = 0, \forall q \in W_h\}$ satisfies (**This condition can be weakened.**)

$$w \in G_h \quad \text{implies} \quad \text{div } w = 0. \quad (16)$$



An example of Finite Element spaces: $\mathbb{RT}_l$ (Raviart-Thomas) finite Element spaces

Example of space discretization: $\mathbb{RT}_l$ MFE

We define the Raviart-Thomas Mixed FE spaces, for $l \in \mathbb{N}$:

$$V_h^{\text{div}} = \{v \in H_{\text{div}}(\Omega) : v|_K \in \mathbb{D}_l, \quad \forall K \in \mathcal{T}_h\}, \quad (17)$$

where $\mathcal{T}_h$ is a family of triangulations of $\bar{\Omega}$ with d -simplex and

$$W_h = \{p \in L^2(\Omega) : p|_K \in \mathbb{P}_l, \quad \forall K \in \mathcal{T}_h\}, \quad (18)$$

where $\mathbb{P}_l$ is the space of d -variate polynomials on K having degree less than or equal to l and

$$\mathbb{D}_l = (\mathbb{P}_l)^d \oplus \mathbf{x}\mathbb{P}_l.$$



Nice property of $\mathbb{RT}_l$

Nice property of $\mathbb{RT}_l$

One of the main properties of the spaces $\mathbb{RT}_l$ is that

$$\operatorname{div} V_h^{\operatorname{div}} \subset W_h.$$



DMFE (Dual Mixed Finite Element) scheme for the Poisson equation

Formulation of the DMFE scheme for the Poisson equation

Find $(p_h, u_h) \in V_h^{\text{div}} \times W_h$ such that

$$(p_h, \psi)_{L^2(\Omega)} - (u_h, \text{div} \psi)_{L^2(\Omega)} = 0, \quad \forall \psi \in V_h^{\text{div}} \quad (19)$$

and

$$(\text{div} p_h, \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)}, \quad \forall \varphi \in W_h. \quad (20)$$



Well-Posedness and convergence result for the DMFES for the Poisson equation

Theorem (cf. Quarteroni and Valli, 2008)

■ *The well-posedness result:*

$$\|p_h\|_{H_{\text{div}}(\Omega)} + \|u_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \quad (21)$$

■ *Error estimate:*

$$\|\nabla u + p_h\|_{H_{\text{div}}(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \leq C\mathbb{E}_h(-\nabla u, u), \quad (22)$$

where $\mathbb{E}_h$ is the error given by

$$\mathbb{E}_h(P, U) = \inf_{\psi \in V_h^{\text{div}}} \|P - \psi\|_{H_{\text{div}}(\Omega)} + \inf_{\varphi \in W_h} \|U - \varphi\|_{L^2(\Omega)}. \quad (23)$$



Space and time discretizations

Space discretization

The FE spaces are Raviart-Thomas elements $\mathbb{RT}_0$ (Lowest Order).

Time discretization

Constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^*$. The mesh points are $t_n = nk$.

■ Discrete temporal derivative

$$\partial^1 v^{n+1} = \frac{v^{n+1} - v^n}{k}.$$



Formulation of a MFE scheme for (1)–(3)

The unknowns of this scheme are the set of the couples

$$\left\{ (p_h^n, u_h^n) \in V_h^{\text{div}} \times W_h; n \in \llbracket 0, N+1 \rrbracket \right\}.$$

These unknowns are expected to approximate the set of the unknowns

$$\{(-\nabla u(t_n), u(t_n)); \quad n \in \llbracket 0, N+1 \rrbracket\}.$$



Formulation of a DMFE scheme for (1)–(3)

Find $(p_h^n, u_h^n) \in V_h^{\text{div}} \times W_h$ such that:

First main discrete equation

For any $n \in \llbracket 0, N \rrbracket$ and for all $\varphi \in W_h$:

$$\begin{aligned} \left(\partial^1 u_h^{n+1}, \varphi \right) &+ \left(\nabla \cdot p_h^{n+1}, \varphi \right) - \left(\nabla \cdot \mathcal{F}(-p_h^{n+1}), \varphi \right) \\ &+ \left(\mathcal{G}(-p_h^{n+1}), \varphi \right) = 0. \end{aligned} \quad (25)$$

Second main discrete equation

For any $n \in \llbracket 0, N + 1 \rrbracket$:

$$(p_h^n, \psi) = (u_h^n, \text{div} \psi), \quad \forall \psi \in V_h^{\text{div}}, \quad (26)$$

where

$$\left(\text{div} p_h^0, \varphi \right) = \left(-\Delta u^0, \varphi \right)_{L^2(\Omega)}, \quad \forall \varphi \in W_h. \quad (27)$$

Formulation of a DMFE scheme for (1)–(3)

Find $(p_h^n, u_h^n) \in V_h^{\text{div}} \times W_h$ such that:

First main discrete equation

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Conclusion and perspectives

Conclusion

We presented a fully discrete MFE scheme for non-linear parabolic equation associated to Bingham Flow type equation. As space discretization, we used lowest order Raviart-Thomas elements $\mathbb{RT}_0$. The discretization in time is given by backward Euler method of first order. A convergence order is proved to be one in time and space.

Perspective

Improve the order in time of the scheme presented here using Crank-Nicolson method



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