Lecture 2: Hecke characters, abelian reciprocity, and Tate's thesis

1. Hecke characters

Now let K be an algebraic number field with ring of integers $\mathcal{O}_K$. We would now like to define a wide class of characters which simultaneouly generalize

(1) characters on invertible residue classes modulo an ideal,
(2) class group characters,

and for which one can associate L-series having nice analytic properties. This is the achievement of Hecke.

1.1. **Notation.**

We begin with some notation.

The ring of integers of K is denoted $\mathcal{O}_K$, and the latter's unit group is $\mathcal{O}_K^\times$. Let J_K denote the group of fractional ideals of K, P_K the subgroup of principal fractional ideals. Let $C_K = J_K/P_K$ denote the class group of K.

For an integral ideal $\mathfrak{q}$ we denote by $\mathcal{O}_K^\times(\mathfrak{q})$ the group of units congruent to 1 modulo $\mathfrak{q}$ and by $J_K^{\mathfrak{q}}$ the group of fractional ideals of K which are prime to $\mathfrak{q}$.

For a finite place v of K let ϖ_v denote a uniformizer at v, and let $\mathfrak{p}_v$ denote the associated prime ideal of K. Write $K_\infty = K \otimes_{\mathbb{Q}} \mathbb{R}$.

1.2. **(Classical) Hecke characters.**

Definition 1. *A character $\chi : J_K^{\mathfrak{q}} \to \mathbb{C}^\times$ is called a* ***(classical) Hecke character with modulus*** $\mathfrak{q}$ *if there exist characters*

$$\chi_f : (\mathcal{O}_K/\mathfrak{q})^\times \to \mathbb{C}^\times \quad \text{and} \quad \chi_\infty : K_\infty^\times \to \mathbb{C}^\times$$

(with χ_∞ continuous) such that

$$\chi((a)) = \chi_f(a)\chi_\infty(a) \tag{1.1}$$

for all $a \in \mathcal{O}_K$ prime to $\mathfrak{q}$. We say that χ is ***unitary*** *if it takes values in S^1. A (classical) Hecke character with modulus $\mathfrak{q}$ is called* ***primitive*** *if there is no proper divisor $\mathfrak{q}' \mid \mathfrak{q}$ such that χ factors through a (classical) Hecke character with modulus $\mathfrak{q}'$. In this case, we call $\mathfrak{q}$ the* ***conductor*** *of χ.*

Note that χ_∞ factors through $K_\infty^\times/\mathcal{O}_K^\times(\mathfrak{q})$, since

$$\chi_\infty(u) = \chi_f(u)\chi_\infty(u) = \chi((u)) = 1, \qquad \forall\, u \in \mathcal{O}_K^\times(\mathfrak{q}).$$

Moreover, the characters χ_f and χ_∞ are uniquely determined by χ. Indeed, if (1.1) holds for all $a \in \mathcal{O}_K$ prime to $\mathfrak{q}$ then in particular it holds for all $a \in \mathcal{O}_K$ congruent to 1 modulo $\mathfrak{q}$, in which case it becomes $\chi((a)) = \chi_\infty(a)$. This can be extended to all $\gamma \in K^\times$ congruent to 1 module $\mathfrak{q}$, the latter being dense in $K_\infty^\times$. This determines the values of χ_∞, by continuity, and hence of χ_f, again by (1.1).

In fact, we have the opposite construction, as follows.

Lemma 1.3. *Given characters*

$$\chi_f : (\mathcal{O}_K/\mathfrak{q})^\times \to \mathbb{C}^\times \quad \textit{and} \quad \chi_\infty : K_\infty^\times/\mathcal{O}_K^\times(\mathfrak{q}) \to \mathbb{C}^\times$$

(with χ_∞ continuous) such that

$$\chi_f(u)\chi_\infty(u) = 1 \quad \textit{for all} \quad u \in \mathcal{O}_K^\times, \tag{1.2}$$

we can find a (classical) Hecke character χ with modulus $\mathfrak{q}$ verifying (1.1) *for all $a \in \mathcal{O}_K$ prime to $\mathfrak{q}$. Any two χ obtained in this way differ by a class group character.*

Proof. If the class group is trivial, then we simply take (1.1) to be the definition of χ on ideals (a) with $a \in \mathcal{O}_K$ prime to $\mathfrak{q}$. This is well-defined, since $\chi_f\chi_\infty$ is trivial on units. We can then extend this multiplicatively to fractional ideals (γ) with $\gamma = a/b \in K^\times$ prime to $\mathfrak{q}$. For the general case, see Proposition 7.7 of Narkiewicz, *Elementary and analytic theory of algebraic numbers.* ☐

1.4. **Examples.** Let us give a few examples:

(1) clearly, any class group character is a (classical) Hecke character (with modulus the trivial ideal $\mathfrak{q} = \mathcal{O}_K$, which we write as $\mathfrak{q} = 1$). In this case, χ_f and χ_∞ are both trivial;

(2) slightly more generally, take $\mathfrak{q} = 1$ so that χ_f is itself trivial. Then the mod 1 (classical) Hecke characters are *archimedean fattenings*, under twists by $\chi_\infty \in \widehat{K_\infty^\times/\mathcal{O}_K^\times}$, of class groups characters. For example, if $K = \mathbb{Q}(i)$, a mod 1 (classical) Hecke character $\chi_\infty : \mathbb{C}^\times \to S^1$ is of the form $(z/|z|)^{4m}$, for an integer m.

(3) on the opposite extreme, take χ_∞ trivial. Then up to multiplication by class group characters, (classical) Hecke characters with trivial χ_∞ are, in view of (1.2), the characters of the quotient of $(\mathcal{O}_K/\mathfrak{q})^\times$ by the subgroup $\mathcal{O}_K^\times$ mod $\mathfrak{q}$. For example, for $K = \mathbb{Q}$ this gives the even Dirichlet characters.

Remark 1. The last example is important, as you can see from it what would happen if we had taken the most naïve generalization of a Dirichlet character to an arbitrary number field (even if the number field has trivial class group). For a character of the invertible residue classes $(\mathcal{O}_K/\mathfrak{q})^\times$ to be well-defined on ideals, we would certainly need for it to be trivial on the image of the unit group $\mathcal{O}_K^\times$. But the size of this image can behave quite erratically as $\mathfrak{q}$ varies: for example, a variation of the Artin primitive root conjecture for real quadratic fields would state that a fundamental unit would *generate* $(\mathcal{O}_K/\mathfrak{q})^\times$ for an infinite number of $\mathfrak{q}$, in which case there would be no non-trivial characters of interest.

On the other hand, an elementary exercise using Poisson summation shows[1] that, if you give yourself an "epsilon bit of room" at the archimedean

[1]See my paper with Valentin Blomer, *Non-vanishing of L-functions, the Ramanujan conjecture, and families of Hecke characters*, Canad. J. Math. 65 (2013), 22-51.

places (allowing for a non-trivial factor χ_∞, as in the definition of Hecke characters, lying in a open neighborhood of the trivial character), then the number of Hecke characters to *any* modulus behaves regularly as a function of $N\mathfrak{q}$ (approximately $N\mathfrak{q}$).

1.5. ***L*-series.** To a (unitary, say) Hecke character χ with modulus $\mathfrak{q}$ we may associate an L-series

$$L(s,\chi) = \sum_{\mathfrak{a}\subset\mathcal{O}_K} \chi(\mathfrak{a})\mathrm{N}\mathfrak{a}^{-s},$$

where we set $\chi(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not coprime to $\mathfrak{q}$. Assuming χ primitive, Hecke proved the nice analytic properties of $L(s,\chi)$ (meromorphic continuation and functional equation), through an adaptation of Riemann's proof. We will not explain this, but rather introduce the adelic perspective.

1.6. **(Adelic) Hecke characters.** We now give an alternative description of a (classical) Hecke character, using the idele class group $K^\times\backslash\mathbb{A}_K^\times$. We begin with the following

Definition 2. *An* ***(adelic) Hecke character*** *is a continuous unitary character $\omega : \mathbb{A}_K^\times \to \mathbb{C}^\times$ which is trivial on $K^\times$. It is* ***unitary*** *if it takes values in S^1.*

Any such ω is a restricted tensor product $\otimes_v'\omega_v$ of characters $\omega_v : K_v^\times \to \mathbb{C}^\times$, where ω_v is obtained via the natural inclusion of $K_v^\times$ into $\mathbb{A}_K^\times$.

We now let $U(\mathfrak{q})$ denote the open compact subgroup of $\mathbb{A}_f^\times$ which at all finite places $v \nmid \mathfrak{q}$ is the local unit group $\mathcal{O}_v^\times$, and at $v \mid \mathfrak{q}$ is are the local units congruent to 1 modulo $\mathfrak{q}\mathcal{O}_v$. These subgroups form a basis of open neighborhoods about 1 in $\prod_{v<\infty}\mathcal{O}_v^\times$. Since ω_f is continuous, and $\mathbb{C}^\times$ has no small subgroups (see later argument), we deduce the existence of a $\mathfrak{q}$ for which ω_f is trivial on $U(\mathfrak{q})$.

Definition 3. *The* ***conductor*** *of an (adelic) Hecke character ω is the smallest integral ideal $\mathfrak{q}$ of $\mathcal{O}_K$ such that χ_f is trivial on $U(\mathfrak{q})$.*

1.7. **Equivalence between definitions.** Given a primitive (classical) Hecke character χ of conductor $\mathfrak{q}$ we may construct an (adelic) Hecke character ω of conductor $\mathfrak{q}$ in the following way. We have only to find ω_v for finite places v, since we will put $\omega_\infty = \chi_\infty^{-1}$. Let S be the finite set of finite places dividing $\mathfrak{q}$. For $v \notin S$ we put $\omega_v(u\varpi_v^n) = \chi(\mathfrak{p}_v^n)$. Note that if S were empty (i.e., $\mathfrak{q} = 1$), then the resulting character ω of $\mathbb{A}_K^\times$ is indeed trivial on $K^\times$, since in that case χ_f is trivial and

$$\omega(\gamma) = \omega_f(\gamma)\omega_\infty(\gamma) = \chi((\gamma))\chi_\infty(\gamma)^{-1} = \chi_\infty(\gamma)\chi_\infty(\gamma)^{-1} = 1,$$

where we have used the compatibility condition of $\chi_f = 1$ and χ_∞ in the definition of a classical Hecke character.

If S is non-empty, to define $\omega_S : K_S^\times \to \mathbb{C}^\times$, we note that the values of ω_S on the image of $K^\times$ inside $K_S^\times$ are uniquely determined by the choice of ω^S and ω_∞. Indeed, for all $\gamma \in K^\times$ it must satisfy

$$1 = \omega(\gamma) = \omega_S(\gamma)\omega^S(\gamma)\omega_\infty(\gamma).$$

But by the *weak approximation* property of the idele group $\mathbb{A}_K^\times$, $K^\times$ is dense in $K_S^\times$. Since ω_S is continuous it is in turn uniquely determined by its values on $K^\times$.

Conversely, given an (adelic) Hecke character ω of conductor $\mathfrak{q}$ we may define a primitive (classical) Hecke character χ of conductor $\mathfrak{q}$, by the composition

$$\chi : J_K^{\mathfrak{q}} \longrightarrow \mathbb{A}_K^\times / K^\times U(\mathfrak{q}) \xrightarrow{\omega} \mathbb{C}^\times, \tag{1.3}$$

where the first map is induced by sending a prime ideal $\mathfrak{p}_v \nmid \mathfrak{q}$ to the class of the uniformizer ϖ_v. (Check that this does not depend on the choice of uniformizer.)

We shall henceforth use the language of (adelic) Hecke characters, dropping the (adelic), and shall generally use χ rather than ω.

2. Reciprocity for GL_1

We now explain the relevance of Hecke characters to the reciprocity laws of local and global class field theory.

2.1. **Finite order characters and Galois characters.** Fix an algebraic closure $\overline{K}$ of K and let $K^{\mathrm{ab}} \subset \overline{K}$ be the maximal abelian extension of K. Recall that global class field theory provides for a natural homomorphism

$$\theta_K : K^\times \backslash \mathbb{A}_K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K), \tag{2.1}$$

which (since we are in the number field case[2]) is surjective with kernel the connected component of the identity in $K^\times \backslash \mathbb{A}_K^\times$. More concretely, the kernel is the closure of the image of K_∞^+ in $K^\times \backslash \mathbb{A}_K^\times$, where $K_\infty^+ = \prod_{v|\infty} K_v^+$ and K_v^+ is the connected component of the identity in $K_v^\times$ (thus equal to either $\mathbb{R}_+^\times$ or $\mathbb{C}^\times$ in the real and complex cases, respectively).

Passing to the Pontryagin duals, we first observe that the characters of

$$\mathrm{Gal}(K^{\mathrm{ab}}/K) = \mathrm{Gal}(\overline{K}/K)/\mathrm{Gal}(\overline{K}/K^{\mathrm{ab}}) = \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$$

are simply the characters of $\mathrm{Gal}(\overline{K}/K)$, since these all factor through the commutator subgroup, which is $\mathrm{Gal}(\overline{K}/K^{\mathrm{ab}})$. We obtain an injective map

$$\mathrm{Hom}(\mathrm{Gal}(\overline{K}/K), \mathbb{C}^\times) \longrightarrow \mathrm{Hom}(K^\times \backslash \mathbb{A}_K^\times, \mathbb{C}^\times) \tag{2.2}$$

given by pull-back.

Lemma 2.2. *The image of the map* (2.2) *consists precisely of the Hecke characters of finite order.*

[2] In the function field case, the map is injective with dense image

Proof. We first note that a Galois character $\mathrm{Gal}(\overline{K}/K) \to \mathbb{C}^\times$ is necessarily of finite order. Indeed, any continuous homomorphism $\chi : G \to \mathbb{C}^\times$, where G is profinite, has finite image. This is a classical argument which is important to go over.

It is based on the "no small subgroups property" of the Lie group $\mathbb{C}^\times$: a sufficiently small neighborhood of 1 in $\mathbb{C}^\times$ contains no non-trivial subgroup (there are, in a sense, too many opens sets in this case). On the other hand, any neighborhood of the identity of a locally profinite group contains an open subgroup (few open sets).

So take a small enough open neighborhood U of $\mathbb{C}^\times$, containing 1. Then the inverse image of U is an open neighborhood V of 1 in G. Since G is locally profinite, there is an open subgroup H contained in V. Its image under χ is a subgroup of $\mathbb{C}^\times$ contained in U, and is therefore trivial. This shows that $\ker\chi$ contains an open subgroup, which then implies[3] that $\ker\chi$ is itself open. But G is in fact profinite, hence compact, so the open subgroup $\ker\chi$ is of finite index. We conclude by $\mathrm{im}\chi \simeq G/\ker\chi$.

We have therefore shown that the image of the map (2.2) is *contained in* the subgroup of finite order Hecke characters. We now have to show that every such Hecke character arises in this way. Since $\chi_f : \mathbb{A}_f^\times \to \mathbb{C}^\times$ is continuous it has finite image. Thus χ is of finite order if, and only if, the archimedean component χ_∞ is. The finite order characters of $K_\infty^\times$ are easy to describe: they are precisely those that are trivial on the finite index subgroup K_∞^+. (For example, when $K = \mathbb{Q}$, we obtain all Dirichlet characters, when K is imaginary quadratic the condition is that $\chi_\infty = 1$, and when K is real quadratic the condition is that $\chi_\infty(x_1, x_2) = \mathrm{sgn}(x_1)^{\epsilon_1}\mathrm{sgn}(x_2)^{\epsilon_2}$.) Since K_∞^+ is exactly the kernel of the Artin map θ_K, we are done. □

2.3. **The Weil group.** From the previous paragraph, we see that there are (many!) more Hecke characters than there are Galois characters. Enter the Weil group W_K of K.

To warm up, recall that local class field theory provides for a homomorphism

$$\theta_v : K_v^\times \longrightarrow \mathrm{Gal}(\overline{K_v}/K_v)^{\mathrm{ab}}$$

which is injective, but not onto. It is clear, in any case, that the two groups cannot be isomorphic, since the Galois group is compact (since profinite) and $K_v^\times$ is not (choosing a uniformizer we get $K_v^\times \simeq \mathcal{O}_v^\times \times \mathbb{Z}$).

To rectify things, we could take the profinite completion $\widehat{K_v^\times} \simeq \mathcal{O}_v^\times \times \widehat{\mathbb{Z}}$ to get an isomorphism (which is one way of expressing local class field theory), but instead we shall give a name to the image of θ_v. Let k_v be the residue field of K_v and q_v its cardinality.

[3]This is true in any topological group. If a subgroup K contains an open subset U containing the identity, then since group multiplication is a homeomorphism the set Uk is open, and contained in K, for all $k \in K$. Thus $\cup_{k\in K} Uk$ is open and contained in K. But since U contains the identity, this same union contains K, showing equality, and openness of K.

Definition 4. *The **Weil group of** K_v, denoted W_{K_v}, is the dense subgroup of* $\mathrm{Gal}(\overline{K_v}/K_v)$ *consisting of* σ *which induce on* $\overline{k_v}$ *the map* $x \mapsto x^{q_v^n}$ *for some* $n \in \mathbb{Z}$.

With this convention, the Artin homomorphism becomes an isomorphism

$$\theta_v : K_v^\times \xrightarrow{\sim} W_{K_v}^{\mathrm{ab}}.$$

By replacing $\widehat{\mathbb{Z}}$ by $\mathbb{Z}$ in the passage from $\mathrm{Gal}(\bar{K}_v/K_v)^{\mathrm{ab}}$ to $W_{K_v}^{\mathrm{ab}}$ we have resolved the problem of incompatible topologies with $K_v^\times$.

The global Weil group W_K of K does not (at present) admit a simple description (only a construction through cocycles, given by Weil), so we rather isolate a few key properties. (For more details, see Tate's article *Number Theoretic Background* in the Corvallis proceedings, the canonical reference for Weil groups.)

The **Weil group of** K (a number field) is given by a triple $(W_K, \varphi_K, \{r_E\})$ satisfying several properties, of which we shall now recall the most salient. To begin with, W_K is a topological group, and

$$\varphi_K : W_K \to \mathrm{Gal}(\overline{K}/K)$$

is a (continuous) surjective homomorphism. Already from these two ingredients we may extract a few related maps. For every finite extension $K \subset E \subset \overline{K}$, we put $W_E = \varphi_K^{-1}(\mathrm{Gal}(\overline{K}/E))$ – an open subgroup, by continuity. For every such extension E, φ_K induces a surjective map

$$\varphi_E^{\mathrm{ab}} : W_E^{\mathrm{ab}} \to \mathrm{Gal}(\overline{K}/E)^{\mathrm{ab}}.$$

Moreover, we obtain bijections

$$W_K/W_E \xrightarrow{\sim} \mathrm{Gal}(\overline{K}/K)/\mathrm{Gal}(\overline{K}/E),$$

which, in the case where E is Galois over F, become group isomorphisms $W_K/W_E \xrightarrow{\sim} \mathrm{Gal}(E/K)$.

We now describe the third ingredient $\{r_E\}$ in the triple defining the Weil group. These r_E, indexed by finite extensions E as above, are isomorphisms of topological groups

$$r_E : E^\times \backslash \mathbb{A}_E^\times \xrightarrow{\sim} W_E^{\mathrm{ab}}$$

such that the composed map

$$\varphi_E^{\mathrm{ab}} \circ r_E : K^\times \backslash \mathbb{A}_E^\times \longrightarrow \mathrm{Gal}(\overline{E}/E)^{\mathrm{ab}}$$

is the reciprocity map θ_K from (2.1).

The triple $(W_K, \varphi_K, \{r_E\})$ is assumed to verify some compatibility properties that we shall not describe.

Remark 1. For every finite Galois extension E we get an exact sequence

$$1 \to W_E/W_E^{\mathrm{c}} \to W_K/W_E^{\mathrm{c}} \to W_K/W_E \to 1.$$

The first group is W_E^{ab}, which is identified with $E^\times \backslash \mathbb{A}_E^\times$ through r_E. Let us call the middle group $W_{E/K}$. The third group is isomorphic to $\mathrm{Gal}(E/K)$

through φ_K. We deduce that $W_{E/K}$ is an extension of $\mathrm{Gal}(E/K)$ by $E^\times\backslash\mathbb{A}_E^\times$. As such, it gives rise to an element in the Galois cohomology group

$$\alpha_{E/K} \in H^2(\mathrm{Gal}(E/K), E^\times\backslash\mathbb{A}_E^\times).$$

This fundamental class plays a fundamental role in class field theory.

Remark 2. Like that of θ_K, the kernel of φ_K is enormous: the inverse limit, under the norm maps, of the connected component of the identity in $E^\times\backslash\mathbb{A}_E^\times$ (i.e., the closure of the image of E_∞^+ in the quotient).

2.4. **Algebraic Hecke characters.** Between the class of the finite order Hecke characters (which correspond to all complex-valued characters of the absolute Galois group) and all Hecke characters (which correspond to all complex-valued characters of the Weil group), there is an intermediate class introduced by Weil in 1956, called the *algebraic Hecke characters*.

We begin by a simple example, taking $K = \mathbb{Q}$. The finite order Hecke characters are the Dirichlet characters, and an arbitrary Hecke character is of the form $\chi|\cdot|^s$ for $s \in \mathbb{C}$, where χ is a Dirichlet character. So the difference between the two classes of characters is entirely accounted for by the simple operation of twisting by $|\cdot|^s$. For an arbitrary Hecke character over $\mathbb{Q}$ to be algebraic, s must be an integer.

To generalize this definition, we now write an arbitrary Hecke character of K uniquely as a product $\chi|\cdot|_{\mathbb{A}_K}^\sigma$, where $\sigma \in \mathbb{R}$ and χ is unitary. We need some notation for the infinity types of unitary Hecke characters. For $v \mid \infty$, write

$$\chi_v(x) = |x|_v^{it_v}(x/|x|_v)^{m_v},$$

where $t_v \in \mathbb{R}$ and $m_v \in \mathbb{Z}$.

Definition 5. *A Hecke character is* ***algebraic*** *if*

(1) $t_v = 0$ *for every* $v \mid \infty$*,*
(2) $\frac{m_v}{2} + \sigma \in \mathbb{Z}$ *for every complex* v*,*
(3) $\sigma \in \mathbb{Z}$ *if there is any real place.*

This class of characters was introduced by Weil in 1956, where they were called *Hecke characters of type* (A_0). He also introduced a slightly larger class called *Hecke characters of type* (A), namely, Hecke characters for which $\sigma \in \mathbb{Q}$ and $t_v = 0$ for all v. He then showed that the (classical) Hecke characters associated with an adelic Hecke character of type A through the map (1.3), despite not necessarily being of finite order, take on algebraic values, whence the name (and the letter A). (Waldschmidt, in 1982, showed that the type (A) condition is necessary for algebraicity.) Moreover, the same result is true for characters of type (A_0), where algebraic is replaced by lying in a finite extension of K (a number field). Compare this to finite order Hecke characters, whose values are roots of unity. In our usage, we will use the type (A_0) condition for our definition of algebraicity. (See Paskunas's thesis, *Variations on a theorem of Tate*, available on the arXiv, for more on the type (A) condition.)

Thus, for any totally real field extension, an algebraic Hecke character is a twist by $|\cdot|_{\mathbb{A}_K}^w$ of a finite order character, where $w \in \mathbb{Z}$. Weil (in his 1957 article) called such algebraic Hecke characters *trivial*. This means that the theory of algebraic Hecke characters is not much richer than the theory of finite order characters, when the underlying field is totally real. This is a reflection of the "large unit group" is such fields: the rank of $\mathcal{O}_K^\times$ is 1 less than the dimension of $K_\infty^\times$.

A more interesting example of an algebraic Hecke character is the one given for $K = \mathbb{Q}(i)$ in Example (2) (or any twist of that example by $|\cdot|_{\mathbb{A}}^w$, where $w \in \mathbb{Z}$). More generally, for any CM field extension (totally imaginary extension of a totally real field F), the infinity types of algebraic Hecke characters are twists by $|\cdot|_{\mathbb{A}_K}^w$ of characters of the form $\prod_{v|\infty}(x_v/|x_v|)^{m_v}$, where $2w \in \mathbb{Z}$ and $m_v \equiv 2w \pmod 2$ for all v. In particular, if the character is unitary (so that $w = 0$), then $m_v \in 2\mathbb{Z}$ for all v. Note that in the case of CM extensions, $K_\infty^\times$ has dimension $2d$ (where $d = [F : \mathbb{Q}]$), whereas the unit group has rank $d - 1$.

Remark 2. The definition we gave of an algebraic Hecke character does not appear, at first glance, very natural. We now give an alternative formulation of the definition. Let $d_v = [K_v : \mathbb{R}]$. Since, on K_∞^+, we have

$$\begin{aligned}\prod_{v=\mathbb{C}}(x_v/|x_v|_v)^{m_v}\prod_{v|\infty}|x_v|_v^{d_v\sigma} &= \prod_{v=\mathbb{C}} x_v^{m_v}|x_v|_v^{2\sigma-m_v}\prod_{v=\mathbb{R}} x_v^\sigma \\ &= \prod_{v=\mathbb{C}} x_v^{m_v}(x_v\overline{x}_v)^{\sigma-\frac{m_v}{2}}\prod_{v=\mathbb{R}} x_v^\sigma \\ &= \prod_{v|\infty} x_v^{\sigma+\frac{m_v}{2}}\overline{x}_v^{\sigma-\frac{m_v}{2}}\prod_{v=\mathbb{R}} x_v^\sigma,\end{aligned}$$

a Hecke character is algebraic if the restriction of χ_∞ to K_∞^+ is of the form

$$\prod_{v=\mathbb{C}} x_v^{p_v}\overline{x}_v^{q_v}\prod_{v=\mathbb{R}} x_v^k$$

for integers $\{p_v, q_v\}_{v=\mathbb{C}}$ such that $\frac{p_v+q_v}{2}$ is independent of v, and equal to k. For example, the character $|\cdot|_{\mathbb{A}_K}$ corresponds to $p_v = q_v = 1$ for every $v \mid \infty$.

2.5. **Correspondence with compatible systems of ℓ-adic Galois characters.** What kind of Weil or Galois group characters do these algebraic Hecke characters correspond to? Note that the finite order property of complex valued characters of the Galois group is no longer forced when $\mathbb{C}$ is replaced by $\overline{\mathbb{Q}_\ell}$, an algebraic closure of $\mathbb{Q}_\ell$. (The two fields $\mathbb{C}$ and $\overline{\mathbb{Q}_\ell}$ are isomorphic as fields, since they are both algebraically closed and of the same transcendence degree. Their topologies, however, are quite different: $\overline{\mathbb{Q}_\ell}$ is not complete.)

Example 1. We recall the definition of the cyclotomic character

$$\omega_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times.$$

Recall that $\mathbb{Z}_\ell^\times$ is the inverse limit of $(\mathbb{Z}/\ell^n\mathbb{Z})^\times$. Fixing a primitive ℓ^n-root of unity ζ_n, all others can be obtained by powers of ζ_n by a full set of representatives of $(\mathbb{Z}/\ell^n\mathbb{Z})^\times$. The Galois group acts on primitive roots of unity, and so $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ determines a unique element in $\omega_\ell(\sigma; n) \in (\mathbb{Z}/\ell^n\mathbb{Z})^\times$. The system $\omega_\ell(\sigma) = (\omega_\ell(\sigma; n))_n$ forms a compatible system, thus an element in $\mathbb{Z}_\ell^\times$. The resulting map $\sigma \mapsto \omega_\ell(\sigma)$ is a continuous surjective homomorphism, so of infinite image.

We would like a correspondence between algebraic characters of $\mathbb{Q}$ and compatible families of ℓ-adic characters of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which, in particular, sends $|\cdot|^a$, for an integer $a \in \mathbb{Z}$, to $\{\omega_\ell^a\}_\ell$. This can be done as follows: if K is an algebraic number field, and E is a finite extension of K, we call a continuous character

$$\mathsf{X} : \mathbb{A}_K^\times \to E^\times$$

a *(universal) algebraic Hecke character* (my terminology) if there is a homomorphism of $\mathbb{Q}$-algebraic tori

$$T : \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m$$

such that $T|_{K^\times} = \mathsf{X}|_{K^\times}$. If this is the case, then this induces a continuous homomorphism

$$\mathsf{X}.T^{-1} : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{A}_E^\times,$$

and we may compose this with the projection to the local completion $E_v^\times$ for any place v.

(1) When v is complex this gives an adelic Hecke character as we have defined it, which (by Waldschmidt's theorem) is algebraic, in the sense of being of type (A_0).
(2) For $v < \infty$, a v-adic Hecke characters with values in the totally disconnected topological group $E_v^\times$. Its kernel therefore contains the connected component of the identity in $\mathbb{A}_K^\times$. In this way, we obtain (via the inverse of the Artin map $\mathrm{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \mathbb{A}_K^\times/K^\times K_\infty^+$) a Galois character.

As v varies over finite place, this in fact forms a compatible system. For more details, see Chapters 0 and 1 of Schappacher's book, *Periods of Hecke characters.*

3. Tate's thesis

Tate's approach to prove the functional equation of $\Lambda(s, \chi)$ can be described succinctly as follows:

(1) prove a functional equation for all local zeta integrals using a multiplicity one result on $K_v^\times$-invariant functionals;
(2) prove the functional equation for a global zeta integral using Poisson summation;

(3) extract the functional equation for $\Lambda(s,\chi)$ from both of the above, using that the local factors of the L-function are constants of proportionality.

3.1. **Local multiplicity one.** . Fix a place v of K. Let $k = K_v$. We shall (for the most part) suppress the subscript v in the notation, and reinsert it later when we work globally.

Let $\mathcal{S}_k$ be the space of Schwartz-Bruhat functions on k. For k archimedean, this means the usual space of Schwartz functions, consisting of (complex-valued) functions all of whose derivatives decay rapidly at infinity. For k non-archimedean, this means locally constant and of compact support. Denote by $\mathcal{S}'_k$ its dual, the space of continuous linear functionals on $\mathcal{S}_k$, i.e., tempered distributions on k. (For finite places, the topology on $\mathcal{S}_k$ comes from the direct limit of its finite dimensional subspaces, so the continuity of a functional on $\mathcal{S}_k$ is automatic. For archimedean places, the topology on $\mathcal{S}_k$ derives from its structure as a Fréchet space.)

There is an action of $k^\times$ on $\mathcal{S}_k$ given by right translation. Correspondingly, there is an action of $k^\times$ on $\mathcal{S}'_k$ in which the pair (x,λ) is sent to the functional $f \mapsto \langle \lambda, f(\cdot x^{-1})\rangle$.

Let Ω be the space of quasicharacters of $k^\times$. We can write an arbitrary quasicharacter as $\omega = \chi|\cdot|^s$, where χ is a (unitary) character. The space Ω is endowed with a natural measure (Lebesgue on every connected component). We decompose $\mathcal{S}_k$ under the $k^\times$-action, obtaining

$$\mathcal{S}'_k = \sum_\chi \int_{\mathbb{C}} \mathcal{S}'_k(s,\chi)\,ds,$$

where

$$\mathcal{S}'_k(\omega) = \{\lambda \in \mathcal{S}'_k : x.\lambda = \omega(x)\lambda \quad \forall x \in k^\times\}$$

is the ω-*isotypic component*. Then we have the following fundamental result, whose proof will be sketched later.

Theorem 1 (Local multiplicity one)**.** *For every quasicharacter ω of $k^\times$ we have*

$$\dim \mathcal{S}'_k(\omega) = 1.$$

We defer the proof to the end of the lecture.

3.2. **Local functional equation.** The local functional equation will now be obtained as a corollary of this theorem. Indeed, we shall define two non-zero elements of $\mathcal{S}'_k(s,\chi)$, and Theorem 1 will then state that they are proportional. The constant of proportionality will be the local epsilon factor.

For s large enough, there is an obvious element of $\mathcal{S}_k(s,\chi)$, obtained by forcing equivariance. For $f \in \mathcal{S}_k$ we put

$$\langle z(s,\chi), f\rangle = \int_{k^\times} f(x)\chi(x)|x|^s d^\times x.$$

This integral converges absolutely for $\mathrm{Re}(s) > 0$. Indeed, it suffices to show this to be the case when $\chi = \chi_0$ is the trivial character and $f = \mathbf{1}_{\mathfrak{o}}$. In this case we have (exercise!)

$$\langle z(s, \chi_0), \mathbf{1}_{\mathfrak{o}} \rangle = \zeta_v(s). \tag{3.1}$$

Thus, for $\mathrm{Re}(s) > 0$, we have $z(s, \chi) \in \mathcal{S}'_k(s, \chi)$.

Now we have a mechanism for producing other elements of $\mathcal{S}'_k(s, \chi)$, namely, via Fourier transform. Let ψ be a non-trivial additive character of k. Let dx be the measure on k which is self-dual with respect to the Fourier transform with respect to ψ. For $\lambda \in \mathcal{S}'_k(\omega)$, we have

$$\begin{aligned}\langle \hat{\lambda}, f(\cdot x^{-1}) \rangle = \langle \lambda, \widehat{f(\cdot x^{-1})} \rangle &= \langle \lambda, |x| \widehat{f}(\cdot x) \rangle \\ &= |x| \langle \lambda, \widehat{f}(\cdot x) \rangle = |x| \omega^{-1}(x) \langle \lambda, \widehat{f} \rangle = |x| \omega^{-1}(x) \langle \hat{\lambda}, f \rangle.\end{aligned}$$

We see that $\hat{\lambda} \in \mathcal{S}'_k(|\cdot|\omega^{-1})$. Taking $\omega = \chi^{-1}|\cdot|^{1-s}$, with χ unitary, we have $|\cdot|\omega^{-1} = \chi|\cdot|^s$, yielding a Fourier transform

$$\mathcal{S}'_k(1 - s, \chi^{-1}) \longrightarrow \mathcal{S}'_k(s, \chi), \qquad \lambda \mapsto \hat{\lambda}.$$

The problem at this point is that we don't yet have any explicit element in $\mathcal{S}'_k(1 - s, \chi^{-1})$ to dualize, since our local zeta integral $z(s, \chi)$ was defined only for $\mathrm{Re}(s) > 0$. So we need to meromorphically continue $z(s, \chi)$ (as a distribution) to all of $\mathbb{C}$. We do this as follows.

Proposition 3.3. *Let*

$$z_0(s, \chi) = \frac{z(s, \chi)}{L(s, \chi)} \in \mathcal{S}_k(s, \chi). \tag{3.2}$$

Then $z_0(s, \chi)$ extends holomorphically to all of $s \in \mathbb{C}$. Thus the expression $z(s, \chi_0) = L(s, \chi_0) z_0(s, \chi_0)$ meromorphically continues $z(s, \chi_0)$ to all of $\mathbb{C}$.

Proof. We sketch the proof for non-archimedean k. We split into two cases, according to whether χ is unramified or not.

For unramified characters $\omega = |\cdot|^s$, we observe that for functions $f \in \mathcal{S}_k$ which are compactly supported in $k^\times$ (i.e., those which vanish in a neighborhood about zero), the zeta integral $\langle z(s, \chi_0), f \rangle$ is analytic in $s \in \mathbb{C}$. Now take any function in $\mathcal{S}_k$. Since it is locally constant, we may simply subtract off its value at zero, by hitting it with the group algebra element $[1] - [\varpi^{-1}] \in \mathbb{Z}[k^\times]$ (the action being induced from the right action of $k^\times$), and the resulting function vanishes in a neighborhood of zero.

With this in mind, we let $z_0(s, \chi_0) \in \mathcal{S}'_k(s, \chi_0)$ be defined by

$$\langle z_0(s, \chi_0), f \rangle = \langle z(s, \chi_0), ([1] - [\varpi^{-1}]) f \rangle.$$

The expression on the right-hand side makes sense for all $s \in \mathbb{C}$, and when $\mathrm{Re}(s) > 0$ we calculate

$$\begin{aligned}\langle z_0(s,\chi_0), f\rangle &= \int_{k^\times} (f(x) - f(\varpi^{-1}x))|x|^s d^\times x \\ &= \langle z(s,\chi_0), f\rangle - q_v^{-s} \int_{k^\times} f(x)|x|^s d^\times x \\ &= L(s,\chi_0)^{-1}\langle z(s,\chi_0), f\rangle,\end{aligned}$$

as desired.

Let χ now be a ramified character. Here the local L-function is 1, so we simply need to prove that $z(s,\chi)$ extends holomorphically. We will profit from the oscillation of χ along units. Let's do a sample computation. Let f_t be the characteristic function of $\mathfrak{p}_v^t$. Let u be a unit on which $\chi(u) \neq 1$. Then, changing variables,

$$\begin{aligned}\langle z(s,\chi), f_t\rangle &= \int_{k^\times} f_t(x)\chi(x)|x|^s d^\times x \\ &= \chi(u) \int_{k^\times} f_t(x)\chi(x)|x|^s d^\times x \\ &= \chi(u)\langle z(s,\chi), f_t\rangle.\end{aligned}$$

Thus $\langle z(s,\chi), f_t\rangle = 0$ for any such f_t. Now take f arbitrary. Then since f is constant on a neighborhood of the origin, there is some t for which the restriction of f to $\mathfrak{p}_v^t$ is constant. The above remark shows that

$$\langle z(s,\chi), f\rangle = \int_{k^\times - \mathfrak{p}_v^t} f(x)\chi(x)|x|^s d^\times x.$$

As the integrand is supported away from zero, the integral is analytic as a function of $s \in \mathbb{C}$. □

We may apply the Fourier transform to $z_0(1-s,\chi^{-1}) \in \mathcal{S}_k(1-s,\chi^{-1})$ and then local multiplicity one to deduce the existence of a constant of proportionality $\epsilon(s,\chi;\psi) \in \mathbb{C}$ such that

$$z_0(\widehat{1-s,\chi^{-1}}) = \epsilon(s,\chi;\psi) z_0(s,\chi), \tag{3.3}$$

valid for all $s \in \mathbb{C}$.

3.4. **Global zeta integral.** . We now return to the global setting. We define the global Schwartz space $\mathcal{S}_{\mathbb{A}}$ as the space generated by factorizable functions $f = \otimes_v f_v$, where $f_v \in \mathcal{S}_{K_v}$ for all v and such that f_v is the characteristic function f_v° of $\mathcal{O}_v$ for almost all v. We denote by $\mathcal{S}'_{\mathbb{A}}$ the space of continuous linear functionals on $\mathcal{S}_{\mathbb{A}}$. We may define an additive Fourier transform on $\mathcal{S}_{\mathbb{A}}$ in the usual way. This involves an unramified non-trivial additive character $\psi_0 : K\backslash\mathbb{A}_K \to \mathbb{C}^\times$.

3.4.1. *Composite functional equation.* We put

$$z_0(s,\chi)=\prod_v z_0(s,\chi_v)\in\mathcal{S}'_{\mathbb{A}},\qquad \epsilon(s,\chi;\psi)=\prod_v \epsilon(s,\chi_v;\psi_v)\in\mathbb{C}.$$

Assuming proper measure normalizations we have $\epsilon(s,\chi_v;\psi_v)=1$ for almost every v, assuring the convergence of the infinite product.

In fact, any non-trivial additive character ψ of $F\backslash\mathbb{A}$ is of the form $\psi_0(a\cdot)$, where $a\in F^\times$. Using the product formula, one can then show that for $\psi=\otimes_v\psi_v$ a non-trivial additive character of $F\backslash\mathbb{A}$, the product $\prod_v\epsilon(s,\chi_v,\psi_v)$ is independent of ψ. We may thus write $\epsilon(s,\chi)$ (suppressing the dependency in ψ) for the global product of epsilon factors.

The local functional equations (3.3) then yield

$$z_0(\widehat{1-s,\chi^{-1}})=\epsilon(s,\chi)z_0(s,\chi) \tag{3.4}$$

in $\mathcal{S}_{\mathbb{A}}$. Note that (3.4) does not use the fact that χ is a Hecke character, only that it is a continuous character of $\mathbb{A}_K^\times$. So this is not a globally very interesting relation, it is simply the patching together of all the local functional equations.

3.4.2. *Global zeta function.* Let χ be a continuous character of $\mathbb{A}_K^\times$. For $f\in\mathcal{S}_{\mathbb{A}}$ and $\mathrm{Re}(s)$ large enough, we define the global zeta integral as

$$\langle z(s,\chi),f\rangle=\int_{\mathbb{A}^\times}f(x)\chi(x)|x|_{\mathbb{A}}^s d^\times x.$$

To check convergence of this integral, it suffices to do so for factorizable $f=\otimes_v f_v$. In this case, we have

$$\langle z(s,\chi),f\rangle=\prod_v\langle z(s,\chi_v),f_v\rangle.$$

Now $f_v=f_v^\circ$ and $\chi_v=1_v$ for all $v\notin S$, for some finite set of places S. Using (3.1), we have

$$\prod_{v\notin S}\langle z(s,1_v),f_v^\circ\rangle=\prod_{v\notin S}\zeta_v(s)=\zeta_K^S(s),$$

we see that $\langle z(s,\chi),f\rangle$ converges for $\mathrm{Re}\,s>1$. For $\mathrm{Re}\,s>1$ we have thereofore defined a non-zero global distribution $z(s,\chi)\in\mathcal{S}'_{\mathbb{A}}$.

Using (3.2), we deduce that, for $\mathrm{Re}(s)>1$,

$$z(s,\chi)=\Lambda(s,\chi)z_0(s,\chi). \tag{3.5}$$

Thus, on this half-plane, the completed L-function $\Lambda(s,\chi)$ is the factor of proportionality of the two elements $z(s,\chi),z_0(s,\chi)\in\mathcal{S}'_{\mathbb{A}}$. Again, this does not use the automorphy of χ.

Henceforth we assume that χ is a Hecke character. The key input to the following result is Poisson summation for the discrete cocompact subgroup K of $\mathbb{A}_K$. This is the statement that for $f\in\mathcal{S}_{\mathbb{A}}$, with Fourier transform

$$\hat f(\xi)=\int_{\mathbb{A}_K}f(y)\psi_0(\xi y)dy,$$

we have

$$\sum_{\gamma\in K} f(\gamma)=\sum_{\xi\in K}\hat{f}(\xi).$$

If $a\in\mathbb{A}_K^\times$, then applying this to the translated function $f(xa^{-1})$ we get

$$|a|^{-1}\sum_{\gamma\in K} f(\gamma a^{-1})=\sum_{\gamma\in K}\hat{f}(\xi a).$$

Proposition 3.5. *Let χ be a Hecke character. The distribution $z(s,\chi)$ extends meromorphically to all of $s\in\mathbb{C}$ and satisfies the functional equation*

$$z(\widehat{1-s,\chi^{-1}})=z(s,\chi) \tag{3.6}$$

in $\mathcal{S}_{\mathbb{A}}$.

Proof. The computation is very similar to Riemann's proof. For $f\in\mathcal{S}(\mathbb{A}_K)$, since $\mathbb{A}_K^\times=\mathbb{A}_K^1\times\mathbb{R}_+$, where $\mathbb{A}_K^1$ are the norm 1 ideles, we have, for $\mathrm{Re}(s)>1$,

$$\begin{aligned}
\langle z(s,\chi),f\rangle &= \int_0^\infty\int_{\mathbb{A}_K^1} f(at)\chi(a)d^\times a\, t^s d^\times t\\
&= \left(\int_0^1+\int_1^\infty\right)\int_{a\in\mathbb{A}_K^1} f(at)\chi(a)d^\times a\, t^s\, d^\times t\\
&= \int_1^\infty\int_{a\in\mathbb{A}_K^1} f(at^{-1})\chi(a)d^\times a t^{-s}\, d^\times t+\int_1^\infty\int_{a\in\mathbb{A}_K^1} f(at)\chi(a)d^\times a\, t^s\, d^\times t.
\end{aligned}$$

By the rapid decay of f_∞, the second of these two terms is absolutely convergent, locally uniformly for any $s\in\mathbb{C}$. Let $\mathcal{F}$ be a fundamental domain for the action of $K^\times$ on $\mathbb{A}_K^1$; note that $\mathcal{F}$ is compact. Unfolding the integral, we have

$$\begin{aligned}
\int_{a\in\mathbb{A}_K^1} f(at^{-1})\chi(a)d^\times a &= \int_{a\in\mathcal{F}}\chi(a)\sum_{\gamma\in K^\times} f(\gamma a t^{-1})d^\times a\\
&= \int_{a\in\mathcal{F}}\chi(a)\Big(\sum_{\gamma\in K} f(\gamma a t^{-1})-f(0)\Big)d^\times a.
\end{aligned}$$

Applying Poisson summation, we obtain

$$\begin{aligned}
\int_{a\in\mathbb{A}_K^1} f(at^{-1})\chi(a)d^\times a &= \int_{a\in\mathcal{F}}\chi(a)\Big(t\sum_{\xi\in K}\hat{f}(\xi a^{-1}t)-f(0)\Big)d^\times a\\
&= \int_{a\in\mathcal{F}}\chi(a)\Big(t\sum_{\xi\in K^\times}\hat{f}(\xi a^{-1}t)+t\hat{f}(0)-f(0)\Big)d^\times a.
\end{aligned}$$

Refolding the integral gives

$$\begin{aligned}\int_{a\in\mathbb{A}_K^1} f(at^{-1})\chi(a)d^\times a &= t\int_{\mathbb{A}_K^1}\chi(a)\hat{f}(a^{-1}t)d^\times a + (t\hat{f}(0)-f(0))\int_{\mathcal{F}}\chi(a)d^\times a\\ &= t\int_{\mathbb{A}_K^1}\chi^{-1}(a)\hat{f}(at)d^\times a + (t\hat{f}(0)-f(0))\int_{\mathcal{F}}\chi(a)d^\times a.\end{aligned}$$

Putting this together with the first display, we obtain

$$\begin{aligned}\langle z(s,\chi),f\rangle = \int_1^\infty\int_{\mathbb{A}_K^1}\chi^{-1}(a)\hat{f}(at)d^\times a t^{1-s}d^\times t + \int_1^\infty\int_{\mathbb{A}_K^1} f(at)\chi(a)d^\times a t^s d^\times t\\ + \int_1^\infty (t\hat{f}(0)-f(0))\int_{\mathcal{F}}\chi(a)d^\times a t^{-s}d^\times t.\end{aligned}$$

By the rapid decay of $\hat{f}$ the first integral is absolutely convergent, locally uniformly for all $s\in\mathbb{C}$. If χ is ramified, the remaining terms vanish since $\int_{\mathcal{F}}\chi(a)d^\times a = 0$. Otherwise, if χ is trivial the supplementary terms are given by

$$\mathrm{vol}(\mathcal{F})\bigg(\hat{f}(0)\int_1^\infty t^{1-s}d^\times t - f(0)\int_1^\infty t^{-s}d^\times t\bigg) = -\mathrm{vol}(\mathcal{F})\bigg(\frac{\hat{f}(0)}{1-s}+\frac{f(0)}{s}\bigg).$$

The functional equation then follows. □

3.5.1. *The L-function.* Let χ be a Hecke character.

We begin by showing the meromorphy of $\Lambda(s,\chi)$. Since $z_0(s,\chi)$ is meromorphic on $\mathbb{C}$, the meromorphic extension of $z(s,\chi)$ of Proposition 3.5, combined with the relation (3.5), implies the memorphic continuation of $\Lambda(s,\chi)$.

Next we deduce the functional equation. From (3.5) and (3.6) (and the linearity of Fourier transform), we have

$$\Lambda(1-s,\chi^{-1})z_0\widehat{(1-s,\chi^{-1})} = z\widehat{(1-s,\chi^{-1})} = z(s,\chi) = \Lambda(s,\chi)z_0(s,\chi).$$

Inserting (3.4) yields

$$\Lambda(1-s,\chi^{-1})\epsilon(s,\chi)z_0(s,\chi) = \Lambda(s,\chi)z_0(s,\chi),$$

from which the functional equation

$$\Lambda(1-s,\chi^{-1})\epsilon(s,\chi) = \Lambda(s,\chi)$$

for the L-function (finally!) follows.

3.6. **Proof of local multiplicity one.** We shall sketch the proof of Theorem 1 for non-archimedean k. To study $\mathcal{S}_k'$ we shall use the exact sequence

$$0\longrightarrow C_c^\infty(k^\times)\longrightarrow\mathcal{S}_k\longrightarrow\mathbb{C}\longrightarrow 0,$$

where the last arrow is evaluation at 0. The space $C_c^\infty(k^\times)$ is, by definition, the subspace of $\mathcal{S}_k$ consisting of functions supported away from 0. This gives rise to the exact sequence on distributions

$$0\longrightarrow(\mathcal{S}_k')_0\longrightarrow\mathcal{S}_k'\longrightarrow C_c^\infty(k^\times)'\longrightarrow 0,$$

where $(\mathcal{S}'_k)_0$ is the subspace of tempered distributions supported at the origin. The map $\mathcal{S}'_k \longrightarrow C_c^\infty(k^\times)'$ is restriction to functions supported away from the origin. Since k is non-archimedean, we have $(\mathcal{S}'_k)_0 = \mathbb{C}\delta_0$.

We now take ω-equivariants. We get

$$0 \longrightarrow (\mathcal{S}'_k)_0(\omega) \longrightarrow \mathcal{S}'_k(\omega) \longrightarrow C_c^\infty(k^\times)'(\omega).$$

Now both the left and right members of the above exact sequence are easy to describe. Indeed, $(\mathcal{S}'_k)_0(\omega) = \mathbb{C}\delta_0$ when ω is the trivial character ω_0 and $(\mathcal{S}'_k)_0(\omega) = 0$ otherwise. Moreover,

$$C_c^\infty(k^\times)'(\omega) = \mathbb{C}\omega(x)d^\times x.$$

When $\omega = \omega_0$ is trivial we obtain

$$0 \longrightarrow \mathbb{C}\delta_0 \longrightarrow \mathcal{S}'_k(\omega_0) \longrightarrow \mathbb{C}d^\times x.$$

Let $\lambda \in \mathcal{S}'_k(\omega_0)$. Let $c \in \mathbb{C}$ be such that the restriction of λ to $C_c^\infty(K_v^\times)$ is $cd^\times x$. In particular, applying λ to $\mathbf{1}_{\mathcal{O}_v} - \mathbf{1}_{\mathfrak{p}_v}$ we have

$$\langle \lambda, \mathbf{1}_{\mathcal{O}_v} - \mathbf{1}_{\mathfrak{p}_v} \rangle = c\langle d^\times x, \mathbf{1}_{\mathcal{O}_v} - \mathbf{1}_{\mathfrak{p}_v} \rangle = c\int_{\mathcal{O}_v^\times} d^\times x.$$

But, by the $k^\times$-invariance of λ, we have $\langle \lambda, \mathbf{1}_{\mathcal{O}_v} \rangle = \langle \lambda, \mathbf{1}_{\mathfrak{p}_v} \rangle$, so $c = 0$.

When ω is non-trivial, we already know that $\mathcal{S}'_k(\omega)$ is non-zero, since we have constructed a non-zero element in it, in Proposition 3.3. So it remains to show that the restriction map $\mathcal{S}'_k(\omega) \to \mathbb{C}\omega(x)d^\times x$ is injective. Let us examine this for ω ramified, leaving the case of ω non-trivial, unramified to the reader. Let $\lambda \in \mathcal{S}'_k(\omega)$ be such that λ restricts to $C_c^\infty(K_v^\times)$ as $c\omega(x)d^\times x$, for some $c \in \mathbb{C}$. We want to show that c determines λ on all of $\mathcal{S}_k$. Following similar reasoning to the proof of Proposition 3.3, we can show that λ is zero on the characteristic function of any $\mathfrak{p}^t$. Now let $f \in \mathcal{S}_k$ be arbitrary. Then by the local constancy of f, for t sufficiently large we have

$$\langle \lambda, f \rangle = \langle \lambda, f - f(0)\mathbf{1}_{\mathfrak{p}^t} \rangle = c\int_{K_v^\times - \mathfrak{p}^t} f(x)\omega(x)d^\times x,$$

as desired. $\square$