Lecture 11: Waldspurger's formula (statement)

1. Hecke's and Waldspurger's formulae (for newforms)

Let F be a number field, with adele ring $\mathbb{A}$. Let $G = \mathrm{GL}_2$ as a group over F. Let $\pi \subset L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ be a cuspidal automorphic representation of $G(\mathbb{A})$. For simplicity, we shall assume that π has trivial central character.

1.1. **Hecke's integral representation.** Let A be the diagonal torus in G. For $\varphi \in \pi$ consider the period integral

$$\mathcal{P}_A(\varphi) = \int_{A(F)Z(\mathbb{A})\backslash A(\mathbb{A})} \varphi(a)da.$$

The quotient $A(F)Z(\mathbb{A})\backslash A(\mathbb{A})$ is non-compact, and is in fact of infinite volume; indeed it is isomorphic to $F^\times\backslash\mathbb{A}^\times$. Nevertheless, the integral is convergent due to the rapid decay of the cusp form φ.

We owe to Hecke the fact that, for a certain well-chosen *new vector* $\varphi^{\mathrm{new}} \in \pi$, we have

$$\mathcal{P}_A(\varphi^{\mathrm{new}}) = \Lambda(1/2, \pi),$$

where $\Lambda(s, \pi)$ is the completed L-function for π, which includes the Gamma factors at infinity.

To prepare the ground for the following paragraphs, note that the diagonal torus A above can be thought of as the torus $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$, where $E = \mathbb{Q}\oplus\mathbb{Q}$ is the split quadratic étale algebra of $\mathbb{Q}$, embedded into G. (Recall the standard way to obtain such embeddings: we view E as a vector space of dimension 2 over $\mathbb{Q}$, and note that multiplication by the unit group $E^\times = \mathbb{Q}^\times \times \mathbb{Q}^\times$ is by $\mathbb{Q}$-endomorphisms. Picking the canonical basis of $E = \mathbb{Q}^2$ yields the embedding $E^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q})$ whose image is A.)

1.2. **Waldspurger's theorem.** Now let T be an elliptic (i.e., non-split, or, to use another term, anisotropic) torus in G, so that T is the image of $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$ under an embedding into G, where E is now a quadratic *field* extension E of F. For $\varphi \in \pi$, we consider the compact period integral

$$\mathcal{P}_T(\varphi) = \int_{T(F)Z(\mathbb{A})\backslash T(\mathbb{A})} \varphi(t)dt. \tag{1.1}$$

Here the measure on $Z(\mathbb{A})\backslash T(\mathbb{A})$ is the quotient of Haar measures which assign volume 1 to maximal compact subgroups at finite places (along with appropriate normalizations at infinity).

In a landmark 1985 paper, Waldspurger related the period squared $|\mathcal{P}_T(\varphi)|^2$ to the central value of the (completed) base change L-function

$$\Lambda(1/2, \mathrm{BC}_{E/F}(\pi)) = \Lambda(1/2, \pi)\Lambda(1/2, \pi \times \eta_E).$$

Here, η_E is the quadratic Dirichlet character associated with the quadratic field extension by class field theory: for a prime p which is unramified in E we have $\eta_E(p) = +1$ or -1 according to whether p splits or is inert in E, respectively.

If π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, with central character ω_π, one knows that[1] $\widetilde{\pi} = \pi \otimes \omega_\pi^{-1}$. Thus if π has trivial central character – as we are in fact assuming – then π is self-contragredient.[2] The same is evidently true of the quadratic twist $\pi \otimes \eta_E$. From this it follows that the global epsilon factors $\epsilon(1/2, \pi)$ and $\epsilon(1/2, \pi \times \eta_E)$, and hence their product

$$\epsilon(1/2, \mathrm{BC}_{E/F}(\pi)) = \epsilon(1/2, \pi)\epsilon(1/2, \pi \times \eta_E),$$

are either $+1$ or -1. Indeed, the functional equation of a self-dual π reads $\Lambda(s, \pi) = \epsilon(s, \pi)\Lambda(1 - s, \pi)$; applying the functional equation again to the right-hand side gives $\epsilon(s, \pi)^2 = 1$. Note that, in particular, the global condition $\epsilon(1/2, \mathrm{BC}_{E/F}(\pi)) = 1$ assures that the base change L-function $L(1/2, \mathrm{BC}_{E/F}(\pi))$ does not vanish *for trivial reasons* (it could vanish for reasons unrelated to the functional equation).

Now the local epsilon factors $\epsilon(1/2, \mathrm{BC}_{E/F}(\pi_v))$ are constrained in the same way as the global one, by the self-duality of π_v.[3] The global condition

$$\epsilon(1/2, \mathrm{BC}_{E/F}(\pi)) = 1 \tag{1.2}$$

is therefore equivalent to $\epsilon(1/2, \mathrm{BC}_{E/F}(\pi_v)) = -1$ at an even (or empty) number of places. For simplicity, we shall for the moment impose the following additional local assumptions

$$\epsilon(1/2, \mathrm{BC}_{E/F}(\pi_v)) = 1 \qquad \text{for all } v. \tag{1.3}$$

Once this formula is refined and explicated for newform data, one obtains the following clean result.

Theorem 1.1 (Waldspurger, Gross–Prasad)**.** *Under the assumption* (1.3) *on π and E, we have*

$$|\mathcal{P}_T(\varphi^{\mathrm{new}})|^2 = C|d_E|^{-1/2}\Lambda(1/2, \pi)\Lambda(1/2, \pi \times \eta_E), \tag{1.4}$$

for an explicit constant $C > 0$, depending on the ramification data of π, which can be explicated in practice.

1.3. **Remarks.** We make a few remarks on Theorem 7.1.

(1) The contribution of Gross–Prasad to the above theorem is the definition of the vector φ^{new} which makes for such a clean statement. In the more general version of the Waldspurger formula that we give later (see Theorem 7.1) the identity is "distributional", valid for any choice of factorizable $\varphi \in \pi$.

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[1] For π irreducible admissible on GL_n over a non-archimedean field, one shows that $\tilde{\pi}$ is equivalent to $\pi({}^t g^{-1})$. Then, for $n = 2$, one shows by an explicit computation that ${}^t g^{-1} = (\det g)^{-1} w^{-1} g w$, where w is the non-trivial Weyl group element. See, for example, Bump, Theorem 4.2.2.

[2] The converse is not true: *monomial representations* satisfy $\pi \simeq \pi \otimes \omega_\pi^{-1}$, with ω_π non-trivial. In this case ω_π is necessarily quadratic. The monomial representations of GL_2 are precisely the self-dual representations of non-trivial central character.

[3] The local epsilon factor depends on a choice of a non-trivial additive character ψ of k, but their values at the central point $s = 1/2$ are independent of this choice.

(2) If we relax the assumption (1.3), while keeping the global condition (1.2), then the L-value will not longer be related to the T-period of a cusp form on GL_2, but rather on $B^\times$, where B is the unique quaternion algebra over F which is non-split precisely at the even number of places v where $\epsilon(1/2, \mathrm{BC}_{E/F}(\pi_v), \psi_v) = -1$.

(3) Note that when $F = \mathbb{Q}$ and E is imaginary quadratic we may identify

$$\begin{aligned} T(\mathbb{Q})Z(\mathbb{A})\backslash T(\mathbb{A})/T(\hat{\mathbb{Z}})T(\mathbb{R}) &= E^\times \mathbb{A}^\times \backslash \mathbb{A}_E^\times / \widehat{\mathcal{O}}_E^\times \mathbb{C}^\times \\ &= E^\times \backslash \mathbb{A}_E^\times / \widehat{\mathcal{O}}_E^\times \mathbb{C}^\times \\ &\simeq E^\times \backslash \mathbb{A}_{E,f}^\times / \widehat{\mathcal{O}}_E^\times \\ &\simeq \mathrm{Cl}_E, \end{aligned}$$

so that the toric period integral (1.1) reduces to a sum over the class group of E. This generalizes (upon taking squares) the period relation, due to Hecke, for the level one spherical (completed) Eisenstein series $\mathrm{Eis}_s = E^*(\cdot, s)$ over $\mathbb{Q}$:

$$\begin{aligned} \mathcal{P}_T(\mathrm{Eis}_s) &= \frac{w}{2} \frac{1}{\sqrt{d_E}} \sum_{\mathfrak{a} \in \mathrm{Cl}_E} E^*(z_{\mathfrak{a}}, s) \\ &= |d_E|^{(s-1)/2} \xi_E(s) \\ &= |d_E|^{(s-1)/2} \Lambda(1/2, \pi_{\mathrm{Eis}_s}), \end{aligned}$$

In particular, at $s = 1/2$, letting $\mathrm{Eis} = \mathrm{Eis}_{1/2}$, we have

$$|\mathcal{P}_T(\mathrm{Eis})|^2 = |d_E|^{-1/2} |\Lambda(1/2, \pi_{\mathrm{Eis}})|^2.$$

Note that the normalization of the Eisenstein series $E^*(\cdot, s)$ is such that the first Fourier–Whittaker coefficient at $W_{s-1/2}(2\pi y)e(x)$ is 1 (while the constant term is $\xi(2s)y^s + \xi(2(1-s))y^{1-s}$). The more general version of Waldspurger's theorem we present later will have the feature of being scale invariant.

(4) When the epsilon factor is -1, in which case the base change L-function necessarily vanishes at $1/2$, the Gross–Zagier formula relates (when π_∞ is discrete series) the derivative of the base change L-function at $1/2$ to height pairings of CM points on modular or Shimura curves.

(5) Note that one can deform the Hecke integral, as in

$$\int_{A(F)Z(\mathbb{A})\backslash A(\mathbb{A})} \varphi(a)|a|^{s-1/2} da = \Lambda(s, \pi),$$

to get formulae for non-central values. This is not so with the Waldspurger formula.

2. Application to positivity

It is clear that Waldspurger's formula (1.4) implies that

$$L(1/2,\pi)L(1/2,\pi\otimes\eta_E)\geq 0.$$

In particular, each of the above central L-values has the same sign. We now follow an argument of Guo (suggested by Sarnak) which allows one to boost this inequality to the non-negativy of both factors individually.

We begin with a lemma. Recall that the Generalized Riemann Hypothesis (GHR) states that the L-function $L(s,\pi)$ is non-vanishing on $\mathrm{Re}(s)>1/2$.

Lemma 2.1. *Let π be a self-contragredient cuspidal automophic representation of $\mathrm{GL}_2(\mathbb{A})$. Then*

(1) *for $\sigma>1$ we have $L(\sigma,\pi)>0$;*
(2) *under GRH, we have $L(1/2,\pi)\geq 0$.*

Proof. Since π is self-contragredient we have $\overline{L(\sigma,\pi)}=L(\sigma,\pi)$ for $\sigma>1$ real, so that $L(\sigma,\pi)$ is a real number. Now locally for every finite place v one has $L(\sigma,\pi_v)>0$ on $\sigma\geq 1/2$. Indeed, this follows from the Jacquet-Shalika bound $|\alpha_v|,|\beta_v|<q_v^{1/2}$ on the Satake parameters (for finite places v at which π_v is unramified). The lemma follows from the Euler product expansion of $L(s,\pi)$ on $\mathrm{Re}(s)>1$.

The second statement follows from the first, since $\sigma\mapsto L(\sigma,\pi)$ is continuous on $\sigma>1/2$. □

As an application of the Waldspurger formula, Guo [Guo, 1996] was able to prove the positivity of $L(1/2,\pi)$ for such π unconditionally.

Corollary 1 (Guo)**.** *Let π be a cuspidal automorphic representation on $\mathrm{GL}_2(\mathbb{A})$ with trivial central character. Then $L(1/2,\pi)\geq 0$.*

Proof. As already remarked, (1.4) implies $L(1/2,\pi)L(1/2,\pi\otimes\eta_E)\geq 0$. Now if $L(1/2,\pi)<0$ then $L(1/2,\pi\otimes\eta_E)<0$ for all E. But a deep result of Bump-Friedberg-Hoffstein shows that

$$\frac{1}{N}\sum_{\mathrm{Disc}(E)\leq N}L(1/2,\pi\otimes\eta_E)\sim C_\pi,$$

for a (strictly) positive constant C_π. □

3. Inner product relations

In this section we calculate the L^2 norm of a normalized newform, both in the local and global senses. The basic form of these identities states that the L^2-norm of an arithmetically normalized newvector is the value at $s=1$ of the *adjoint L-function*. The latter is the L-function whose local Euler factor at an unramified finite place v is

$$(1-\alpha_v\beta_v^{-1}\mathrm{N}\mathfrak{p}_v^{-s})^{-1}(1-\mathrm{N}\mathfrak{p}_v^{-s})^{-1}(1-\alpha_v^{-1}\beta_v\mathrm{N}\mathfrak{p}_v^{-s})^{-1},$$

where α_v,β_v are the Satake parameters of π_v.

The computations that follow depend on an important subgroup of $G = \mathrm{GL}_2$, called *the mirabolic subgroup*. It is given by

$$P = \left\{ \begin{pmatrix} x & y \\ & 1 \end{pmatrix} : \ x \in \mathbb{G}_m, \ y \in \mathbb{G}_a \right\}.$$

Note that P is the stabilizer subgroup of the row vector e_2 under right-multiplication. If B denotes the standard Borel subgroup of G then we have $B = ZP$.

3.1. **Local inner product relation.** Let v be a finite place of F. Let π_v be an irreducible unitary representation of GL_2, which is of infinite dimension (such representations are called *generic*). We shall furthermore assume that π_v has trivial central character. Let ψ be a non-trivial additive character of F_v, trivial on $\mathcal{O}_v$. Then π_v has a Whittaker model $\mathcal{W}(\pi_v, \psi)$. We put

$$\langle W_v, W_v' \rangle_v = \int_{F_v^\times} W_v \begin{pmatrix} a & \\ & 1 \end{pmatrix} \overline{W_v' \begin{pmatrix} a & \\ & 1 \end{pmatrix}} da,$$

where we have normalized the measure on $F_v^\times$ to be as in Tate's thesis, namely, $da = \zeta_v(1) dx/|x|$, where dx is the self-dual Haar measure on F_v. Then $\pi_v \mapsto \mathcal{W}(\pi_v, \psi)$, $\varphi \mapsto W_\varphi$ is an isometry.

Definition 1. If $\varphi_v, \varphi_v' \in \pi_v$ we shall denote by

$$\langle \varphi_v, \varphi_v' \rangle_v$$

the inner product of $W_{\varphi_v}, W_{\varphi_v'}$ in the Whittaker model.

Let us work out the unramified computation. An important ingredient will be the Casselman-Shalika formula (due to Casselman for GL_2), which states that for π_v an unramified admissible representation of $\mathrm{GL}_2(F_v)$ with trivial central character, we have

$$W_{\varphi_v^\circ} \begin{pmatrix} \varpi_v^k & \\ & 1 \end{pmatrix} = \begin{cases} \sum_{i+j=k} \alpha_v^i \beta_v^j, & k \geq 0; \\ 0, & \text{else}, \end{cases} \tag{3.1}$$

where α_v, β_v are the Satake parameters of π_v.

Lemma 3.1. *In the unramified case, we have the formula*

$$\langle W_v^\circ, W_v^\circ \rangle_v = \frac{\zeta_v(1)}{\zeta_v(2)} L(1, \pi_v, \mathrm{Ad}).$$

Proof. See Jacquet–Shalika, *On Euler Products and the Classification of Automorphic Representations I*, American Journal of Mathematics, 1981, Proposition 2.3, where in fact it is shown that

$$\int_{N(F_v)\backslash G(F_v)} W_v(g) \overline{W_v'(g)} |\det(g)|^s 1_{\mathcal{O}_v^2}(e_2 g) dg = \frac{\zeta_v(s)}{\zeta_v(2s)} L(s, \pi_v, \mathrm{Ad}). \tag{3.2}$$

We should actually do this for ourselves in this case. □

3.2. **Global inner product relation.** For $\varphi, \varphi' \in \pi$ we shall always write

$$\langle \varphi, \varphi' \rangle = \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g)\overline{\varphi'(g)}dg$$

for the Petersson inner product. We shall take the measure dg on $\mathrm{GL}_2(\mathbb{A})$ to be the *Tamagawa measure*, which gives $K_{0,v}$ measure $\zeta_v(2)^{-1}$ for finite places v.

Proposition 3.1. *For factorizable $\varphi = \otimes_v \varphi_v, \varphi' = \otimes_v \varphi'_v \in \pi$ these global and local inner products are related by the following formula:*

$$\langle \varphi, \varphi' \rangle = \frac{\mathrm{Res}_{s=1}\zeta^S(s)}{\zeta^S(2)} L^S(1, \pi, \mathrm{Ad}) \prod_{v \in S} \langle \varphi_v, \varphi'_v \rangle_v.$$

Proof. To relate the Petersson inner product with local inner products, we introduce the mirabolic Eisenstein series. For a Schwartz–Bruhat function Φ on $\mathbb{A}^2$ (viewed as row vectors), the function $g \mapsto \Phi(e_2 g)$ on $G(\mathbb{A})$ is left-invariant under $P(\mathbb{A})$. We consider the Mellin transform

$$F(g, \Phi; s) = |\det(g)|^s \int_{\mathbb{A}^\times} \Phi(ae_2 g)|a|^{2s} d^\times a.$$

Note the following transformation property of $F(g, \Phi; s)$: for

$$b = zp = \begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} x & y \\ & 1 \end{pmatrix} \in B(\mathbb{A}) = Z(\mathbb{A})P(\mathbb{A})$$

we have

$$\begin{aligned} F(bg, \Phi; s) &= |\det(bg)|^s \int_{\mathbb{A}^\times} \Phi(ae_2 bg)|a|^{2s} d^\times a \\ &= |xz^2|^s |\det(g)|^s \int_{\mathbb{A}^\times} \Phi(aze_2 g)|a|^{2s} d^\times a \\ &= |x|^s F(g, \Phi; s). \end{aligned}$$

Thus $F(bg, \Phi; s) = \delta_B(b)^s F(g, \Phi; s)$, where δ_B is the modular character of B, so that $F(\cdot, \Phi; s) \in \mathrm{Ind}_B^G(\delta_B^{s-1/2})$. We may therefore introduce the Eisenstein series

$$E(g, \Phi; s) = \sum_{\gamma \in B(F)\backslash G(F)} F(\gamma g, \Phi; s).$$

Using Poisson summation, one can show the basic analytic properties of this Eisenstein series. In particular, it has a meromorphic continuation to the entire complex plane, with a simple pole at $s = 1$ of residue $\widehat{\Phi}(0)$.

For $\mathrm{Re}\, s > 1$ consider the absolutely convergent integral

$$Z(s, \varphi, \varphi', \Phi) = \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g)\overline{\varphi'(g)} E(g, \Phi; s) dg.$$

Then the residue at $s = 1$ yields the inner product of the lemma times $\widehat{\Phi}(0)$.

On the other hand, for $\operatorname{Re} s > 1$, unfolding over the $B(F)\backslash G(F)$, inserting the definition of $F(\cdot, \Phi; s)$, and unfolding over the center, we find

$$\begin{aligned} Z(s,\varphi,\varphi',\Phi) &= \int_{B(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g)\overline{\varphi'(g)}F(g,\Phi;s)dg \\ &= \int_{P(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g)\overline{\varphi'(g)}|\det(g)|^s \int_{\mathbb{A}^\times} \Phi(ae_2g)|a|^{2s}d^\times a dg \\ &= \int_{P(F)\backslash G(\mathbb{A})} \varphi(g)\overline{\varphi'(g)}|\det(g)|^s\Phi(e_2g)dg. \end{aligned}$$

To continue, we recall the Fourier-Whittaker expansion of the cuspidal $\varphi \in \pi$. Let $\psi : N(F)\backslash N(\mathbb{A}) \to \mathbb{C}^\times$ be a non-trivial character. We normalize the measure on $N(F)\backslash N(\mathbb{A})$ to be of volume 1. Then

$$\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) = \sum_{\gamma\in F^\times} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)\psi(\gamma x),$$

where

$$W_\varphi(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)dn$$

is the global (adelic) Fourier-Whittaker coefficient. Note that $W_\varphi(ng) = \psi(n)W_\varphi(g)$. Setting $x = e$ this gives

$$\varphi\,(g) = \sum_{\gamma\in F^\times} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right) = \sum_{\gamma\in N(F)\backslash P(F)} W_\varphi(\gamma g).$$

We now unfold the Fourier expansion of φ over $P(F)\backslash G(\mathbb{A})$ to obtain

$$\begin{aligned} Z(s,\varphi,\varphi',\Phi) &= \int_{N(F)\backslash G(\mathbb{A})} W_\varphi(g)\overline{\varphi'(g)}|\det(g)|^s\Phi(e_2g)dg \\ &= \int_{N(\mathbb{A})\backslash G(\mathbb{A})}\int_{N(F)\backslash N(\mathbb{A})} W_\varphi(ng)\overline{\varphi'(ng)}|\det(ng)|^s\Phi(e_2ng)dndg \\ &= \int_{N(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi(g)\int_{N(F)\backslash N(\mathbb{A})} \overline{\varphi'(ng)}\psi(n)dn|\det(g)|^s\Phi(e_2g)dg \\ &= \int_{N(\mathbb{A})\backslash G(\mathbb{A})} W_\varphi(g)\overline{W_{\varphi'}(g)}|\det(g)|^s\Phi(e_2g)dg \end{aligned}$$

If all data is factorizable, we obtain, for $\operatorname{Re} s > 1$,

$$Z(s,\varphi,\varphi',\Phi) = \prod_v Z(s, W_{\varphi_v}, W_{\varphi'_v}, \Phi_v),$$

where

$$Z(s,W_v,W'_v,\Phi_v) = \int_{N(F_v)\backslash G(F_v)} W_v(g)\overline{W'_v(g)}|\det(g)|^s\Phi_v(e_2g)dg.$$

For unramified data we of course have $\widehat{\Phi_v}(0) = 1$. From (3.2) we have

$$Z(s, \varphi, \varphi', \Phi) = \widehat{\Phi^S}(0) \frac{\zeta^S(s)}{\zeta^S(2s)} L^S(s, \pi, \mathrm{Ad}) \prod_{v \in S} Z(s, W_{\varphi_v}, W_{\varphi'_v}, \Phi_v).$$

We now take the residue at $s = 1$, noting that since

$$\int_{P(F_v)\backslash G(F_v)} \Phi(e_2 g) |\det g| dg = \zeta_v(2) \widehat{\Phi_v}(0),$$

we have

$$\begin{aligned} Z(1, W_v, W'_v, \Phi_v) &= \zeta_v(2) \widehat{\Phi_v}(0) \int_{N(F_v)\backslash P(F_v)} W(p) \overline{W'(p)} dp \\ &= \widehat{\Phi_v}(0) \langle W_v, W_v \rangle_v. \end{aligned}$$

This establishes the formula. □

4. Local multiplicity at most one

We take k to be a local field of characteristic zero and write π throughout for a smooth irreducible infinite dimensional representation of $G = \mathrm{GL}_2(k)$ with trivial central character. We write A for the diagonal torus in G and T for an elliptic torus in G. Let Z be the center of G.

Lemma 4.1. *Let H be either T or A. For any character χ of H whose restriction to Z is trivial we have* $\dim \mathrm{Hom}_H(\pi, \chi) \leq 1$.

Let $\bar{H} = Z\backslash H$. Then Lemma 4.1 states that $(\mathrm{PGL}_2, \bar{H})$ forms a Strong Gelfand pair. This is the basic theoretical underpinning of Waldspurger's formula. For a beautiful discussion on the topic of Gelfand pairs in number theory, see Gross's 1991 Bulletin of the AMS article.

Proof. For simplicity we assume that k is non-archimedean. We continue to write $\bar{H} = Z\backslash H$ as above, and put $\bar{G} = \mathrm{PGL}_2(k)$. Let $\mathcal{H}$ denote the diagonally embedded subgroup $\bar{H}$ inside the product $\mathcal{G} = \bar{G} \times \bar{H}$. Note that $(\bar{G}, \bar{H})$ is a Strong Gelfand pair precisely when $(\mathcal{G}, \mathcal{H})$ is a Gelfand pair. We shall prove the latter statement.

The proof is a simple application of the lemma of Gelfand–Kazhdan, giving a sufficient condition under which $(\mathcal{G}, \mathcal{H})$ is a Gelfand pair. Namely, if $S(\mathcal{G})$ denotes the space of locally constant functions of compact support, and $S(\mathcal{G})^*$ its linear dual (i.e., the space of tempered distributions on $\mathcal{G}$), then $(\mathcal{G}, \mathcal{H})$ is a Gelfand pair as soon as there exists an anti-involution ι of $\mathcal{G}$, which stabilizes $\mathcal{H}$, and acts trivially on the subspace of bi-$\mathcal{H}$-invariant distributions in $\mathcal{S}(\mathcal{G})^*$.

To verify the Gelfand–Kazhdan property, we take ι to be the involution sending $(g, h) \in \mathcal{G} = \bar{G} \times \bar{H}$ to $(jg^{-1}j^{-1}, h)$, where j is any element of the normalizer of $\bar{H}$ which does not lie in $\bar{H}$. □

Remark 1. The lemma of Gelfand–Kazhdan generalizes the original observation of Gelfand, valid in the setting of compact groups, where the involution ι is only asked to act trivially on the double quotient space $H\backslash G/H$. (In the non-compact case, Gelfand–Kazhdan allow ι to fix *almost all* cosets in $H\backslash G/H$.) In particular, we could use the simpler version to show that a maximal torus inside the projective units of the unique division quaternion algebra over the non-archimedean local field k is a Gelfand pair. In this case one takes for ι the involution $\iota(g,h) = (j\bar{g}j^{-1}, h)$, where $\bar{g} = \mathrm{tr}(g) - g$.

In what follows, for the sake of simplicity, we shall only be interested in the case when $\chi = 1$. We now want to understand under what conditions the local Hom space $\mathrm{Hom}_H(\pi, \mathbb{C})$ is non-zero. We can exhibit an explicit element in $\mathrm{Hom}_H(\pi, \mathbb{C})$ by the integral

$$\varphi \mapsto \int_{Z\backslash H} \langle \pi(h)\varphi, \varphi\rangle dh,$$

provided it converges, in which case we write

$$\alpha(\varphi,\varphi) = \frac{\int_{Z\backslash H} \langle \pi(h)\varphi, \varphi\rangle dh}{\langle \varphi, \varphi\rangle}. \tag{4.1}$$

The question then becomes whether or not the integral is identically zero. We shall examine these questions in the following paragraphs.

5. The Hecke integral, revisited

While we presented Hecke's and Waldspurger's formulae in compact form, using a particular choice of automorphic vector, we would now like to extend it to a formula for arbitrary $\varphi \in \pi$. We begin, for simplicity, with Hecke's formula, writing it in a way which will anticipate the more complicated Waldspurger's formula.

5.1. **Local Hecke formula.** We begin with a local version of the Hecke formula, which for unramified data expresses (4.1) in terms of the local central L-value of π_v. For $s \in \mathbb{C}$, we put

$$\beta_v(\varphi_v; s) = \int_{A_v} W_{\varphi_v}(a)|a|^{s-1/2} da.$$

and shorten this to β_v when $s = 1/2$.

Lemma 5.1. *We have* $\dim \mathrm{Hom}_A(\pi_v, \mathbb{C}) = 1$. *Moreover,*

$$\varphi_v \mapsto \alpha_v(\varphi_v, \varphi_v)$$

is a non-zero element in $\mathrm{Hom}_A(\pi_v, \mathbb{C})$. *Moreover, when* π_v *is unramified and* φ_v° *is the new vector, we have*

$$\alpha(\varphi_v^\circ, \varphi_v^\circ) = \frac{\zeta_v(2)}{\zeta_v(1)} \frac{L(1/2, \pi_v)^2}{L(1, \pi_v, \mathrm{Ad})}.$$

Proof. Note that $\beta_v \in \mathrm{Hom}_{A_v}(\pi_v, \mathbb{C})$. See Gelbart, *Automorphic Forms on Adele Groups*, Theorem 6.12, for a proof that β_v is non-zero. Using the identification

$$|\beta_v(\varphi_v)|^2 = \int_{Z_v \backslash A_v} \langle \pi_v(a)\varphi_v, \varphi_v\rangle da, \tag{5.1}$$

we deduce that α_v is non-zero.

For the second statement we take π_v unramified and let $\varphi_v^\circ \in \pi_v^{K_v}$ be the new vector. Then, by Lemma 3.1, we have

$$\alpha_v(\varphi_v^\circ, \varphi_v^\circ) = \frac{|\beta_v(\varphi_v^\circ)|^2}{\langle \varphi_v^\circ, \varphi_v^\circ\rangle} = \frac{\zeta_v(2)}{\zeta_v(1)} \frac{1}{L(1, \pi_v, \mathrm{Ad})} |\beta_v(\varphi_v^\circ)|^2.$$

Finally, we compute $\beta_v(\varphi_v^\circ) = L(1/2, \pi_v)$ by inserting the Casselman–Shalika formula (3.1) into the A_v integral. □

5.2. **Global Hecke formula.**

Lemma 5.2. *Let S be a finite set of places of F outside of which π is unramified. Then*

$$\frac{|\mathcal{P}_A(\varphi)|^2}{\langle \varphi, \varphi\rangle} = \frac{\zeta^S(2) L^S(1/2, \pi)^2}{\mathrm{Res}_{s=1}\zeta^S(s) L^S(1, \pi, \mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi_v).$$

Proof. The central idea of the proof is this: "unfold to Whittaker". We need to do this in a right half-plane and then analytically continue.

Let $\mathrm{Re}(s) > 1$. Unfolding the period integral, for factorizable $\varphi = \otimes_v \varphi_v \in \pi$, we get

$$\begin{aligned}
\int_{F^\times \backslash \mathbb{A}^\times} \varphi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) |a|^{s-1/2} da &= \int_{F^\times \backslash \mathbb{A}^\times} \sum_{\gamma \in F^\times} W_\varphi\left(\begin{pmatrix} \gamma a & \\ & 1\end{pmatrix}\right) |a|^{s-1/2} da \\
&= \int_{\mathbb{A}^\times} W_\varphi\left(\begin{pmatrix} a & \\ & 1\end{pmatrix}\right) |a|^{s-1/2} da \\
&= \prod_v \int_{F_v^\times} W_{\varphi_v}\left(\begin{pmatrix} a_v & \\ & 1\end{pmatrix}\right) |a_v|^{s-1/2} da_v.
\end{aligned}$$

From this it follows that

$$|\mathcal{P}_{A, |\cdot|^{s-1/2}}(\varphi)|^2 = \prod_v |\beta_v(\varphi_v; s)|^2.$$

Using Proposition 3.1 we deduce

$$\frac{|\mathcal{P}_{A, |\cdot|^{s-1/2}}(\varphi)|^2}{\langle \varphi, \varphi\rangle} = \frac{\zeta^S(2)}{\mathrm{Res}_{s=1}\zeta^S(s) L(1, \pi, \mathrm{Ad})} \prod_{v \notin S} |\beta_v(\varphi_v^\circ; s)|^2 \prod_{v \in S} \frac{|\beta_v(\varphi_v; s)|^2}{\langle \varphi_v, \varphi_v\rangle}.$$

Inserting the identity $\beta_v(\varphi_v^\circ; s) = L(s, \pi_v)$ at the unramified places and (5.1) at the remaining places, and taking $s = 1/2$, yields the result. □

Corollary 2. *With the notations as above,*

$$\mathcal{P}_A|_\pi \neq 0 \quad \textit{if, and only if,} \quad L(1/2, \pi) \neq 0.$$

6. Local obstruction for T non split

The situation is very different for $\mathrm{Hom}_{T_v}(\pi_v, \mathbb{C})$, where T_v is a non-split torus in $\mathrm{GL}_2(F_v)$ associated with a quadratic field extension E_v of F_v. A famous result of Tunnell and Saito (following earlier ideas of Vigneras from 1979-80) relates the existence of a non-zero T_v-invariant functional on π_v to the sign of the "base change" epsilon factor.

Theorem 6.1 (Tunnell, Saito)**.** *Let π_v be a generic irreducible unitary representation of $\mathrm{GL}_2(F_v)$, with trivial central character. Let B_v be the non-trivial quaternion division algebra over F_v and let π'_v denote the Jacquet-Langlands lift to $G'_v = B_v^\times$. Let E_v be a quadratic field extension of F_v which embeds into both $M_2(F_v)$ and B_v. Write T_v for the associated torus. Then*

$$\dim \mathrm{Hom}_{T_v}(\pi_v, \mathbb{C}) + \dim \mathrm{Hom}_{T_v}(\pi'_v, \mathbb{C}) = 1.$$

More precisely,

$$\dim \mathrm{Hom}_{T_v}(\pi_v, \mathbb{C}) = 1 \;\; \textit{if, and only if,} \;\; \epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = 1$$

and

$$\dim \mathrm{Hom}_{T_v}(\pi'_v, \mathbb{C}) = 1 \;\; \textit{if, and only if,} \;\; \epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = -1.$$

Remark 2. In the above theorem, the convention is that $\pi'_v = 0$ if π_v is not a discrete series representation. In this case, the result states that $\dim \mathrm{Hom}_{T_v}(\pi_v, \mathbb{C}) = 1$ and that $\epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = 1$.

Remark 3. The above theorem is often phrased more uniformly, without assuming that E_v is a field. Letting η_{E_v} denote the character associated with the quadratic étale extension E_v over F_v, the result is that

$$\dim \mathrm{Hom}_{T_v}(\pi_v, \mathbb{C}) = 1 \text{ if, and only if, } \epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = \eta_{E_v}(-1)$$

and

$$\dim \mathrm{Hom}_{T_v}(\pi'_v, \mathbb{C}) = 1 \text{ if, and only if, } \epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = -\eta_{E_v}(-1).$$

When E_v is the split algebra, we have $\eta_{E_v}(-1) = 1$. Note that in this case $\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = \epsilon(1/2, \pi_v)$ so that

$$\epsilon(1/2, \pi_v)\epsilon(1/2, \pi_v \otimes \eta_{E_v}) = \epsilon(1/2, \pi_v)^2 = 1$$

always holds, meaning that $\dim \mathrm{Hom}_{T_v}(\pi_v, \mathbb{C}) = 1$. This recovers the first statement of Lemma 5.1.

Proof. This is essentially a case-by-case computation (Tunnell's proof was made more uniform by Saito, who also extended it to dyadic fields).

Let us illustrate with principal series representations, again under the hypothesis of trivial central character. Thus $\pi = \mathrm{Ind}_B^G(\chi, \chi^{-1})$. In this case we have

$$\epsilon(\pi)\epsilon(\pi \otimes \eta_E) = \epsilon(\chi)\epsilon(\chi^{-1})\epsilon(\chi\eta_E)\epsilon(\chi^{-1}\eta_E).$$

Now we have a general fact about epsilon factors, that $\epsilon(\pi)\epsilon(\pi^\wedge) = \omega_\pi(-1)$. We apply this with $\pi = \chi$ in which case $\pi^\wedge$ is χ^{-1} so that $\epsilon(\chi)\epsilon(\chi^{-1}) = \chi(-1)$

and $\epsilon(\chi\eta_E)\epsilon(\chi^{-1}\eta_E) = \chi(-1)\eta_E(-1)$. We thus obtain $\epsilon(\pi)\epsilon(\pi\otimes\eta_E) = \chi(-1)^2\eta_E(-1) = \eta_E(-1)$, as claimed. □

7. Waldspurger formula

We now state Waldspurger's formula, in the version he originally gave.

Let G' be an inner form of G, possibly equal to G. Let T' be an *elliptic* torus in G', associated with a quadratic field extension E of F. For π' a cuspidal automorphic representation of $G'(\mathbb{A})$ with trivial central character, and $\varphi' \in \pi'$, we consider the compact period integral

$$\mathcal{P}_{T'}(\varphi') = \int_{T'(F)Z'(\mathbb{A})\backslash T'(\mathbb{A})} \varphi'(t)dt.$$

Recall the definition of the local functional α_v in (4.1).

Theorem 7.1 (Waldspurger, 1985, Proposition 7)**.** *Let π' be a cuspidal automorphic representation of $G'(\mathbb{A})$, with trivial central character. Let $\varphi' = \otimes_v\varphi'_v \in \pi$ be factorizable. Then, for a sufficiently large finite set of places S of F, containing all archimedean places, we have*

$$\frac{|\mathcal{P}_{T'}(\varphi')|^2}{\langle\varphi',\varphi'\rangle} = \frac{\zeta_F^S(2)}{4}\frac{L^S(1/2,\pi)L^S(1/2,\pi\otimes\eta_E)}{L^S(1,\eta_E)L^S(1,\mathrm{Ad},\pi)}\prod_{v\in S}\alpha_v(\varphi'_v,\varphi'_v),$$

where π is the Jacquet–Langlands transfer of π' to $G = \mathrm{GL}_2$.

We have $\mathrm{Hom}_T(\pi,\mathbb{C}) \neq 0$ if, and only if, α_v is non-zero, considered as an element of $\mathrm{Hom}_T(\pi,\mathbb{C})$. From this and

$$\mathrm{Hom}_{T'(\mathbb{A})}(\pi',\mathbb{C}) \simeq \prod_v \mathrm{Hom}_{T'_v}(\pi'_v,\mathbb{C}),$$

this establishes the following corollary.

Corollary 3. *Let π' be a cuspidal automorphic representation of $G'(\mathbb{A})$, with trivial central character. Let π be the Jacquet–Langlands lift of π' to $G = \mathrm{GL}_2$. The following statements are equivalent:*

(1) $\mathrm{Hom}_{T'}(\pi',\mathbb{C}) \neq 0$ *and* $L(1/2,\pi)L(1/2,\pi\otimes\eta_E) \neq 0$*;*
(2) $\mathcal{P}_{T'}|_{\pi'} \neq 0$.

Remark 4. Here is one way of thinking about the forward direction: to ensure that the automorphic period $\mathcal{P}_{T'} \in \mathrm{Hom}_{T'}(\pi',\mathbb{C})$ is non-vanishing on π', we of course need to have the everywhere local non-vanishing $\mathrm{Hom}_{T'}(\pi',\mathbb{C}) \neq 0$. The theorem states that one must furthermore assume the arithmetic condition $L(1/2,\pi)L(1/2,\pi\otimes\eta_E) \neq 0$. The latter can therefore be viewed as a global obstacle.

As usual, let π be a cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Recall the Hecke period P_A. We continue to let E

denote a quadratic field extension of F. We introduce the twisted version given by

$$\mathcal{P}_{A,\eta_E}(\varphi) = \int_{A(F)Z(\mathbb{A})\backslash A(\mathbb{A})} \varphi(a)\eta_E(a)da.$$

Now let $X(E:F)$ denote the set of isomorphism classes of pairs (G', T'), where G' is an inner form of G (possibly equal to G) and T' is a maximal torus of G' such that $T'(F) = E^\times$. Let $X(\pi)$ be the set of (isomorphism classes of) triplets (G', T', π') such that $(G', T') \in X(E:F)$ and π' is the Jacquet–Langlands lift of π. This set is non-empty since it contains (G, T, π).

Corollary 4. *The following two statements are equivalent:*

1. $P_A|_\pi \neq 0$ *and* $\mathcal{P}_{A,\eta_E}|_\pi \neq 0$*;*
2. *there is* $(G', T', \pi') \in X(\pi)$ *such that* $P_{T'}|_{\pi'} \neq 0$.

Proof. Note that from Corollary 2 we may replace the first condition, that $L(1/2, \pi)L(1/2, \pi \otimes \eta_E) \neq 0$, by the double condition (1). Observe, furthermore, that the non-vanishing of the central value of the base change L-function implies $\epsilon(1/2, \pi)\epsilon(1/2, \pi \otimes \eta_E) = 1$.

Let S be set of places v of F such that $\epsilon(\pi_v)\epsilon(\pi_v \otimes \eta_{E_v}) = -\eta_{E_v}(-1)$. Then $|S|$ is necessarily even; indeed, using $\eta_E(-1) = 1$, we have

$$1 = \epsilon(\pi)\epsilon(\pi \otimes \eta_E) = \prod_{v \in S}(-\eta_{E_v}(-1)) \prod_{v \notin S} \eta_{E_v}(-1) = (-1)^{|S|}\eta_E(-1).$$

We let $G' = B^\times$ for the unique quaternion algebra ramified at S. Then π is discrete series at all places in S (see Remark 2) and therefore admits a Jacquet–Langlands lift π' to G'. The quadratic étale exension E_v/F_v at $v \in S$ is necessarily a field, and by Theorem 6.1 we have $\mathrm{Hom}_{T'}(\pi', \mathbb{C}) \neq 0$. Putting all of these together yields the equivalence. $\square$