Lecture 4: Automorphic forms on real Lie groups and adelization

Contents

1. Left-invariant differential operators

In the last lecture, we relied heavily on the left-invariant differential operators $\mathbf{L}, \mathbf{R}, \mathbf{H}$ on G, whose explicit definitions were taken out of thin air. We now reinterpret them naturally, using the infinitesimal action of the Lie algebra.

1.1. **Left derivation.** If $h \in G$ and f is a function on G then we write ${}^h f$ for the function ${}^h f(g) = f(hg)$. If D is a differential operator on G, then we write ${}^h D$ for the differential operator $({}^h D)(f)(g) = (D\,{}^h f)(h^{-1}g)$. We say that D is left-G-invariant if ${}^h D = D$ for all $h \in G$. We denote by $\mathcal{D}(G)$ the algebra of left-invariant differential operators on G.

Let $\mathfrak{g}_0 = \mathrm{Lie}(G) = \mathfrak{sl}_2(\mathbb{R})$ be the Lie algebra of G and denote by $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{sl}(\mathbb{C})$ its complexification. For $X \in \mathfrak{g}$, we put, for any $f \in C(G)$,

$$\mathcal{L}_X f(g) = \lim_{t\to 0} \left.\frac{f(ge^{tX}) - f(g)}{t}\right|_{t=0} = \frac{d}{dt} f(g\exp(tX))|_{t=0}. \tag{1.1}$$

Example 1. Let

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then W generates the Lie algebra of K. It will not be surprising that

$$\mathcal{L}_W = \frac{\partial}{\partial\theta}.$$

Indeed, $\exp(tW) = k_t$ so that, using our Iwasawa coordinates, we have

$$g\exp(tW) = \begin{pmatrix} uy^{-1/2} & \\ & uy^{-1/2} \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_{\theta+t},$$

from which the claim follows.

Then $\mathcal{L}_X$ is a derivation on functions on G; it is left-G-invariant, since

$${}^h\mathcal{L}_X f(g) = \frac{d}{dt}({}^h f)(h^{-1}g\exp(tX))|_{t=0} = \frac{d}{dt} f(g\exp(tX))|_{t=0} = \mathcal{L}_X f(g).$$

We have

$$\mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]},$$

which could in fact be taken as the definition of the Lie bracket $[X, Y]$. Thus, when $\mathcal{D}(G)$ is given its natural structure as a Lie algebra, the linear map

$$\mathcal{L} : \mathfrak{g} \to \mathcal{D}(G) \tag{1.2}$$

is a Lie algebra homomorphism.

1.2. **Recovering L, R, H.** We shall now show that the operators $\mathbf{L}$, $\mathbf{R}$, $\mathbf{H}$ from the last section arise from the action of the Lie algebra $\mathfrak{g}$ on $C^1(G)$ as left-invariant derivations. First we give ourselves the basis

$$H_+ = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad R_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

of $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$, which verifies the following bracket relations

$$[H_+, R_+] = 2R_+, \quad [H_+, L_+] = -2L_+, \quad [R_+, L_+] = H_+. \tag{1.3}$$

Let

$$c = -\frac{1+i}{2} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

be the Cayley matrix giving rise to the map

$$\mathbb{H} \to \mathbb{D}, \; z \mapsto \frac{z-i}{z+i} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.z = c.z.$$

Conjugating by c we get another triple $c^{-1}(H_+, R_+, L_+)c = (H, R, L)$, where

$$H = -iW, \quad R = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad L = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

The triple (H, L, R) is a $\mathbb{C}$-basis of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and satisfies the same commutativity relations as (H_+, L_+, R_+).

Lemma 1.1. *We have $\mathcal{L}_H = \mathbf{H}$, $\mathcal{L}_L = \mathbf{L}$, $\mathcal{L}_R = \mathbf{R}$.*

Proof. We have already calculated $\mathcal{L}_H$. For the others, notice that we only have to prove the formula for $\mathcal{L}_L$, since L and R are complex conjugate.

Notice that $L = -iR_+ + \frac{i}{2}W + \frac{1}{2}H_+$. We leave it as an exercise (see Bump, *Automorphic forms and representations*, Proposition 2.2.5) to prove the following formulae, in Iwasawa coordinates:

$$\mathcal{L}_{R_+} = y \cos 2\theta \frac{\partial}{\partial x} + y \sin 2\theta \frac{\partial}{\partial y} + \sin^2 \theta \frac{\partial}{\partial \theta}$$
$$\mathcal{L}_{H_+} = -2y \sin 2\theta \frac{\partial}{\partial x} + 2y \cos 2\theta \frac{\partial}{\partial y} + \sin 2\theta \frac{\partial}{\partial \theta}$$

Inserting everything gives the stated formula for $\mathcal{L}_L$. $\square$

2. Higher order derivatives

Higher order left-invariant differential operators do not come from the Lie algebra of G but rather the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$. This algebra of operators can be realized by an extension of the rule (1.1) to the *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

This is an associative algebra, along with a linear map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$, which is a Lie algebra homomorphism when $\mathcal{U}(\mathfrak{g})$ is endowed with its natural bracket, satisfying the following universal property: for any linear map $\varphi : \mathfrak{g} \to A$ into an associative algebra, which is a Lie algebra homomorphism when A is endowed with the bracket $[x, y] = xy - yx$, there is a unique algebra homomorphism $h : \mathcal{U}(\mathfrak{g}) \to A$ such that φ factors through h via i. We can construct $\mathcal{U}(\mathfrak{g})$ explicitly as the tensor algebra

$$T^{\bullet}(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$$

of $\mathfrak{g}$ modulo the ideal generated by the relations $X \otimes Y - Y \otimes X - [X, Y]$; we then take i to be the natural map $\mathfrak{g} \to T^{\bullet}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$.

We denote the product of two elements $X, Y \in \mathcal{U}(\mathfrak{g})$ by $X \circ Y$, or more simply by XY if the context is clear.

Theorem 2.1 (Poincaré–Birkhoff–Witt)**.** *For any ordered basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, the lexographically ordered monomials $X_1^{a_1} X_2^{a_2} X_3^{a_3}$, with integers $a_i \geq 0$, form a basis for $\mathcal{U}(\mathfrak{g})$. In particular, the natural map $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.*

From the universal property, we may extend the Lie algebra homomorphism $\mathcal{L} : \mathfrak{g} \to \mathcal{D}(G)$ from (1.2) to an algebra homomorphism

$$\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{D}(G) \tag{2.1}$$

under which $X \in \mathfrak{g}$ is sent to $\mathcal{L}_{i(X)}$. The homomorphism (2.1) is, in fact, an isomorphism (see Knapp, *Representation theory of semisimple groups*, Theorem 3.6).

2.1. Casimir operator.

Let $\mathfrak{g}_0^* = \mathrm{Hom}(\mathfrak{g}_0, \mathbb{R})$ be vector space dual of $\mathfrak{g}_0 = \mathfrak{sl}_2(\mathbb{R})$. We identity $\mathfrak{g}_0^*$ with $\mathfrak{g}_0$ via the Killing form on $\mathfrak{g}_0$:

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}X \circ \mathrm{ad}Y) = 4\mathrm{Tr}(X.{}^tY), \tag{2.2}$$

a non-degenerate bilinear form. Let $\{H_+^*, L_+^*, R_+^*\}$ be the dual basis of $\{H_+, L_+, R_+\}$ in $\mathfrak{g}_0$ relative to B. Then the element

$$\Omega = 2(H_+ H_+^* + L_+ L_+^* + R_+ R_+^*) \in \mathcal{U}(\mathfrak{g}) \tag{2.3}$$

is called the *Casimir element*.

Now let $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ be the *center* of the universal enveloping algebra.

Lemma 2.1. *We have $\Omega \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$.*

Proof. We do this by direct calculation. Firstly, one calculates $H_+^* = \frac{1}{8}H_+$, $R_+^* = \frac{1}{4}L_+$ and $L_+^* = \frac{1}{4}R_+$. Thus we have

$$\Omega = \frac{1}{4}(H_+^2 + 2R_+L_+ + 2L_+R_+).$$

By the Poincaré–Birkhoff–Witt theorem it suffices to show that Ω commutes with L_+, R_+, and H_+. This will follow from the commutation relations (1.3). For example, to show Ω commutes with L_+, we compute the following relations in $\mathcal{U}(\mathfrak{g})$:

$$\begin{aligned} H_+^2L_+ = H_+(L_+H_+ - 2L_+) &= H_+L_+H_+ - 2H_+L_+ \\ &= (L_+H_+ - 2L_+)H_+ - 2H_+L_+ \\ &= L_+H_+^2 - 2L_+H_+ - 2H_+L_+ \end{aligned}$$

and

$$\begin{aligned} (R_+L_+)L_+ &= (L_+R_+ + H_+)L_+ = L_+(R_+L_+) + H_+L_+ \\ (L_+R_+)L_+ &= L_+(L_+R_+ + H_+) = L_+(L_+R_+) + L_+H_+, \end{aligned}$$

showing that $\Omega L_+ = L_+\Omega$. The other computations are similar. □

Remark 1. We have taken a somwhat *ad hoc* approach above. In particular, we have not shown that the definition of Ω is independent of the chosen basis, and the verification that $\Omega \in \mathcal{Z}(\mathcal{U}(g))$ is clearly too computational. Here is a better viewpoint.

We have an isomorphism of vector spaces $\mathrm{End}(\mathfrak{g}_0) \simeq \mathfrak{g}_0 \otimes \mathfrak{g}_0^*$ induced by the bilinear map $\mathfrak{g}_0 \times \mathfrak{g}_0^* \to \mathrm{End}(\mathfrak{g}_0)$ sending (X, ℓ) to $f(x) = \ell(x)X$. If we choose a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}_0$ and let $\{\ell_1, \ell_2, \ell_2\}$ denote the dual basis of $\mathfrak{g}_0^*$, then the identity endomorphism $\mathrm{id}_{\mathfrak{g}_0}$ is sent to $\sum_i X_i \otimes \ell_i$, making this last expression independent of the chosen basis. The Killing form B induces an isomorphism $\mathfrak{g}_0 \simeq \mathfrak{g}_0^*$ so that $\mathrm{End}(\mathfrak{g}_0) \simeq \mathfrak{g}_0 \otimes \mathfrak{g}_0$. These maps are in fact isomorphisms as $\mathfrak{g}_0$-modules, since the Killing form is Ad-invariant.

Now $\mathfrak{g}_0 \otimes \mathfrak{g}_0$ maps to $\mathcal{U}(\mathfrak{g})$ by sending the tensor $X \otimes Y$ to XY, and the image of $\mathrm{id}_{\mathfrak{g}_0}$ under the composition $\mathrm{End}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g})$ is seen to be the Casimir. If we endow $\mathcal{U}(\mathfrak{g})$ with the $\mathfrak{g}_0$-module structure given by $X.A = XA - AX$, where $X \in \mathfrak{g}_0$ and $A \in \mathcal{U}(\mathfrak{g})$, then $\mathfrak{g}_0 \otimes \mathfrak{g}_0 \to \mathcal{U}(\mathfrak{g})$ is a $\mathfrak{g}_0$-module homomorphism. Under the composition $\mathrm{End}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g})$ the identity $\mathrm{id}_{\mathfrak{g}_0}$ clearly maps to the center.

Since Ω is central, it is fixed under conjugation by $\mathrm{SL}_2(\mathbb{C})$. It is fixed, in particular, under conjugation by the Cayley matrix, which sends the ordered basis (H_+, R_+, L_+) to (H, R, L). We deduce that

$$\Omega = \frac{1}{4}(H^2 + 2RL + 2LR). \tag{2.4}$$

Recall from the last lecture that

$$\mathbf{\Delta} = \frac{1}{4}\left(\mathbf{H} \circ \mathbf{H} + 2(\mathbf{R} \circ \mathbf{L}) + 2(\mathbf{L} \circ \mathbf{R})\right).$$

It follows from this and Lemma 1.1 that the Laplace-Beltrami operator $\boldsymbol{\Delta}$ is realized as the image of Ω under the isomorphism (2.1). We also see, using the definition (2.3), that the non-ellipticity of Δ is a reflection of the fact that the Killing form on $\mathfrak{g}_0$ is not definite (the Killing form is only definite for the Lie algebras of *compact* Lie groups).

2.2. **The Harish-Chandra homomorphism.** Let

$$\mathfrak{h} = \mathbb{C}H, \quad \mathfrak{u} = \mathbb{C}R, \quad \text{and} \quad \bar{\mathfrak{u}} = \mathbb{C}L,$$

so that $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{h} \oplus \mathfrak{u}$. Let $\mathcal{U}(\mathfrak{h})$ denote the universal enveloping of $\mathfrak{h}$, a commutative algebra, since the Lie bracket is trivial on $\mathfrak{h}$.

Lemma 2.2. *We have*

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathcal{U}(\mathfrak{g})\mathfrak{u} + \bar{\mathfrak{u}}\mathcal{U}(\mathfrak{g})). \tag{2.5}$$

Proof. We apply the Poincaré–Birkhoff–Witt theorem, with respect to the ordered basis $\{L, H, R\}$, to obtain $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) + (\mathcal{U}(\mathfrak{g})\mathfrak{u} + \bar{\mathfrak{u}}\mathcal{U}(\mathfrak{g}))$. We must show that the sum is direct.

Firstly, an arbitrary element of $\mathcal{U}(\mathfrak{g})\mathfrak{u}$ is a linear combination of terms of the form $(L^a H^b R^c)R = L^a H^b R^{c+1}$, with $a, b, c \geq 0$. Similarly an arbitrary element of $\mathcal{U}(\mathfrak{g})\mathfrak{u}$ is a linear combination of PWB basis elements $L^{a+1}H^b R^c$, with $a, b, c \geq 0$. Thus, any element in $\mathcal{U}(\mathfrak{h}) \oplus (\mathcal{U}(\mathfrak{g})\mathfrak{u} + \bar{\mathfrak{u}}\mathcal{U}(\mathfrak{g}))$ is a linear combination of monomials $L^a H^b R^c$, with $a, c \geq 1$. By contrast, an arbitrary element of $\mathcal{U}(\mathfrak{h})$ is a polynomial in H; each power H^b can be written in the PWB basis as $L^a H^b R^c$, with $a = c = 0$. This shows that the intersection is reduced to zero, and the sum is direct. □

Let $\gamma : \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \to \mathcal{U}(\mathfrak{h})$ denote the restriction to $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ of the natural projection with respect to the decomposition in Lemma 2.2.

We will be interested in a shift of γ by the element ρ of $\mathfrak{h}^*$, defined (in this case) as the unique linear functional sending H to 1. Define a linear map

$$\sigma : \mathfrak{h} \to \mathcal{U}(\mathfrak{h}), \qquad h \mapsto h - \rho(h).1. \tag{2.6}$$

Then by the universal property of $\mathcal{U}(\mathfrak{h})$, we may extend σ to an algebra homomorphism from $\mathcal{U}(\mathfrak{h})$ to itself, again denoted by σ.

Definition 1. The **Harish-Chandra homomorphism** is the composition

$$\gamma_{HC} = \sigma \circ \gamma : \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \to \mathcal{U}(\mathfrak{h}).$$

Remark 2. The Harish-Chandra homomorphism γ_{HC} is indeed an algebra homomorphism. To see this, it suffices to check that γ is multiplicative. For $z_1, z_2 \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ we have (using that z_1 lies in the center and $\mathcal{U}(\mathfrak{h})$ is commutative)

$$\begin{aligned} z_1 z_2 - \gamma(z_1)\gamma(z_2) &= z_1 z_2 - z_1\gamma(z_2) + z_1\gamma(z_2) - \gamma(z_1)\gamma(z_2) \\ &= z_1(z_2 - \gamma(z_2)) + \gamma(z_2)(z_1 - \gamma(z_1)). \end{aligned}$$

From the definition of γ, the first term in $\mathcal{U}(\mathfrak{g})\mathfrak{u} + \bar{\mathfrak{u}}\mathcal{U}(\mathfrak{g})$.

For the second term we observe that $z_1 - \gamma(z_1)$ (and thus the product $\gamma(z_2)(z_1 - \gamma(z_1)))$ lies in $\mathcal{U}(\mathfrak{g})\mathfrak{u}$. To see this, we use the ad action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$ given by $\mathrm{ad}(X)D = XD - DX$, where $X \in \mathfrak{g}$ and $D \in \mathcal{U}(\mathfrak{g})$. Let $z \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ be arbitrary. We expand z in the basis of ordered monomials $L^a H^b R^c$. Using the commutation relations (1.3), the action of $\mathrm{ad}(H)$ on this monomial is $-2a + 2c$. Since $\mathrm{ad}(H)(z) = Hz - zH = 0$, it follows that $a = c$. Therefore, whenever L appears in such a monomial, so does R. This means that $\gamma(z) \in \mathcal{U}(\mathfrak{z}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{u}$, so that $z - \gamma(z) \in \mathcal{U}(\mathfrak{g})\mathfrak{u}$ as claimed.

From the above we deduce that $z_1 z_2 - \gamma(z_1)\gamma(z_2) \in \mathcal{U}(\mathfrak{g})\mathfrak{u} + \bar{\mathfrak{u}}\mathcal{U}(\mathfrak{g})$. Since $\gamma(z_1)\gamma(z_2) \in \mathcal{U}(\mathfrak{h})$, using the definition of γ again (and uniqueness of the direct sum decomposition), we see that the desired multiplicative property $\gamma(z_1 z_2) = \gamma(z_1)\gamma(z_2)$ holds.

Example 2. We calculate the image of γ_{HC} of the Casimir operator Ω. For this we must first write Ω in the form required by the Poincaré–Birkhoff–Witt theorem, with respect to the *ordered* basis $\{L, H, R\}$: using the commutativity relation $RL = LR + H$, and recalling the expression (2.4), we have

$$\Omega = \frac{1}{4}(H^2 + 2H + 4LR).$$

Thus γ sends Ω to $\frac{1}{4}(H^2 + 2H)$. Now, recalling the map (2.6), we have

$$\sigma(H^2+2H) = \sigma(H)^2+2\sigma(H) = (H-1)^2+2(H-1) = (H-1)(H+1) = H^2-1.$$

Thus

$$\gamma_{HC}(\Omega) = \frac{1}{4}(H^2 - 1). \tag{2.7}$$

Notice that the latter expression is invariant under $H \mapsto -H$, which is just the action of the non-trivial element of the Weyl group W of $\mathfrak{h}$. In other words $\gamma_{HC}(\Omega) \in \mathcal{U}(\mathfrak{h})^W$. The shift by $\rho(H)$ is what allowed us to obtain a Weyl group invariant polynomial on $\mathfrak{h}$.

The following is one of many famous theorems of Harish-Chandra.

Theorem 2.2 (Harish-Chandra)**.** *The homomorphism γ_{HC} is an algebra isomorphism onto $\mathcal{U}(\mathfrak{h})^W$.*

Remark 3. Notice that $\gamma : \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \to \mathfrak{h}$ depends on the choice of a choice of positivity in the root system, which is reflected in the ordering of the basis of $\mathfrak{g}$. Part of the Harish-Chandra's theorem is that the shifted map γ_{HC} is independent of this choice.

Remark 4. Since $\mathcal{U}(\mathfrak{h})$ is the symmetric algebra $S(\mathfrak{h})$ of $\mathfrak{h}$, and the latter can be identified with the polynomial algebra on $\mathfrak{h}^*$, the above theorem states that the center of $\mathcal{U}(\mathfrak{g})$ is sent isomorphically onto the Weyl group invariant polynomials on $\mathfrak{h}^*$.

Proof. That γ_{HC} lies in the Weyl group invariants is the statement that, for all $z \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ and $\lambda \in \mathfrak{h}^*$, we have

$$\gamma(z)(\lambda - \rho) = \gamma(z)(w\lambda - \rho), \tag{2.8}$$

where, as in Remark 4, we are viewing $\gamma(z) \in \mathcal{U}(\mathfrak{h}) = S(\mathfrak{h})$ as a polynomial function on $\mathfrak{h}^*$. As this is an equality on polynomials $\gamma(z)$, it is enough to show this for integral dominant weights λ. Recall that $\lambda \in \mathfrak{h}^*$ is integral dominant if it restricts to real values on $\mathfrak{h}_0$ and satisfies $\lambda(H_+) \in \mathbb{Z}_{\geq 0}$.

To show (2.8), we admit some knowledge of the theory of Verma modules (see Knapp, *Representation theory of semisimple groups*, Ch IV, §8). These are infinite dimensional cyclic highest weight modules for $\mathcal{U}(\mathfrak{g})$, denoted $V(\lambda)$ for $\lambda \in \mathfrak{h}^*$, with H acting by the scalar $(\lambda - \rho)(H)$ on the highest weight vector. Their quotients (for integral dominant λ) are used to construct all the finite dimensional representations V_λ of G.

Now any $z \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$ acts by a scalar on the Verma module $V(\lambda)$. Since $z - \gamma(z) \in \mathcal{U}(\mathfrak{g})\mathfrak{u}$ (see Remark 2) and $\mathfrak{u}$ kills the highest weight vector, one may evaluate this scalar as $\gamma(z)(\lambda - \rho)$, the left-hand side of (2.8). The statement then follows from the inclusion $V(w\lambda) \subset V(\lambda)$ (see Knapp, Ch. IV, Lemma 4.40), which uses the assumption of λ dominant integral.

The surjectivity is clear. Indeed, since γ_{HC} is an algebra homomorphism, and the Weyl group invariants of $\mathcal{U}(\mathfrak{h})$ are just the polynomials in H^2, it suffices to show that H^2 has a preimage under γ_{HC}. From (2.7) such a preimage is given by $4\Omega + 1$.

Finally, to prove injectivity, it is enough to show that if $\gamma_{HC}(z) = 0$ then z acts by 0 as a differential operator on $C^\infty(\mathrm{SU}(2))$, since the algebra homomorphism $\mathcal{U}(\mathfrak{g}) \to D(\mathrm{SU}(2))$ from (2.1) (here we use the fact that $\mathfrak{g}$ is also the Lie algebra of the compact group SU(2)) is an isomorphism. To show this, by the Peter-Weyl theorem, it is enough to show that z acts by zero as a differential operator on matrix coefficients of irreducible finite dimensional representations of SU(2). Let (π_λ, V_λ) be the irreducible representation associated with the integral dominant weight λ as above (see Knapp, Ch IV, §7). Since $\gamma(z)(\lambda - \rho)$ is the scalar by which z acts on V_λ, the assumption is that $\pi_\lambda(z)$ acts by zero. The formula

$$z.(\pi_\lambda(g)v_1, v_2) = (\pi_\lambda(g)\pi_\lambda(z)v_1, v_2)$$

then shows that z kills the matrix coefficients of V_λ, as desired. □

Remark 5. In the more general statement and proof of the above fact, the surjectivity of γ_{HC} becomes much harder.

We had already established, in Lemma 2.1, that $\mathbb{C}[\Omega] \subset \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$. We may now upgrade this to an equality.

Corollary 1. *We have* $\mathcal{Z}(\mathcal{U}(\mathfrak{g})) = \mathbb{C}[\Omega]$.

Proof. From (2.7) it follows that $\gamma_{HC}(\Omega)$ generates $\mathcal{U}(\mathfrak{h})^W = \mathrm{Im}\,\gamma_{HC}$. Since γ_{HC} is injective, it follows that Ω generates $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$. □

3. Automorphic forms on G

We arrive at the definition of an automorphic form on G.

Definition 2 (K-finite automorphic forms)**.** Let Γ be a lattice in G. A smooth function $\phi : G \to \mathbb{C}$ is called a Γ-*automorphic form* if the following conditions are satisfied

(1) ϕ is left-Γ-invariant,
(2) ϕ is right-K-finite (the space of right translates of ϕ by elements in $K = \mathrm{SO}(2)$ is finite dimensional),
(3) ϕ is $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$-finite (the space of translates of ϕ by Ω is finite dimensional)
(4) ϕ is of moderate growth (there is $A \in \mathbb{R}$ such that $\phi(g_{\mathfrak{a}} g) \ll y^A$, where $g.i = x + iy$ and $g_{\mathfrak{a}}(\infty) = \mathfrak{a}$)

We denote by $\mathcal{A}(\Gamma)$ the space of all Γ-automorphic forms.

Remark 6. From the K-finite and $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$-finite conditions it follows that automorphic forms are real analytic.

Definition 3. A Γ-automorphic form ϕ is called *cuspidal* if the constant term of ϕ along every $\mathbb{Q}$-Borel subgroup of G vanishes almost everywhere. The space of all cuspidal Γ-automorphic forms is denoted $\mathcal{A}_0(\Gamma)$.

4. Adelization of automorphic forms

Under the assumption that Γ is a *congruence* subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we may view automorphic forms in $\mathcal{A}(\Gamma)$ as automorphic forms on an adelic quotient. We briefly examine this passage here.

4.1. The additive group.

We recall strong approximation for the additive group $\mathbb{G}_a$.

Theorem 4.1. *$\mathbb{Q}$ is dense in $\mathbb{A}_f$*

Proof. This is a slight upgrade of the Chinese Remainder Theorem. Recall that a basis of open sets of $\mathbb{A}_f$ is given by $V_S \times \prod_{p \notin S} \mathbb{Z}_p$, where $V_S \subset \mathbb{Q}_S$ is open, as S varies over finite subsets of the primes. Density of $\mathbb{Q}$ inside $\mathbb{A}_f$ (our goal here) is then the statement that $\mathbb{Q}$ has non-empty intersection with every $V_S \times \prod_{p \notin S} \mathbb{Z}_p$. By killing denominators, we can make this last property integral (at the cost of making the archimedean valuation large): $\mathbb{Z}$ has non-empty intersection with every $U_S \times \prod_{p \notin S} \mathbb{Z}_p$ for an open subset U_S of $\mathbb{Z}_S$. We can take U_S to be a tuple of $\epsilon_p > 0$ neighborhoods about a point $x_p \in \mathbb{Z}_p$. We know that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ for every $p \in S$, so we can find $y_p \in \mathbb{Z}$ such that y_p is in the ϵ_p neighborhood of x_p. Let $n \in \mathbb{N}$ be such that $p^n > y_p$ for all $p \in S$. Then the Chinese remainder theorem gives $y \in \mathbb{Z}$ such that $y \equiv y_p \mod p^n$ for all $p \in S$. Therefore y itself lies in the prescribed neighborhood. $\square$

4.2. **The group** SL_2**.** We now show the analogous statement for SL_2. Recall that the topology of $\mathrm{SL}_2(\mathbb{A}_f)$ admits a basis of open sets of the origin given by

$$U_S \times \prod_{p \notin S} \mathrm{SL}_2(\mathbb{Z}_p),$$

where $U_S \subset \mathrm{SL}_2(\mathbb{Q}_S)$ is open and S varies among finite sets of primes of $\mathbb{Q}$.

Theorem 4.2. $\mathrm{SL}_2(\mathbb{Q})$ *is dense in* $\mathrm{SL}_2(\mathbb{A}_f)$.

Proof. Let $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ be the closure of $\mathrm{SL}_2(\mathbb{Q})$ in $\mathrm{SL}_2(\mathbb{A}_f)$. We of course want to show that $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q})) = \mathrm{SL}_2(\mathbb{A}_f)$.

Note that $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ contains $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Z})) = \mathrm{SL}_2(\hat{\mathbb{Z}})$. It thus suffices to show that $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ contains $\mathrm{SL}_2(\mathbb{Q}_p)$ (naturally embedded in $\mathrm{SL}_2(\mathbb{A}_f)$), for every p. Indeed in that case $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ contains $\mathrm{SL}_2(\mathbb{Q}_S)$, for any finite set of primes S. Thus $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ contains $\mathrm{SL}_2(\mathbb{Q}_S) \times \prod_{p \notin S} \mathrm{SL}_2(\mathbb{Z}_p)$ for any S. As the latter cover $\mathrm{SL}_2(\mathbb{A}_f)$ we are done.

Recall that $\mathrm{SL}_2(k)$ for k any field is generated by the upper and lower unipotent triangular subgroups $U_\pm$. Thus it suffices to prove that $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ contains $U_\pm(\mathbb{Q}_p)$. Now $\mathrm{cl}(\mathrm{SL}_2(\mathbb{Q}))$ contains the closure of $U_\pm(\mathbb{Q})$, so we are reduced to showing that the closure of $U_\pm(\mathbb{Q})$ inside $U_\pm(\mathbb{A}_f)$ contains $U_\pm(\mathbb{Q}_p)$. But $U_\pm$ are isomorphic to $\mathbf{G}_a$, the classical strong approximation says that $\mathrm{cl}(U_\pm(\mathbb{Q})) = U_\pm(\mathbb{A}_f)$, and we're done. □

Corollary 2. *Let* K_f *be a compact open subgroup of* $\mathrm{SL}_2(\mathbb{A}_f)$. *Put* $\Gamma = \mathrm{SL}_2(\mathbb{Q}) \cap K_f$. *Then the embedding of* $\mathrm{SL}_2(\mathbb{R})$ *into* $\mathrm{SL}_2(\mathbb{A})$ *induces a homeomorphism*

$$\Gamma \backslash \mathrm{SL}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}) / K_f.$$

Proof. We first prove that the map $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{Q}) \backslash \mathrm{SL}_2(\mathbb{A}) / K_f$, in which g_∞ is sent to its double class, is surjective. Take an element $(g_f K_f, g_\infty)$ in $(\mathrm{SL}_2(\mathbb{A}_f)/K_f) \times \mathrm{SL}_2(\mathbb{R})$. Strong approximation states that there is $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that $\gamma K_f = g_f K_f$. Thus $\gamma^{-1}.(g_f K_f, g_\infty) = (K_f, \gamma^{-1} g_\infty)$: every double class is therefore represented by a pair in $\mathrm{SL}_2(\mathbb{A}_f) \times \mathrm{SL}_2(\mathbb{R})$ with 1 in the first coordinate. This is the desired surjectivity.

Now we prove that if $g_\infty, g'_\infty \in \mathrm{SL}_2(\mathbb{R})$ are sent to the same double class, then they differ by a left-translate of Γ. Indeed, if $g'_\infty = \gamma g_\infty k$, where $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ and $k \in K_f$, then writing $\gamma = (\gamma_f, \gamma_\infty)$ as an element in $\mathrm{SL}_2(\mathbb{A}_f) \times \mathrm{SL}_2(\mathbb{R})$ we have $g'_\infty = \gamma_\infty g_\infty$ and $\gamma_f = k^{-1}$. Thus $\gamma \in K_f \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma$, as desired. □

Note that if $K_f = \mathrm{SL}_2(\widehat{\mathbb{Z}})$ then $\mathrm{SL}_2(\mathbb{Q}) \cap K_f = \mathrm{SL}_2(\mathbb{Z})$. More generally, the Γ which arise as intersections $\mathrm{SL}_2(\mathbb{Q}) \cap K_f$, where K_f is a compact open subgroup of $\mathrm{SL}_2(\mathbb{A}_f)$, are precisely the congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

4.3. **Adelic automorphic forms.** We now give a definition for adelic automorphic forms on a connected reductive group G over $\mathbb{Q}$, such as $G = \mathrm{SL}_2$, GL_2, or $B^\times$, where B is the unit group of a quaternion algebra defined over $\mathbb{Q}$. Note that GL_2 is a special case of $B^\times$, when B is the split matrix algebra.

Definition 4. A complex valued function ϕ on $G(\mathbb{A})$ is said to be an automorphic form if the following conditions are satisfied

(1) ϕ is smooth ;
(2) ϕ is left-invariant under $G(\mathbb{Q})$;
(3) ϕ admits a central character;
(4) ϕ is right-invariant under an open compact subgroup of $G(\mathbb{A}_f)$;
(5) ϕ is right K_∞-finite;
(6) ϕ is $\mathcal{Z}(\mathcal{U}(\mathfrak{g}))$-finite;
(7) ϕ is of moderate growth.

Remark 7. We comment on some of the above conditions.

The smoothness condition in (1) has the standard meaning at the real place, and means locally constant at the finite places.

The central character condition in (3) means that there is a continuous unitary character $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ such that $\phi(zg) = \omega(z)\phi(g)$, for all $g \in G(\mathbb{A})$, $z \in Z(\mathbb{A})$.

The group K_∞ in condition (5) is any choice of maximal compact subgroup of $G(\mathbb{R})$.

The final condition on moderate growth has two types of formulations, either through Siegel sets (in the style of condition (4) in Definition 2), or by using a matrix norm. We shall not go into this.